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Teaching and Learning Trigonometry with Technology

Barry Kissane
b.kissane@murdoch.edu.au
School of Education, Murdoch University
Australia

Marian Kemp
m.kemp@murdoch.edu.au
Student Learning Centre, Murdoch University
Australia

Abstract: *Modern school classrooms have access to a range of potential technologies, ranging from calculators to computers to the Internet. This paper explores some of the potential for such technologies to affect the curriculum and teaching of trigonometry in the secondary school. We identify some of the ways in which the teaching of trigonometry might be supported by the availability of various forms of technology. We consider circular measures, graphs of functions, trigonometric identities, equations and statistical modeling and focus on activities that are not possible without the use of technology. Modern technology provides an excellent means of exploring many of the concepts associated with trigonometry, both trigonometric and circular functions. Many of these opportunities for learning were not available before technology development and access within schools we enjoy today. This paper suggests some of the avenues for exploration.*

1. Introduction

Trigonometry has long been a standard component of the secondary school curriculum in all countries, usually in the latter half of the secondary years. As an ancient branch of mathematics, trigonometry was revolutionized by the invention and publication of trigonometric tables hundreds of years ago, and facilitated more recently by the availability of trigonometric tables on, firstly, slide rules and then secondly, 'scientific' calculators. Indeed, one of the distinguishing differences between scientific and less sophisticated calculators is the availability of instant access to trigonometric ratios and inverse trigonometric ratios, of interest not only to 'scientists'.

While such advances in technology were significant, they offered mathematics education little more than a chance to avoid the tedium of locating trigonometric values in table books, and did not have an appreciable effect on how trigonometry was taught and learned or even what was regarded as important about trigonometry to justify its place in the school curriculum. This paper describes and analyses some of the ways in which the teaching of trigonometry and the curriculum in trigonometry are potentially affected by the recent availability of technologies for both students and teachers. The paper focuses for the most part on opportunities for learning and teaching that were not readily available before the availability of technology in the classroom.

This topic has been surprisingly neglected in the research literature. For example, our search of recent ATCM conferences revealed no work at all focusing on school trigonometry. Even the wider research literature contains very few references to teaching and learning trigonometry. An exception is [1] at the college level, providing evidence of the lack of success of 'traditional' lecture-based teaching methods. In another study involving year 10 students [2], it was found that all but 4 of 178 students gained a score of zero on a test of trigonometry a year after it was first taught, which suggests that there might be a good case for looking at alternatives to traditional instruction.

2. The definition of trigonometric ratios

An approach used by many teachers to introduce trigonometric ratios is to have students draw many right triangles with the same angles and to explore the ratios of sides. Such an activity continues to be worthwhile, although is tedious and, of course, somewhat error-prone when student measurements are used. An excellent means of using this idea with the aid of technology involves constructing a dynamic triangle using a geometry program and hence automating the exploration to some extent. Figure 1 shows one way of doing this with *Cabri Geometry II Plus* [3]. After constructing triangle ADE, with angle E constrained to be 90 degrees, changing the dimensions of the triangle by dragging point D (constructed on line AC) changes all the side lengths but does not change the (automatically calculated) ratio of DE/AD, the sine of angle A. Dragging C allows students to explore an angle of a different size. Through manipulating such a triangle, students can experience the key relationship that the ratio is independent of the size of the triangle, of fundamental importance to defining trigonometric ratios. (Of course, students who already understand similar triangle relationships will not be surprised by this, but the point of this sort of activity is precisely that students may not yet fully appreciate the idea of similarity. Indeed, part of learning about similarity may well involve explorations of this kind.). Once this idea is clear to students, the concept of a sine of an angle *size* (such as 23.7 degrees), rather than a particular angle, becomes accessible, and the tabulation of these in either table books or a calculator is meaningful.
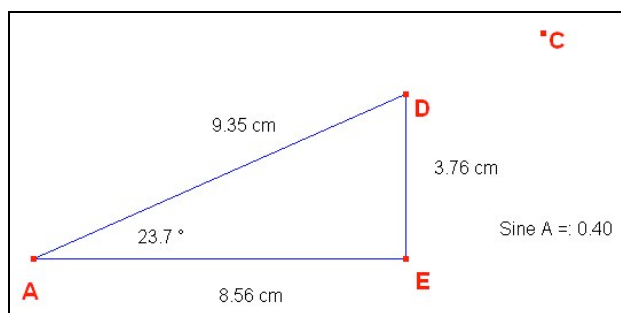

