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1. Fundamentals of mechanics

1.1 Lagrangian mechanics

[LL]

1.1.1 Generalized coordinates

- → Particle: point mass
- → Particle position vector $\mathbf{r}$. In Cartesian components $\mathbf{r} = (x, y, z)$.
- → Particle velocity $\mathbf{v} = \mathrm{d}\mathbf{r}/\mathrm{d}t = \dot{\mathbf{r}}$. In Cartesian coordinates $v_x = \mathrm{d}x/\mathrm{d}t$, etc...
- → Particle acceleration $\mathbf{a} = \mathrm{d}^2\mathbf{r}/\mathrm{d}t^2 = \dot{\mathbf{v}} = \ddot{\mathbf{r}}$. In Cartesian coordinates $a_x = \mathrm{d}^2x/\mathrm{d}t^2$, etc...
- → N particles $\Longrightarrow s = 3N$ degrees of freedom
- → Generalized coordinates: any s quantities q_i that define the positions of the N-body system ($\mathbf{q} = q_1, ..., q_s$)
- → Generalized velocities: $\dot{q}_i$ ($\dot{\mathbf{q}} = \dot{q}_1, ..., \dot{q}_s$)
- → We know from experience that, given $\mathbf{q}$ and $\dot{\mathbf{q}}$ for all particles in the system at a given time, we are able to predict $\mathbf{q}(t)$ at any later time t. In other words, if all $\mathbf{q}$ and $\dot{\mathbf{q}}$ are specified $\Longrightarrow \ddot{\mathbf{q}}$ are known.
- → Equations of motion are ODE for $\mathbf{q}(t)$ that relate $\ddot{\mathbf{q}}$ with $\mathbf{q}$ and $\dot{\mathbf{q}}$. The solution $\mathbf{q}(t)$ is the path (orbit).

1.1.2 Principle of least action & Euler-Lagrange equations

- → Given a mechanical system, we define the *Lagrangian* function $\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$. $\mathcal{L}$ does not depend on higher derivatives, consistent with the idea that motion is determined if $\mathbf{q}$ and $\dot{\mathbf{q}}$ are given.
- → Given two instants t_1 and t_2, we define the *action* $S = \int_{t_1}^{t_2} \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t)dt$
- → The system occupies positions $\mathbf{q}_1$ and $\mathbf{q}_2$ at time t_1 and t_2, respectively. Note that in this formalism instead of fixing position and velocity at the initial time t_1, we fix positions at the initial and final times.

→ *Principle of least action* (or Hamilton's principle): from t_1 to t_2 the system moves in such a way that S is a minimum (extremum) over all paths, i.e. (for 1 degree of freedom)

$$\delta S = \delta \int_{t_1}^{t_2} \mathcal{L}(q,\dot{q},t)\mathrm{d}t = \int_{t_1}^{t_2} \left[\mathcal{L}(q+\delta q,\dot{q}+\delta\dot{q},t) - \mathcal{L}(q,\dot{q},t)\right]\mathrm{d}t = \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial q}\delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}}\delta \dot{q}\right)\mathrm{d}t = 0,$$

Now, we have

$$\frac{\partial \mathcal{L}}{\partial q}\delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{\mathrm{d}(\delta q)}{\mathrm{d}t} = \frac{\partial \mathcal{L}}{\partial q}\delta q + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\delta q\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)\delta q,$$

so the above equation can be rewritten as

$$\delta S = \left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\delta q\right]_{t_1}^{t_2} + \int_{t_1}^{t_2}\left(\frac{\partial \mathcal{L}}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)\delta q \mathrm{d}t = 0,$$

which is verified for all δq only when

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}} = 0,$$

because $\delta q(t_1) = \delta q(t_2) = 0$, as all possible paths are such that $q(t_1) = q_1$ and $q(t_2) = q_2$.

→ Generalizing to the case of s degrees of freedom we have the Euler-Lagrange (E-L) equations:

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0, \qquad i = 1, ..., s$$

→ Transformations like $\mathcal{L} \to A\mathcal{L}$, with A constant, or $\mathcal{L} \to \mathcal{L} + \mathrm{d}F/dt$, where $F = F(\mathbf{q}, t)$ do not affect the particles' motion, because

$$\mathcal{L}' = A\mathcal{L} \implies \delta S' = \delta AS = A\delta S = 0 \iff \delta S = 0,$$

and

$$\mathcal{L}' = \mathcal{L} + \mathrm{d}F/dt \implies S' = S + \int_{t_1}^{t_2}\frac{\mathrm{d}F}{\mathrm{d}t}\mathrm{d}t = S + F(\mathbf{q}_2, t_2) - F(\mathbf{q}_1, t_1) = S + C,$$

where C is a constant (independent of $\mathbf{q}$, $\dot{\mathbf{q}}$).

1.1.3 Inertial frames

→ *Inertial reference frame*: such that space is homogeneous and isotropic and time is homogeneous. For instance, in any inertial reference frame a particle that is at rest at a given time will remain at rest at all later times.

→ Galileo's relativity principle: laws of motion are the same in all inertial reference frames (moving at constant velocity w.r.t. one another)

→ Free particle: particle subject to no force.

→ Lagrangian of a free particle cannot contain explicitly the position vector $\mathbf{r}$ (space is homogeneous) or the time t (time is homogeneous) and cannot depend on the direction of $\mathbf{v}$ (space is isotropic) $\implies \mathcal{L} = \mathcal{L}(v^2)$

→ More specifically, it can be shown (see LL) that for a free particle

$$\mathcal{L} = \frac{1}{2}mv^2,$$

where m is particle mass. $T = (1/2)mv^2$ is the particle kinetic energy.

