1 The dynamics of extensive growth

1.1 A snapshot of a simple economy

1. It is an economy the aggregate output of which is the quantity of a staple commodity that supports a population of given size at a given point in time:

G is such a quantity.

2. This economy produces G according to a well established technology, the result of a given knowledge base, of a given institutional set-up (what is permissible and what is not), of given organisation and routinised procedures. Labour is assumed to be the only prime productive input. The labour technical norm of this society is summed up by a coefficient stating the amount of labour, specified in terms of a head count of labourers, per unit of output:

l_g is such a coefficient

No matter how simple this economy may be, a further input is required next to labour by this economy: seeds. Seeds are retrieved from the available output G and are required on the grounds of a technical coefficient per unit of output:

k_g is such a coefficient.

It follows that the technology is specified by

$$(l_g, k_g)$$

3. Production is carried out on a given, historically acquired, plot of land of given natural characteristics. *Acquired* means that this land had been appropriated by free settlement, act of force, discovery. Land, as such defined, is the natural resource of this economy.

4. For simplicity's sake, population and size of the labour force coincide. Furthermore, in this simple, non-market economy the entire labour force is *efficiently* employed.

L is the size of the population-labour force.

It follows that

$$L = l_g G$$

The necessary stock of seeds is:

$$K = k_g G$$

5. A distributional principle: each member of the labour force receives an amount of consumption that is institutionally fixed: the leader, the chief or the elders' caucus

sets it to be equal to w. It follows that aggregate consumption equals:

$$C = wL$$

6. The simple computation of this economy's net product:

$$NP = (1 - k_g)G$$

7. The investable surplus, I, that can be earmarked to support extensive growth, namely growth with an invariant technology is

$$I = (1 - k_g)G - wl_gG$$

and since $I = \Delta K = k_g \Delta G$

define the achievable growth rate at a given point in time as:

$$\frac{\Delta G}{G} = g = \frac{1 - k_g}{k_g} - w\frac{l_g}{k_g}$$

This relationship states the economy's fundamental trade-off (g, w).

8. Note that (g, w) is a social option wholly constrained by the extant technology (k_g, l_g). Let it be supposed that this economy's covenant is to keep the labour force fully employed. Thus if the latter growth rate is $n = \frac{\Delta L}{L}$, then

$$n = g$$

if no change in technology occurs.

9. A useful definition: the economy's investable rate. It is simply defined as:

$$I(t) = \frac{dK}{dt} = rK(t)$$

$$r(t) = \frac{dK}{dt}\frac{1}{K(t)} = \frac{1 - k_g}{k_g} - w(t)\frac{l_g}{k_g}$$

Hence

$$g = r$$

Questions

a) Can $n = g$ be sustained?

b) What social arrangements are likely to hinder the growth of this economy?

c) How is the knowledge base generated?

2 Growth with decreasing returns

1. Growth demands that output be expanded on new plots of land: more of the natural resource is required. This is the most immediate hurdle of the growth process.

2. Even if, by the simplest possible assumption, it is assumed that land can be freely grabbed, it cannot be assumed that its natural characteristics remain unaltered.

Historical observation bears out the view that, given a society's geographical and spatial knowledge and its means of mobility and discovery, by and large early settlements occur on the lands known to be the most fertile. As population grows and more land is required, expansion takes place on lands of decreasing fertility.

Hence, decreasing returns due to natural resources are the norm; this can be stated as: more labour is required on newly exploited lands, given the same amount of seeds, to obtain one unit of output.

3. In discrete terms, as the economy moves from land A to B to C to D.....the labour coefficient applying therein changes as:

$$l_{gA} < l_{gB} < l_{gC} < l_{gD} \cdots\cdots\cdots\cdots$$

3. More formally and to retain flexibility and simplicity, a simple function can be assumed: labour per unit of

output in the last land. i.e. the marginal land, is of the form:

$$l_g = l(G)$$

It is a function of G in the sense that it is the volume of production that has pushed cultivation onto lands of decreasing fertility requiring a higher labour input.

