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UNDERSTANDING WHAT YOU ARE DOING: A NEW ANGLE ON CAS?*

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Powerful Computer Algebra Systems (CAS) are often used only with reluctance in early undergraduate mathematics teaching, partly because of concerns that they may not encourage students to understand what they are doing. In this exploratory study, a version of a CAS that has been designed for secondary school students was used, with a view to considering the value of this sort of student learning support for first year undergraduate students enrolled in degree programs other than mathematics. Workshops were designed to help students understand aspects of elementary symbolic manipulation, through the use of the Algebra mode of an algebraic calculator, the Casio Algebra FX 2.0. The Algebra mode of this calculator allows a user to undertake elementary algebraic manipulation, routinely providing all intermediate results, in contrast to more powerful CAS software, which usually provides simplified results only. The students were volunteers from an introductory level unit, designed to provide a bridge between school and university studies of mathematics and with a focus on algebra and calculus. The two structured workshop sessions focussed respectively on the solution of linear equations and on relationships between factorising and expanding; attention focussed on using the calculators as personal learning devices. Following the workshops, structured interviews were used to systematically record student reactions to the experience. As a result of the study, the paper offers advice on the merits of using algebraic calculators in this sort of way.

Computer algebra systems (CAS's) have been available on computers for at least two decades and on hand-held calculators for around a decade, and yet have not been widely accepted into the early undergraduate or senior secondary years of schooling.

A possible reason for this slow introduction of a powerful new technology may be an ambivalence among mathematics teachers regarding the place of symbolic manipulation in elementary algebra. Indeed, as Heid and Edwards note, "The sheer power of CAS may have slowed their introduction into mathematics instruction." [2, p.135]. Once the necessary syntax has been learned, modern CAS's can be used to perform all of the necessary symbolic manipulation expected of unsophisticated students. Many teachers have expressed concern that students will "not really understand what they are doing" if they use a CAS to perform what once was intended to be completed by hand.

Nor are these sentiments restricted to teachers. In a recent study, Povey and Ransom [6] reported similar unease among undergraduate mathematics students regarding various kinds of information technologies. They noted two themes, both of central importance to this paper, one concerned with understanding mathematics and the other with who is in control of the mathematics. In reflecting on the feelings of many students, they noted:

> The implication is that if you don't know 'how the computer did it', then you don't 'understand' and it is out of your control. This particular choice of words is interesting because, of course, computers and calculators do not do calculations the same way that people do. Even when people realise that this phrase is not to be taken literally, they can find it hard to let it go because it serves as a metaphor for 'understanding'.[6, p.57]

Most concerns with CAS seem to be related to the issue of whether students ought be permitted to use it in their everyday mathematical work. Less extensive uses of CAS involve educational or instructional uses, which are the focus of the present paper. In exploring four possible roles for CAS in the secondary school, Heid and Edwards [2] identified one of these as:

> … to create and generate symbolic procedures, giving the user access to symbolic procedures of almost any "chunk size." For example, the CAS can perform traditional symbolic manipulations in a step-by-step fashion, with students issuing commands to transform an equation until it is solved. [2, p.131]

Consistent with this suggestion, Kutzler [5] described in detail how a hand-held CAS system can be used to solve a linear equation with a succession of equivalence transformations. Similarly, Klein and Kertay [3] also described some ways of doing this with simple linear equations and a TI-89 calculator, observing that the most popular approach with their students involved the use of an applet downloaded into the calculator. The purpose of the applet was to allow students to choose various symbolic manipulations for themselves and explore their consequences.

Consistent with the concerns that excessive uses of CAS may even be harmful, educational uses of CAS are concerned with the development of what Pierce [5] has described as 'algebraic insight' and what Arcavi [1] has described as 'symbol sense'. This paper describes an exploratory study of some ways in which a hand-held CAS might be used to help students in these directions.

Using CAS for learning

Most CAS devices have been programmed on the assumption that users are already confident with elementary algebra, and hence tend to automatically provide a simplified version of any algebraic results. Presumably, the reason for this is that authors of CAS programs design the software to produce the results most likely to be useful to the users. If users are mostly advanced undergraduates, or professional users of mathematics, such a strategy seems sensible; however, if the users themselves are not yet fluent with symbolic manipulation techniques or their meanings, the situation may be different.