→ In Cartesian coordinates $\mathbf{q} = \mathbf{r} = (x, y, z)$ and $\dot{\mathbf{q}} = \mathbf{v} = (v_x, v_y, v_z)$, so for a free particle the Lagrangian is $\mathcal{L} = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2)$. The E-L equations for a free particle are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \mathbf{v}} = 0,$$

so for the x component

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial v_x} = m\frac{\mathrm{d}v_x}{\mathrm{d}t} = 0 \qquad \Longrightarrow \qquad v_x = const,$$

and similarly for y and z components. $\Longrightarrow \mathrm{d}\mathbf{v}/\mathrm{d}t = 0$, which is the law of inertia (*Newton's first law of motion*).

1.1.4 Lagrangian of a free particle in different systems of coordinates

→ Let's write the Lagrangian of a free particle in different systems of coordinates. Note that, if $\mathrm{d}l$ is the infinitesimal displacement, $v^2 = (\mathrm{d}l/\mathrm{d}t)^2 = \mathrm{d}l^2/\mathrm{d}t^2$.

→ In Cartesian coordinates $\mathrm{d}l^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$, so

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

→ In cylindrical coordinates $\mathrm{d}l^2 = \mathrm{d}R^2 + R^2\mathrm{d}\phi^2 + \mathrm{d}z^2$, so

$$\mathcal{L} = \frac{1}{2}m(\dot{R}^2 + R^2\dot{\phi}^2 + \dot{z}^2)$$

→ In spherical coordinates $\mathrm{d}l^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2$, so

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2)$$

→ The above can be derived also by taking the expression of the position vector $\mathbf{r}$, differentiating and squaring (see e.g. BT08 app. B). For instance, in cylindrical coordinates

$$\mathbf{r} = R\mathbf{e}_R + z\mathbf{e}_z$$

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \dot{R}\mathbf{e}_R + R\dot{\mathbf{e}}_R + \dot{z}\mathbf{e}_z,$$

because $\dot{\mathbf{e}}_z = 0$. Now,

$$\mathbf{e}_R = \cos\phi\mathbf{e}_x + \sin\phi\mathbf{e}_y$$

and

$$\mathbf{e}_\phi = -\sin\phi\mathbf{e}_x + \cos\phi\mathbf{e}_y$$

So

$$\mathrm{d}\mathbf{e}_R = (-\sin\phi\mathbf{e}_x + \cos\phi\mathbf{e}_y)\mathrm{d}\phi$$

$$\mathrm{d}\mathbf{e}_R = \mathbf{e}_\phi \mathrm{d}\phi$$

$$\dot{\mathbf{e}}_R = \dot{\phi}\mathbf{e}_\phi$$

so

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \dot{R}\mathbf{e}_R + R\dot{\phi}\mathbf{e}_\phi + \dot{z}\mathbf{e}_z,$$

$$v^2 = \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|^2 = \dot{R}^2 + R^2\dot{\phi}^2 + \dot{z}^2.$$

→ Alternatively, we can derive the same expressions for the cylindrical and spherical coordinates, starting from the Cartesian coordinates. For instance, in spherical coordinates r, θ, ϕ we have

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta,$$

so

$$\dot{x} = \dot{r}\sin\theta\cos\phi + r\cos\theta\cos\phi\dot{\theta} - r\sin\theta\sin\phi\dot{\phi},$$

$$\dot{y} = \dot{r}\sin\theta\sin\phi + r\cos\theta\sin\phi\dot{\theta} + r\sin\theta\cos\phi\dot{\phi},$$

$$\dot{z} = \dot{r}\cos\theta - r\sin\theta\dot{\theta},$$

then

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2.$$

→ The Lagrangian of a free particle in spherical coordinates is

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right).$$

1.1.5 Lagrangian of a system of particles

→ Additivity of the Lagrangian: take two dynamical systems A and B. If each of them were an isolated system, they would have, respectively, Lagrangians $\mathcal{L}_A$ and $\mathcal{L}_B$. If they are two parts of the same system (but so distant that the interaction is negligible) the total Lagrangian must be $\mathcal{L} = \mathcal{L}_A + \mathcal{L}_B$.

→ So, for a *system of non-interacting particles*

$$\mathcal{L} = \sum_a \frac{1}{2}m_a v_a^2,$$

were the subscript a identifies the a-th the particle.

→ Closed system: system of particles that interact, but are not affected by external forces.

→ Lagrangian for a closed *system of interacting particles*:

$$\mathcal{L} = T - V$$

where T is the kinetic energy and V is potential energy.

→ The potential energy V depends only on the position of the particles: $V = V(\mathbf{q})$. This is a consequence of the assumption that the interaction is instantaneously propagated: a change in position of one of the particles instantaneously affects the force experienced by the other particles.

→ We have seen that in general $T = T(\mathbf{q}, \dot{\mathbf{q}})$: see, for instance, the expression of T in cylindrical or spherical coordinates. In other words, in generalized coordinates the kinetic energy can depend also on the $\mathbf{q}$, not only on $\dot{\mathbf{q}}$:

$$\mathcal{L} = \frac{1}{2}\sum_{i,k} A_{ik}(\mathbf{q})\dot{q}_i\dot{q}_k - V(\mathbf{q}),$$

where $i = 1, \ldots, s$ and $k = 1, \ldots, s$, where s is the number of degrees of freedom ($s = 3N$ for a system of N particles).

