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2. The gravitational two-body problem

2.1 The reduced mass

[LL]

→ Two-body problem: two interacting particles.

→ Lagrangian

$$\mathcal{L} = \frac{1}{2}m_1|\dot{\mathbf{r}}_1|^2 + \frac{1}{2}m_2|\dot{\mathbf{r}}_2|^2 - V(|\mathbf{r}_1 - \mathbf{r}_2|)$$

→ Now take the origin in the centre of mass, so $\mathbf{r}_1m_1 + \mathbf{r}_2m_2 = 0$, and define $\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2$, so

$$\mathbf{r}_1 = \frac{m_2}{m_1 + m_2}\mathbf{r}$$

$$\mathbf{r}_2 = -\frac{m_1}{m_1 + m_2}\mathbf{r}$$

→ Substituting these in the Lagrangian, we get

$$\mathcal{L} = \frac{1}{2}\mu^*|\dot{\mathbf{r}}|^2 - V(r),$$

where

$$\mu^* \equiv \frac{m_1m_2}{m_1 + m_2} = \frac{m_1m_2}{M}$$

is the reduced mass, where $M \equiv m_1 + m_2$ is the total mass.

→ So the two-body problem is reduced to the problem of the motion of one particle of mass μ^* in a central field with potential energy $V(r)$.

2.2 Kepler's problem: integration of the equations of motion

[LL; R05]

→ In the case of gravitational potential

$$V(|\mathbf{r}_1 - \mathbf{r}_2|) = -\frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|},$$

so

$$V(r) = -\frac{Gm_1m_2}{r} = -\frac{G\mu^*M}{r}$$

→ A widely used notation in celestial mechanics is $\mu \equiv GM = G(m_1 + m_2)$, where μ is called the “gravitational mass”. Note that μ does not have units of mass, while μ^* and M have units of mass.

→ The problem is reduced to the motion of a particle of mass $m = \mu^*$ in a central field with potential energy $\propto 1/r$: this is known as *Kepler's problem*. Newtonian gravity: attractive. Coulomb electrostatic interaction: attractive or repulsive.

→ Let's focus on the attractive case $\implies V = -\alpha/r$ with constant $\alpha > 0$. We are describing the motion of a particle m moving in a central potential $V = -\alpha/r$.

→ In the case of the gravitational two-body problem $m = \mu^*$ (reduced mass) and $\alpha = G(m_1 + m_2)\mu^* = GM\mu^* = \mu\mu^*$

→ We have seen that for motion in a central field, the radial motion is like 1-D motion with effective potential energy

$$V_{\text{eff}}(r) = -\frac{\alpha}{r} + \frac{L^2}{2mr^2},$$

→ See plot of V_{eff} and energy levels (See fig 10 of LL; FIG CM2.1)

→ Minimum of V_{eff} at $r = L^2/m\alpha$. $V_{\text{eff},\text{min}} = -\alpha^2m/2L^2$.

→ Motion is possible only when $E > V_{\text{eff}}$. If $E < 0$ motion is finite. If $E > 0$ motion is infinite.

→ Path: $\phi = \phi(r)$. Take

$$d\phi = \frac{Ldr}{r^2\sqrt{2m[E - V_{\text{eff}}(r)]}} = \frac{Ldr}{r^2\sqrt{2m[E - V(r)] - \frac{L^2}{r^2}}}$$

and substitute $V = -\alpha/r$.

→ We get

$$d\phi = \frac{Ldr}{r^2\sqrt{2m\left[E + \frac{\alpha}{r}\right] - \frac{L^2}{r^2}}},$$

which can be integrated analytically to obtain:

$$\phi = \arccos \frac{(L/r) - (m\alpha/L)}{\sqrt{2mE + \frac{m^2\alpha^2}{L^2}}} + \phi_0 = \arccos \frac{\frac{L^2}{m\alpha r} - 1}{\sqrt{1 + \frac{2EL^2}{m\alpha^2}}} + \phi_0,$$

with ϕ_0 constant (verified by differentiation). Note that

$$\frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}}$$

→ Defining $\ell \equiv L^2/m\alpha$ and $e \equiv \sqrt{1 + (2EL^2/m\alpha^2)}$ we get

$$\frac{\ell}{r} = 1 + e \cos f,$$

where $f = \phi - \phi_0$ is called the *true anomaly*.

→ This is the equation of a conic section where ℓ is the semi-latus rectum and e is the eccentricity. r is the distance from one focus. ϕ_0 is such that $\phi = \phi_0$ at the perihelion.

→ In the two-body problem each orbit is a conic section with one focus in the centre of mass (see plot of conic sections: fig. 2.4 of MD; FIG CM2.2).

→ We have seen that for motion in a central field the time dependence of the coordinates is given by

$$dt = \frac{\sqrt{m} dr}{\sqrt{2[E - V(r)] - \frac{L^2}{mr^2}}},$$

which for a Kepler potential can be integrated analytically (see below, for instance for elliptic orbits).

