PDE - Poisson Equation

$$-\Delta u = f$$

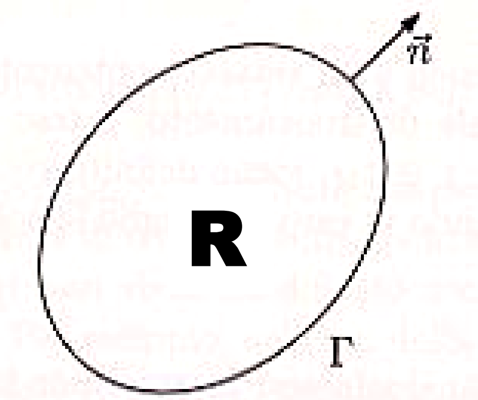
$$(-\nabla^2 u = f) \quad \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

f=0 Laplace Equation

Dirichlet B.C. $\quad u = g$

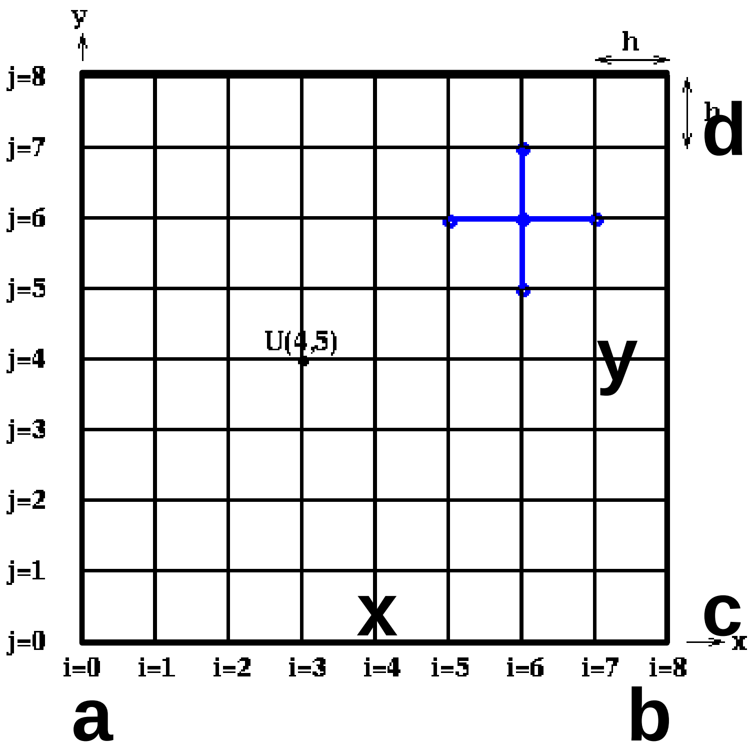
Neumann B.C. on Γ: $\quad \frac{\partial u}{\partial \vec{n}} = g$

Mixed B.C. $\quad au + b\frac{\partial u}{\partial \vec{n}} = g$



PDE - Poisson Equation

$$\begin{cases} -\Delta u(x,y) = f(x,y) \\ u(a,y) = g_1(y) \\ u(b,y) = g_2(y) \\ u(x,c) = g_3(x) \\ u(x,d) = g_4(x) \end{cases}$$



mesh:

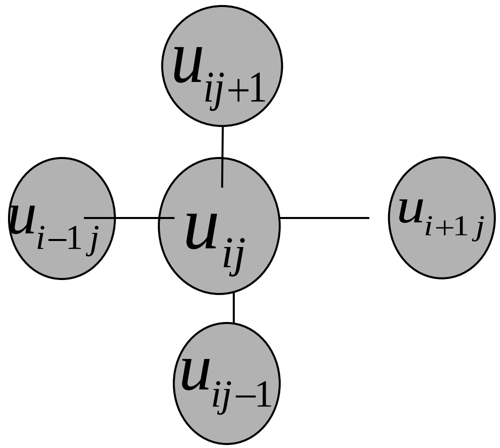
$$x_i = a + ih \qquad h = (b - a)/N$$

$$y_j = c + jk \qquad k = (d - c)/M$$

PDE - Poisson Equation

We want u(x,y) at inner nodes, with h=k.