→ In Cartesian coordinates we have $\mathbf{q} = \mathbf{r}_a = (x_a, y_a, z_a)$ positions and $\dot{\mathbf{q}} = \mathbf{v}_a = (v_{x,a}, v_{y,a}, v_{z,a})$ (velocities), so the Lagrangian for a closed system of N particles is

$$\mathcal{L} = \frac{1}{2}\sum_{a=1,..,N} m_a v_a^2 - V(\mathbf{r}_1, \mathbf{r}_2, ...\mathbf{r}_N)$$

→ Applying Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \mathbf{r}_a} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \mathbf{v}_a} = 0,$$

we get the equations of motion:

$$m_a\dot{\mathbf{v}}_a = -\frac{\partial V}{\partial \mathbf{r}_a}, \qquad i.e. \qquad m_a\ddot{x}_a = -\frac{\partial V}{\partial x_a}, \qquad etc.$$

i.e. $m_a\ddot{\mathbf{r}} = \mathbf{F}_a$ (*Newton's second law of motion*), where $\mathbf{F}_a = -\partial V/\partial \mathbf{r}_a$ is the force acting on the a-th particle.

→ For a particle moving in an external field

$$\mathcal{L} = \frac{1}{2}mv^2 - V(\mathbf{r}, t).$$

If the external field is uniform $\implies V = -\mathbf{F}(t) \cdot \mathbf{r}$ ($\mathbf{F}$ dependent of time, but independent of position)

1.2 Conservation laws

→ *Constant of motion:* quantity that remains constant during the evolution of a mechanical system $C = C[\mathbf{q}(t), \dot{\mathbf{q}}(t), t] = const$ (i.e. $\mathrm{d}C/\mathrm{d}t = 0$).

→ *Integral of motion:* a constant of motion that depends only on $\mathbf{q}$ and $\dot{\mathbf{q}}$ (in other words, it does not depend explicitly on time) $I = I[\mathbf{q}(t), \dot{\mathbf{q}}(t)] = const$ (i.e. $\mathrm{d}I/\mathrm{d}t = 0$). The value of the integral for a system equals the sum of the values for sub-systems that interact negligibly with one another.

→ Integrals of motion derive from fundamental properties (symmetries): isotropy/homogeneity of time and space. Among constants of motions, only integrals of motions are important in mechanics. Example of a constant of motion that is not an integral of motion: for a 1-D free particle $x(t) = x_0 + \dot{x}_0 t$ (where x_0 and $\dot{x}_0$ are the initial conditions and $\dot{x} = \dot{x}_0 = const$), so $x_0(x,t) = x(t) - \dot{x}_0 t$ is a constant of motion, but not an integral of motion (it depends explicitly on t).

→ There are seven integrals of motions: total energy E, momentum $\mathbf{P}$ (3 components), angular momentum $\mathbf{L}$ (3 components).

1.2.1 Energy

→ Homogeneity of time $\Longrightarrow$ Lagrangian of a closed system does not depend explicitly on time $\partial\mathcal{L}/\partial t = 0 \Longrightarrow$

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} = \sum_i \left[\frac{\partial\mathcal{L}}{\partial q_i}\dot{q}_i + \frac{\partial\mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i\right],$$

where $i = 1, \ldots, s$, with s number of degrees of freedom. Using E-L equations:

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} = \sum_i \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial \dot{q}_i}\right)\dot{q}_i + \frac{\partial\mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i\right] = \sum_i \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial \dot{q}_i}\dot{q}_i\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i \frac{\partial\mathcal{L}}{\partial \dot{q}_i}\dot{q}_i - \mathcal{L}\right) = 0$$

→ Energy

$$E \equiv \sum_i \frac{\partial\mathcal{L}}{\partial \dot{q}_i}\dot{q}_i - \mathcal{L}$$

$\Longrightarrow \mathrm{d}E/\mathrm{d}t = 0$

→

$$\mathcal{L} = T - V \implies E = \sum_i \frac{\partial T}{\partial \dot{q}_i}\dot{q}_i - T + V$$

→ We have seen that T is a quadratic function of $\dot{q}_i$, so by Euler theorem on homogeneous functions $\dot{q}_i \partial T/\partial \dot{q}_i = 2T$, so

$$E = 2T - \mathcal{L} = T(\mathbf{q}, \dot{\mathbf{q}}) + V(\mathbf{q}),$$

i.e. total energy E is the sum of kinetic and potential energy.

→ Euler theorem on homogeneous functions: if $f(tx) = t^n f(x)$ then $xf'(x) = nf(x)$.

1.2.2 Momentum

Homogeneity of space $\Longrightarrow$ conservation of momentum. Lagrangian must be invariant if the system is shifted in space by $\boldsymbol{\epsilon}$.

Cartesian coordinates

→ Lagrangian must be invariant if the system is shifted in space by $\delta\mathbf{r}_a = \boldsymbol{\epsilon}$:

$$\delta\mathcal{L} = \mathcal{L}(\mathbf{q} + \mathrm{d}\mathbf{q}, \dot{\mathbf{q}}) - \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) = \sum_a \frac{\partial \mathcal{L}}{\partial \mathbf{r}_a} \cdot \delta\mathbf{r}_a = \boldsymbol{\epsilon} \cdot \sum \frac{\partial \mathcal{L}}{\partial \mathbf{r}_a}$$

$\Longrightarrow$

$$\sum \frac{\partial \mathcal{L}}{\partial \mathbf{r}_a} = 0 \Longrightarrow \frac{\mathrm{d}}{\mathrm{d}t} \sum \frac{\partial \mathcal{L}}{\partial \mathbf{v}_a} = 0$$

or

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = 0,$$

where

$$\mathbf{P} \equiv \sum \frac{\partial \mathcal{L}}{\partial \mathbf{v}_a} = \sum m_a \mathbf{v}_a$$

is momentum

→ Momentum is additive $\mathbf{P} = \sum_a \mathbf{p}_a$, where $\mathbf{p}_a = \partial\mathcal{L}/\partial\mathbf{v}_a = m_a\mathbf{v}_a$ is the momentum of the individual particles.