→ Depending on the sign of E (and therefore on the value of e) we distinguish:

- $E < 0$ ($e < 1$): elliptic orbits
- $E = 0$ ($e = 1$): parabolic orbits
- $E > 0$ ($e > 1$): hyperbolic orbits

2.2.1 Elliptic orbits

[LL; R05; MD]

→ $E < 0 \implies e < 1 \implies$ elliptic orbit. Brief summary of properties of the ellipse (see fig. 4.3 of R05; FIG CM2.3). S focus, S' other focus C centre, P any point on ellipse, $CA = a$, $CB = b$, $CS = ae$, $SQ = \ell$,

$$SP/PM = e < 1 \quad (\text{eccentricity}),$$

$$SP + PS' = 2a,$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$a = \ell/(1 - e^2) \quad (\text{Semi - major axis})$$

$$b = \ell/\sqrt{1 - e^2} \quad (\text{Semi - minor axis}),$$

$$b = a\sqrt{1 - e^2} \quad (I)$$

→ From the relations among e , ℓ , L and E , we get

$$a = \ell/(1 - e^2) = \alpha/2|E| \quad (II)$$

$$b = \ell/\sqrt{1 - e^2} = L/\sqrt{2m|E|} \quad (III)$$

→ *Pericentre and apocentre.* We recall that the equation for the distance r from one of the focus is

$$r = \frac{\ell}{1 + e \cos \phi} = \frac{a(1 - e^2)}{1 + e \cos \phi},$$

where we have assumed $\phi_0 = 0$, so $f = \phi$. Therefore the apocentre ($\cos \phi = -1$) is $r_{\text{apo}} = a(1 + e)$ and the pericentre ($\cos \phi = 1$) is $r_{\text{peri}} = a(1 - e)$.

→ We have seen that for motion in a central field the sectorial velocity dA/dt is constant (*Kepler's second law*).

→ *Kepler's third law.* Using conservation of angular momentum

$$L = mr^2\dot{\phi} = 2m \frac{dA}{dt} = \text{const}$$

we get period T for elliptic orbit:

$$Ldt = 2mdA \implies TL = 2mA = 2mab\pi,$$

where $A = \pi ab$ is the area of the ellipse.

\implies

$$T = 2\pi a^{3/2} \sqrt{m/\alpha} = \pi \alpha \sqrt{m/2|E|^3},$$

which is *Kepler's third law* $T \propto a^{2/3}$. Note that period depends on energy only. We have used definitions of a , b and L as functions of ℓ (semi-latus rectum): $a = \ell/(1 - e^2) = \alpha/2|E|$, $b = \ell/\sqrt{1 - e^2} = L/\sqrt{2m|E|}$, $L = \sqrt{m\alpha\ell}$.

→ In the case of the gravitational two-body problem we have $m = \mu^*$ and $\alpha = GM\mu^*$, so

$$T^2 GM = 4\pi^2 a^3 \quad \text{or} \quad GM = n^2 a^3,$$

with $n \equiv 2\pi/T$ is the *mean motion* (i.e. the mean angular velocity),

$$E = -\frac{GM\mu^*}{2a}$$

$$e = \sqrt{1 + \frac{2EL^2}{G^2 M^2 \mu^{*3}}} = \sqrt{1 - \frac{\tilde{L}^2}{GMa}}$$

where $\tilde{L} \equiv L/\mu^* = r^2\dot{\phi}$ is the modulus of the angular momentum per unit mass.

→ In the limit $m_2 \ll m_1$ (where m_1 and m_2 are the masses of the two bodies, for instance Sun and planet), then $\mu^* \approx m_2$, $M \approx m_1$, $r_1 \approx 0$, $r_2 \approx r$, so we have *Kepler's first law*: the orbit of each planet is an ellipse with the Sun in one of its foci.

→ Using the expression for the orbital energy we can relate the velocity modulus v to r and a as follows:

$$E = T + V = \frac{1}{2}\mu^*v^2 - \frac{GM\mu^*}{r} = -\frac{GM\mu^*}{2a},$$

so

$$v^2 = GM \left(\frac{2}{r} - \frac{1}{a} \right),$$

or

$$a = \left(\frac{2}{r} - \frac{v^2}{GM} \right)^{-1}.$$

Kepler's equation

→ From the time dependence of radial coordinate (see above) we have, in the case of elliptic orbit:

$$dt = \frac{rdr}{\sqrt{2|E|/m}\sqrt{-r^2 + \alpha r/|E| - L^2/2m|E|}},$$

→ Note that

$$\frac{L^2}{2m|E|} = b^2 = a^2(1 - e^2) = a^2 - a^2e^2$$

and

$$\alpha r/|E| = 2ar, \quad \text{because } \alpha = 2a|E|$$

so

$$dt = \frac{rdr}{\sqrt{2|E|/m}\sqrt{a^2e^2 - (r - a)^2}}.$$

→ Let us introduce the angular variable ξ , known as the *eccentric anomaly*. We substitute

$$r = a(1 - e \cos \xi),$$

$$dt = \sqrt{a^2m/2|E|}(1 - e \cos \xi)d\xi$$

$$t = \sqrt{ma^3/\alpha}(\xi - e \sin \xi),$$

where we have used $\alpha = 2|E|a$. Note that $0 < \xi < 2\pi$: we do the calculation for $[0, \pi]$ (so $\sin \xi = \sqrt{1 - \cos^2 \xi}$). The calculation for $[\pi, 2\pi]$ is similar, with $\sin \xi = -\sqrt{1 - \cos^2 \xi}$.

