Introducing Erasures in Decision-Feedback Equalization to Reduce Error Propagation

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***Abstract*—A simple modification of the decision feedback equalizer (DFE) slicer is proposed to reduce the effect of error propagation. A comparison of the performance of the modified DFE and conventional DFE is made for specific channels. On these channels, modified DFE performs only marginally better than the conventional DFE in terms of average error probability, but may offer some advantages in terms of error probability conditioned on specific input sequences and in terms of the distribution of error burst lengths. Some examples are given, concerning binary PAM and multilevel quadrature amplitude modulation (*M*–QAM) systems.**

***Index Terms*—Decision feedback equalization, digital communications.**

I. Introduction

Decision-feedback equalization (DFE) is a well-known technique that finds applications in many areas of communications [1]. An example is the transmission of high data rates over a linear channel causing intersymbol interferences (ISI) which can be compensated on the receiver side by a DFE. One of the major disadvantages of DFE is the effect of error propagation [2]–[6]. In the present letter, a simple modification of the decision device is proposed to achieve better system performance. In the binary case, the analysis of the modified DFE is developed analytically for a short memory channel. Finally, the system is generalized to channels with longer memory and higher level constellations, with some results obtained by computer simulation.

II. DFE With Erasures

Let us refer to the DFE system shown in Fig. 1. We assume a zero-forcing criterion to design the forward filter (FFF) [1] so that

$$x_k = a_k + \sum_{j=1}^{\infty} h_j a_{k-j} + n_k \tag{1}$$

where a_k is the current symbol to be detected, h_j is the jth channel response sample, and n_k is the additive noise. The previous expression implicitly means that the overall channel impulse response is causal. In Sections II and III, we will refer, for the sake of simplicity, to binary PAM, i.e., $a_k \in \{+1, -1\}$, whereas, some simulation results will be presented in Section IV for higher order constellations. For our purposes, we also will suppose that both $\{n_k\}$ and $\{a_k\}$ are sequences of independent identically distributed random variables. We define $p = \text{Prob}\,\{a_k = 1\}$ and $q = \text{Prob}\,\{a_k = -1\}$. The cumulative distribution function $N(n)$ of the r.v.'s n_k is arbitrary: in this paper, we assume it is the Gaussian distribution, with zero mean and variance σ^2. In the following, we will refer our results to the signal-to-noise ratio $\rho = 1/2\sigma^2$. The key observation of our proposal is that if the sample at the input of the decision device (γ_k in Fig. 1) is close to the threshold, the corresponding output of the slicer $\hat{a}_k$ is not very reliable: so, using this unreliable symbol can reduce the noise margin of future symbols instead of enhancing it. This suggests the possibility of reducing error propagation by avoiding the feedback of the less reliable symbols. On this basis, we propose a DFE in which erasures are introduced in the feedback chain: for the hard-decisor scheme, it is sufficient to use a different nonlinear function for the slicer, as shown in Fig. 1, where the input–output characteristic of the dead-zone limiter used as decision device for the feedback loop is shown. A symbol is called unreliable if the absolute value of the corresponding sample γ_k is below the threshold A. In this case, the symbol is not fed back, but an erasure is introduced. The erasure criterion can be stated as follows:

$$\begin{cases} \gamma_k \geq A & \longrightarrow \text{feedback “1”} \\ |\gamma_k| < A & \longrightarrow \text{unreliable symbol: feedback “0”} \\ \gamma_k \leq -A & \longrightarrow \text{feedback “}-1.\text{”} \end{cases}$$

The resulting scheme is a "decision feedback with erasures" equalizer, and will be indicated as "E-DFE."

III. Error Probability Analysis for 1-Bit Memory Channels

Let us start with the 1-bit memory channels $h_1 \neq 0$, $h_j = 0$ for $j > 1$. At the input of the slicer, the receiver computes

$$\gamma_k = x_k - h_1 b_{k-1} = a_k + z_k + n_k \tag{2}$$

$$z_k = h_1 e_{k-1}, e_k = a_k - b_k \tag{3}$$

where e_k assumes a value in the set $\{-2, -1, 0, 1, 2\}$. The random sequence $\{z_k\}$ in (3) represents a finite-state, discrete-time Markov chain with state diagram as shown in Fig. 2. Moreover, the Markov chain given by the sequence of $\{z_k\}$ is homogeneous. It is possible to show that the error probability