Properties: the function is continuous and differentiable.

$$l_g^{'} > 0; \quad l_g^{''} > 0; \quad l(0) = 0$$

The capital coefficient k_g is kept constant

$$k_g = cons$$

4. Implication: consider all variables to be time functions.

3 Full employment and decreasing returns

1. The 'full employment' assumption implies that labour be absorbed as population rises:

$$L(t) = L^s(t)$$

where $L^s(t)$ is population and $L(t)$ the work force absorbed in the production process.

2. Absorption:

$$L(t) = \int_0^{G(t)} l_g(z)dz$$

Population:

$$L^s(t) = L^s(0)e^{nt}$$

assuming a constant population growth rate n.

It follows that:

$$\int_0^{G(t)} l_g(z)dz = L^s(0)e^{nt}$$

Given the assumptions, this relation can also be read as

$$G(t) = \Theta(L(t)) \qquad \Theta^{'} > 0; \quad \Theta^{''} < 0$$

4 Rent

1. An historically complex problem. Rent is a pervasive social phenomenon stemming from the paucity of natural resources and from their appropriation being turned into ownership rights. It is defined by an asymmetrical social positioning.

2. In the case discussed above plots of land of different fertility have an owner presumably enjoying exclusive rights upon them.

3. In plain and cold terms, the net product of each plot of land per unit of labour, $np(t) = \frac{1-k_g}{l_g(t)}$, is obviously different and such that:

$$np_A > np_B > np_C > ....$$

Moreover, if each 'owner' invested according to the same rate :

$$w_A > w_B > w_C > ....$$

or if each consumed the same w then:

$$r_A > r_B > r_C > ....$$

4. It is clearly the case that new labourers (as population increases) be forced on marginal lands of ever decreasing fertility. 'Owners' of differently fertile lands can thus propose a *contract* to these labourers enticing them to till their lands instead. On the grounds of at least weak rationality, the wage rate would equal the real consumption per head obtainable on the historically reached marginal

land and it would be the same across the whole economy. If, on average, all land owners invested to achieve the same growth, g, then all would appropriate an 'income' of decreasing magnitude from the land of highest fertility to that of the lowest, namely the marginal one. In the latter case such 'income' would be zero. Define such 'income': rent.

5. As the economy grows, rent rises.

6. In these circumstances, the surplus rate, defined as the ratio of the investable surplus I to the required stock of inputs K and reckoned on the marginal land:

$$r(t) = \frac{I(t)}{K(t)} = \frac{1 - k_g}{k_g} - w(t)\frac{l_g(G(t))}{k_g}$$

Since,

$$
\begin{aligned}
I(t) &= \frac{dK}{dt} = r(t)K(t) \\
\frac{dG}{dt} &= r(t)G(t) \\
g(t) &= r(t)
\end{aligned}
$$

Note that reckoning $r(t)$ on the marginal land implies a consumption per head that is accordingly therein defined.

7. **Question**: how can $L(t) = L^s(t)$ be achieved?

5 Surplus and growth rate dynamics

1. The answer implies checking that

$$\frac{dL}{dt} = \frac{dL^s}{dt}$$

be satisfied at all times.

$$\begin{aligned}
\frac{dL}{dt} &= \frac{d}{dt}\int_0^{G(t)} l_g(z)dz = l_g(G(t))\frac{dG}{dt} \\
\frac{dL^s}{dt} &= nL^s(0)e^{nt} \\
l_g(G(t))\frac{dG}{dt} &= nL^s(0)e^{nt}
\end{aligned}$$

from which

$$l_g(G(t))r(t)G(t) = nL^s(0)e^{nt}$$

and finally

$$r(t) = \frac{nL^s(0)e^{nt}}{l_g(G(t))G(t)}$$

This is the required surplus rate to keep the population-work force fully and efficiently employed. It is also the economy's growth rate.

