# AUTOMATIC CONTROL 2
# DIGITAL CONTROL SYSTEMS

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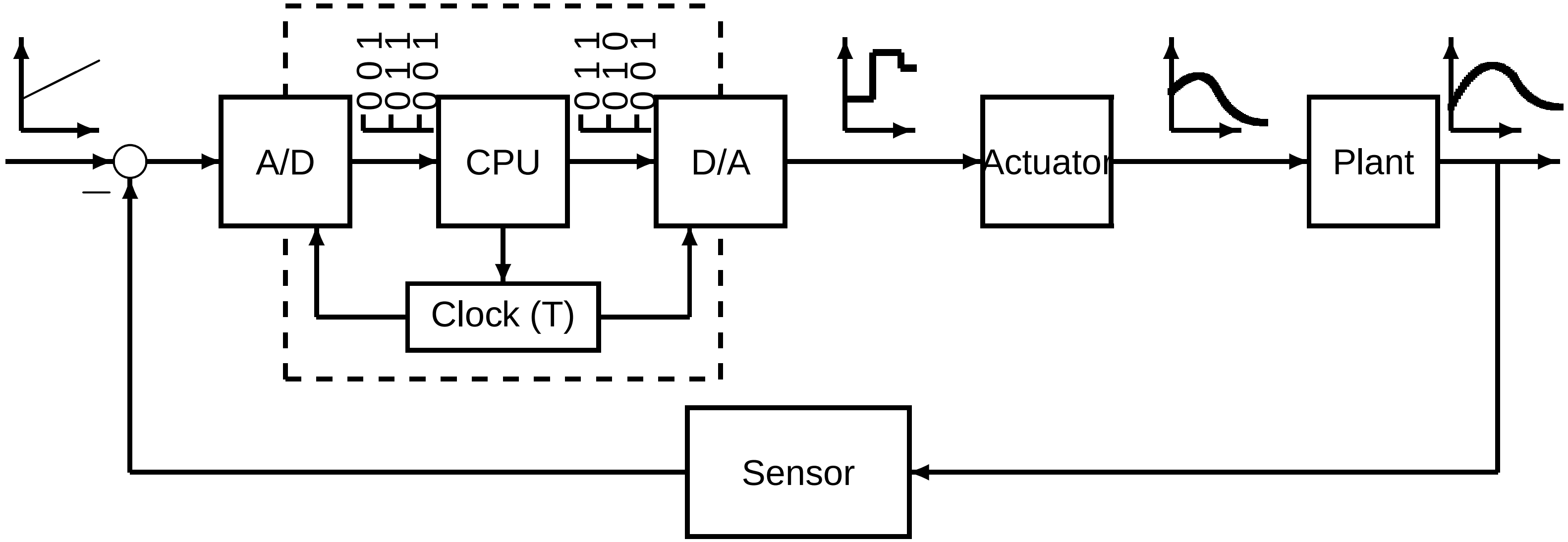
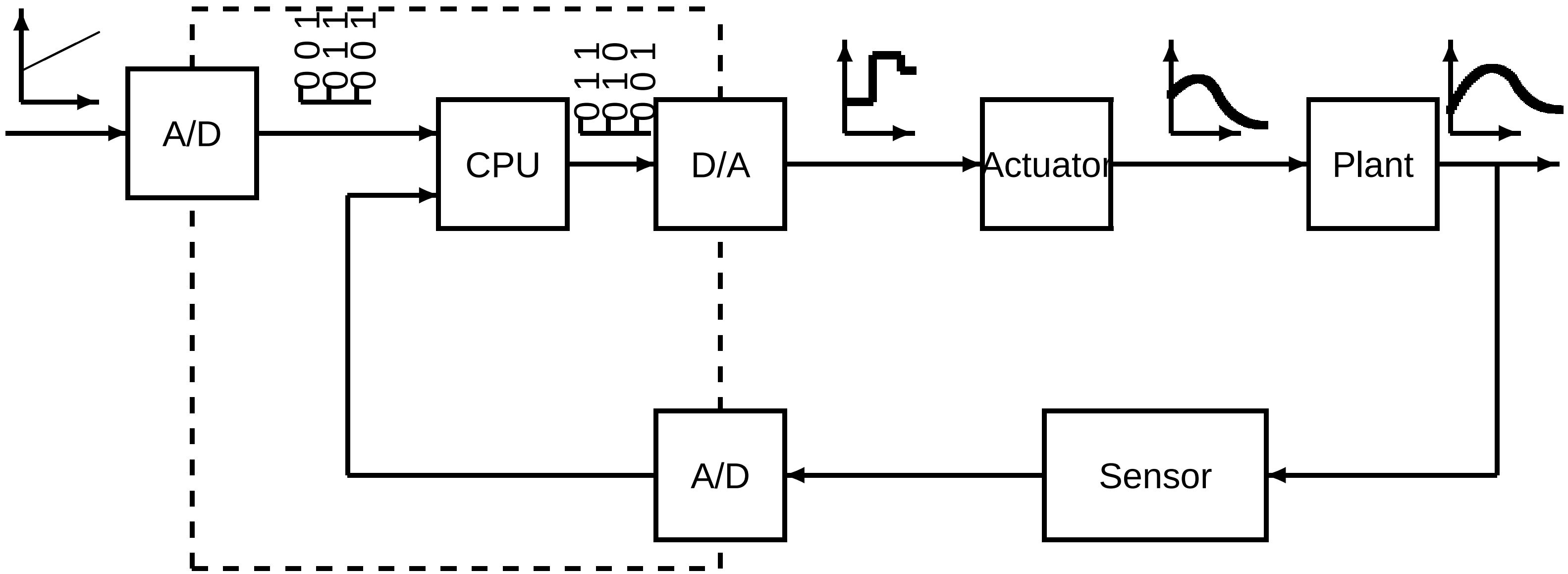
# DIGITAL CONTROL SYSTEMS INTRODUCTION