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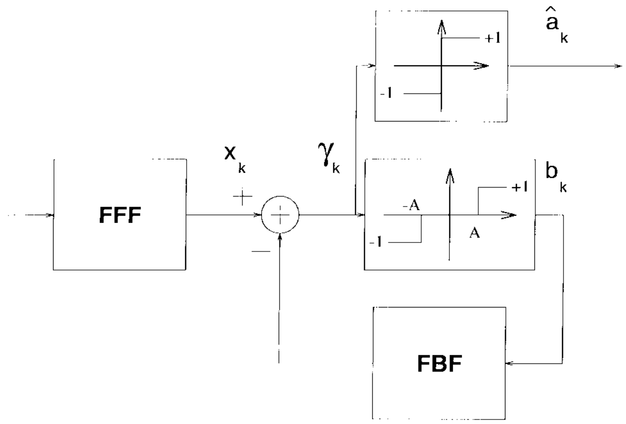

Fig. 1. Modified DFE system.

at time k is given by [7]

$$P_e(k) = \underline{S}^{(k)}\underline{W}^T \tag{4}$$

where $\underline{S}^{(k)} = [p_1^{(k)}\ p_2^{(k)}\ \cdots\ p_5^{(k)}]$, $p_i^{(k)}$ is the probability to be in state i at time k; the elements of vector $\underline{W}$ are given by $W_i = pN(-1-\phi_i) + q[1-N(1-\phi_i)]$, $i = 1, \cdots, 5$, and ϕ_i are the possible values of z as reported in Fig. 2.

Finally, we obtain the asymptotic bit error probability as

$$P_e = \lim_{k\to\infty} \underline{S}^{(k)}\underline{W}^T = \lim_{k\to\infty} \underline{S}^{(0)}\underline{\underline{P}}^k\underline{W}^T \tag{5}$$

where $\underline{\underline{P}}$ is the transition matrix associated with the state diagram in Fig. 2 and $\underline{S}^{(0)}$ is the initial state vector (arbitrary). The elements of $\underline{\underline{P}}$, i.e., the transition probabilities $p_{i,j}$, can be calculated as $p_{i,j} = \text{Prob}\{\phi^{(k+1)} = \phi_j/\phi^{(k)} = \phi_i\} = p_j(\phi_i)$, where $p_j(z)$ is the probability of going into state j given the current state z [7]:

$$\begin{aligned}
p_1(z) &= p\cdot N(-A-z-1)\\
p_2(z) &= q\cdot[1-N(A-z+1)]\\
p_3(z) &= p\cdot[N(A-z-1)-N(-A-z-1)]\\
p_4(z) &= q\cdot[N(A-z+1)-N(-A-z+1)]\\
p_5(z) &= p\cdot[1-N(A-z-1)] + q\cdot N(-A-z+1).
\end{aligned} \tag{6}$$

Note that the system proposed is coincident with the conventional DFE if we let $A = 0$,[1] and becomes the unequalized system (except for the FFF) if we let A tend to infinity.

A. Conventional DFE

For $A = 0$, (5) can be reduced [7] to the following closed form:

$$P_e = \frac{N(-1)[1-2pq(\gamma-\beta)]}{N(-1) - 2pqN(1)(\gamma-\beta) + pq(\gamma^2-\beta^2)+\gamma} \tag{7}$$

where $\gamma = N(1+2h_1)$ and $\beta = N(1-2h_1)$. For $p = q = \frac{1}{2}$, the last formula is coincident with the result presented in [3]. It is worthwhile observing that this formula gives the exact error probability of conventional DFE, whatever p and q, for a one-symbol memory channel: by comparing it with the ideal case $P_e = N(-1)$, we can evaluate the effect of error propagation.

[1] In this case, the states ϕ_3 and ϕ_4 are not reachable.

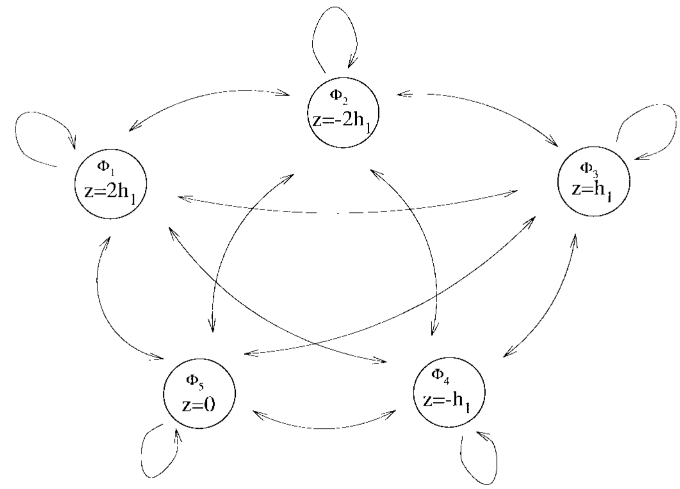

Fig. 2. Markov chain state diagram for the DFE with erasure.

