PDE – 1st order linear equation

$$\begin{cases} u_t + a u_x = 0 \\ u(x,0) = u_0(x) \\ u(0,t) = f_1(t) \\ u(1,t) = f_2(t) \end{cases}$$

a **positive constant**

$$\boldsymbol{R} = \{0 < x < 1\} \quad t > 0$$

Transport Equation in one space dimension

The solution is a «wave» that propagates with velocity *a*:

$u(x, t) = u_0(x - at)$

$t > 0$

a > 0 from left to right;

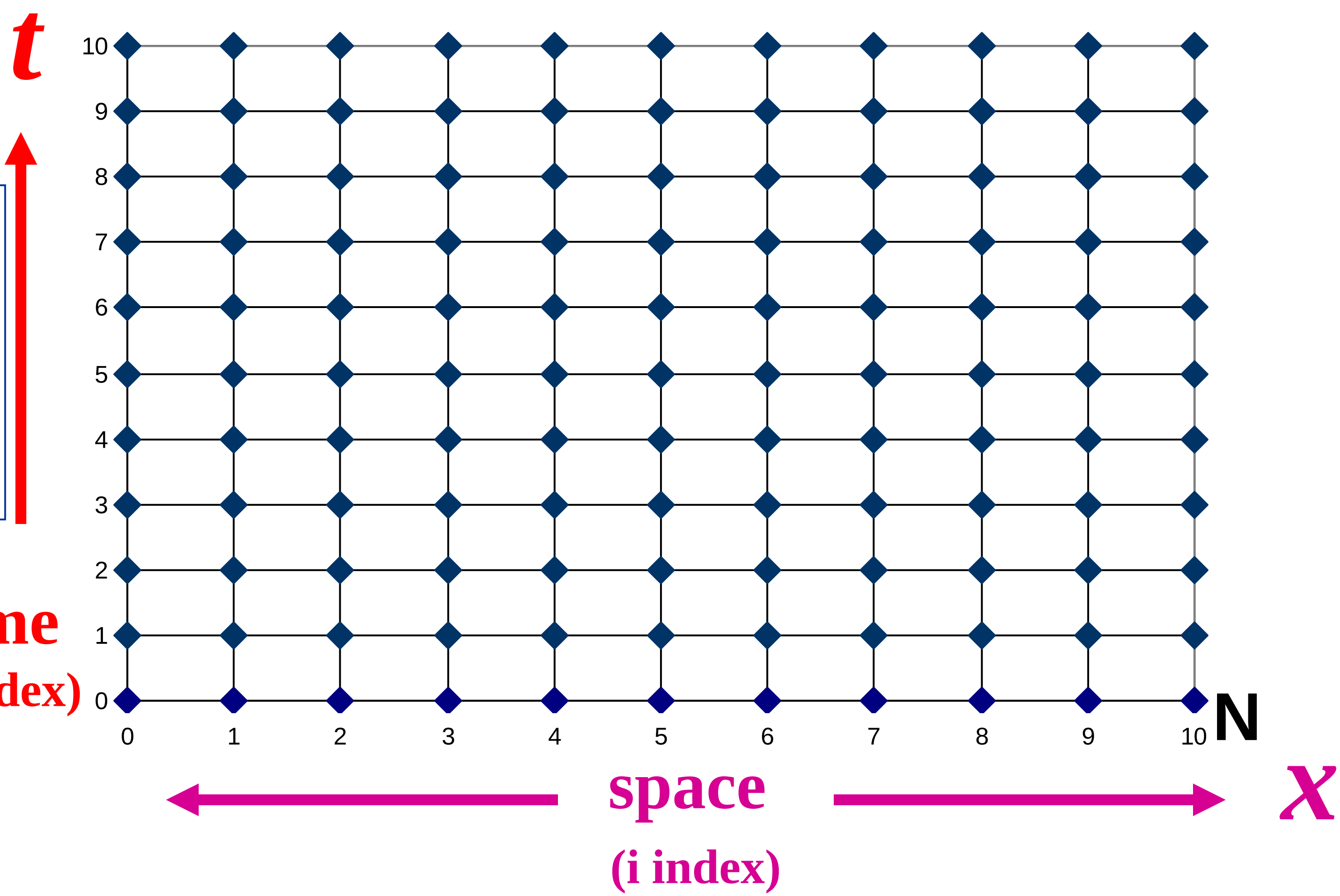
a < 0 from right to left;

$$h = \Delta x$$

$$k = \Delta t$$

time

(j index)