→ So, for an elliptic orbit

$$r = a(1 - e \cos \xi),$$

$$t - \tau = \sqrt{ma^3/\alpha}(\xi - e \sin \xi),$$

where τ is the time of pericentric passage (because when $t = \tau$ $\xi = 0$, so $r = a(1 - e) = r_{\text{peri}}$). The latter equation is known as *Kepler's equation*. Here $0 \leq \xi \leq 2\pi$ for one period. Note that $\sqrt{ma^3/\alpha} = T/2\pi$. To obtain ξ (then r) as a function of t Kepler's equation must be solved numerically. Note that $\sqrt{ma^3/\alpha} = T/2\pi$.

→ Note that often the eccentric anomaly is indicated with E , instead of ξ (see, e.g., R05, MD).

Mean anomaly, true anomaly and eccentric anomaly.

→ In Kepler's equation ξ is the *eccentric anomaly*. Kepler's equation can be written as

$$\mathcal{M} = \xi - e \sin \xi,$$

where $\mathcal{M} = n(t - \tau)$ is the *mean anomaly*, with $n = 2\pi/T$ mean motion and $T = 2\pi a^{3/2} \sqrt{m/\alpha}$ is the period.

→ The *geometric interpretation of the eccentric anomaly* ξ is given in fig. 2.7b of MD (FIG CM2.4). So:

$$x = a \cos \xi$$

$$y^2 = b^2 - \frac{b^2}{a^2} x^2 = b^2(1 - \cos^2 \xi) = b^2 \sin^2 \xi = a^2(1 - e^2) \sin^2 \xi$$

$$\begin{aligned} r^2 &= (x - ae)^2 + y^2 = x^2 - 2aex + a^2e^2 + y^2 = a^2 \cos^2 \xi - 2a^2e \cos \xi + a^2e^2 + a^2 \sin^2 \xi - a^2e^2 \sin^2 \xi = \\ &= a^2 - 2a^2e \cos \xi + a^2e^2 \cos^2 \xi = a^2(1 - e \cos \xi)^2 \end{aligned}$$

so

$$r = a(1 - e \cos \xi)$$

→ f as a function of ξ . It is useful to derive relations between the eccentric anomaly ξ and the *true anomaly* f (see, e.g., VK 3.8). We know that

$$r = \frac{a(1 - e^2)}{1 + e \cos f}$$

so

$$1 - e \cos \xi = \frac{1 - e^2}{1 + e \cos f}$$

$$e \cos f = \frac{1 - e^2}{1 - e \cos \xi} - 1$$

$$\cos f = \frac{\cos \xi - e}{1 - e \cos \xi},$$

and, using $\sin f = \sqrt{1 - \cos^2 f}$,

$$\sin f = \sqrt{1 - e^2} \frac{\sin \xi}{1 - e \cos \xi}.$$

→ In summary, we recall that there are three different “anomalies”: *true anomaly* f (or $\phi - \phi_0 = \phi - \omega$), *mean anomaly* $\mathcal{M} = n(t - \tau)$ and *eccentric anomaly* ξ (see definitions above).

2.3 Systems of coordinates and orbital elements

2.3.1 Celestial systems of coordinates

- See fig. 2.5 of R05 (FIG CM2.5).
- Celestial sphere: fictitious sphere of arbitrary radius surrounding the Earth, on which the celestial bodies are projected.
- Celestial equator: great circle obtained intersecting the plane of the Earth equator and the celestial sphere.
- Celestial poles: intersections of Earth axis with celestial sphere
- Celestial meridians: great circles on the celestial sphere joining the poles
- Plane of the ecliptic: plane of the Earth orbit around the Sun
- Ecliptic: great circle obtained intersecting the plane of the ecliptic with the celestial sphere
- Vernal and autumnal equinoctial points: intersections between ecliptic and celestial equator. Also known as first point of Aries and Libra.
- Vernal equinox or first point of Aries (Υ) reference point on the ecliptic and on the celestial equator
- Angle between ecliptic and celestial equator (i.e. inclination of Earth axis w.r.t. Earth orbit) is $23^{\circ}26'$

Equatorial coordinates

- Right ascension α in hours (0-24) or degrees (0-360) from Υ eastwards.
- Declination δ : angle along the meridian (in degrees) 0 at the equator, 90 at the north pole, -90 at the south pole.

Ecliptic coordinates

- Ecliptic longitude λ : in degrees (0-360) or hours (0-24) from Υ eastwards. Also known as celestial longitude.
- Ecliptic latitude β : in degrees from 0 (ecliptic) to 90° (at the North pole of the ecliptic K). Also known as celestial latitude.