Figure 1: Use of *Cabri Geometry II* to explore Sine ratios

Although there seems merit in students constructing this for themselves, there are versions of this idea available as applets on the Internet as well. Figure 2 shows an example from the Austrian *Maths Online* website [4], which has some excellent educational materials, with sound pedagogical advice for their use (including commentary for teachers and exercises with solutions for students). Some of the applets in the Gallery relate directly to trigonometry, using Java applets, with powerful connections between the unit circle and various graphs.
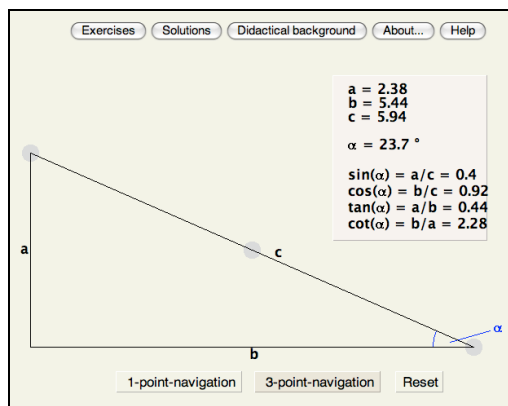

Figure 2: Exploring a triangle to see the invariance of trigonometric ratios with scaling [4]

In the example shown here, students can change the triangle size (by dragging the circle at c) and will see that all of the side lengths change, but that none of the ratios change. Such experiences were not readily accessible prior to the advent of technology so that students were obliged to either rely on their teacher's assertions or to understand a formal deductive argument (using similarity). Although both of these have their place in mathematics education, the experience offered by the technology seems to be powerful and different and likely to be conceptually helpful. This seems to be especially the case for modern students, who seem to be motivated by working with technology.

3. Circular functions and unit circles

Although curricula in various countries do not consistently make a distinction, right-angled triangle trigonometry (with angles necessarily less than 180 degrees) leads to *trigonometric* functions while unit circle definitions lead naturally to the idea of radian measure and functions whose domain is the real number line. In some countries these are called *circular* functions and are routinely used in most applications of trigonometry to periodic phenomena as well as routinely used in the study of calculus. In this paper, including in its title, we do not emphasise this subtlety of nomenclature, but note that the idea of a unit circle is a very powerful one and is critical to helping students understand the meaning of the ratios (with both meanings of the ratios, in fact).

Unit circle applets

A number of excellent applets are available online to help students experience the unit circle. These might be used by students themselves or might be used in a classroom as demonstration devices, much more powerful than similar versions drawn by hand on a blackboard. A very good example comes from a Flash applet on the NRICH site in the UK [5], with a point moving around a unit circle. Knowledgeable teachers can save this applet for use offline in a classroom, such as on an interactive whiteboard. This applet is illustrated in Figure 3.
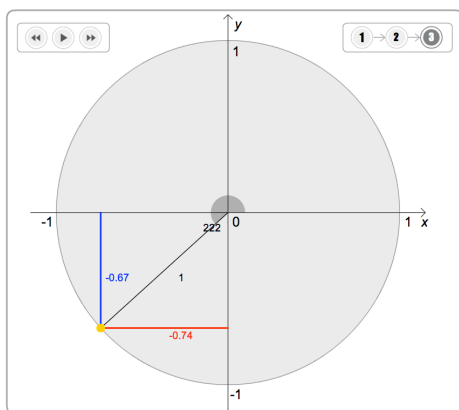

Figure 3: Nrich's *Round and round a circle* applet [5]

A *GeoGebra* applet at the Lawrenceville School in the USA [6] similarly provides a nice example of a unit circle that would also be helpful for classroom display or student use, with angle measures both in degrees and in radians, making connections between trigonometric and circular functions.

