Finite Elements - Homework

1. **Nearly Incompressible Linear Elasticity**

Let $\Omega := (0,1) \times (0,1)$. Consider the displacement formulation of the isotropic, homogeneous, linear elasticity problem: find u such that

$$\begin{cases} 2\mu \,\mathrm{div}(\nabla^s u) + \lambda \nabla(\mathrm{div} u) + f = 0 & \text{in } \Omega \,, \\ u = 0 & \text{on } \Gamma_D \,, \\ (2\mu \nabla^s u + \lambda (\mathrm{div} u)\mathbb{I})n = 0 & \text{on } \Gamma_N \,, \end{cases} \tag{1}$$

where μ and λ are the Lamé's constants, f is a given load density, $\Gamma_D = \{0\} \times (0,1) \cup (0,1) \times \{0\}$ (left vertical and lower horizontal sides) and $\Gamma_N = \{1\} \times (0,1) \cup (0,1) \times \{1\}$ (right vertical and upper horizontal sides). Recall that ∇^s denotes the symmetric gradient and $\mathbb{I}$ is the second-order identity tensor.

Set $\mu = 0.5$ and $f = [1,1]$. Define $V := H^1_{\Gamma_D}(\Omega)^2$. The variational formulation of problem (1) reads as follows: find $\in V$ such that, for all $v \in V$,

$$2\mu \int_\Omega \nabla^s u : \nabla^s v \, dx + \lambda \int_\Omega (\mathrm{div} u)(\mathrm{div} v) \, dx = \int_\Omega f \cdot v \, dx \,. \tag{2}$$

Introducing the new variable $p := \lambda \,\mathrm{div} u$, problem (1) can be written as follows: find (u,p) such that

$$\begin{cases} 2\mu \,\mathrm{div}(\nabla^s u) + \nabla p + f = 0 & \text{in } \Omega \,, \\ \mathrm{div} u - \dfrac{1}{\lambda} p = 0 & \text{in } \Omega \,, \\ u = 0 & \text{on } \Gamma_D \,, \\ (2\mu \nabla^s u + \lambda (\mathrm{div} u)\mathbb{I})n = 0 & \text{on } \Gamma_N \,. \end{cases} \tag{3}$$

Define $Q := L^2(\Omega)$. The variational formulation of problem (3) is: find $(u,p) \in V \times Q$ such that, for all $(v,q) \in V \times Q$,

$$\begin{cases} 2\mu \displaystyle\int_\Omega \nabla^s u : \nabla^s v \, dx + \int_\Omega p \,\mathrm{div} v \, dx & = \displaystyle\int_\Omega f \cdot v \, dx \,, \\ \displaystyle\int_\Omega \mathrm{div} u \, q \, dx - \frac{1}{\lambda} \int_\Omega p \, q \, dx = 0 \,. \end{cases} \tag{4}$$

(a) Implement in *FreeFem++* a discretization of the variational formulation (2) with continuous linear elements with structured meshes `Th=square(n,n,...)`, with `n=16`, `n=32` and `n=64`, for $\lambda = 10$, $\lambda = 10^4$ and $\lambda = 10^7$; plot the first and second components of the computed displacement u_h separately, the mesh after displacement (use `movemesh`) and report the norm of the computed displacement vector u_h at the point $(1,1)$. [*Hint:* modify `lame.edp`.]

(b) Implement in *FreeFem++* a discretization of the variational formulation (4) with $P_1^b - P_1^c$ elements with structured meshes with `n=4`, `n=8`, `n=16`, `n=32` and `n=64`, for $\lambda = 10^7$; plot the first and second components of the computed displacement u_h separately, the mesh after displacement and report the norm of the computed displacement vector u_h at the point $(1,1)$. [*Hint:* modify `stokes.edp`.]

(c) For the analytical solution u, $u(1,1) \simeq 0.1866$. Observe that the error $|u(1,1) - u_h(1,1)|$ decreases to zero linearly in h.

(d) Run the *FreeFem++* code of (b) with $P_2 - P_1^c$, $P_2 - P_1^d$ and $P_1 - P_0$ elements, and compare the obtained discrete solutions with those obtained in (a) and in (b). Which methods are affected by numerical locking?

2. **Membrane problem in mixed form**

Let $\Omega := (0,1) \times (0,1)$ and let the usual variational spaces

$$\Sigma = H_{\text{div}}(\Omega) , \qquad U = L^2(\Omega) .$$

Consider the following problem (in variational form) of the elastic membrane in mixed form.

Find $\sigma \in \Sigma, u \in U$ such that

$$\begin{cases} \displaystyle\int_\Omega \sigma \cdot \tau \, dx + \int_\Omega (\text{div}\tau) \, u \, dx = 0 \quad \forall \tau \in \Sigma \, , \\ \displaystyle\int_\Omega (\text{div}\sigma) \, v = - \int_\Omega f \, v \, dx \quad \forall v \in U \, . \end{cases} \tag{5}$$

where $f(x,y) = 1$ is a given loading function and where we assume the material tensor $\mathbb{K}$ equal to the identity.

(a) Implement in *FreeFem++* a discretization of the variational formulation (5) with structured meshes `Th=square(n,n,...)`, with `n=8`, `n=16`, `n=32` and `n=64`, using the Raviart-Thomas element (as already done in the laboratory class).

(b) Implement in *FreeFem++* a discretization of the variational formulation (5) with the same structured meshes introduced above, but using the discrete spaces

$$\Sigma_h = \{\tau_h \in [C^0(\Omega)]^2 \text{ such that } \tau_h|_K \in P_2 \; \forall K \in \mathcal{T}_h\}$$
$$U_h = \{v_h \in U \text{ such that } v_h|_K \in P_0 \; \forall K \in \mathcal{T}_h\} \, .$$

Note that above we are requiring the functions of Σ_h to be continuous in *all* components.

Compare the plots (and values) of the discrete stresses σ_h with the results obtained with the Raviart-Thomas element. Is the behavior of this second element satisfactory? If not, what is the cause of such bad behavior?

(c) Implement in *FreeFem++* a discretization of the variational formulation (5) with the same structured meshes introduced above, but using the discrete spaces

$$\Sigma_h = \{\tau_h \in [C^0(\Omega)]^2 \text{ such that } \tau_h|_K \in P_1 \; \forall K \in \mathcal{T}_h\}$$
$$U_h = \{v_h \in U \text{ such that } v_h|_K \in P_0 \; \forall K \in \mathcal{T}_h\} \, .$$

Compare the plots (and values) of the discrete stresses σ_h with the results obtained with the Raviart-Thomas element. Is the behavior of this second element satisfactory? If not, what is the cause of such bad behavior?