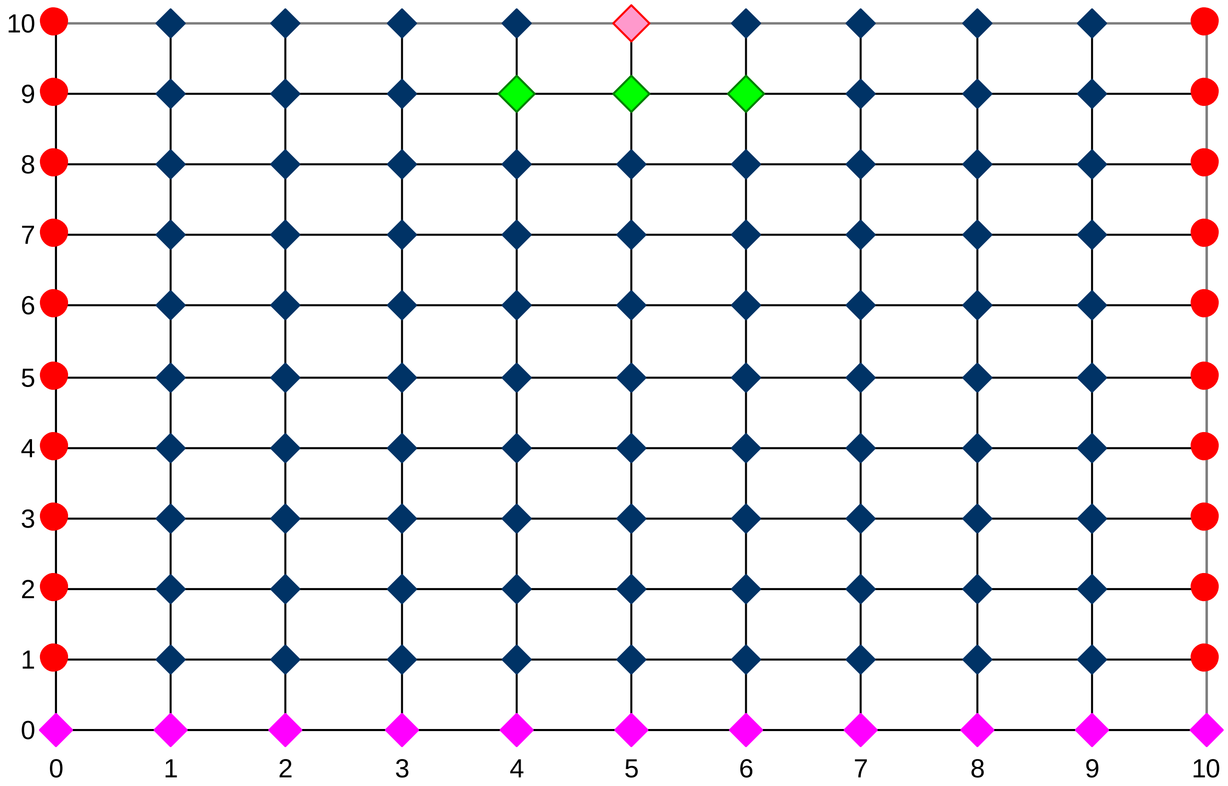


Initial and boundary conditions

$$u_t + a u_x = 0$$

Boundary Condition (left):