The unit circle on a graphics calculator

The technology of the graphics calculator is potentially more accessible to many students, as it requires neither computers nor Internet access [7]. A dynamic unit circle can be drawn on a

graphics calculator and traced. Figure 4 shows this done on a Casio fx-9860G II graphics calculator in parametric mode with the illustrated settings chosen to suit the calculator screen.
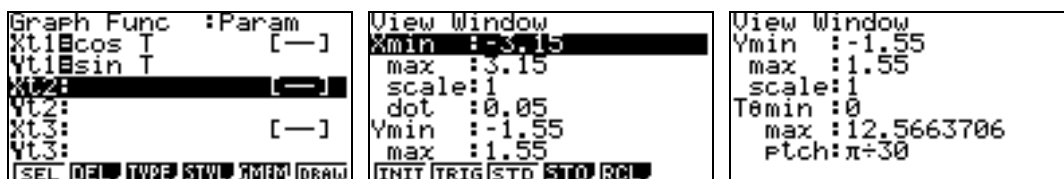

Figure 4: defining a unit circle on a graphics calculator [8]

With the calculator set to radians, students can trace the graph to see for themselves values for sin T and cos T as y and x values respectively, as shown in Figure 5, rather than relying on a demonstration by the teacher or a static image in a textbook.
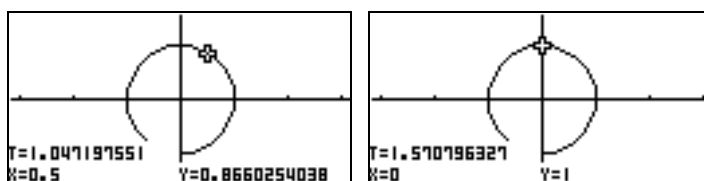

Figure 5: Tracing a unit circle on a graphics calculator

As well as tracing, particular values can be shown. The example below shows the result of using calculator functions to obtain the sine and cosine of $\pi/4$, each about $\sqrt{2}/2$ in this case.
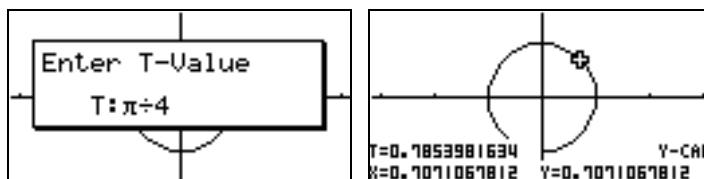

Figure 6: Using a unit circle to evaluate circular functions

A table can be traced too, and will allow negative values for T. Table settings -2π to 2π and step $\pi/60$ shown below will ensure that results for the main exact values for T (such as $\pi/3$, $\pi/4$ etc.) will be displayed correctly. In Figure 7, $T = \pi/3$ with sin $T = \frac{1}{2}$ and cos $T = \sqrt{3}/2$ are shown.
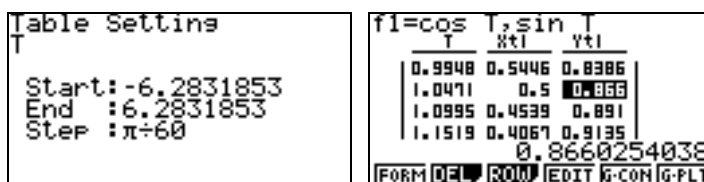

Figure 7: Table of values from a unit circle

In summary, access to technology provides students with the possibility of interacting with unit circles to help develop the important conceptual ideas associated with the sine and cosine functions, in ways that were not previously available.

4. Graphs of trigonometric functions
Prior to the availability of technology, graphing trigonometric functions was a tedious undertaking for students, with little practical purpose. Technology provides a mechanism for graphs to be drawn much more quickly, which allows students to compare graphs in order to gain some insights into the

nature and differences between functions. Ready access to graphs on demand also allows students to use graphs for practical purposes, such as considering solutions to equations.

