1 Introducing investment

Recent economic literature has persuasively argued that efforts directed at seeking ways to upgrade technologically the productive apparatus is a bounded search in a space of opportunities that become visible as the process unfolds. This search is necessarily local and subject to evolution as skills and competence are acquired and learning from experience takes place.

The first important observation is that searching is a process cast into agents' bounded rationality. The second and by no means less relevant one is that agents can neither fully scan the entire domain of technological opportunities in theory available in the whole economic system nor can they immediately or instantaneously translate actual observation of better or applicable techniques into adoptable plans to upgrade and invest.

Identifying informative sources and collecting information are uncertain activities which depend on acquired capabilities, established technological prowess, consolidated knowledge.

The purpose of this paper is to investigate the evolutionary process of imitation and innovation

as a process of searching in a given neighbourhood of firms.

Consider the following technological sequence:

$$T^j(1) < T^j(2) < ... < T^j(N_t^j) \equiv \overline{T}_{j,t}$$

N_t^j being the number of techniques that have been sequentially adopted by sector j up to time $T^j(N_t^j)$. Hence, N_t^j is the most recent technique extant at time t while the $n-th$ technique had been adopted at time $T^j(n)$.

Let then a simple economy be described.

It is assumed that in each sector j there is a *leader* pursuing innovation, and $F_j - 1$ *followers* trying to imitate it.

It is a well established fact that information is broadcast by leading and innovating firms: they allow the diffusion of new principles and technological paradigms across sectors and industries. Normally, when a new technological breakthrough is achieved its content can find applications in other sectors of activity.

Since the capability of understanding and processing information coming from a different sector and a different technological context depends on the common knowledge basis, the transmission of such information depends on the strength of this shared knowledge which measures the potential intensity of their interaction and the probability of actually passing on relevant information.

Let this measure be defined, in general, by $\epsilon^{i,j} \in [0,1]$for any two leaders belonging to different sectors i and j. Consider, however, its average:

Assumption 1 *The average probability for the economy as a whole that information be passed on is*

$$\hat{\epsilon} = 1 - \alpha \frac{1}{J}$$

J has an important role to play: it is the number of sectors or industries that are present in the economy. Thus, the higher is this number, that is, the denser is the economy's structure, the more overlapping are the neighbourhoods within which firms search, hence the greater is the strength of information transmission.

The concept of technological search. Leading firms collect information packets from other leading firms within a small cognitive neighbourhood. Let these packets be called *bits*. Collection of information is idiosyncratic and gradual; it is sequential.

Consider the following assumption

Assumption 2 *For each innovation, the generic innovator* i *stands probability* p_z *of locking into a technological search* z*, requiring an informative sequence of* S_z *bits,* $z = 1, 2, ..., Z$*. Obviously* $\sum_{z=1}^{Z} p_z = 1$*.*

Without loss of generality we rank these possible technological searches as

$$S_1 < S_2 < ... < S_Z$$

We refer to $S_j(n)$ to denote the *information content* of the $n-th$ innovation and for notational convenience we set $\overline{S}_{j,t} \equiv S_j(N_t^j)$. It is a fairly well established fact that, at least on average, innovations requiring a greater effort, which is here measured by the number of bits collected, yield a greater productivity increase.

Assumption 3 *Let us denote by λ_z the productivity growth rate resulting from the innovation produced by a search of type z, then*

$$\lambda_1 < \lambda_2 < ... < \lambda_Z$$

When a sector is involved in an innovation wave, productivity gains occur at a rate which is specific of the sector and depends on the amount of information gathered. Let $\lambda_j(n)$, for $j = 1, 2...J$; $n = 1, 2...N_t^j$, stand for such a rate. Since information is mainly retrieved from neighbours, we are led to assume that the longer are the informative sequences, the larger is the number of neighbours S^*, which each leader contacts in its search. We formalize this fact in the following

Assumption 4 *The number of neighbours is equal to the mean value of the random variable length of the informative sequence*

$$S^* = \sum_{z=1}^{Z} \pi_z S_z$$

The search process, however, requires that autonomous searching be also conducted since if there were no primary innovative input, no informative interaction could take place. Consider, therefore, an entirely exogenous innovative event occurring with a Poisson arrival rate equal to h. There accordingly are two overlapping processes: an exogenous one and an interactive one.