The present study uses the Casio Algebra FX 2.0 calculator, which is an algebraic calculator (i.e., incorporating a CAS) designed for a relatively unsophisticated audience of secondary school students. A distinct difference provided between this calculator and other CAS devices is that it offers students an 'Algebra' mode, intended to provide details of the symbolic manipulation steps used.

To illustrate the difference, consider the simple linear equation, $3x - 4 = x + 2$. A standard CAS system allows for students to 'do the same thing to both sides' in a systematic manner in order to duplicate the thinking associated with traditional by-hand procedures. Both Kutzler [5] and Klein & Kertay [3] describe such a procedure, used to help beginners understand the solution of simple linear equations. Kutzler [4] describes the use of a CAS for this purpose as 'scaffolding', and outlines a theoretical rationale for it:

> Solving an equation by applying equivalence transformations means the alternation of two tasks: Choosing an equivalence transformation, then applying it to the equation. … The main advantage of the … approach is that students can fully concentrate on the higher level task of choosing an equivalence transformation, then let the calculator apply it, then study its effects. Once they are comfortable with choosing equivalence transformations, they should solve equations by hand. [4, p.25]

Figure 1 shows the kind of facility offered by the calculator in the CAS mode of the Algebra FX 2.0, before and after the operation of subtracting x from each side of the equation has been completed. On the calculator, the user enters only the operation '– x', which the calculator interprets as 'subtract x from the previous Answer', in this case, subtracting x from each side of the equation previously entered:

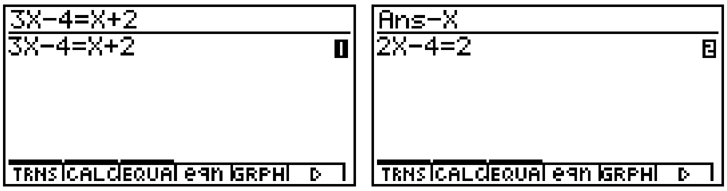

FIGURE 1. Using CAS with automatic simplification

Further steps of adding 4 to each side of the equation, and then dividing each side by 2 yield the conventional solution on the calculator, as shown in Figure 2. The final screen dump conveniently summarises the various solution steps.

FIGURE 2. Completing the solution with automatic simplification

The major focus of this study involves the use of the calculator's Algebra mode, for which automatic simplifications are not provided, and users have to decide for themselves which symbolic steps are to be performed next. To illustrate the difference, consider the same equation as previously, in Algebra mode instead of CAS mode. As Figure 3 illustrates, the calculator responds correctly to the command to subtract x from each side, but leaves the result unsimplified, until the *simplify* command is used.

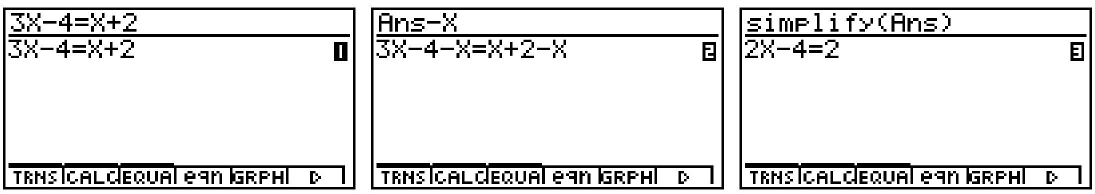

Figure 3: Using Algebra mode to solve an equation

While an experienced user might regard this as unnecessarily tedious, a student struggling with the logic of the process of doing the same thing to both sides of an equation is provided with an opportunity to really see what is happening. The screen dumps in Figure 4 show a final summary of the operations used to solve the equation.