Graphing applets on computers

A number of technology applications offer an opportunity for students to explore the nature of trigonometric graphs and an understanding of why they are periodic. An example is shown in Figure 8 from [4]. Importantly for many teachers, materials from this site can be downloaded for use on computers when classroom Internet access is problematic.
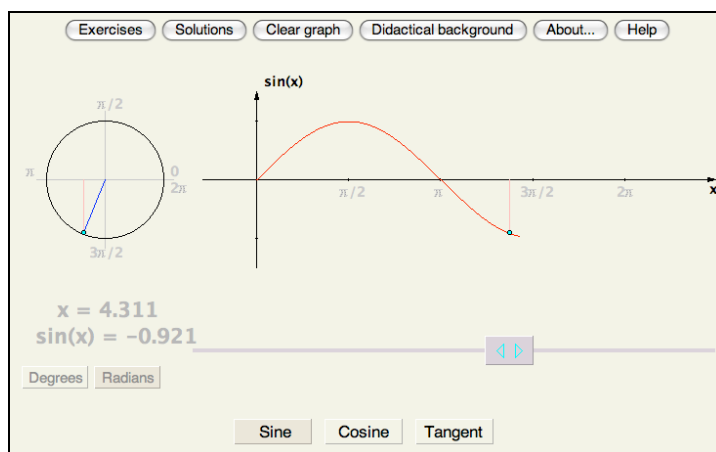

Figure 8: Graphing the sine function from a unit circle [4]

The Japanese *Manipula Math* collection [9] can be used online or can be purchased for individual or class use offline. This collection includes a significant number of interesting and educationally sound applets related to trigonometry.

Graphing on graphics calculators

Graphs of trigonometric and circular functions are readily produced on graphics calculators, and thus accessible routinely to students themselves in classrooms with access to that sort of technology. Students are able to easily draw any trigonometric graphs and explore them by tracing or using automatic procedures to explore key points, noting periodicity, amplitude, and so on. Before such access to technology, students usually had less experience with using graphs. Figure 9 shows an example of exploring a graph to find maximum point(s) and zeroes of a function.
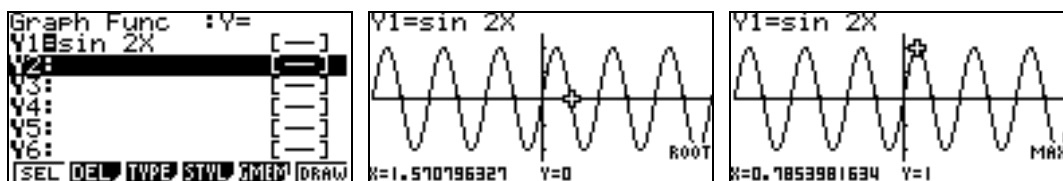

Figure 9: Using graphs to understand periodicity properties

On some calculators, such as Casio's fx-9860G II, families of functions can be drawn and animated, allowing students to appreciate the relationships between function parameters and graphs in a way that was not possible previously. Figure 10 shows calculator settings to illustrate the process, which allows only one parameter at a time to be varied on the calculator.
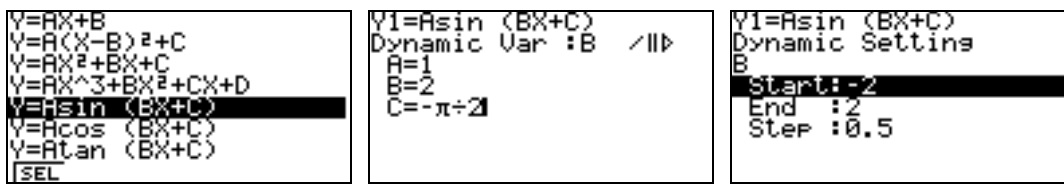
Figure 10: Defining dynamic sine functions

Figure 11 shows some of the series of graphs produced, related to the linear parameter *B* in this case. Although teachers can use this kind of facility for demonstration purposes, it seems most likely to be a powerful new opportunity for students themselves to explore.
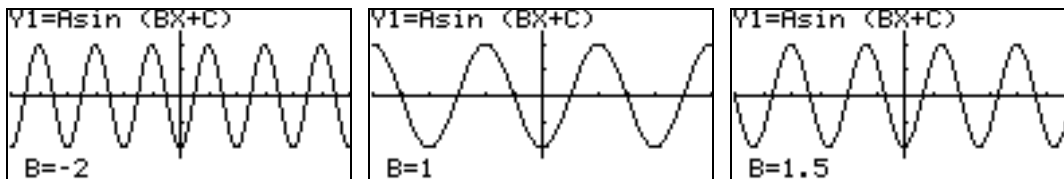

Figure 11: Dynamic graphs showing effects of varying coefficient *B*

Statistical modeling on a graphics calculator

As noted earlier, trigonometry is useful for modeling periodic phenomena. If periodic data are obtained, they can be entered into lists and graphed in statistics mode of a graphics calculator. One of the regression functions on the Casio fx-9860G II calculator is *Sine*, which will suit some data quite well. For example, the predicted times of sunrise in Fremantle, Western Australia, on the first day of each month of 2009 are available online. [10].