Centered differences:



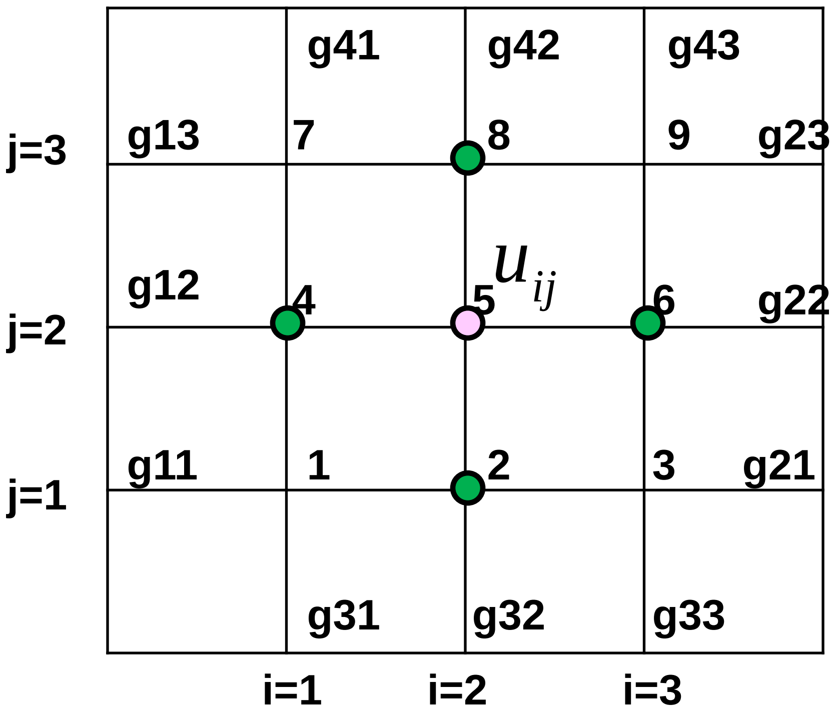
$$-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}=f$$

$$-\left[\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2}\right]-\left[\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{h^2}\right]=f(x_i,y_j)$$

$$\boldsymbol{-u_{i-1,j}-u_{i+1,j}+4u_{i,j}-u_{i,j-1}-u_{i,j+1}=h^2 f_{ij}}$$

PDE - Poisson Equation

N=M=4 ... 4x4 mesh (grid):



$(N-1)^2$ unknowns u_{ij}
i,j=1,..,N-1

Unknowns:

$$(u_{1,1}, u_{2,1}, u_{3,1}, \ldots, u_{N-1,1}, u_{1,2}, \ldots, u_{N-1,N-1})$$

$$-u_{i-1,j} - u_{i+1,j} + 4u_{i,j} - u_{i,j-1} - u_{i,j+1} = h^2 f_{ij}$$

PDE - Poisson Equation

One equation for each inner node

$$
\begin{bmatrix}
4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4
\end{bmatrix}
\begin{bmatrix}
u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{1,2} \\ u_{2,2} \\ u_{3,2} \\ u_{1,3} \\ u_{2,3} \\ u_{3,3}
\end{bmatrix}
=
\begin{bmatrix}
h^2 f_{1,1} + g11 + g31 \\
h^2 f_{2,1} + g32 \\
h^2 f_{3,1} + g21 + g33 \\
h^2 f_{1,2} + g12 \\
h^2 f_{2,2} \\
h^2 f_{3,2} + g22 \\
h^2 f_{1,3} + g13 + g41 \\
h^2 f_{2,3} + g42 \\
h^2 f_{3,3} + g43 + g23
\end{bmatrix}
$$

Symmetric, diagonally dominant (not strictly), positive definite. M-matrix!

$$a_{ij} \le 0 \quad i, j = 1, 2, \ldots, n \quad i \ne j \quad \text{and} \quad A^{-1} \ge 0$$

PDE - Poisson Equation

How to solve the linear system (N-1)x(N-1)?

Iterative method: Gauss-Seidel: (step k)

$$x^{(k+1)} = D^{-1}(b - Lx^{(k+1)} - Ux^{(k)})$$

$$\boxed{-u_{i,j-1} - u_{i-1,j}} + 4u_{i,j} \boxed{-u_{i+1,j} - u_{i,j+1}} = h^2 f_{ij}$$

We do not need to build the matrix...solution row by row

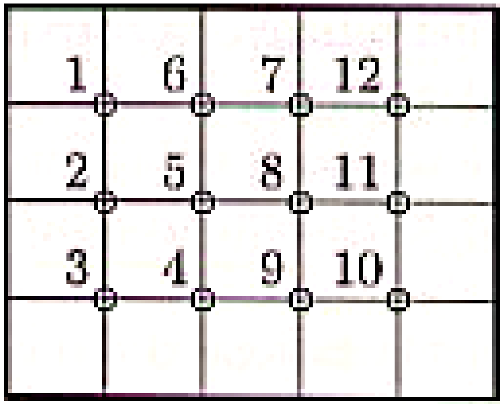
PDE - Poisson Equation

$$-u_{i-1,j}-u_{i+1,j}+4u_{i,j}-u_{i,j-1}-u_{i,j+1}=h^2 f_{ij}$$

Gauss – Seidel iteration

$$
\begin{aligned}
& for \quad k = 1, 2, \ldots \\
& \quad for \quad i = 2, .., N \\
& \quad\quad for \quad j = 2, .., N \\
& \quad\quad\quad u^{(k+1)}{}_{i,j} = \frac{1}{4}(u^{(k+1)}{}_{i-1,j} + u^{(k)}{}_{i+1,j} + u^{(k+1)}{}_{i,j-1} + u^{(k)}{}_{i,j+1} + h^2 f_{ij}) \\
& \quad\quad end \\
& \quad end \\
& end
\end{aligned}
$$