→ We also have

$$\sum \frac{\partial \mathcal{L}}{\partial \mathbf{r}_a} = 0 \Longrightarrow \sum \frac{\partial V}{\partial \mathbf{r}_a} = \sum \mathbf{F}_a = 0,$$

where $\mathbf{F}_a$ is the force acting on the a-th particle. When the bodies are two, is $\mathbf{F}_1 + \mathbf{F}_2 = 0$ or $\mathbf{F}_1 = -\mathbf{F}_2$ i.e. *Newton's third law of motion.*

Generalized coordinates

→ We define

$$p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

as *generalized momenta*. Note that $p_i = mv_i$ in Cartesian coordinates, but in general p_i depend on both q_i and $\dot{q}_i$.

→ The E-L equation can be written as

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = F_i,$$

where $F_i = \partial\mathcal{L}/\partial q_i$ is the generalized force.

→ Then

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_i p_i = \sum_i F_i = 0.$$

Centre of mass

→ For a system of particles there is a special inertial reference frame in which $\mathbf{P} = 0$: this is the reference frame in which the centre of mass is at rest.

$$\mathbf{P} = \sum m_a \mathbf{v}_a = \frac{\mathrm{d}}{\mathrm{d}t}\sum m_a \mathbf{r}_a = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}_{\mathrm{cm}}\sum m_a = \sum m_a \frac{\mathrm{d}\mathbf{r}_{\mathrm{cm}}}{\mathrm{d}t} = 0,$$

where

$$\mathbf{r}_{\mathrm{cm}} \equiv \frac{\sum m_a \mathbf{r}_a}{\sum m_a}$$

is the position of the centre of mass.

→ In a general inertial frame the centre of mass moves with a velocity

$$\mathbf{v}_{\mathrm{cm}} = \frac{\mathrm{d}\mathbf{r}_{\mathrm{cm}}}{\mathrm{d}t} = \frac{\sum m_a \mathbf{v}_a}{\sum m_a} = \frac{\mathbf{P}}{\sum m_a} = const$$

→ If the total energy of the system in the centre-of-mass reference frame is E_{int}, in a general inertial frame the total energy is

$$E = \frac{1}{2}\sum_a m_a v_{\mathrm{cm}}^2 + E_{int}$$

→ Note: the components of the centre of mass $\mathbf{r}_{\mathrm{cm}}$ are not constants of motion. The components of $\mathbf{r}_{\mathrm{cm}}(0) = \mathbf{r}_{\mathrm{cm}}(t) - t\mathbf{v}_{\mathrm{cm}}$ are constants of motion, but they are not integrals of motion (they depend explicitly on time).

1.2.3 Angular momentum

→ Isotropy of space $\Longrightarrow$ conservation of angular momentum

→ The Lagrangian is invariant under rotation. Apply a rotation represented by a vector $\delta\boldsymbol{\phi}$ (with magnitude $\delta\phi$, which is the angle of rotation, and direction along the rotation axis) $\Longrightarrow \delta\mathbf{r} = \delta\boldsymbol{\phi} \times \mathbf{r}$ and $\delta\mathbf{v} = \delta\boldsymbol{\phi} \times \mathbf{v}$

$$\delta\mathcal{L} = \sum_a \left(\frac{\partial\mathcal{L}}{\partial\mathbf{r}_a}\cdot\delta\mathbf{r}_a + \frac{\partial\mathcal{L}}{\partial\mathbf{v}_a}\cdot\delta\mathbf{v}_a\right) = 0$$

$$\delta\mathcal{L} = \sum_a \left[\dot{\mathbf{p}}_a\cdot(\delta\boldsymbol{\phi}\times\mathbf{r}_a) + \mathbf{p}_a\cdot(\delta\boldsymbol{\phi}\times\mathbf{v}_a)\right] = 0.$$

Using the vector identity $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})$ we get

$$\delta\mathcal{L} = \delta\boldsymbol{\phi}\cdot\sum(\mathbf{r}_a\times\dot{\mathbf{p}}_a + \mathbf{v}_a\times\mathbf{p}_a) = \delta\boldsymbol{\phi}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\sum\mathbf{r}_a\times\mathbf{p}_a = 0,$$

because

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}_a\times\mathbf{p}_a) = \mathbf{r}_a\times\dot{\mathbf{p}}_a + \mathbf{v}_a\times\mathbf{p}_a$$

→ As this must be satisfied for all $\delta\boldsymbol{\phi}$, we must have

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = 0,$$

where

$$\mathbf{L} \equiv \sum \mathbf{r}_a \times \mathbf{p}_a$$

is the angular momentum, which (as well as linear momentum) is additive.

→ The angular momentum in a reference frame in which the system is at rest ($\mathbf{P} = 0$) is $\mathbf{L}_{int}$ (intrinsic angular momentum)

→ In a general inertial frame the angular momentum is

$$\mathbf{L} = \mathbf{L}_{int} + \mathbf{r}_{\rm cm} \times \left(\sum m_a\right) \mathbf{v}_{\rm cm} = \mathbf{L}_{int} + \mathbf{r}_{\rm cm} \times \sum m_a \mathbf{v}_a = \mathbf{L}_{int} + \mathbf{r}_{\rm cm} \times \mathbf{P}$$

1.3 Integration of the equations of motion

1.3.1 Motion in one dimension

→ One dimension = one degree of freedom = one coordinate q

→ Lagrangian:

$$\mathcal{L} = \frac{1}{2} a(q) \dot{q}^2 - V(q).$$

If $q = x$ is a Cartesian coordinate

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 - V(x).$$

→ Energy is integral of motion:

$$E = \frac{1}{2} m \dot{x}^2 + V(x),$$

then (taking $\dot{x} \geq 0$)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{\frac{2}{m}[E - V(x)]},$$

$$\mathrm{d}t = \frac{\sqrt{m}}{\sqrt{2[E - V(x)]}} \mathrm{d}x,$$

$$t = t_0 + \sqrt{\frac{m}{2}} \int_{x_0}^{x} \frac{1}{\sqrt{[E - V(x)]}} \mathrm{d}x.$$

→ Motion in region of space such that $V(x) < E$. If this interval is bounded, motion is finite. From the above equations and the E-L equations (equations of motion) it is clear that motion is oscillatory ($\dot{x}$ changes sign only at turning points x such that $V(x) = E$) $\Longrightarrow$ motion is periodic with period

$$T = \sqrt{2m} \int_{x_1}^{x_2} \frac{1}{\sqrt{[E - V(x)]}} \mathrm{d}x,$$

where x_1 and x_2 are the turning points at which $E = V(x_1) = V(x_2)$. T is twice the time to go from x_1 to x_2 (see Fig. 6 LL. FIG CM1.1).