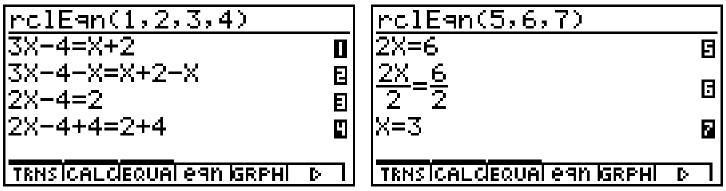

Figure 4: Summary of the solution in Algebra mode

A distinct advantage of performing the manipulations with the calculator is that they will always be correct, even if a poor choice is made. For example, a common misconception is that numbers or variables can be ‘removed’ by ‘taking them away’. The consequence of such a move to solve the equation $2x = 6$, in the final step of the solution of $3x - 4 = x + 2$, can be seen to be incorrect, or at least unhelpful, when the subtraction operation is performed on the calculator, as shown in Figure 5.

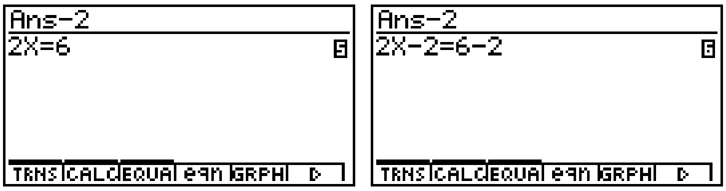

Figure 5: A poor choice of equivalence transformation is accepted and executed

Several authors [2,3,4] regard this inherent capacity of a CAS as a distinct advantage for learners. The particular additional advantage offered in the present context by Algebra mode is that the calculator leaves the result unsimplified, so that the result is completely transparent to the learner, and its limitations clear.

As for equations, a CAS will automatically provide a simplified version of an algebraic expansion. Figure 6 shows the binomial expansion of $(3x - 5)(2x + 7)$ in CAS mode. While this provides an immediate result, suitably simplified in descending order of powers of the variable, it seems unlikely to be of much help to a student who has not yet understood the processes involved. Indeed, such a screen seems to provide no more assistance to a student grappling with this idea than looking up the correct answer in the back of a textbook.

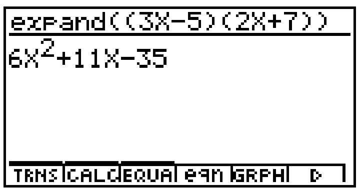

Figure 6: An expansion in CAS mode

In contrast, the first two screen dumps in Figure 7 show that the Algebra mode of the calculator provides an opportunity for students to see precisely what is involved in such an expansion. The simplified version is only available after the user decides to ask the calculator to provide it, as shown in the second screen. In addition, Figure 7 shows that the calculator in Algebra mode uses less conventional algebraic notation (*xx* instead of x^2) in order to make clear what operations are involved in expanding some expressions.

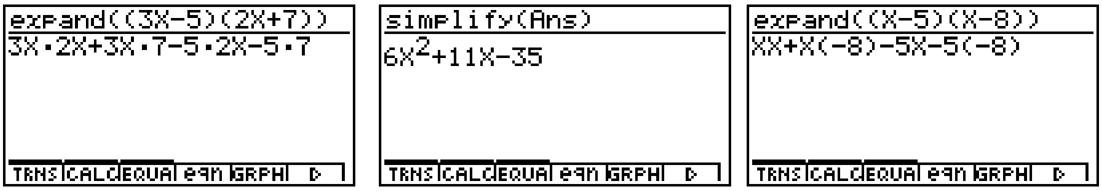

Figure 7: Expansions in Algebra mode, with unsimplified results and use of *xx* instead of x^2

Design of the Study

These potentials for learning were the basis of an exploratory study involving students in a first year undergraduate mathematics service unit at Murdoch University. Students in this unit are not mathematics majors, but are usually hoping to major in science or computer science. Many students have not done mathematics for some years; many of them are mature age students, returning to study. Most of these students have weak algebraic backgrounds and take the unit as a transition to undergraduate mathematics. Students are expected to make routine use of graphics calculators in the unit, and are helped to do so, but do not generally have access to an algebraic calculator with CAS capabilities.