Data are entered as day of the year and hours past midnight. A resulting scatter graph shows a roughly sinusoidal shape. The graph shows that a sine function fits the data well in this case
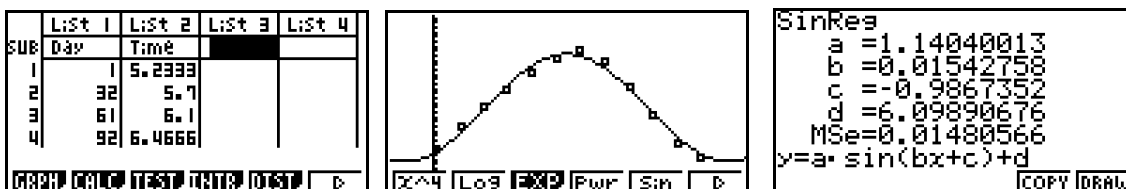

Figure 12: Modeling sunset times in Fremantle, Western Australia

Although the model is a consequence of choosing to use only some of the data (one day per month), it provides a reasonably good fit and the corresponding formula can be used to predict tolerably well the sunrise at any date during the year. One way of doing this is to tabulate the trigonometric function as in Figure 13.
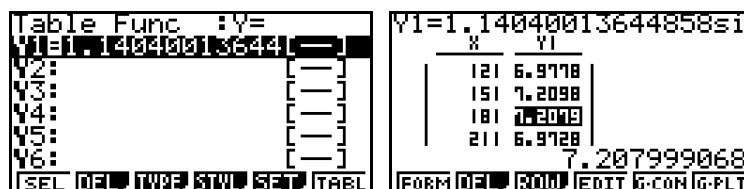

Figure 13: Using the model to predict sunrise time.

Work of this kind will help students to both appreciate the value of trigonometric functions to interpret and model everyday phenomena (which have nothing to do with triangles) as well as to

appreciate the power of a mathematical formula to make reasonable predictions. Other real data (such as tidal data, or daily hours of sunshine over the year) can also be used to show the cyclical nature of circular functions and their appropriateness for modeling periodic phenomena. While the website [10] is a good source of data for Australian locations, other sources will be needed in other countries to obtain local data. With access to technology, the curriculum might be extended to provide students with the opportunities to address a wider range of practical situations than previously and, incidentally, make some nice connections between the study of trigonometry and that of statistics.

5. Trigonometric identities and equations

Although there is a tendency to describe anything with an equals sign as an equation, it is important to be more careful with the language and so we consider separately trigonometric identities and trigonometric equations. In the former case, the mathematical concern is whether or not two expressions are universally equivalent, which normally requires a mathematical proof of some sort. In the latter case, the concern is to find whether there are particular values that render two expressions equal and, if there are any such, how many of them there are.

Identities on graphics calculators

Identities can be checked on a graphics calculator, but not proved. In either Graph or Table mode it is necessary to enter each side of the identity as a separate function, *Y1(x)* and *Y2(x)*. If a relation is an identity, graphs of *Y1* and *Y2* will overlap, so that only a single graph is shown. Figure 14 shows an example of this, verifying that, over the domain chosen, the two expressions seem to be the same, as they have a common graph. The common graph is clear when the graphs are traced in the usual way.
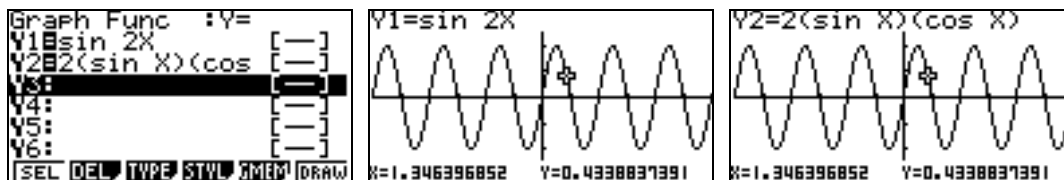