B. Worst Case Sequence

It is well known that, to operate satisfactorily, DFE needs the input sequence to be as random as possible: in fact, for some input sequences, performance greatly deteriorates by error propagation [4], [5]. So, there is some interest in determining the system behavior when the worst case input sequence is given. In our model, the worst case is simply obtained by putting in the previous analysis $h_1 < 0$ and $p = \text{Prob}\{a_k = 1\} = 1$.

IV. Numerical Results

A. Short-Memory Case

First, we consider it convenient to check whether there is some improvement by setting the threshold $A \neq 0$. For this purpose, in Fig. 3 we report the error probability normalized to the values referred to $A = 0$ (i.e., the conventional DFE) as a function of threshold A for $\rho = 5$, 6, 7, 8, and 9 dB. So, in this figure, the points having ordinates lower than one signify an improvement with respect to conventional DFE. We can conclude that the best performance is obtained with a threshold $A > 0$, and that the optimum threshold decreases when increasing the signal-to-noise ratio. It is also important to note that increasing the value of A beyond the optimum value deteriorates the performance; this is because, with high A, we have almost only erasures and the FBF becomes ineffective. Furthermore, we observe that the optimum value of A, as shown in Fig. 3, depends on ρ: therefore, in order to implement the E-DFE, an estimate of the signal-to-noise ratio is required.

B. Long-Memory Case

To investigate E-DFE behavior with channel memory higher than 1, we will proceed by computer simulation. The assessment of the optimum A value for the E-DFE also has been pursued by simulation. In Fig. 4, the error probability for a channel with $h_1 = -0.6, h_2 = -0.3, h_3 = -0.2, h_4 = -0.2, h_5 = -0.1, h_j = 0$ for $j > 5$ is investigated with a random input sequence. The signal-to-noise ratio has been fixed at 9 dB, and the simulation obtained with $3 \cdot 10^7$ iterations. As we are interested not only in the average error probability, but in the distribution of the error length too, we

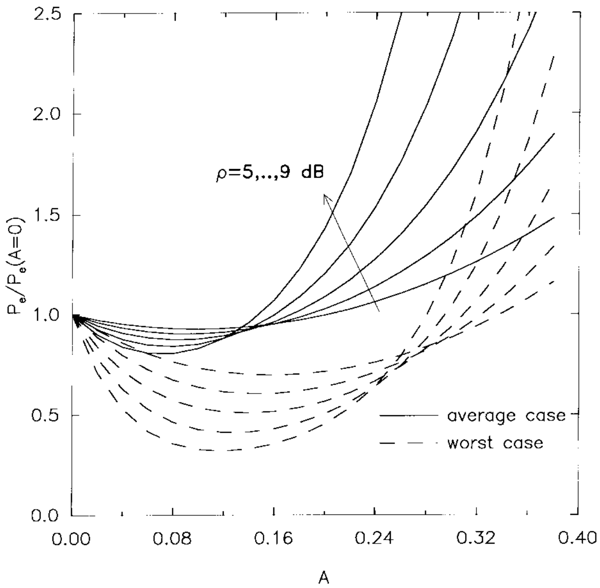

Fig. 3. Normalized bit-error probability as a function of threshold A, binary PAM. $h_1 = -0.7$, analytical results.

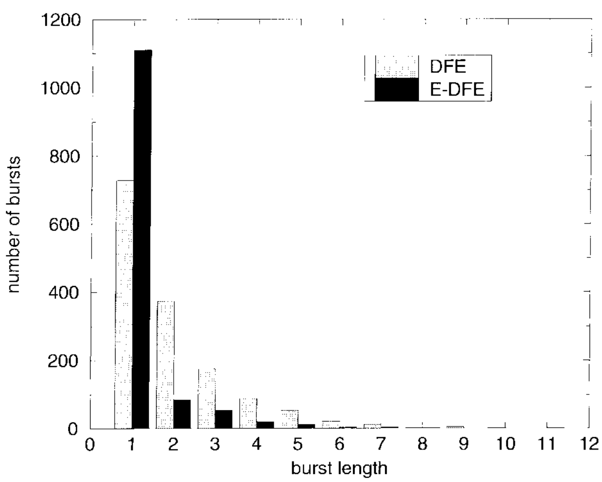

Fig. 4. Channel as specified, $\rho = 9$ dB, burst percentage versus burst length. Comparison between DFE ($A = 0$, $P_e = 9.9 \cdot 10^{-5}$) and E-DFE ($A = 0.1$, $P_e = 5.7 \cdot 10^{-5}$), binary PAM.

report in this figure the histograms of the number of error events versus the length (in bits) of the bursts. The two set of bars refer to conventional DFE ($A = 0$) and E-DFE with $A = 0.1$. The analysis of this figure makes clear how the statistics of the error length is modified by the introduction of erasures: it can be concluded that the average number of errors decreases, resulting in a lower bit-error probability, and also that the average error length is lowered. In fact, by introducing erasures, we increase the number of bursts of length 1, but obtain a reduction of longer bursts. For example, this effect can be exploited by means of error-correcting codes designed to counteract short error bursts: in this case, the use of E-DFE could give an improvement similar to that obtainable by using interleaving techniques.