1.3.2 Motion in a central field

→ Motion in a central field: motion of a single particle in an external field such that its potential energy depends only on the distance r from a fixed point: $V = V(r) \implies$

$$\mathbf{F} = -\frac{\partial V}{\partial \mathbf{r}} = -\frac{\mathrm{d}V}{\mathrm{d}r}\frac{\mathbf{r}}{r}$$

→ For instance, in Cartesian coordinates:

$$F_x = -\frac{\partial V}{\partial x} = -\frac{dV}{dr}\frac{\partial r}{\partial x} = -\frac{dV}{dr}\frac{x}{r},$$

etc., because

$$\frac{\partial r}{\partial x} = \frac{\partial\sqrt{x^2+y^2+z^2}}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

→ Take center of the field as origin: angular momentum $\mathbf{L}$ is conserved (even in the presence of the field), because the field does not have component orthogonal to position vector.

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \dot{\mathbf{r}} \times \mathbf{p} + \mathbf{r} \times \dot{\mathbf{p}} = 0.$$

→ $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is conserved and is orthogonal to $\mathbf{r}$, so $\mathbf{r}$ stays always in the same plane $\implies$ motion is planar.

→ Using polar coordinates (r, ϕ) in the plane of the motion, the Lagrangian reads (see kinetic energy in cylindrical coordinates: Section 1.1.4)

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - V(r)$$

→ *Motion in ϕ*. E-L equations for coordinate $\phi \implies$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\mathrm{d}(mr^2\dot{\phi})}{\mathrm{d}t} = 0,$$

where

$$L_z = L = mr^2\dot{\phi} = const$$

is the modulus of the angular momentum. ϕ is ciclic coordinate: it does not appear in $\mathcal{L}$. Associated generalized momentum is constant.

→ *Kepler's second law*: let's define an infinitesimal sector bounded by the path as

$$\mathrm{d}A = \frac{1}{2}r^2\mathrm{d}\phi$$

(show Fig. 8 LL FIG CM1.2). $\mathrm{d}A/\mathrm{d}t = r^2(\mathrm{d}\phi/\mathrm{d}t)/2 = L/(2m) = const$ is the sectorial velocity $\implies$ the particle's position vector sweeps equal areas in equal times (Kepler's second law).

→ *Motion in r.*

$$E = 2T - \mathcal{L} = const,$$

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + V(r) = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + V(r),$$

$\Longrightarrow$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \sqrt{\frac{2}{m}[E - V_{\mathrm{eff}}(r)]}$$

$$\mathrm{d}t = \frac{\mathrm{d}r}{\sqrt{\frac{2}{m}[E - V_{\mathrm{eff}}(r)]}}$$

(time t as a function of r), where

$$V_{\mathrm{eff}}(r) = V(r) + \frac{L^2}{2mr^2}.$$

→ The radial part of the motion behaves like a motion in one-dimension with effective potential energy $V_{\mathrm{eff}}(r)$, defined above, where $L^2/2mr^2$ is called the centrifugal energy.

→ The radii r such that $E = V_{\mathrm{eff}}(r)$ are the radial turning points, corresponding to $\dot{r} = 0$: if motion is finite, pericentre (r_{peri}) and apocentre (r_{apo}). If motion is infinite $r_{\mathrm{apo}} = \infty$

→ Substituting $mr^2\mathrm{d}\phi/L = \mathrm{d}t$

$$\mathrm{d}\phi = \frac{L\mathrm{d}r}{r^2\sqrt{2m[E - V_{\mathrm{eff}}(r)]}}$$

(angle ϕ as a function of r, i.e. path or trajectory).

→ Consider variation of ϕ for finite motion in one radial period:

$$\Delta\phi = 2\int_{r_{\mathrm{peri}}}^{r_{\mathrm{apo}}} \frac{L\mathrm{d}r}{r^2\sqrt{2m[E - V_{\mathrm{eff}}(r)]}}$$

→ Closed orbit only if $\Delta\phi = 2\pi m/n$ with m, n integers. In general orbit is not closed (rosette). All orbits are closed only when $V \propto 1/r$ (Kepler's potential) or $V \propto r^2$ (harmonic potential).

1.4 Hamiltonian mechanics

[LL; VK]

1.4.1 Hamilton's equations

→ In Lagrangian mechanics generalized coordinates (q_i) and generalized velocities ($\dot{q}_i$), $i = 1, ..., s$, where s is the number of degrees of freedom.

→ In Hamiltonian mechanics generalized coordinates (q_i) and generalized momenta (p_i), $i = 1, ..., s$.

→ The idea is to transform $\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$ into a function of $(\mathbf{q}, \mathbf{p}, t)$, where $p_i = \partial\mathcal{L}/\partial\dot{q}_i$ are the generalized momenta. This can be accomplished through a *Legendre transform*.