Figure 14: Graphical checking of the identity $\sin 2X = 2 \sin X \cos X$

Similarly, table values for *Y1* and *Y2* will coincide for an identity; otherwise they won't. Such uses of technology do not provide a proof of course, but a sort of verification as well as a way of clarifying the nature of an identity (distinguishing it from an equation). They also provide a quick means of determining whether or not an expression appears to be an identity (to decide whether a proof is worth searching for).
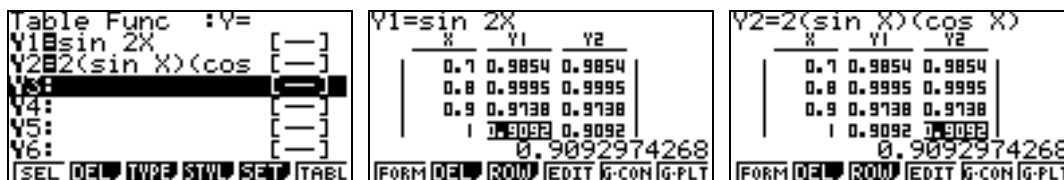

Figure 15: Numerical checking of the identity $\sin 2X = 2 \sin X \cos X$

As a further illustration of this use of technology, consider the identity $\sin^2 x + \cos^2 x = 1$; first graph $Y1(x) = \sin^2 x$ and then $Y2(x) = \cos^2 x$, as in Figure 16. Trace the graph, flipping up and down between the two graphs using the up/down cursor. Notice that the sum of the *y*-coordinates appears

to be constant, regardless of *x*. Then Graph *Y3 = Y1 + Y2*. The result is a constant, 1. Turning off graphs of Y1 and Y2 will show only the sum of 1 for all values of *x* in the domain of the screen.
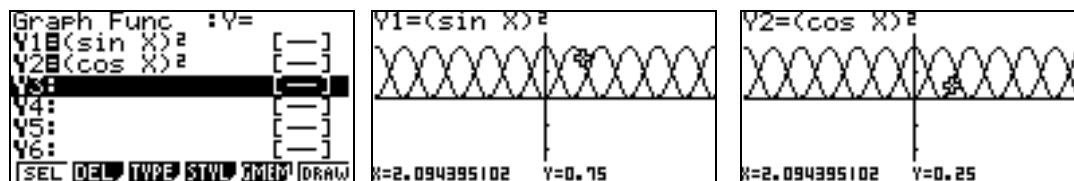

Figure 16: Checking the identity $\sin^2 x + \cos^2 x = 1$

Equations on graphics calculators

On a graphics calculator, only numerical solutions to equations are usually possible. The easiest way to solve equations numerically is to graph each side of the equation separately, and to then use automatic calculator procedures to locate the points of intersection. The periodic nature of solutions is displayed, providing a motivation to develop a way of describing a general solution.

For example, consider the sample equations shown in recent New South Wales (Australia) syllabus documents for senior secondary school:

| $3 + 2\cos x = 15 \cos x$ | $2 \sin x = \cos x$ | $2\sin x = \tan x$ |
| $\sec^2 x = 3$ | $2 - 2\cos 2t = 0$ | $\sin 2x = x/2$ |

Graphs can be used to find approximate solutions to equations like these. For example, Figure 17 shows the first three solutions to $3 + 2\cos x = 15 \cos x$ for $x > 0$. Careful inspection shows that the first and the third solutions are in similar places on the graphs, while the second is in a different place, apparently 2π further to the right of the first solution. Such observations might help students to see the likely pattern of solutions and help them to conceptualise the idea of a *general* solution.
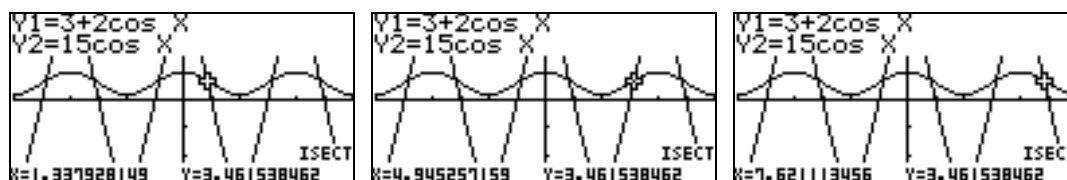