2.3.2 Orbital elements

- For simplicity it is convenient to refer to a body orbiting the Solar System, but the same formalism applies to any body orbiting another body, when a reference plane is fixed.
- See fig. 2.6 of R05 (FIG CM2.6).
- Position and orbit of a celestial body (e.g. planet) defined by 6 quantities called elements. Let us specialize to the case of the elliptic orbit.

- 3 elements define the orientation of the orbit (Ω, i, ω)
- 2 elements define size and shape of the orbit (a, e)
- 1 element defines position of the body at a given time (τ)
- Line of nodes: intersection between body orbital plane and ecliptic plane
- Nodes: intersections between ecliptic and line of nodes
- Ascending node N : when in this node the body goes from the south ecliptic hemisphere to the north ecliptic hemisphere
- (Ω) Longitude of the ascending node Ω : angle from Υ to N measured eastward on the ecliptic (in degrees from 0 to 360).
- (i) Inclination i : angle between body orbital plane and ecliptic plane (in degrees)
 - Line of apses: line joining the apocentre (aphelion) and pericentre (perihelion), intercepting the celestial sphere in B (proj. of pericentre) and B' (proj. of apocentre)
- (ω) Argument of pericentre ω : angle between N and B
- (a) Semi-major axis of the elliptic orbit $a = GM\mu^*/2|E| = GM_{\odot} m_{planet}/2|E|$ (size of orbit)
- (e) Eccentricity of the orbit e : distance between focus and centre is ae (shape of the orbit)
- (τ) Time of pericentre passage τ : epoch at which the body was at pericentre.
 - Orbital elements for a body orbiting in the Solar System: longitude of the ascending node (Ω), inclination (i), argument of perihelion (ω), semi-major axis (a), eccentricity (e), time of perihelion passage (τ)
 - When the 6 elements are given, position of body known at any time.
 - Similar orbital elements are taken for satellites of the Earth (taking the equatorial plane as reference) or for satellites of other planets (with planet's equator as reference)
 - Also used as element the "longitude of pericentre" $\varpi = \omega + \Omega$ (dog-leg angle, because Ω and ω lie in different planes if $i \neq 0$)
 - Also used $\chi = -n\tau$ (*mean anomaly at epoch*, sometimes indicated with \mathcal{M}_0), instead of τ (see R05 page 211). The element χ is related to the value of the mean anomaly $\mathcal{M} \equiv n(t - \tau) = nt + \chi$ at $t = 0$.
 - Also used *mean longitude at epoch* $\epsilon = \varpi - n\tau = \varpi + \chi$ (see R05 211; MD 6.8) ϵ is the value of the *mean longitude* $\lambda \equiv \mathcal{M} + \varpi = n(t - \tau) + \varpi = nt + \epsilon$ at $t = 0$.
 - R05 uses the pair (ω, χ) . MD use the pair (ϖ, ϵ) .

2.4 Kepler's problem in Hamiltonian mechanics

2.4.1 Kepler's problem in two dimensions

[VK 4.10]

→ Let us consider for simplicity the planar problem: ϕ and r are polar coordinates in the plane of the orbit. In other words, we are assuming that the reference plane of our coordinate system coincides with the plane of the orbit (inclination $i = 0$).

→ The kinetic energy is

$$T = \frac{1}{2}\mu^* (\dot{r}^2 + r^2\dot{\phi}^2)$$

→ The potential energy is

$$V = -\frac{\mu\mu^*}{r}$$

with $\mu = GM = G(m_1 + m_2)$

→ The Lagrangian is

$$\mathcal{L} = \frac{1}{2}\mu^* (\dot{r}^2 + r^2\dot{\phi}^2) - V(r).$$

→ The generalized momenta are

$$p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = \mu^* \dot{r},$$

$$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \mu^* r^2 \dot{\phi}.$$

→ The Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\mu^* (\dot{r}^2 + r^2\dot{\phi}^2) - \frac{\mu\mu^*}{r},$$

$$\mathcal{H} = \frac{1}{2\mu^*} \left(p_r^2 + \frac{p_\phi^2}{r^2} \right) - \frac{\mu\mu^*}{r},$$

which does not depend on ϕ (i.e., ϕ is a cyclic coordinate)

→ We want to transform $\mathbf{q} = (r, \phi)$ and $\mathbf{p} = (p_r, p_\phi)$ into a new set of canonical coordinates (\mathbf{Q}, \mathbf{P}) , for which the Hamiltonian is zero. So we take the action S as the generating function in the form

$$S = S(\mathbf{q}, \mathbf{P}, t) = S(r, \phi, P_1, P_2, t),$$

with P_1 and P_2 new momenta, which must be constants (because the new Hamiltonian $\mathcal{H}' = 0$).