→ Example: Legendre transform for functions of two variables. Start from $f = f(x, y)$, $u \equiv \partial f/\partial x$ and $v \equiv \partial f/\partial y$. The total differential of f is

$$\mathrm{d}f = u\mathrm{d}x + v\mathrm{d}y.$$

We want to replace y with v, so we use

$$\mathrm{d}(vy) = v\mathrm{d}y + y\mathrm{d}v$$

so

$$\mathrm{d}f = u\mathrm{d}x + \mathrm{d}(vy) - y\mathrm{d}v,$$

$$\mathrm{d}(vy - f) = -u\mathrm{d}x + y\mathrm{d}v,$$

so $g(x, v) \equiv vy - f(x, v)$ with $\partial g/\partial x = -u$ and $\partial g/\partial v = y$.

→ We can do the same starting from $\mathcal{L}$:

$$\mathrm{d}\mathcal{L} = \sum_i \frac{\partial \mathcal{L}}{\partial q_i}\mathrm{d}q_i + \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\mathrm{d}\dot{q}_i + \frac{\partial \mathcal{L}}{\partial t}\mathrm{d}t = \sum_i \dot{p}_i\mathrm{d}q_i + \sum_i p_i\mathrm{d}\dot{q}_i + \frac{\partial \mathcal{L}}{\partial t}\mathrm{d}t$$

$$\mathrm{d}\mathcal{L} = \sum_i \dot{p}_i\mathrm{d}q_i + \sum_i \mathrm{d}(p_i\dot{q}_i) - \sum_i \dot{q}_i\mathrm{d}p_i + \frac{\partial \mathcal{L}}{\partial t}\mathrm{d}t$$

$$\mathrm{d}\left(\sum_i p_i\dot{q}_i - \mathcal{L}\right) = \sum_i \dot{q}_i\mathrm{d}p_i - \sum_i \dot{p}_i\mathrm{d}q_i - \frac{\partial \mathcal{L}}{\partial t}\mathrm{d}t$$

→ So the Legendre transform of $\mathcal{L}$ is the Hamiltonian:

$$\mathcal{H}(p, q, t) \equiv \sum_i p_i\dot{q}_i - \mathcal{L}$$

→ The differential of $\mathcal{H}$ is:

$$d\mathcal{H} = \sum_i \dot{q}_i\mathrm{d}p_i - \sum_i \dot{p}_i\mathrm{d}q_i - \frac{\partial \mathcal{L}}{\partial t}\mathrm{d}t$$

→ It follows

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i},$$

which are called *Hamilton's equations* or canonical equations. We have replaced s 2nd-order equations with $2s$ first-order equations. We also have

$$\frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}$$

→ **p** and **q** are called canonical coordinates.

→ The time derivative of $\mathcal{H}$ is

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial t} + \sum_i \frac{\partial \mathcal{H}}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial \mathcal{H}}{\partial p_i}\dot{p}_i = \frac{\partial \mathcal{H}}{\partial t}$$

→ Hamiltonian is constant if $\mathcal{H}$ does not depend explicitly on time. This is the case for closed system, for which $\mathcal{L}$ does not depend explicitly on time. This is a reformulation of energy conservation, because we recall that

$$E \equiv \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{q}_i - \mathcal{L} = \sum_i p_i\dot{q}_i - \mathcal{L} = \mathcal{H}$$

1.4.2 Canonical transformations

[VK 4.7-4.8]

→ Given a set of canonical coordinates $(\mathbf{p}, \mathbf{q})$, we might want to change to another set of coordinates $(\mathbf{P}, \mathbf{Q})$ to simplify our problem.

→ We can consider general transformations of the form $\mathbf{Q} = \mathbf{Q}(\mathbf{p}, \mathbf{q}, t)$ and $\mathbf{P} = \mathbf{P}(\mathbf{p}, \mathbf{q}, t)$: it is not guaranteed that Hamilton's equations are unchanged.

→ A transformation $\mathbf{Q} = \mathbf{Q}(\mathbf{p}, \mathbf{q}, t)$ and $\mathbf{P} = \mathbf{P}(\mathbf{p}, \mathbf{q}, t)$ is called *canonical* if in the new coordinates

$$\dot{Q}_i = \frac{\partial \mathcal{H}'}{\partial P_i}, \qquad \dot{P}_i = -\frac{\partial \mathcal{H}'}{\partial Q_i},$$

with some Hamiltonian $\mathcal{H}' = \mathcal{H}'(\mathbf{P}, \mathbf{Q}, t)$.

→ In any canonical coordinate system the variation of the action is null

$$\delta S = \delta \int_{t_1}^{t_2} \mathcal{L} \mathrm{d}t = \delta \int_{t_1}^{t_2} \left(\sum_i p_i \dot{q}_i - \mathcal{H} \right) \mathrm{d}t = 0$$

$$\delta S' = \delta \int_{t_1}^{t_2} \mathcal{L}' \mathrm{d}t = \delta \int_{t_1}^{t_2} \left(\sum_i P_i \dot{Q}_i - \mathcal{H}' \right) \mathrm{d}t = 0$$

This means that the difference between the two Lagrangians $\mathcal{L} - \mathcal{L}' = \mathrm{d}F/\mathrm{d}t$ must be a total time derivative, because

$$\delta \int_{t_1}^{t_2} \frac{\mathrm{d}F}{\mathrm{d}t} \mathrm{d}t = \delta [F]_{t_2}^{t_1} = 0.$$

→ Let us impose the above condition, i.e.,

$$\mathcal{L} = \mathcal{L}' + \frac{\mathrm{d}F}{\mathrm{d}t},$$

$$\mathcal{L}\mathrm{d}t = \mathcal{L}'\mathrm{d}t + \mathrm{d}F,$$

$$\left(\sum_i p_i \frac{\mathrm{d}q_i}{\mathrm{d}t} - \mathcal{H} \right) \mathrm{d}t = \sum_i P_i \mathrm{d}Q_i - \mathcal{H}'\mathrm{d}t + \mathrm{d}F$$

$$\sum_i p_i \mathrm{d}q_i - \mathcal{H}\mathrm{d}t = \sum_i P_i \mathrm{d}Q_i - \mathcal{H}'\mathrm{d}t + \mathrm{d}F,$$

where $F = F(\mathbf{q}, \mathbf{Q}, t)$.