Finally, it is interesting to extend the E-DFE concept to higher order constellations. To this aim, in Fig. 5, we show, as an example, the constellation for a 16-QAM modulation, where the symbols are complex-valued (the real and imaginary parts taking values in the set $\{\pm 1, \pm 3\}$). For M-QAM systems, the E-DFE scheme is similar to that shown in Fig. 1 for binary PAM, except that it is necessary to define how the decision devices work: for this purpose, in Fig. 5, the thresholds defining the decision regions for symbols $\hat{a}_k = \hat{a}_{pk} + j\hat{a}_{qk}$ are indicated with continuous lines (placed at $0, \pm 2$ both for the real and imaginary parts), whereas dead zones (dashed lines) are introduced around these thresholds for symbols $b_k = b_{pk} + jb_{qk}$. Similarly to binary PAM, if the complex received sample γ_k falls inside one of these zones, the intermediate symbol b_k, as shown in Fig. 5, is fed back. In Fig. 6, the performance of the in-phase component of a 64-QAM system with DFE and E-DFE over the same five-tap channel of Fig. 4 is reported.

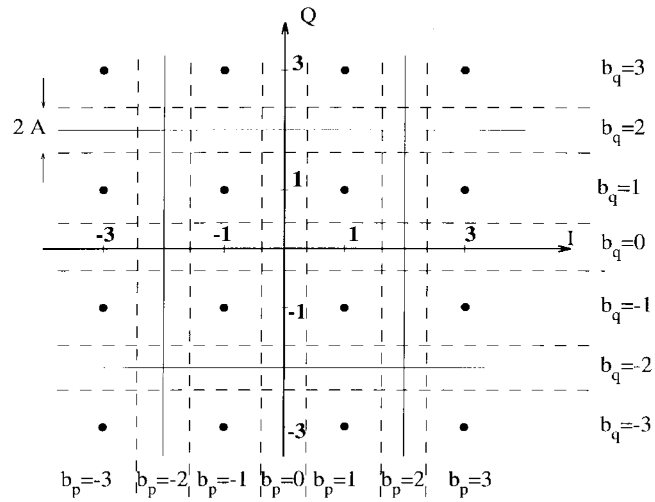

Fig. 5. Decision regions for a 16-QAM constellation.

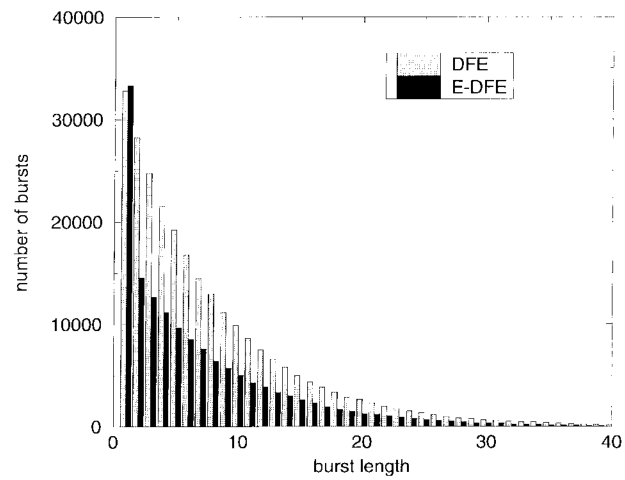

Fig. 6. Performance of 64-QAM, channel as in Fig. 4, $\rho = 9$ dB, number of bursts versus burst length, $3 \cdot 10^8$ iterations. Comparison between DFE ($A = 0$, $P_e = 7.3 \cdot 10^{-3}$) and E-DFE ($A = 0.1$, $P_e = 3.7 \cdot 10^{-3}$).

It can be noted that, by introducing a threshold $A = 0.1$, the number of bursts of length 1 does not practically change, whereas longer bursts are significantly decreased: as a result, the number of errors is halved by using E-DFE.

V. Conclusions

In this paper, we have described a modification of the classic decision feedback equalizers. In this new system, the nonlinearity of the feedback chain is different from the classic one-threshold slicer, allowing the possibility of error propagation to be reduced. In the binary case, an exact analytical approach has been carried out to investigate system performance for the 1-bit memory channels. System performance for channels with higher memory and higher order constellations (M-QAM) has been investigated by means of computer simulations. The analysis shows that the proposed system allows a reduction of error propagation, resulting in an improvement in performance, particularly when the channel exhibits long memory.

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