→ Let us write the H-J equation. We have $p_i = \partial S / \partial q_i$, so $p_r = \partial S / \partial r$, $p_\phi = \partial S / \partial \phi$, and the H-J equation $\mathcal{H} + \partial S / \partial t = 0$ reads

$$\frac{1}{2\mu^*} \left[\left(\frac{\partial S}{\partial r} \right)^2 + \left(\frac{1}{r} \frac{\partial S}{\partial \phi} \right)^2 \right] - \frac{\mu\mu^*}{r} + \frac{\partial S}{\partial t} = 0.$$

→ Using the method of separation of variables, we look for a solution in the form

$$S(r, \phi, t) = S_r(r) + S_\phi(\phi) + S_t(t)$$

→ We get

$$\frac{1}{2\mu^*} \left[\left(\frac{dS_r}{dr} \right)^2 + \left(\frac{1}{r} \frac{dS_\phi}{d\phi} \right)^2 \right] - \frac{\mu\mu^*}{r} = -\frac{dS_t}{dt}.$$

For this to be satisfied we must have

$$\frac{dS_t}{dt} = -\alpha_1 = \text{const}$$

$$\frac{1}{2\mu^*} \left[\left(\frac{dS_r}{dr} \right)^2 + \left(\frac{1}{r} \frac{dS_\phi}{d\phi} \right)^2 \right] - \frac{\mu\mu^*}{r} = \alpha_1,$$

which can be written as

$$\left(\frac{dS_\phi}{d\phi} \right)^2 = r^2 \left[2\mu^* \left(\alpha_1 + \frac{\mu\mu^*}{r} \right) - \left(\frac{dS_r}{dr} \right)^2 \right].$$

For this to be true we must have

$$\frac{dS_\phi}{d\phi} = \alpha_2$$

$$\frac{dS_r}{dr} = \sqrt{2\mu^* \left(\alpha_1 + \frac{\mu\mu^*}{r} \right) - \frac{\alpha_2^2}{r^2}}.$$

→ Thus, the generating function is

$$S = -\alpha_1 t + \alpha_2 \phi + \int dr \sqrt{2\mu^* \left(\alpha_1 + \frac{\mu\mu^*}{r} \right) - \frac{\alpha_2^2}{r^2}}.$$

→ We take the new generalized momenta as

$$P_1 = \alpha_1, \quad P_2 = \alpha_2$$

so the new generalized coordinates are

$$Q_1 = \beta_1 = \frac{\partial S}{\partial P_1} = \frac{\partial S}{\partial \alpha_1}$$

$$Q_2 = \beta_2 = \frac{\partial S}{\partial P_2} = \frac{\partial S}{\partial \alpha_2}$$

→ The new Hamiltonian is

$$\mathcal{H}' = \mathcal{H} + \frac{\partial S}{\partial t} = \mathcal{H} - \alpha_1 = 0,$$

so the integral of motion

$$\alpha_1 = \mathcal{H} = \mu^* \tilde{E} = E = -\frac{\mu^* \mu}{2a}$$

is the total energy.

→ We have introduced the total energy per unit mass

$$\tilde{E} = E/\mu^* = \frac{1}{2}v^2 - \frac{\mu}{r}.$$

We focus on the elliptic case, so

$$\tilde{E} = -\frac{\mu}{2a}$$

→ We also have

$$\frac{\partial S}{\partial \phi} = p_\phi,$$

so

$$\alpha_2 = p_\phi = \mu^* r^2 \dot{\phi} = L = \mu^* \tilde{L} = \mu^* \sqrt{a\mu(1-e^2)},$$

which is the modulus of the angular momentum.

→ We have introduced the angular momentum per unit mass

$$\tilde{L} \equiv r^2 \dot{\phi} = \sqrt{a\mu(1-e^2)},$$

where we have used the definition of eccentricity:

$$e \equiv \sqrt{1 + (2EL^2/\mu^{*3}\mu^2)} = \sqrt{1 - L^2/a\mu^2\mu}.$$

→ So

$$P_1 = -\frac{\mu^*\mu}{2a}$$

$$P_2 = \mu^* \sqrt{a\mu(1-e^2)}$$

→ We now derive Q_1 and Q_2 :

$$Q_1 = \frac{\partial S}{\partial \alpha_1} = -t + I_1$$

where

$$I_1 = \int \frac{\mu^* dr}{\sqrt{2\mu^* \left(\alpha_1 + \frac{\mu\mu^*}{r} \right) - \frac{\alpha_2^2}{r^2}}}$$

$$= \int \frac{\mu^* dr}{\sqrt{2\mu^* \left(-\frac{\mu^*\mu}{2a} + \frac{\mu\mu^*}{r} \right) - \frac{\mu^{*2}a\mu(1-e^2)}{r^2}}}$$

$$= \frac{1}{\sqrt{\mu}} \int \frac{r dr}{\sqrt{-\frac{r^2}{a} + 2r - a(1-e^2)}},$$

and

$$Q_2 = \frac{\partial S}{\partial \alpha_2} = \phi - \frac{\alpha_2}{\mu^*} I_2,$$

where

$$I_2 = \int \frac{\mu^* dr}{r^2 \sqrt{2\mu^* \left(\alpha_1 + \frac{\mu\mu^*}{r} \right) - \frac{\alpha_2^2}{r^2}}}$$

$$\begin{aligned}
&= \int \frac{\mu^* dr}{r^2 \sqrt{2\mu^* \left(-\frac{\mu^* \mu}{2a} + \frac{\mu \mu^*}{r}\right) - \frac{\mu^{*2} a \mu (1-e^2)}{r^2}}} \\
&= \frac{1}{\sqrt{\mu}} \int \frac{dr}{r \sqrt{-\frac{r^2}{a} + 2r - a(1-e^2)}}
\end{aligned}$$

→ The integrals I_1 and I_2 can be solved analytically with the change of variable

$$r = a(1 - e \cos \xi), \quad dr = ae \sin \xi d\xi,$$

where ξ is the eccentric anomaly.