1.0.1 Dynamics

The following system of equations describes the gradual process of gleaning information. Denote by ρ_a the share of firms that have reached a complete set of bits and consequently innovate. These firms then spread information to others that are still in the process of collecting the required ones and begin again their searching and learning process to attempt achieving the next innovation. ρ_c is the share of firms that are in the critical stage, that require just an additional bit to complete the sequence; thus, the number of firms that will eventually reach the active stage a and become innovators depends on this critical number. ρ_k denotes the share of firms that are still in stage k $(k = 1, 2, ..c, a)$ of the sequence.

$$
\begin{gathered}
\dot{\rho}_a = -\rho_a + (h + S^{*}\hat{\epsilon}\rho_a)\,\rho_c \\
\dot{\rho}_c = -\,(h + S^{*}\hat{\epsilon}\rho_a)\,\rho_c + (h + S^{*}\hat{\epsilon}\rho_a)\,\rho_{c-1} + p\,(h + S^{*}\hat{\epsilon}\rho_a)\,\rho \\
\dot{\rho}_{c-1} = -\,(h + S^{*}\hat{\epsilon}\rho_a)\,\rho_{c-1} + (h + S^{*}\hat{\epsilon}\rho_a)\,\rho_{c-2} \\
\dots \\
\dot{\rho}_{k+1} = -\,(h + S^{*}\hat{\epsilon}\rho_a)\,\rho_{k+1} + (1 - p)\,(h + S^{*}\hat{\epsilon}\rho_a)\,\rho_k \\
\dot{\rho}_k = -\,(h + S^{*}\hat{\epsilon}\rho_a)\,\rho_k + (h + S^{*}\hat{\epsilon}\rho_a)\,\rho_{k-1} \\
\dots \\
\dot{\rho}_0 = -\,(h + S^{*}\hat{\epsilon}\rho_a)\,\rho_0 + \rho_a
\end{gathered}
$$

It is important to note that the exogenous shock occurs with probability h while the endogenous one with probability $S^{*}\hat{\epsilon}\rho_{a}$.

It is clear that the sum of all shares must be normalised to 1.

$$\rho_c + ... + \rho_0 = 1$$

The dynamics of this system can be very complex. It is, however, interesting to observe it in the stationary state. In this case, solutions for the shares of firms in the various stages of information collection are

$$\rho_c = \rho_k = \rho_{k-1} = ... = \rho_0$$
$$\rho_{S_2-2} = \rho_{S_2-3} = ... = \rho_{k+1} = (1-p)\,\rho_k$$

and

$$\rho_c = \frac{1}{S^{*}}$$

Let this system be considered in the limit for $h \longrightarrow 0$. This assumption is required in order to generate the number of firms that are involved in a process of innovation thanks to an initial shock that is then spent and through the interactive process of information collection. The expected value, V_T , is

$$E(V_T) = \frac{\partial}{\partial h}\rho_a\Big|_{h=0} = \frac{1}{(1-\hat{\epsilon})\,S^*}.$$

reflecting the following question: how does the share of innovating firms change for an increase in the probability of a shock occurring when there is initially none, i.e. when $h = 0$?

Taking into account the components of $\hat{\epsilon}$

$$E(V_T) = J\frac{1}{\alpha S^*}$$

This can be described as an innovation wave or avalanche. It is clear that this is a wave of an average, expected dimension: waves of all sizes can actually occur.

It is also straightforward to compute the average waiting time between two avalanches

$$\omega = E\left(T\left(N_t + 1\right) - T\left(N_t\right)\right) \sim \frac{S^*}{h}$$

Since the average size of an avalanche is $E\left(V_T\right)$, the expected average number of firms in each sector is $\frac{1}{\alpha S^*}$ and since the average waiting time between avalanches is ω, then the expected number on innovations in any sector J at time t is

$$E(N_t^j) \ \sim \ \frac{h}{\alpha S^{*2}} t \text{ as } t \to +\infty$$

that is, in the very long period.

Productivity growth

$$
\begin{aligned}
a_{j,N_t^j} &= a_{j,N_t^j - 1} e^{-\lambda_j \left(N_t^j\right)} \\
a_{k_j,N_t^j} &= a_{k_j,N_t^j - 1} e^{-\lambda_j \left(N_t^j\right)}
\end{aligned} \qquad (1)
$$

Consider the average long-term growth of such an economy. It is simply

$$\gamma_I = \frac{\lambda^*}{\alpha(S^*)^2} ht$$

That is the number of expected innovations per sector over a period t times the average productivity growth rate λ^*.