PDE - Poisson Equation

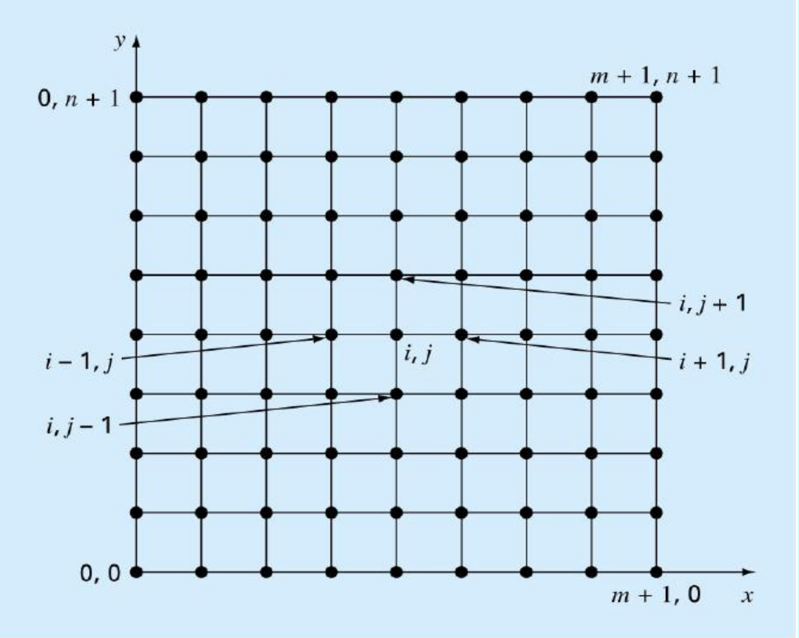


Block Tridiagonal

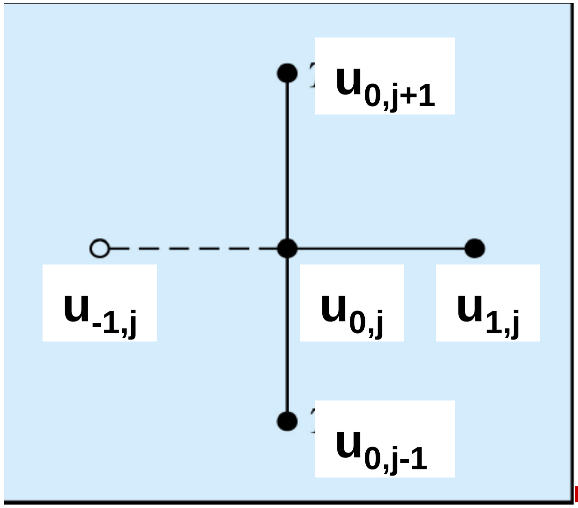
The structure of the matrix depends on the nodes indexing

$$A = \left(\begin{array}{ccc|ccc|ccc|ccc}
4 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 4
\end{array}\right)$$

Poisson Equation: Neumann B.C.



$$-u_{i-1,j} - u_{i+1,j} + 4u_{i,j} - u_{i,j-1} - u_{i,j+1} = h^2 f_{ij}$$



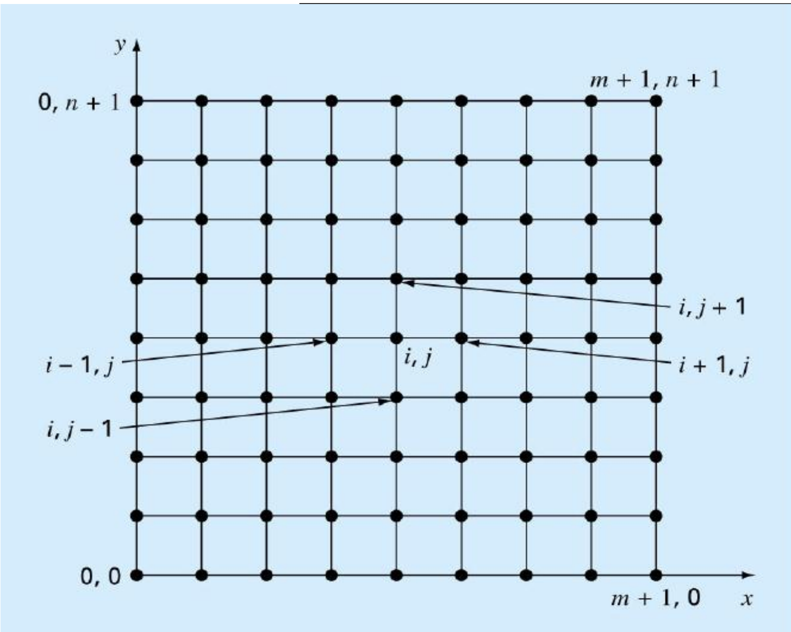
$$-u_{1,j} - u_{-1,j} + 4u_{0,j} - u_{0,j-1} - u_{0,j+1} = h^2 f_{0,j}$$