→ F is called the generating function of the transformation.

→ Taking F in the form $F = F(\mathbf{q}, \mathbf{Q}, t)$, the above equation can be written as

$$\sum_i p_i \mathrm{d}q_i - \mathcal{H}\mathrm{d}t = \sum_i P_i \mathrm{d}Q_i - \mathcal{H}'\mathrm{d}t + \frac{\partial F}{\partial t}\mathrm{d}t + \sum_i \frac{\partial F}{\partial q_i}\mathrm{d}q_i + \sum_i \frac{\partial F}{\partial Q_i}\mathrm{d}Q_i,$$

$$\frac{\partial F}{\partial t}\mathrm{d}t + \sum_i \frac{\partial F}{\partial q_i}\mathrm{d}q_i + \sum_i \frac{\partial F}{\partial Q_i}\mathrm{d}Q_i = \sum_i p_i \mathrm{d}q_i + (\mathcal{H}' - \mathcal{H})\mathrm{d}t - \sum_i P_i \mathrm{d}Q_i$$

Clearly the above is verified when

$$p_i = \frac{\partial F}{\partial q_i}$$

$$P_i = -\frac{\partial F}{\partial Q_i}$$

$$H' = H + \frac{\partial F}{\partial t}.$$

The above relations can be combined (and when necessary inverted) to give $\mathbf{Q} = \mathbf{Q}(\mathbf{q}, \mathbf{p}, t)$ and $\mathbf{P} = \mathbf{P}(\mathbf{q}, \mathbf{p}, t)$, i.e. the canonical transformation in terms of the generating function F.

→ Sometimes is it convenient to have a generating function that is not in the form $F = F(\mathbf{q}, \mathbf{Q}, t)$, but depends on other combinations of new and old canonical coordinates: other possible choices are $(\mathbf{q}, \mathbf{P}, t)$, $(\mathbf{p}, \mathbf{Q}, t)$, $(\mathbf{p}, \mathbf{P}, t)$.

→ We distinguish four classes of generating functions F, differing by the variables on which F depends:

$$F = F_1(\mathbf{q}, \mathbf{Q}, t), \quad F = F_2(\mathbf{q}, \mathbf{P}, t), \quad F = F_3(\mathbf{p}, \mathbf{Q}, t), \quad F = F_4(\mathbf{p}, \mathbf{P}, t).$$

→ We derive here the canonical transformation for a generating function F_2, depending on $(\mathbf{q}, \mathbf{P}, t)$. In order to do so we use the Legendre transform. Start from

$$\mathrm{d}F_1 = \sum_i p_i \mathrm{d}q_i - \sum_i P_i \mathrm{d}Q_i + (\mathcal{H}' - \mathcal{H})\mathrm{d}t$$

where $F_1(\mathbf{q}, \mathbf{Q}, t)$ is the generating function considered above.

$$\mathrm{d}F_1 = \sum_i p_i \mathrm{d}q_i - \sum_i \mathrm{d}(P_i Q_i) + \sum_i Q_i \mathrm{d}P_i + (\mathcal{H}' - \mathcal{H})\mathrm{d}t$$

$$\mathrm{d}(F_1 + \sum_i P_i Q_i) = \sum_i p_i \mathrm{d}q_i + \sum_i Q_i \mathrm{d}P_i + (\mathcal{H}' - \mathcal{H})\mathrm{d}t$$

so the generating function is now

$$F_2 = F_2(\mathbf{q}, \mathbf{P}, t) \equiv F_1(\mathbf{q}, \mathbf{Q}, t) + \sum_i P_i Q_i$$

and the change of variables is as follows

$$p_i = \frac{\partial F_2}{\partial q_i}$$

$$Q_i = \frac{\partial F_2}{\partial P_i}$$

$$\mathcal{H}' = \mathcal{H} + \frac{\partial F_2}{\partial t},$$

where $F_2 = F_2(\mathbf{q}, \mathbf{P}, t)$

→ Similarly (exploiting Legendre transform) we can obtain transformation equations for $F = F_3(\mathbf{p}, \mathbf{Q}, t) = F_1 - \sum_i q_i p_i$ and $F = F_4(\mathbf{p}, \mathbf{P}, t) = F_1 + \sum_i Q_i P_i - \sum_i q_i p_i$. In summary the canonical transformations are

$$F = F_1(\mathbf{q}, \mathbf{Q}, t), \qquad p_i = \frac{\partial F_1}{\partial q_i} \qquad P_i = -\frac{\partial F_1}{\partial Q_i}$$

$$F = F_2(\mathbf{q}, \mathbf{P}, t), \qquad p_i = \frac{\partial F_2}{\partial q_i} \qquad Q_i = \frac{\partial F_2}{\partial P_i}$$

$$F = F_3(\mathbf{p}, \mathbf{Q}, t), \qquad q_i = -\frac{\partial F_3}{\partial p_i} \qquad P_i = -\frac{\partial F_3}{\partial Q_i}$$

$$F = F_4(\mathbf{p}, \mathbf{P}, t), \qquad q_i = -\frac{\partial F_4}{\partial p_i} \qquad Q_i = \frac{\partial F_4}{\partial P_i}.$$

In addition we have $\mathcal{H}' = \mathcal{H} + \partial F_i/\partial t$, for $i = 1, \ldots, 4$.