$u(0, t) = f_1(t)$



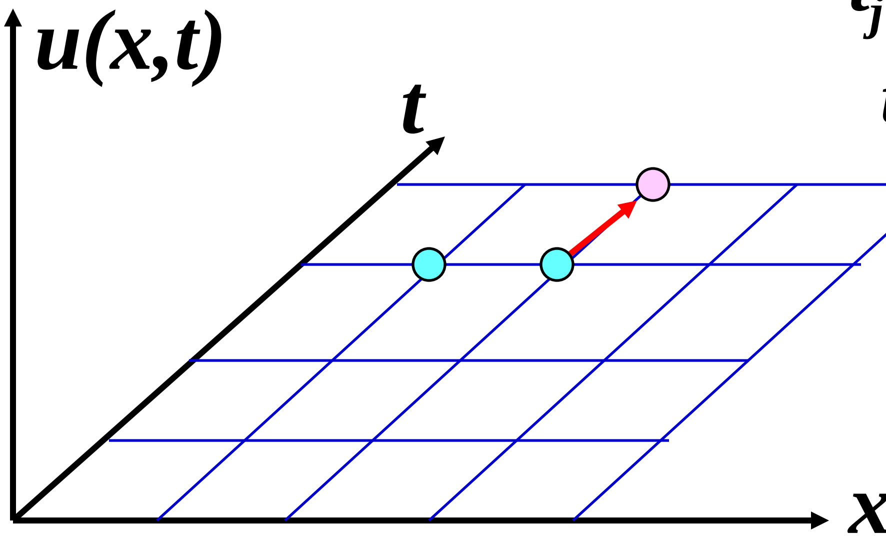
Boundary Condition (right):

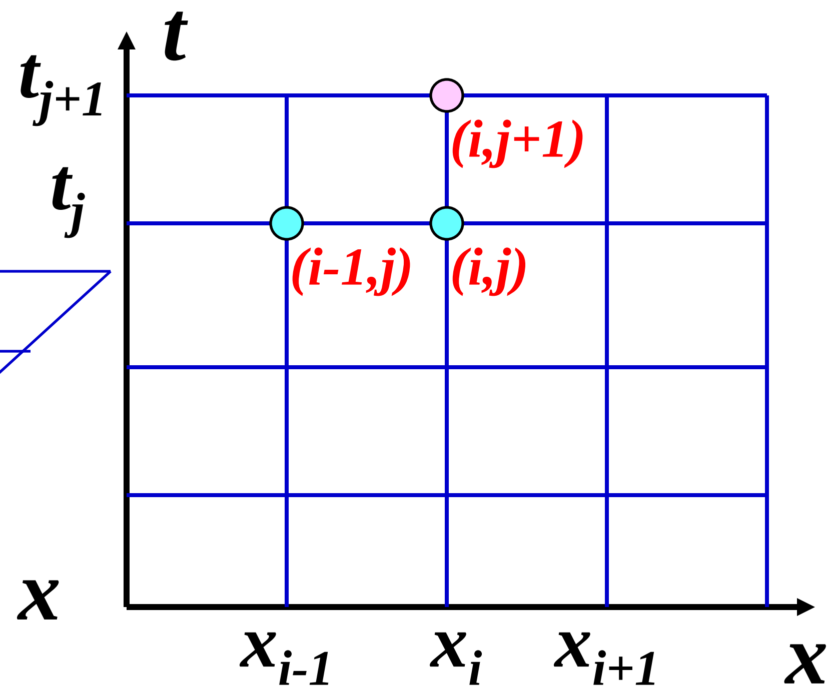
$u(1, t) = f_2(t)$

Initial Conditions : $u(x,0) = u_0(x)$

Numerical Solution: Explicit Scheme

$$\frac{u_i^{j+1} - u_i^j}{k} + a\,\frac{u_i^j - u_{i-1}^j}{h} = 0 \qquad u_i^j = u(x_i, t_j)$$





space (x, i, h): backward differences

$$u_x = \frac{1}{h}\left(u_i^j - u_{i-1}^j\right) + O(h)$$

time (t, j, k): forward differences

$$u_t = \frac{1}{k}\left(u_i^{j+1} - u_i^j\right) + O(k)$$

Numerical Solution: Explicit Scheme

$$\frac{u_i^{j+1} - u_i^j}{k} + a\frac{u_i^j - u_{i-1}^j}{h} = 0 \qquad \text{notation: } u_i^j = u(x_i, t_j)$$