Two voluntary workshops were organised early in the semester to help students use the CAS calculator to support their algebraic learning. The first of these focused on exploring the strategy of ‘doing the same thing to both sides’ of a linear equation, while the second was concerned with expanding and factorising binomial expressions. The first workshop included about 24 of the 80 enrolled students, in two separate groups of 12, a week apart early in the semester, as well as two of the authors. Students were provided with a Casio Algebra FX 2.0 calculator during the workshop, and the group had access to an overhead model of the same calculator. The second workshop was conducted mid-way through the semester, involving about 10 students and the first author. All workshops were quite informal in nature.

For each workshop, a series of relevant algebraic exercises was devised and given in advance to students. In the workshop itself, students and instructors together used the calculators to explore the algebraic procedures and their meanings.

Some weeks after the second workshop, interviews were conducted with eight students from the group of twelve students who had attended the workshop on solving equations; some of these students also attended the second workshop on factorising. Students volunteered to be interviewed and for the interviews to be tape recorded. The main focus of the interviews and this paper is the first workshop.

The interviews were somewhat structured but allowed students some freedom to make comments and give their opinions and thoughts on related matters. Each student was asked why they had attended the workshop, how many equations they could solve in their head from a given list, whether they felt that the workshop had helped them and why or why not it had helped. Other comments were sought concerning the timing of the workshop. The intention was to get feedback about their perceptions of the usefulness of using the algebra mode on the calculator for them, how they thought the workshop on the use of the calculators could be improved and their opinions on the timing of the workshops in the semester.

Results

At the time of the workshops, it seemed that the students in the workshops naturally fell into three main groups. The first included those for whom using the calculators was useful for learning the steps of how to solve equations when 'doing the same thing to both sides'. A second group comprised those who found that the use of the calculators helped them to understand and clarify what they had already been doing previously. A third group consisted of those who were ambivalent about the use of calculators in a situation where they thought that they should be solving the equations by hand, akin to the sentiments reported by Povey and Ransom [6].

The student interviews confirmed that these were the main groups with some overlap. A brief description of these three kinds of responses is given below, with some supporting comments from students.

Useful for learning the process of solving equations

The students that fell into this category had little confidence about solving equations beforehand; they were comfortable about solving them in their heads only while the variable was on one side of the equation. They had weak mathematical backgrounds and were unsure of the procedure. The working through of all the steps was useful to them as this highlighted how they needed to be thinking at the various stages. For example, one student said:

> So I could understand how it works, the idea of how it works. When it showed the steps it was useful. I always did it another way-put the x on one side but I couldn't do it all the time so need another way. The calculator helped to see the other way.

Another student also reacted positively to the details provided by the Algebra mode of the calculator, seeing a parallel with doing the same sorts of things by hand:

> Yes, I think the most beneficial thing to me was being able to see the process. The important thing for me in learning and re-learning is being able to actually fully write everything out so I can follow it all the time. …I have got to have all the bits to follow. I can look at that now and it's just there but it wasn't always like that.

In a later remark, this student seemed to appreciate that they were doing the necessary thinking by themselves, while the calculator was merely doing as it was instructed:

> Doing what you would have been doing by hand, thinking it through, but without having to write it all on paper and use up sheets. Seeing every step made a huge difference to me.

Clarifying what the procedure does

It appeared that some students had been using a procedure of 'changing sides and changing the sign' for their solving of equations. Unfortunately for them this does not cater for the more complex tasks of multiplication and division and so the method as a whole has flaws; indeed, it is a good example of what Skemp [7] referred to as 'instrumental understanding'. The workshop seemed to help such students to see the reason behind some of the steps they had been taking and to draw attention to some of their misconceptions. Thus, one student remarked:

> I never understood what it was about before. I didn't take away, I just did the opposite on each side. I followed the rules of changing sides and changing signs but I didn't know *why* until this session.

Similarly, another student spoke about moving something to the other side of an equations in order to add or subtract, and was surprised to learn of the connections with doing the same thing to both sides:

> I was taught it this way. I would not think about doing the same thing to each side, but you know it ends up the same.