→ We have

$$\begin{aligned}
I_1 &= \frac{1}{\sqrt{\mu}} \int \frac{r dr}{\sqrt{-\frac{r^2}{a} + 2r - a(1-e^2)}} \\
&= \frac{1}{\sqrt{\mu}} \int \frac{a(1-e \cos \xi) ae \sin \xi d\xi}{\sqrt{-a(1-e \cos \xi)^2 + 2a(1-e \cos \xi) - a(1-e^2)}} \\
&= \frac{a^{3/2}}{\sqrt{\mu}} \int \frac{(1-e \cos \xi) e \sin \xi d\xi}{\sqrt{-1 + 2e \cos \xi - e^2 \cos^2 \xi + 2 - 2e \cos \xi - 1 + e^2}} \\
&= \frac{a^{3/2}}{\sqrt{\mu}} \int \frac{(1-e \cos \xi) e \sin \xi d\xi}{\sqrt{e^2 \sin^2 \xi}} \\
&= \frac{a^{3/2}}{\sqrt{\mu}} \int (1-e \cos \xi) d\xi = \frac{a^{3/2}}{\sqrt{\mu}} (\xi - e \sin \xi),
\end{aligned}$$

so

$$Q_1 = -t + I_1 = -t + \frac{a^{3/2}}{\sqrt{\mu}} (\xi - e \sin \xi) = -t + \frac{1}{n} \mathcal{M} = -t + (t - \tau) = -\tau.$$

→ We have

$$\begin{aligned}
I_2 &= \frac{1}{\sqrt{\mu}} \int \frac{dr}{r \sqrt{-r^2/a + 2r - a(1-e^2)}} \\
&= \frac{1}{\sqrt{\mu}} \int \frac{ae \sin \xi d\xi}{a(1-e \cos \xi) \sqrt{-a(1-e \cos \xi)^2 + 2a(1-e \cos \xi) - a(1-e^2)}} \\
&= \frac{1}{\sqrt{a\mu}} \int \frac{e \sin \xi d\xi}{(1-e \cos \xi) \sqrt{-1 + 2e \cos \xi - e^2 \cos^2 \xi + 2 - 2e \cos \xi - 1 + e^2}} \\
&= \frac{1}{\sqrt{a\mu}} \int \frac{e \sin \xi d\xi}{(1-e \cos \xi) \sqrt{e^2 \sin^2 \xi}} \\
&= \frac{1}{\sqrt{a\mu}} \int \frac{d\xi}{1-e \cos \xi} \\
&= \frac{1}{\sqrt{a\mu(1-e^2)}} \int \frac{\sqrt{1-e^2} d\xi}{1-e \cos \xi}
\end{aligned}$$

→ We recall that the relation between true anomaly f and eccentric anomaly ξ is

$$\sin f = \sqrt{1 - e^2} \frac{\sin \xi}{1 - e \cos \xi},$$

so

$$\cos f df = \sqrt{1 - e^2} \frac{\cos \xi (1 - e \cos \xi) - e \sin^2 \xi}{(1 - e \cos \xi)^2} d\xi,$$

$$\cos f df = \sqrt{1 - e^2} \frac{\cos \xi - e}{(1 - e \cos \xi)^2} d\xi,$$

$$\cos f df = \sqrt{1 - e^2} \frac{\cos f}{(1 - e \cos \xi)} d\xi,$$

because

$$\cos f = \frac{\cos \xi - e}{1 - e \cos \xi},$$

so

$$df = \frac{\sqrt{1 - e^2}}{(1 - e \cos \xi)} d\xi,$$

thus

$$I_2 = \frac{1}{\sqrt{a\mu(1 - e^2)}} \int df = \frac{f}{\sqrt{a\mu(1 - e^2)}}$$

So

$$\begin{aligned} Q_2 &= \phi - \frac{\alpha_2}{\mu^*} I_2 = \phi - \frac{\alpha_2}{\mu^*} \frac{f}{\sqrt{a\mu(1 - e^2)}} \\ &= \phi - \sqrt{a\mu(1 - e^2)} \frac{f}{\sqrt{a\mu(1 - e^2)}} = \phi - f = \phi - (\phi - \omega) = \omega \end{aligned}$$

where we have used

$$\alpha_2 = \mu^* \sqrt{a\mu(1 - e^2)}.$$

→ In summary, the new generalized coordinates are $Q_1 = \beta_1 = -\tau$ (minus the time of pericentric passage) and $Q_2 = \beta_2 = \omega$ (argument of pericentre). The new generalized momenta are $P_1 = \alpha_1 = \mu^* \tilde{E}$ (total energy) and $P_2 = \alpha_2 = \mu^* \sqrt{a\mu(1 - e^2)}$ (angular momentum modulus). All of these are constants. The new Hamiltonian is $\mathcal{H}' = 0$.