$$u_0^j = f_1(t_j) \quad u_N^j = f_2(t_j) \qquad j = 1, 2, \ldots, M$$

$$u_i^0 = u_0(x_i) \qquad i = 0, 1, \ldots, N$$

$$\alpha = ak/h$$

$for \quad j = 1, 2, \ldots, M$

$\quad for \quad i = 1, .., N-1$

$$u_i^{j+1} = (1-\alpha)u_i^j + \alpha u_{i-1}^j$$

$\quad end$

end

1^{st} order linear PDE: stability

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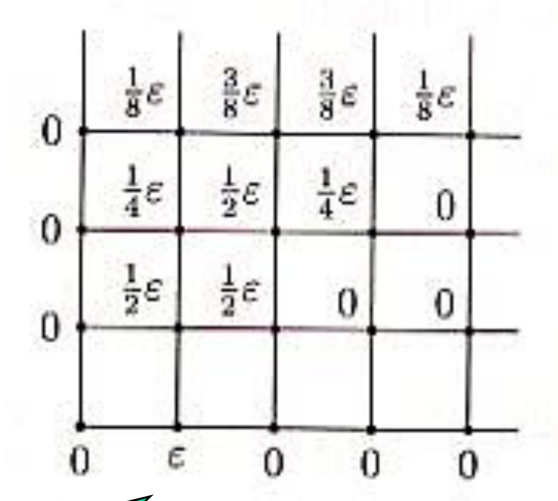
We assume to perturbate the data of the problem (initial and/or boundary conditions)...we want to examinate how the numerical scheme propagates the perturbations (stability of the scheme)

We say that the numerical scheme is **stable** if, given a generic time $t = \tau > 0$, for all sufficiently small space and time discretization steps **h** and **k**, the perturbations remain limited in $(0,\tau]$, uniformly with respect to h and k; otherwise unstable.

Explicit Scheme: stability

$$\alpha = \frac{1}{2}$$

stable

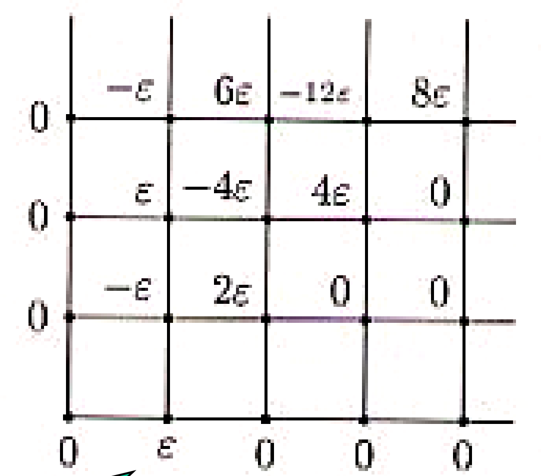


$$u_i^{j+1} = (1-\alpha)u_i^j + \alpha u_{i-1}^j$$

$$\alpha = a\frac{k}{h}$$

$$\alpha = 2$$

unstable



Explicit Scheme: stability

For which values of the parameter α the scheme is stable?

Consider the nodes (x_i,t_j) $i=1,2,\ldots,N-1$ $j \geq 0$

Assume, for simplicity, to perturbate only the initial condition, that is the data u_i^0 (not the boundary data, at x=0, x=1).

$$U_j = \begin{bmatrix} u_1^j \\ u_2^j \\ .. \\ .. \\ u_{N-1}^j \end{bmatrix} \qquad E_j = \begin{bmatrix} \varepsilon_1^j \\ \varepsilon_2^j \\ .. \\ .. \\ \varepsilon_{N-1}^j \end{bmatrix}$$