2. It is clear that the growth rate (surplus rate) cannot remain constant. Its time pattern is set by

$$\frac{dr}{dt} = nr - (1 + \mu)r^2$$

This differential equation is obtained by the following simplifying assumption on the shape of $l_g = l(G)$:

$$\frac{dl_g}{dG}\frac{G}{l_g} = \mu$$

that is a constant elasticity of l_g in respect of G. The differential equation solves for

$$r(t) = \left\{ \left[r(0)^{-1} - \frac{1+\mu}{n} \right] e^{-nt} + \frac{1+\mu}{n} \right\}^{-1}$$

a solution that converges: $\lim_{t \to \infty} r(t) = \frac{n}{1+\mu}$

3. This result can be obtained by immediately seeking the surplus rate stationary state:

$$\frac{dr}{dt} = 0 \qquad \text{from which} \quad r^* = \frac{n}{1+\mu}$$

It is interesting to note that $g^* = r^* = \frac{n}{1+\mu}$ is smaller than n, i.e. the output growth rate is smaller than the population growth rate.

4. Given the (r, w) trade-off, it is straightforward to check the consumption per head dynamics:

$$w(t) = \frac{1 - k_g}{l_g(G)} - \frac{k_g}{l_g(G)} r(t)$$

and differentiating, taking as constant the elasticity μ:

$$\frac{dw}{dt} = -w(t)\mu r(t) - \frac{k_g}{l_g(G)} \frac{dr}{dt}$$

Given the stationary state surplus rate $g^* = r^*$, it is:

$$\frac{dw}{dt} = -w(t)\mu r^*$$

implying a constant decrease of the affordable consumption per head:

$$w(t) = w_0 e^{-\mu \frac{n}{1+\mu} t}$$

6 The magnitude of rent and its dynamics

1. Aggregate rent equals

$$R(t) = (1 - k_g)G(t) - [w(t)L(t) + r(t)k_g G(t)]$$

or

$$R(t) = w(t) \int_0^{G(t)} [l_g(G) - l_g(z)] dz$$

or

$$R(t) = w(t)[l_g(G)G(t) - L(t)]$$

2. By the above assumptions, in the stationary state with a constant growth rate $g^* = r^*$,

$$\frac{dR}{dt} = \mu r^* w(t) L(t) > 0$$

3. As $w(t)$ declines, $R(t)$ rises.

7 The decline of the $w(t)$ and the economy's stalemate

1. It is clear that consumption per head (or real wage rate) cannot constantly fall, be driven to nought and even become negative. Yet, the economy's structure and its functioning forebode an unavoidable doom as long as it remains incapable of increasing productivity.

2. The stumbling block hindering any improvement and exposing the economy to decreasing returns is its failure to improve the method (l_g, k_g) that rules the production process. Note that whilst the appearance and entrenchment of rent creates a hideous distributional problem, average consumption per head would nevertheless decrease.

3. Consider, however, that in each society it can be possible to define what may be called a minimum amount

of consumption required to support life. There is clearly a biological dimension in this concept: a bare amount of calories and proteins is indeed required. Nevertheless, in any given social context, a sustainable train of life over and above food and shelter is also necessary. Studies of famines, drought and calamities have highlighted this point. Let a subsistence consumption per head, $\overline{w}$, be assumed.

In this framework, it is obviously made up of the same commodity as G.

4. Inevitably, $w(t) \rightarrow \overline{w}$; the point t when this occurrence comes to pass can, theoretically, be established:

$$\overline{w} = w_0 e^{-\mu \frac{n}{1+\mu} t}$$

from which

$$\bar{t} = \frac{\log \frac{w_0}{\overline{w}}}{\mu \frac{n}{1+\mu}}$$

5. It is quite conceivable that when $\overline{w}$ be reached the economy undergoes a structural change. If it remains stuck in a no productivity increase trap, then some social arrangement must be found to prevent the economy from falling below $\overline{w}$. Historically, population pressure coupled with decreasing returns has prompted migrations, wars and sharp distributional conflicts (here, w-R strife).