Figure 17: Using graphs to approximate the first three solutions of $3 + 2\cos x = 15 \cos x$

On many calculators, numerical solvers are available. On the Casio fx-9860G II, the EQUA mode allows for numerical solutions to be found directly, as shown in Figure 18, but it's hard to see the periodic nature of the solutions. Again, in this case, the first and third solutions are 2π apart, but students are unlikely to reach conclusions of this kind without teacher help.
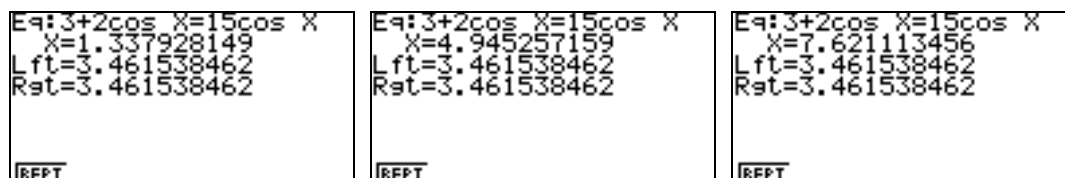

Figure 18: Using a numerical solver to solve trigonometric equations approximately

Note in the examples of equations above, that the equation $\sin 2x = x/2$ requires the use of radians. Equations of this kind also highlight a significant advantage of a graphics calculator over older by-hand methods: that most elementary equations (such as $\sin 2x = x/2$) can be solved (albeit approximately), not only those for which suitable trigonometric transformations can be found, thus allowing a wider range of equations to be considered in the school trigonometry curriculum than

previously. Various solution methods are possible; e.g., to solve $\sec^2 x = 3$ (listed above), students can also graph $Y1(x) = \sec^2 x$ and then use an automatic procedure to find the points for which $Y=3$.

Trigonometry and proof

On a calculator equipped with a computer algebra system (CAS), similar kinds of equation solving and work with identities can be undertaken; an important difference is that the algebraic capabilities of CAS calculators allow for these to be examined *symbolically*, as well as graphically and numerically, as shown in Figure 19 on the Casio *ClassPad*, using commands to expand and collect trigonometric terms. These are key components of standard by-hand approaches to constructing proofs of identities. As the second screen in Figure 19 shows, commands *tExpand* and *tCollect* are both regarded in the *ClassPad* environment as kinds of transformations of algebraic (in this case, trigonometric) expressions.
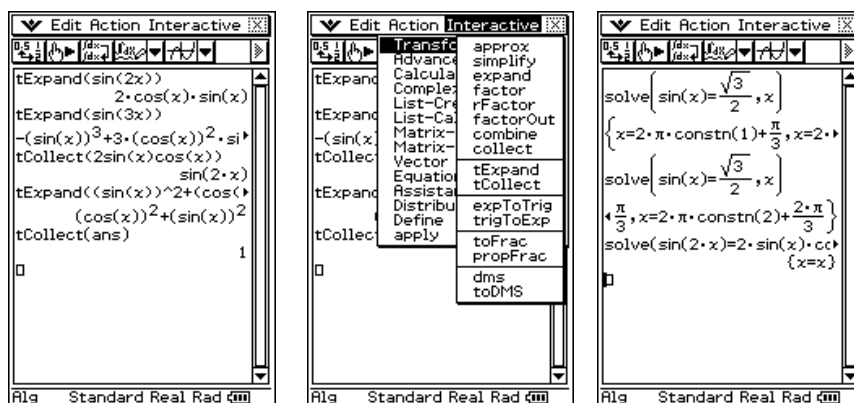

Figure 19: Use of CAS for symbolic trigonometric transformations

A difference for equations on a CAS calculator is that *exact* solutions to equations can be found and, in the case of trigonometric equations, *general* solutions might also be found. Figure 19 shows an example for the Casio *ClassPad*, indicating that there is an infinite number of solutions to the equation, $\sin x = \sqrt{3}/2$. For convenience, these are shown on two separate lines in the screen dump, representing the two families of solutions, $x = 2\pi k + \pi/3$ and $x = 2\pi k + 2\pi/3$. (Notice that the *ClassPad* has used a different symbol (*constn*(1) and *constn*(2) respectively) for the parameter k for each of the two parts of the general solution). Figure 19 also shows an important relationship between equations and identities: an identity such as $\sin 2x = 2 \sin x \cos x$ might be regarded as an equation for which there are an infinite number of solutions, represented on the *ClassPad* with a solution set of $\{x=x\}$.