Solution vector Perturbation vector

Explicit Scheme: stability

$$
\begin{cases}
u_0^j = f_1(t_j) \quad u_N^j = f_2(t_j) & j = 1,2,....,M \\
u_i^0 = u_0(x_i) & i = 0,1,...,N \\
u_i^{j+1} = (1-\alpha)u_i^j + \alpha\, u_{i-1}^j \quad j = 0,1,...,M; & i = 1,2,...,N\text{-}1
\end{cases}
$$

$$U_{j+1} = AU_j + \alpha v_j$$

$$
U_0 = \begin{bmatrix} u^0(x_1) \\ u^0(x_2) \\ .. \\ .. \\ u^0(x_{N-1}) \end{bmatrix}
\qquad
v_j = \begin{bmatrix} u_0^j \\ 0 \\ .. \\ .. \\ u_N^j \end{bmatrix}
$$

$$
A = \begin{bmatrix}
1-\alpha & 0 & . & . & 0 \\
\alpha & 1-\alpha & & & . \\
0 & \alpha & .. & & . \\
.. & .. & .. & .. & 0 \\
0 & & & \alpha & 1-\alpha
\end{bmatrix}
\qquad
U_j = \begin{bmatrix} u_1^j \\ u_2^j \\ .. \\ .. \\ u_{N-1}^j \end{bmatrix}
$$

Explicit Scheme: stability

Error satisfies:

$$E_{j+1} = AE_j \qquad j = 0,1,\ldots$$

Hence:

$$E_{j+1} = A^{j+1}E_0$$

For which values of α the matrix $\mathbf{A}^{j+1}$ is a finite-norm matrix, uniformly with respect to j and N?

Explicit Scheme: stability

$$E_{j+1} = A^{j+1} E_0$$

A sufficient condition for $\|E_{j+1}\|_p \le C$,
with C indipendent from j and N, is:

$$\|A\|_1 \le 1 \qquad \text{that is:} \qquad |\alpha| + |1-\alpha| \le 1$$

In our case $\|A\|_1 = \|A\|_\infty$ and a theorem says: $\|A\|_p \le \|A\|_1 \quad 1 \le p \le \infty$

$$\|E_{j+1}\|_p \le \|A\|_1^{j+1} \|E_0\|_p$$

Explicit Scheme: stability

When $0 < \alpha \leq 1$ we have:

$$\left\| E_{j+1} \right\|_p \leq \left\| E_0 \right\|_p \qquad \text{for all} \quad 1 \leq p \leq \infty$$

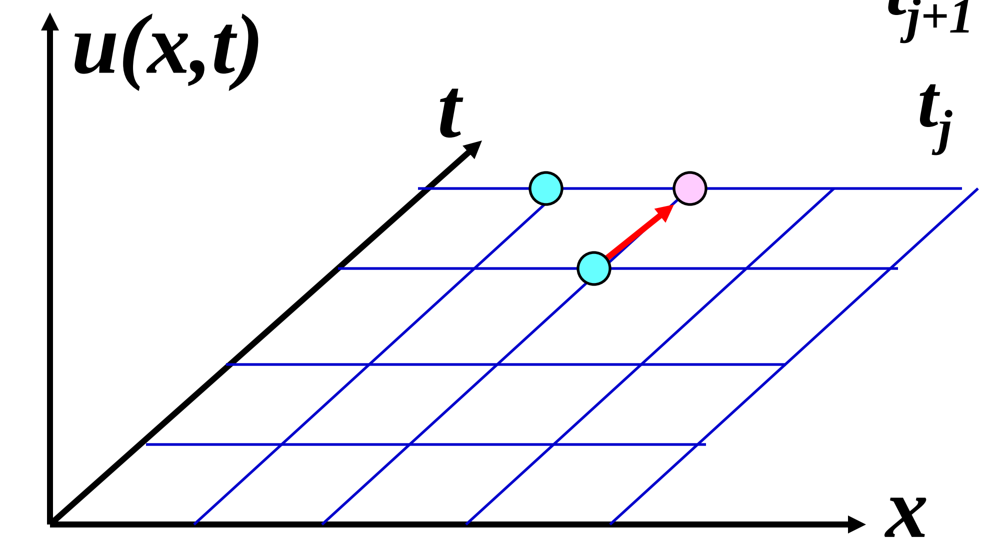
Hence, given the PDE parameter *a* (velocity), the explicit scheme is stable when the chosen space and time discretization steps, h and k, satisfy the condition:

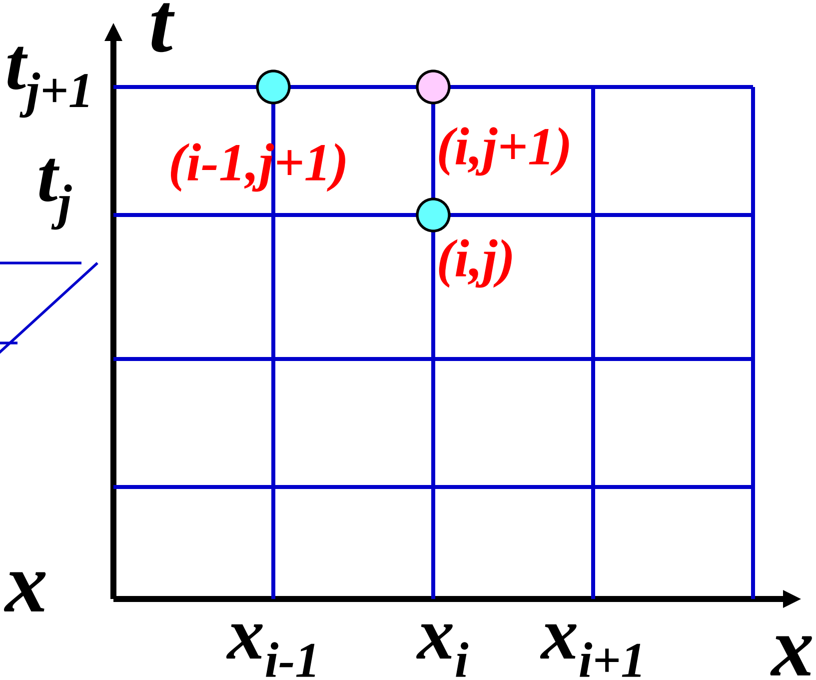
$$a \frac{k}{h} \leq 1$$

The scheme is conditionally stable

Numerical Solution: Implicit Scheme

$$\frac{u_i^{j+1} - u_i^j}{k} + a\,\frac{u_i^{j+1} - u_{i-1}^{j+1}}{h} = 0$$





space (x, i, h): backward difference at time j + 1

$$u_x = \frac{1}{h}\left(u_i^{j+1} - u_{i-1}^{j+1}\right) \quad \boxed{+O(h)}$$

time (t, j, k): forward differences

$$u_t = \frac{1}{k}\left(u_i^{j+1} - u_i^j\right) \quad \boxed{+O(k)}$$

Implicit Scheme: stability

$$\frac{u_i^{j+1}-u_i^j}{k}+a\frac{u_i^{j+1}-u_{i-1}^{j+1}}{h}=0$$

$$AE_{j+1}=E_j \qquad j=0,1,\ldots$$

$$\beta=\frac{\alpha}{1+\alpha}$$

$$A=\begin{bmatrix}1+\alpha & 0 & . & . & 0\\ -\alpha & 1+\alpha & & & .\\ 0 & -\alpha & .. & & .\\ .. & .. & .. & .. & 0\\ 0 & & & -\alpha & 1+\alpha\end{bmatrix}=(1+\alpha)\begin{bmatrix}1 & 0 & . & . & 0\\ -\beta & 1 & & & .\\ 0 & -\beta & .. & & .\\ .. & .. & .. & .. & 0\\ 0 & & & -\beta & 1\end{bmatrix}$$

Implicit Scheme: stability

$$E_{j+1} = A^{-1} E_j$$

$$A^{-1} = \frac{1}{1+\alpha} \begin{bmatrix} 1 & 0 & 0 & 0 & .. & 0 \\ \beta & 1 & & & & \\ \beta^2 & \beta & 1 & & & \\ \beta^3 & \beta^2 & \beta & 1 & & \\ .. & .. & .. & .. & .. & \\ \beta^{N-1} & \beta^{N-2} & \beta^{N-3} & \beta^{N-4} & .. & 1 \end{bmatrix}$$

$$\left\| A^{-1} \right\|_1 \leq \frac{1}{1+\alpha} \sum_{k=0}^{\infty} \beta^k = 1$$

The scheme is unconditionally stable (norm p: $1 \leq p \leq \infty$)