6. Whilst all these processes have indeed been observed, a less dramatic ploy has been found to cope with this stalemate. As still occurring in many developing countries to this very day and widely resorted to in Europe until a few decades ago, production is reorganised to slow down the exploitation of lands of dubious fertility by trying to keep population as much as possible in those lands that have been so far cultivated. This ploy implies abandoning

efficiency in a technical sense, basically producing the same quantity of output with a work force of increasing size, and some redistribution away from rent to keep per capita consumption fixed at $\overline{w}$. The basic assumption here is that population is still exogenously growing at a constant rate n.

7. This assumption is warranted by the fact that demography is ruled by deeply embedded cultural mores.

8. Redefining labour absorption:

$$L(t) = \int_0^{G(t)} l_g(z)dz + L^r(t)$$

$L^r(t)$ is simply an additional term designed to absorb labour in a population of given size; again,

$$\int_0^{G(t)} l_g(z)dz + L^r(t) = L(0)e^{nt}$$

dynamically:

$$nL(0)e^{nt} = l(G)r(t)G(t) + \frac{dL^r}{dt}$$

9. The surplus rate cannot be determined from the above since $w = \overline{w}$. Rent is still being paid and the dynamics of $r(t)$ can be obtained directly from the (w, r) trade-off on the marginal land, namely

$$r(t) = \frac{1 - k_g}{k_g} - \overline{w}\frac{l_g(G)}{k_g}$$

from which

$$\frac{dr}{dt} = \mu r^2 - \mu \frac{1-k_g}{k_g} r < 0$$

In fact, its analytical solution is:

$$r(t) = \left[\left(\frac{1-k_g}{k_g} \right)^{-1} + \left(r_0^{-1} - \left(\frac{1-k_g}{k_g} \right)^{-1} \right) e^{\frac{1-k_g}{k_g} \mu t} \right]^{-1}$$

obviously $\lim_{t \to \infty} r(t) = 0$

10. Basically, this result simply tells that the economy will finally ground to a halt. Hence, the solution can only be temporary.

11. There is a further reason to hold this view. The rent that now goes to the landlords is equal to:

$$R(t) = \overline{w}\,[l_g(G)G(t) - L^s(t)]$$

stating a necessarily distributive process to support the subsistence consumption per head.

From this

$$\frac{dR}{dt} = \overline{w}\,[(1+\mu)r(t)l_g(G)G(t) - nL^s]$$

and given the fact that $r(t) \longrightarrow 0$ it is certainly negative at least from some t onwards. Indeed, when $r(t) = 0$ the entire increase in consumption due to population growth is taken away from rents. Rents are entirely exhausted when $l_g(G)G(t) = L^s(t)$, namely when the entire work force equals the one that would be employed if all output were to be produced according to the production conditions of the marginal land. Since the latter grows at an ever slower rate whilst the former rises at a constant one, $R \to 0$ at which point no redistribution is actually feasible.

12. The surplus labour phase of this economy is a temporary, if a long-drawn, phenomenon.

7.1 The struggle to survive: some simple heuristics

1. An economy that can no longer count on redistribution from rent to a minimum subsistence consumption per head and yet fettered by decreasing returns must, again, undergo a structural change: hoarding surplus labour is no longer a feasible way out.

2. Historically, this is when migration movements, wars and territorial expansion have intensified.

3. In what follows, some simple heuristics of an economy caught in the grips of this stalemate will be described. It is clear that two of the previous assumptions must be forsaken.

- the economy can no longer afford to be inefficient: whatever output growth can be conjured up, it must be achieved by the efficient use of the pristine, unchanged technology (l_g, k_g).

- the population growth rate can no longer be exogenous: w is likely to drop below $\overline{w}$ and subsistence no longer assured.