While it seems that such students have benefited from the workshop experience with the calculators, we are reluctant to conclude that a brief experience of this kind can change a long-standing habit. Indeed, when asked in the interview to describe what she would do to solve $3x = -5$, this student said that she would 'do the opposite thing'.

The calculator is doing all the work

The reluctance of some students to embrace the calculator use seemed to be based on the notion that the only way that you should do algebra is by hand, reminiscent of some of the students described by Povey and Ransom [6]. They were resistant to use the calculator for several reasons which included the fact that they would not be able to use them in the exam at the end of the unit; they thought you should solve the equations by hand before using a calculator; they were also concerned that a calculator could give you the wrong answer. Thus, one student noted, when asked if the workshops had been helpful to her:

> No. We can't use the algebra calculator. I can't see the point if we can't use the calculators. I suppose it was a little bit. [helpful]

Another student felt that the calculator was not helpful, and seemed convinced that by-hand algebraic manipulation was the only credible evidence of understanding:

> It wasn't showing you on paper how to do the steps. It was just banging numbers in the calculator which doesn't show you anything. Unless you have a fundamental idea of how to shift things around you can't use the calculator. Maybe down the track a bit when you can see that they are getting the concepts then you can use this as an adjunct for working things out, but you still need to know how to shift them across there. You can use it to check the hard ones. Once you've got the concept of equations by structuring it yourself then you can use it to check.

This same student showed an awareness of the pitfalls of calculator use when later asked about using the graphics calculators to solve equations using a numerical 'solve' command rather than using the algebraic steps on the calculator. In the interview, it was clear that he was motivated to learn how to solve equations algebraically, so that he would not be forced into relying on the calculator. When referring to routine use of the numerical solve command, he said:

> That would be fraught with danger. You need to be able to look at it and see if the answer is reasonable. Can't just take what the calculator or computer tells you.

Reluctance of this kind is not surprising; nor indeed is it unhealthy, in our view. It may merely reflect a strong view that equations ought to be solved by hand by humans whenever possible. It may also suggest, however, that we needed to be clearer to students what roles the calculators might play in developing their understanding of the solution processes.

Conclusions and implications

This was an exploratory study, quite informal in nature, and we make no strong claims for generalizability of our observations. Nonetheless, the reactions from students were sufficient for us to conclude that further work of this kind seems to be warranted. The students in this unit have particularly weak algebraic backgrounds, and yet will be required to develop considerable manual competence with algebraic manipulation in the early years of their undergraduate study. As mature learners, some of whom are well beyond high school age, the students seem able to learn from this kind of use of CAS on a calculator, especially when the details of the calculator's work are available to them through the Algebra mode.

Student reactions also made it clear that care is needed to conduct work of this kind at an appropriate time in the semester, and to make clear the relationship of the calculator to their other work (which did not involve the use of an algebraic calculator). The purposes of the workshops needed to be made clear to students, so that they realised that the calculators were to be used as an aid to learning how to solve equations rather than a substitute for doing so. Future workshop activities need to be carefully planned to optimise student learning with the calculators, especially as there is not yet a plan to allow students to have uncontrolled access to this sort of technology.

References

1. Arcavi, A (1994). Symbol sense: Informal sense-making in formal mathematics. *For The Learning of Mathematics, 14*(3), 24-35.
2. Heid, M.K. & Edwards, M.T. (2001). Computer algebra systems: Revolution or retrofit for today's mathematics classrooms? *Theory Into Practice, 40*(2), 128-136.
3. R. Klein, R. & Kertay, P. (2002). An introduction to simple linear equations: CASs with the TI-89. *The Mathematics Teacher, 95*(8), 646-649.
4. Kutzler, B. (1997). *Solving Linear Equations with the TI-92*. Hagenberg (Austria): bk teachware.
5. Pierce, R. (2001). Using CAS-calculators requires algebraic insight. *Australian Senior Mathematics Journal, 15*(2), 59-63.
6. Povey, H. & Ransom, M. (2000). Some undergraduate students' perceptions of using technology for mathematics: Tales of resistance. *International Journal of Computers for Mathematical Learning, 5*, 47-63.
7. Skemp, R.R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77*, 20-26.

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