→ With the above canonical transformation we have obtained 4 constant canonical coordinates, i.e. 4 integrals of motion. These integrals of motions fully constrain the orbit. The solution of the equations of motion, i.e. $r = r(t)$ and $\phi = \phi(r)$, is given by the equations

$$\beta_1 = \frac{\partial S}{\partial \alpha_1}, \quad \beta_2 = \frac{\partial S}{\partial \alpha_2}.$$

2.4.2 Kepler's problem in three dimensions

[VK 4.11]

→ We now consider Kepler's problem in 3D: in practice we take a system of coordinates in which the reference plane does not coincide with the orbital plane. We recall that the motion is planar: however it is often convenient to describe it in 3D (for instance, when describing the orbit of a body in the Solar System, it is useful to take as reference plane the ecliptic: see also Chapter on perturbation theory).

→ The Lagrangian in spherical coordinates is

$$\mathcal{L} = \frac{1}{2}\mu^* \left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right) + \frac{\mu^* \mu}{r}.$$

Note that when $\sin \theta = 1$ and $\dot{\theta} = 0$ we reduce the problem to the 2D case.

→ The generalized momenta are

$$\begin{aligned} p_r &= \frac{\partial \mathcal{L}}{\partial \dot{r}} = \mu^* \dot{r}, \\ p_\theta &= \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \mu^* r^2 \dot{\theta}, \\ p_\phi &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \mu^* r^2 \sin^2 \theta \dot{\phi}. \end{aligned}$$

→ The Hamiltonian of the Kepler problem in spherical coordinates is

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}\mu^* (\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) - \frac{\mu^* \mu}{r}. \\ \mathcal{H} &= \frac{1}{2\mu^*} \left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2 \theta} \right) - \frac{\mu^* \mu}{r}. \end{aligned}$$

→ The H-J equation reads

$$\frac{1}{2\mu^*} \left[\left(\frac{\partial S}{\partial r} \right)^2 + \left(\frac{1}{r} \frac{\partial S}{\partial \theta} \right)^2 + \left(\frac{1}{r \sin \theta} \frac{\partial S}{\partial \phi} \right)^2 \right] - \frac{\mu^* \mu}{r} + \frac{\partial S}{\partial t} = 0,$$

→ The action as generating function is in the form $S = S(\mathbf{q}, \mathbf{P}, t) = S(r, \theta, \phi, P_1, P_2, P_3, t)$ with $P_i = \alpha_i = \text{const.}$

Using the method of separation of variables, we look for a solution in the form

$$S(r, \theta, \phi, t) = S_r(r) + S_\theta(\theta) + S_\phi(\phi) + S_t(t)$$

→ We have

$$\begin{aligned} \frac{\partial S_t}{\partial t} &= -\alpha_1 \\ \frac{\partial S_\phi}{\partial \phi} &= \alpha_3 \\ \left(\frac{\partial S_\theta}{\partial \theta} \right)^2 + \frac{\alpha_3^2}{\sin^2 \theta} &= \alpha_2^2 \\ \left(\frac{\partial S_r}{\partial r} \right)^2 + \frac{\alpha_2^2}{r^2} &= 2\mu^* \left(\alpha_1 + \frac{\mu^* \mu}{r} \right) \end{aligned}$$

→ Performing calculations analogous to the 2-D case we get the (constant) generalized coordinates

$$Q_1 = \beta_1 = -\tau, \quad (\text{time of pericentric passage})$$

$$Q_2 = \omega, \quad (\text{argument of pericentre})$$

$$Q_3 = \Omega, \quad (\text{longitude of the ascending node}),$$

and the (constant) generalized momenta

$$P_1 = \mu^* \tilde{E} = -\frac{\mu^* \mu}{2a}, \quad (\text{total energy}),$$

$$P_2 = \mu^* \sqrt{a\mu(1-e^2)}, \quad (\text{angular momentum modulus}),$$

$$P_3 = \mu^* \sqrt{a\mu(1-e^2)} \cos i, \quad (z\text{-component of angular momentum}).$$

The corresponding Hamiltonian is $\mathcal{H}' = 0$.

→ With the above canonical transformation we have obtained 6 constant canonical coordinates, i.e. 6 integrals of motion. These integrals of motions fully constrain the orbit. The solution of the equations of motion is given by the equations

$$\beta_i = \frac{\partial S}{\partial \alpha_i}, \quad i = 1, 2, 3.$$

→ It is also possible to obtain another set of canonical coordinates, maintaining the generalized coordinates as they are, but mass-normalizing the generalized momenta and the Hamiltonian. So, we start from Q_i, P_i and define a new set of variables $\tilde{q}_i = Q_i, \tilde{p}_i = P_i/\mu^*$. The equations of motion keep the canonical form with the Hamiltonian $\tilde{\mathcal{H}} = \mathcal{H}'/\mu^*$ (i.e. this is a canonical transformation; see G09). This can be seen also by noting that

$$\dot{P}_i = -\frac{\partial \mathcal{H}'}{\partial Q_i} \implies \frac{\dot{P}_i}{\mu^*} = -\frac{\partial(\mathcal{H}'/\mu^*)}{\partial Q_i} \implies \dot{\tilde{p}}_i = -\frac{\partial \tilde{\mathcal{H}}}{\partial \tilde{q}_i}$$

and

$$\dot{Q}_i = \frac{\partial \mathcal{H}'}{\partial P_i} = \frac{\partial(\mathcal{H}'/\mu^*)}{\partial P_i/\mu^*} \implies \dot{\tilde{q}}_i = \frac{\partial \tilde{\mathcal{H}}}{\partial \tilde{p}_i}.$$