Some technology applets provide insights into possible proofs of trigonometric relations, although of course the proofs themselves must still be constructed by students. A good example from [9] at http://www.ies.co.jp/math/products/trig/applets/seigen/seigen.html is the applet concerning a proof of the Sine Rule.

Exact values on a calculator

It is of interest here to note that recent versions of both scientific and graphics calculators have included some exact values of trigonometric ratios. There are (very) few values for which exact trigonometric ratios are known, so that students will be surprised to sometimes see results such as these. Figure 20 shows two examples from the Casio fx-9860G II graphics calculator.
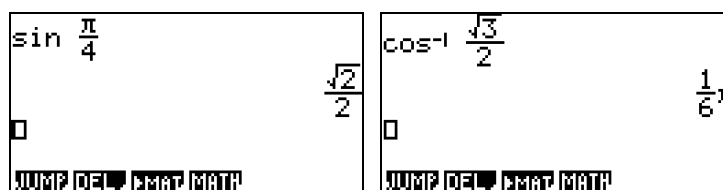
Figure 20: Exact values of trigonometric ratios on a graphics calculator

What distinguishes the exact results from other results is that we can provide a mathematical justification for the values; there seems some prospect that having some exact values appear on calculators may encourage students to distinguish between exact and approximate values (an important end in itself), but may also stimulate them to look for mathematical justifications for the values, which would of course have the forms of proof.

6. Conclusion

This paper has sampled from the available technologies that learners or teachers may have access to for teaching and learning trigonometry in the early years of the twenty-first century, to illustrate the claim that there are many ways in which teaching and learning might change, when compared with earlier approaches. One major change is the opportunity provided by technology to help students engage with trigonometric concepts; a second major change is that students can interact directly with trigonometrical ideas through the medium of technology in a more active way than is possible with paper and pencil alone. While the possibilities in any classroom will of course be dependent on which particular technologies are available, there seems to be much to be gained from the use of the least sophisticated technology, that of the graphics calculator.

References

[1]   Weber, K. (2005) Students' understanding of trigonometric functions. *Mathematics Education Research Journal*, 17(3), 91-112.

[2]   Kendal, M., & Stacey, K. (1996) Trigonometry: Comparing Ratio and Unit Circle Methods In P. Clarkson (Ed) *Technology in Mathematics Education: Proceedings of the Nineteenth Annual Conference of the Mathematics Education Research Group of Australasia*, (pp. 322-329) Melbourne: Mathematics Education Research Group of Australasia.

[3]   Laborde, J-M. & Bellemain, F. (2005) *Cabri Geometry II Plus*. CabriLog. www.cabri.com.

[4]   Embacher, F. & Oberhuemer, P. (2009) *Maths Online* (Accessed 27 July 2009) http://www.univie.ac.at/future.media/moe/galerie.html

[5]   NRICH (2009) *Round and round a circle*. (Accessed 27 July 2009) http://nrich.maths.org/public/viewer.php?obj_id=6084.

[6]   The Lawrenceville School (2009) *Unit Circle*. (Accessed 27 July 2009) http://mathplotter.lawrenceville.org/mathplotter/mathPage/unitCircle.htm

[7]   Kissane, B. (2007) Exploring the place of hand-held technology in secondary mathematics education. *Proceedings of 12th Asian Technology Conference on Mathematics*, (pp. 63-81) Taiwan, December 2007.

[8]   Kissane, B. & Kemp, M. 2006, *Mathematics with a graphics calculator: Casio fx-9860G AU*, Mirabooka: Mathematical Association of Western Australia, p. 55.

[9]   International Education Software (2009) *Manipula Math with Java*. (Accessed 27 July 2009) http://www.ies.co.jp/math/java/.

[10]   Geoscience Australia (2009) *Sunrise and sunset results*. (Accessed 27 July 2009). http://www.ga.gov.au/bin/astro/sunrisenset.