$$\frac{\partial u}{\partial x} \cong \frac{u_{1,j} - u_{-1,j}}{2\Delta x}$$

$$u_{-1,j} = u_{1,j} - 2\Delta x \frac{\partial u}{\partial x}$$

$$-2u_{1,j} + 2\Delta x \frac{\partial u}{\partial x} + 4u_{0,j} - u_{0,j-1} - u_{0,j+1} = h^2 f_{0,j}$$

Poisson Equation: Neumann B.C.



$$
\begin{bmatrix}
4 & -1 & 0 & -2 & & & & & & & & \\
-1 & 4 & -1 & 0 & -2 & & & & & & & \\
0 & -1 & 4 & 0 & 0 & -2 & & & 0 & & & \\
-1 & 0 & 0 & 4 & -1 & 0 & -1 & & & & & \\
 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & & & & \\
 & & -1 & 0 & -1 & 4 & 0 & 0 & -1 & & & \\
 & & & -1 & 0 & 0 & 4 & -1 & 0 & -1 & & \\
 & & & & -1 & 0 & -1 & 4 & -1 & 0 & -1 & \\
 & & 0 & & & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\
 & & & & & & -1 & 0 & 0 & 4 & -1 & 0 \\
 & & & & & & & -1 & 0 & -1 & 4 & -1 \\
 & & & & & & & & -1 & 0 & -1 & 4
\end{bmatrix}
$$

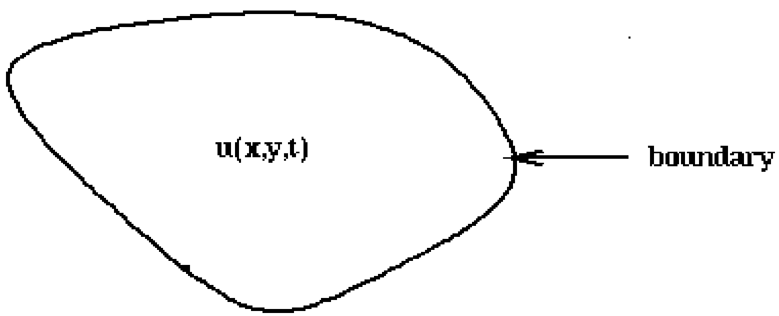
$$-2u_{1,j} + 2\Delta x \frac{\partial u}{\partial x} + 4u_{0,j} - u_{0,j-1} - u_{0,j+1} = h^2 f_{0,j}$$

Multi-dimensional Heat Equation

1 – Dimensional Heat Equation

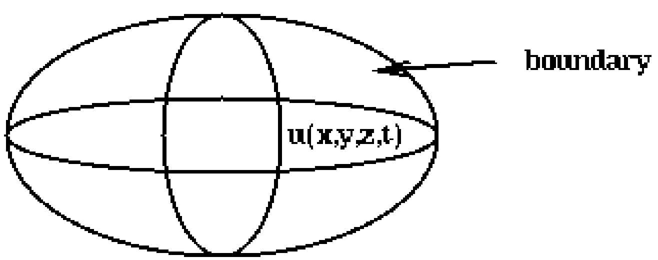
$$\frac{d\,u(x,t)}{dt} = C\left(\frac{d^2 u(x,t)}{d x^2}\right)$$

2 – Dimensional Heat Equation



$$\frac{d\,u(x,y,t)}{dt} = C\left(\frac{d^2 u(x,y,t)}{d x^2} + \frac{d^2 u(x,y,t)}{d y^2}\right)$$

3 – Dimensional Heat Equation



$$\frac{d\,u(x,y,z,t)}{dt} = C\left(\frac{d^2 u(x,y,z,t)}{d x^2} + \frac{d^2 u(x,y,z,t)}{d y^2} + \frac{d^2 u(x,y,z,t)}{d z^2}\right)$$

Multi-dimensional Heat Equation

Discretization of the 2D Heat Equation