→ The (constant) mass-normalized canonical coordinates are

$$\tilde{\beta}_1 = \tilde{q}_1 = -\tau, \quad \tilde{\beta}_2 = \tilde{q}_2 = \omega, \quad \tilde{\beta}_3 = \tilde{q}_3 = \Omega$$

$$\tilde{\alpha}_1 = \tilde{p}_1 = -\frac{\mu}{2a}, \quad \tilde{\alpha}_2 = \tilde{p}_2 = \sqrt{a\mu(1-e^2)}, \quad \tilde{\alpha}_3 = \tilde{p}_3 = \sqrt{a\mu(1-e^2)} \cos i.$$

The mass-normalized Hamiltonian is $\tilde{\mathcal{H}} = \mathcal{H}'/\mu^* = 0$.

2.4.3 Delaunay's variables

[VK 4.12]

→ We can perform a further canonical transformation to transform the above set in a new set of angle-action coordinates, known as Delaunay's variables (coordinates l_D , g_D , h_D , and momenta L_D , G_D , H_D).

→ While \tilde{q}_2 and \tilde{q}_3 are angles, $\tilde{q}_1 = -\tau$ is not. So we replace $-\tau$ with the mean anomaly $\mathcal{M} = n(t - \tau) = n(t + \tilde{q}_1)$, which is an angle. We want to find the transformation to have

$$l_D = n(t + \tilde{q}_1)$$

$$g_D = \tilde{q}_2 = \omega$$

$$h_D = \tilde{q}_3 = \Omega$$

$$L_D = ? \quad (\text{to be determined})$$

$$G_D = \tilde{p}_2 = \sqrt{a\mu(1 - e^2)}$$

$$H_D = \tilde{p}_3 = \sqrt{a\mu(1 - e^2)} \cos i$$

where L_D must be found consistently with the choice of l_D .

→ A generating function that does the job is

$$F(\tilde{q}_1, \tilde{q}_2, \tilde{q}_3, L_D, G_D, H_D, t) = \left(nL_D - \frac{3\mu}{2a} \right) (t + \tilde{q}_1) + \tilde{q}_2 G_D + \tilde{q}_3 H_D,$$

where the term $-3\mu/2a$ appears to simplify the form of L_D and of the Hamiltonian.

→ We have

$$l_D = \frac{\partial F}{\partial L_D} = n(t + \tilde{q}_1)$$

$$g_D = \frac{\partial F}{\partial G_D} = \tilde{q}_2$$

$$h_D = \frac{\partial F}{\partial H_D} = \tilde{q}_3$$

$$\tilde{p}_1 = \frac{\partial F}{\partial \tilde{q}_1} = nL_D - \frac{3\mu}{2a}$$

$$\tilde{p}_2 = \frac{\partial F}{\partial \tilde{q}_2} = G_D,$$

$$\tilde{p}_3 = \frac{\partial F}{\partial \tilde{q}_3} = H_D,$$

so

$$L_D = \frac{1}{n} \left(\tilde{p}_1 + \frac{3\mu}{2a} \right) = \frac{a^{3/2}}{\mu^{1/2}} \left(-\frac{\mu}{2a} + \frac{3\mu}{2a} \right) = \frac{a^{3/2}}{\mu^{1/2}} \frac{\mu}{a} = \sqrt{a\mu}$$

→ The new (Delaunay's) Hamiltonian is

$$\mathcal{K}_D = 0 + \frac{\partial F}{\partial t} = nL_D - \frac{3\mu}{2a} = \sqrt{\mu a}^{-3/2} \sqrt{a\mu} - \frac{3\mu}{2a} = -\frac{\mu}{2a} = -\frac{\mu^2}{2a\mu} = -\frac{\mu^2}{2L_D^2}$$

→ In summary the Delaunay's variables are

$$l_D = n(t - \tau) = \mathcal{M}$$

$$g_D = \tilde{q}_3 = \omega$$

$$h_D = \tilde{q}_3 = \Omega$$

$$L_D = \sqrt{a\mu}$$

$$G_D = \tilde{p}_2 = \sqrt{a\mu(1 - e^2)}$$

$$H_D = \tilde{p}_3 = \sqrt{a\mu(1 - e^2)} \cos i$$

and the corresponding Hamiltonian is

$$\mathcal{K}_D = -\frac{\mu^2}{2L_D^2}.$$

→ Five of the Delaunay's variables are constants, but l_D (mean anomaly) is not constant and varies linearly with time.

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