Example of computation of an integral with methods of complex variables

Compute $\int_{\mathbb{R}} \frac{e^{-ix}}{(x^2+1)^2} dx$.

Solution We begin by observing that the function we want to integrate is summable, as its abolsute value equals (in $x \in \mathbb{R}$) $\frac{1}{(x^2+1)^2}$, which has a behaviour like x^{-4}, as $x \to \pm\infty$. The function $f(z) = \frac{e^{-iz}}{(z^2+1)^2}$ is holomorphic in $\{z \in \mathbb{C} : z^2 + 1 \neq 0\} = \mathbb{C} \setminus \{i, -i\}$. Given $n \in \mathbb{N}$, $n \geq 2$, we indicate with α_n a piecewise C^1 path, describing once, in clockwise sense, first the interval $[-n, n]$, then the semicircle $\{z \in \mathbb{C} : |z| = n, Im(z) \leq 0\}$. By the residue theorem, we have

$$\int_{\alpha_n} f(z)dz = -2\pi i Res(f; -i).$$

f has in $-i$ a pole of order 2, because $e^{-iz} \neq 0 \ \forall z \in \mathbb{C}$ and, if we set $g(z) = (z^2 + 1)^2$, $g(-i) = g'(-i) = 0$, $g''(-i) = -8$. So,

$$Res(f; -i) = \lim_{z\to -i} \tfrac{d}{dz}[(z + i)^2 f(z)] = \lim_{z\to -i} \tfrac{d}{dz}[\tfrac{e^{-iz}}{(z-i)^2}]$$

$$= \lim_{z\to -i} \tfrac{-ie^{-iz}(z-i)^2 - e^{-iz}2(z-i)}{(z-i)^4} = \tfrac{-ie^{-1}(-2i)^2 - e^{-1}2(-2i)}{(-2i)^4} = \tfrac{i}{e}.$$

Hence,

$$\int_{\alpha_n} f(z)dz = \frac{2\pi}{e}.$$

Moreover,

$$\int_{\alpha_n} f(z)dz = \int_{-n}^{n} \frac{e^{-ix}}{(x^2+1)^2} dx - \int_{C_n^-(0)} \frac{e^{-iz}}{(z^2+1)^2} dz. \qquad (1)$$

The first integral converges, as $n \to +\infty$, to what we want to compute. Moreover,

$$\left|\int_{C_n^-(0)} \frac{e^{-iz}}{(z^2+1)^2} dz\right| \leq n\pi \cdot \sup_{|z|=n, Im(z)\leq 0} \frac{|e^{-iz}|}{|z^2+1|^2}.$$

We have $|e^{-iz}| = e^{Im(z)} \leq 1$. Next, if $|z| = n$, $|z^2 + 1|^2 = n^4|1 + z^{-2}|^2 \geq n^4/2$, if $n \geq n_0$. So, if $n \geq n_0$, and $|z| = n$,

$$\frac{|e^{-iz}|}{|z^2+1|^2} \leq \frac{2}{n^4},$$

so that

$$\left|\int_{C_n^-(0)} \frac{e^{-iz}}{(z^2+1)^2} dz\right| \leq \pi n \frac{2}{n^4} \to 0 \quad (n \to +\infty).$$

Then, passing o the limit as $n \to +\infty$ in (1), we obtain

$$\frac{2\pi}{e} = \int_{\mathbb{R}} \frac{e^{-ix}}{(x^2+1)^2} dx.$$