→ *Example of canonical transformations: extended point transformations.* This is of the kind $F = F_2(\mathbf{q}, \mathbf{P})$:

$$\mathbf{Q} = \mathbf{G}(\mathbf{q}), \qquad F(\mathbf{q}, \mathbf{P}) = \sum_k P_k G_k(\mathbf{q}),$$

where $\mathbf{G} = (G_1, ..., G_s)$, and the G_i are given functions. Then

$$p_i = \frac{\partial F}{\partial q_i} = \sum_k P_k \frac{\partial G_k}{\partial q_i}(\mathbf{q}),$$

$$Q_i = \frac{\partial F}{\partial P_i} = G_i(\mathbf{q}).$$

For instance, for a system with 1 degree of freedom, we have

$$Q = G(q), \qquad F(q, P) = PG(q),$$

so

$$p = \frac{\partial F}{\partial q} = P\frac{\partial G}{\partial q}(q) \implies P = p\left(\frac{\partial G}{\partial q}\right)^{-1},$$

$$Q = \frac{\partial F}{\partial P} = G(q).$$

→ *Special classes of Canonical Coordinates.* Among canonical coordinates there are two special classes that are particularly important:

- Sets of canonical coordinates in which both $\mathbf{q}$ and $\mathbf{p}$ are integrals of motion (they remain constant during the evolution of the system): these are the coordinates obtained by solving the *Hamilton-Jacobi equation.*

- Sets of canonical coordinates in which the $\mathbf{q}$ are not constant, but they are cyclic coordinates (they can be interpreted as angles, so they are called *angles*), while the $\mathbf{p}$ are integrals of motion (they are constants; they are called *actions*). These are called *angle-action coordinates.*

1.4.3 Hamilton-Jacobi equation

[VK 4.9]

→ The action $S = \int \mathcal{L} \mathrm{d}t$ can be seen as a generating function. The corresponding canonical transformations are very useful because they are such that $\mathcal{H}' = 0$. We have

$$\mathrm{d}S = \mathcal{L}\mathrm{d}t = \sum_i p_i \frac{\mathrm{d}q_i}{\mathrm{d}t}\mathrm{d}t - \mathcal{H}\mathrm{d}t = \sum_i p_i \mathrm{d}q_i - \mathcal{H}\mathrm{d}t.$$

→ It follows that

$$p_i = \frac{\partial S}{\partial q_i},$$

$$\mathcal{H} = -\frac{\partial S}{\partial t}.$$

→ So $S = S(\mathbf{q}, t)$, but S can be seen also as a generating function $S = S(\mathbf{q}, \mathbf{P}, t)$, with P_i constants (i.e. $\mathrm{d}P_i = 0$ for all i). So, we have

$$Q_i = \frac{\partial S}{\partial P_i}$$

→ The new Hamiltonian is null:

$$\mathcal{H}' = \mathcal{H} + \frac{\partial S}{\partial t} = 0,$$

consistent with the fact that the new canonical coordinates are constant:

$$\dot{P}_i = -\frac{\partial \mathcal{H}'}{\partial Q_i} = 0 \qquad \Longrightarrow \qquad P_i = \alpha_i = const$$

$$\dot{Q}_i = \frac{\partial \mathcal{H}'}{\partial P_i} = 0 \qquad \Longrightarrow \qquad Q_i = \beta_i = const.$$

→ So the action can be seen as a generating function in the form $S = S(\mathbf{q}, \boldsymbol{\alpha}, t)$ where $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_s) = \mathbf{P}$ are constants.

→ Exploiting the fact that $p_i = \partial S / \partial q_i$, the above equation $\mathcal{H}' = 0$ can be written as

$$\mathcal{H}\left(q_i, \frac{\partial S}{\partial q_i}, t\right) + \frac{\partial S}{\partial t} = 0.$$

This is known as the *Hamilton-Jacobi equation.*

→ If the solution S to the H-J equation is obtained, the solution of the equations of motions can be written explicitly as follows. The variables $(\mathbf{p}, \mathbf{q})$ are related to $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ by

$$p_i = \frac{\partial S}{\partial q_i}(\mathbf{q}, \boldsymbol{\alpha}, t), \qquad \beta_i = \frac{\partial S}{\partial \alpha_i}(\mathbf{q}, \boldsymbol{\alpha}, t),$$

which can be combined and inverted to give $q_i = q_i(\boldsymbol{\alpha}, \boldsymbol{\beta}, t)$ and $p_i = p_i(\boldsymbol{\alpha}, \boldsymbol{\beta}, t)$.

→ If $\mathcal{H}$ does not depend explicitly on time then $\mathcal{H} = E = const$ and the H-J equation can be written

$$\mathcal{H}\left(q_i, \frac{\partial S}{\partial q_i}\right) = E,$$

and we also have

$$\frac{\partial S}{\partial t} = -E.$$

→ *Example: free particle.* Take Cartesian coordinates x, y, z as generalized coordinates q_i and p_x, p_y, p_z as generalized momenta p_i. The Hamiltonian of a free particle of mass m is

$$\mathcal{H} = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right).$$

The H-J equation is

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2\right] + \frac{\partial S}{\partial t} = 0.$$

Separation of variables $S(x, y, z, t) = X(x) + Y(y) + Z(z) + T(t)$ then

$$\frac{1}{2m}\left[\left(\frac{\partial X}{\partial x}\right)^2 + \left(\frac{\partial Y}{\partial y}\right)^2 + \left(\frac{\partial Z}{\partial z}\right)^2\right] + \frac{\partial T}{\partial t} = 0,$$

so $X = \alpha_x x$, $Y = \alpha_y y$, $Z = \alpha_z z$ and $T = -(\alpha_x^2 + \alpha_y^2 + \alpha_z^2)t/2m$, therefore $p_x = \alpha_x$, $p_y = \alpha_y$, $p_z = \alpha_z$, $\beta_x = x - \alpha_x t/m$, $\beta_y = y - \alpha_y t/m$, $\beta_z = z - \alpha_z t/m$, which is the solution (the values of the constants depend on the initial conditions at $t = 0$).

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