4. This economy is subject to a Malthusian vicious circle!

5. Consider the following equations

- the population growth rate depends on the availability of food:

$$\frac{dL^s}{dt} = L^s\,[n - F(z)]$$

this equation renders the idea that the potentially autonomous growth rate n is now weakened by food availability at a rate that depends on how much w drops below $\overline{w}$, that is on $(\overline{w} - w) = z$. Some simple properties of $F(z)$:

$$\frac{dF}{dz} > 0; \;\; \frac{dF}{dw} = -\frac{dF}{dz} < 0; \; F(0) = 0$$

- Consumption per head, or the wage rate, increases if absorption of labour (demand for labour) rises above the available labour force and vice-versa. For simplicity's sake assume a linear function:

$$\frac{dw}{dt} = \alpha(L - L^s); \quad \alpha > 0$$

- Absorption of labour now depends strictly on how much this economy can invest to augment the seed-capital stock: there is no way to accumulate surplus labour. Hence, the crucial variable is the investable rate r:

$$\frac{dL}{dt} = l_g(G)rG$$

and by taking into account previous equations:

$$\frac{dL}{dt} = l_g\left[\Theta(L)\right]\left[\frac{1-k_g}{k_g} - w\frac{l_g\left[\Theta(L)\right]}{k_g}\right]\Theta(L)$$

6. This is a system of differential equations of the following type:

$$\begin{aligned}\dot{L^s} &= f_1(L^s, w)\\ \dot{w} &= f_2(L^s, L)\\ \dot{L} &= f_3(L, w)\end{aligned}$$

a system of three equations and three variables.

It basically portrays a movement such that when w increases there is some push for the population to rise. As w rises the investment rate tends to fall diminishing the capability to absorb labour. As a consequence, this tends to create unemployment lowering w which in turn allows for a higher r restoring higher demand for labour and higher w: back to step one.

7. The actual dynamics depend much on the various functions and in particular on $l_g(G)$, $F(z)$.

A relevant question is if there is a stationary state: $(\dot{L^s}, \dot{L}, \dot{w}) = 0$.

8. Solving the system for this particular case:

$$\begin{aligned}
\dot{L}^s &= 0 \text{ when } n = F(z) \\
\dot{w} &= 0 \text{ when } L = L^s \\
\dot{L} &= 0 \text{ when } w = \frac{1 - k_g}{l_g[\Theta(L)]}
\end{aligned}$$

a system that should return three solutions: L^{s*}, L^*, z^*, this last being $z^* = \overline{w} - w^*$.

9. If the economy by chance happens to be on such an equilibrium, it remains there: at a consumption per head that absorbs the entire net product but at the bare minimum that keeps the population from increasing, leaving nothing to invest to increase output although keeping the entire work force employed.

10. A further question concerns local stability. It is, of course, quite unlikely that the economy would ever settle on this stationary state point, but should it ever happen to be there and locally veer away from such a point, would it return thence?

11. It is possible to employ the usual tools to check for local stability by linearising the system:

$$\dot{L}^s = \gamma_1 w$$

$$\dot{w} = -\alpha L^s + \alpha L$$

$$\dot{L} = \beta_1 w + \beta_2 L$$

where the coefficients are the first derivatives at the stationary point. More specifically,

$\gamma_1 = -L^* \frac{dF}{dw} > 0$

$\alpha > 0$

$\beta_1 = -\frac{l_g [\Theta(L^*)]^2}{k_g} \Theta(L^*) < 0$

$\beta_2 = -\Theta(L^*) \frac{1-k_g}{k_g} \frac{dl_g}{d\Theta} \frac{d\Theta}{dL} < 0$

Consider the Jacobian

$$J = \begin{vmatrix} 0 & \gamma_1 & 0 \\ -\alpha & 0 & \alpha \\ 0 & \beta_1 & \beta_2 \end{vmatrix}$$

from which the trace:

$$tr(J) = \beta_2 < 0$$

Thus there is a *prima facie* reason to suppose that the system is locally stable.

The Routh-Hurwitz conditions establish the sufficient ones for stability to hold.