# Typical digital control systems feedback loop

Sampling of the error signal:

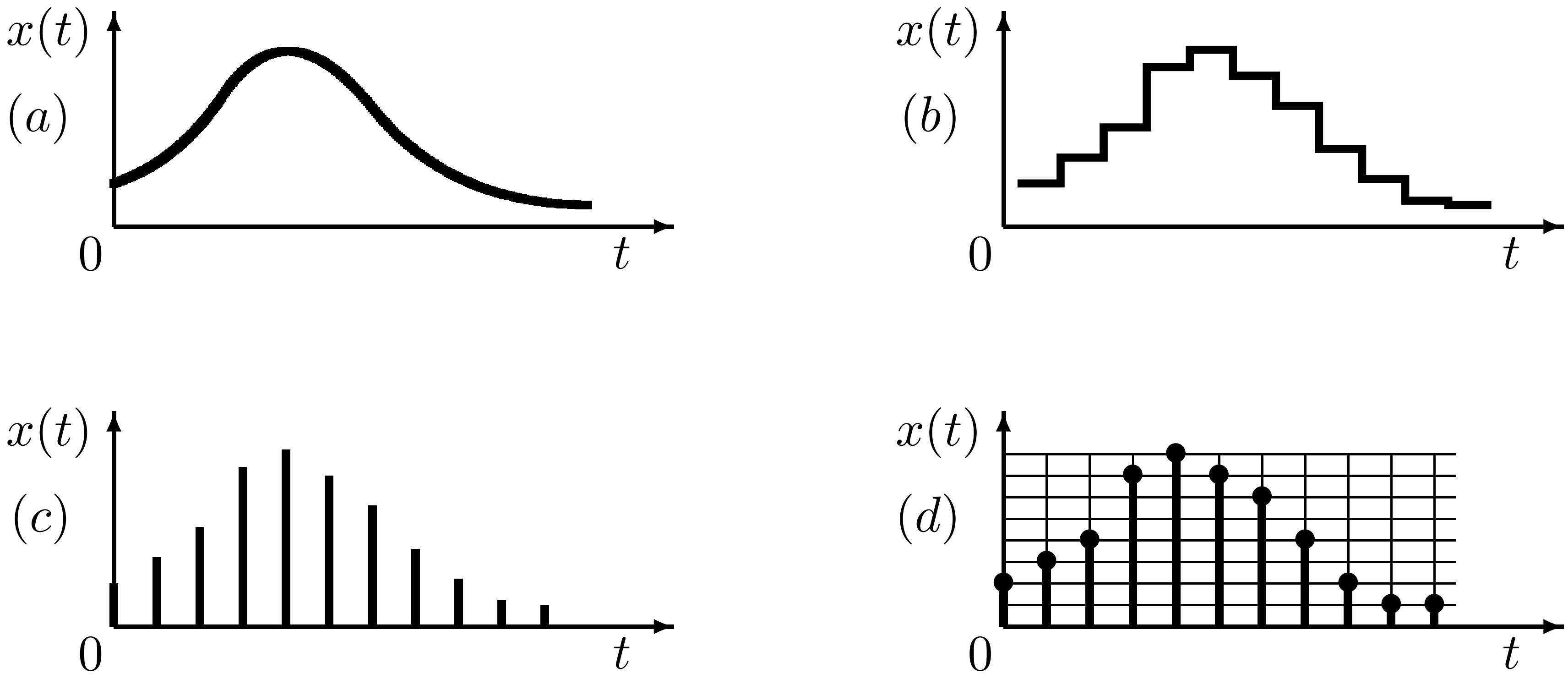



Sampling of the measure:



# Signal classification

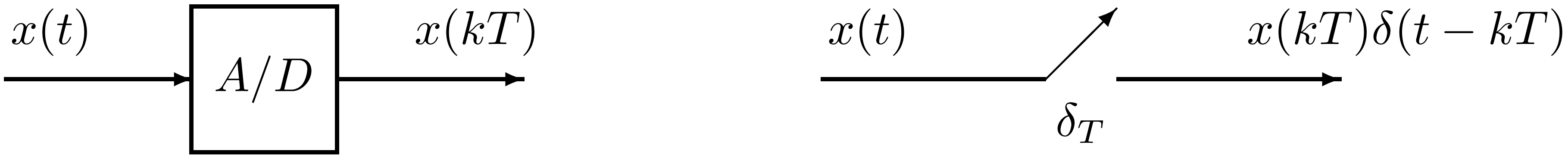
(a) Analogical signal; b) Quantized signal; c) Sampled signal; d) Digital signal



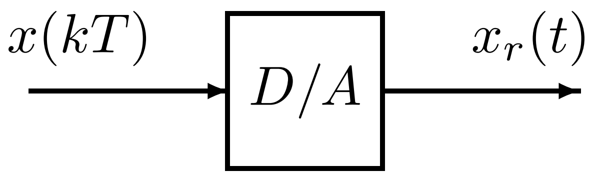
# Interface devices

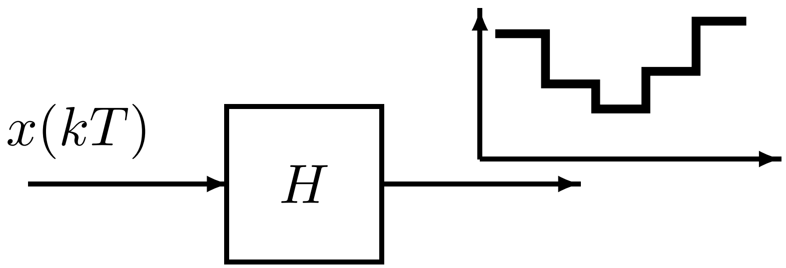
A/D, Analog/Digital converter:


Model: impulsive sampling

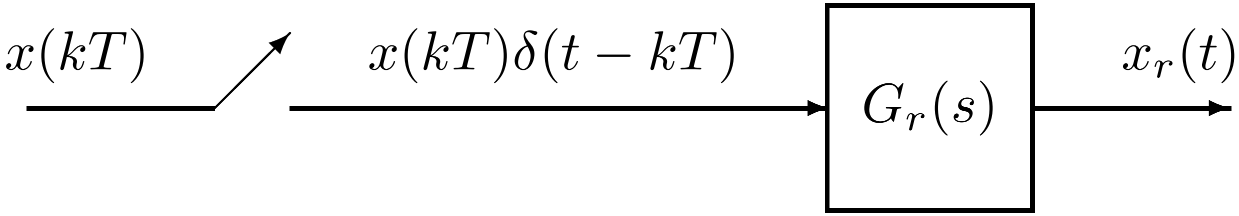


D/A, Digital/Analog converter




Model: zero order hold



# MATHEMATICAL TOOLS

# Difference equations

$$u_k = f(e_0, e_1, \ldots, e_k; u_0, u_1, \ldots, u_{k-1})$$

If $f(\cdot)$ is linear:

$$u_k = -a_1 u_{k-1} - \ldots - a_n u_{k-n} + b_0 e_k + \ldots + b_m e_{k-m}$$

Example:

$$u_k = -a_1 u_{k-1} - a_2 u_{k-2} + b_0 e_k$$

Defining $\nabla$ as the delay operator

$$\begin{aligned} u_k &= u_k \\ u_{k-1} &= u_k - \nabla u_k \\ u_{k-2} &= u_k - 2\nabla u_k + \nabla^2 u_k \end{aligned}$$

we obtain

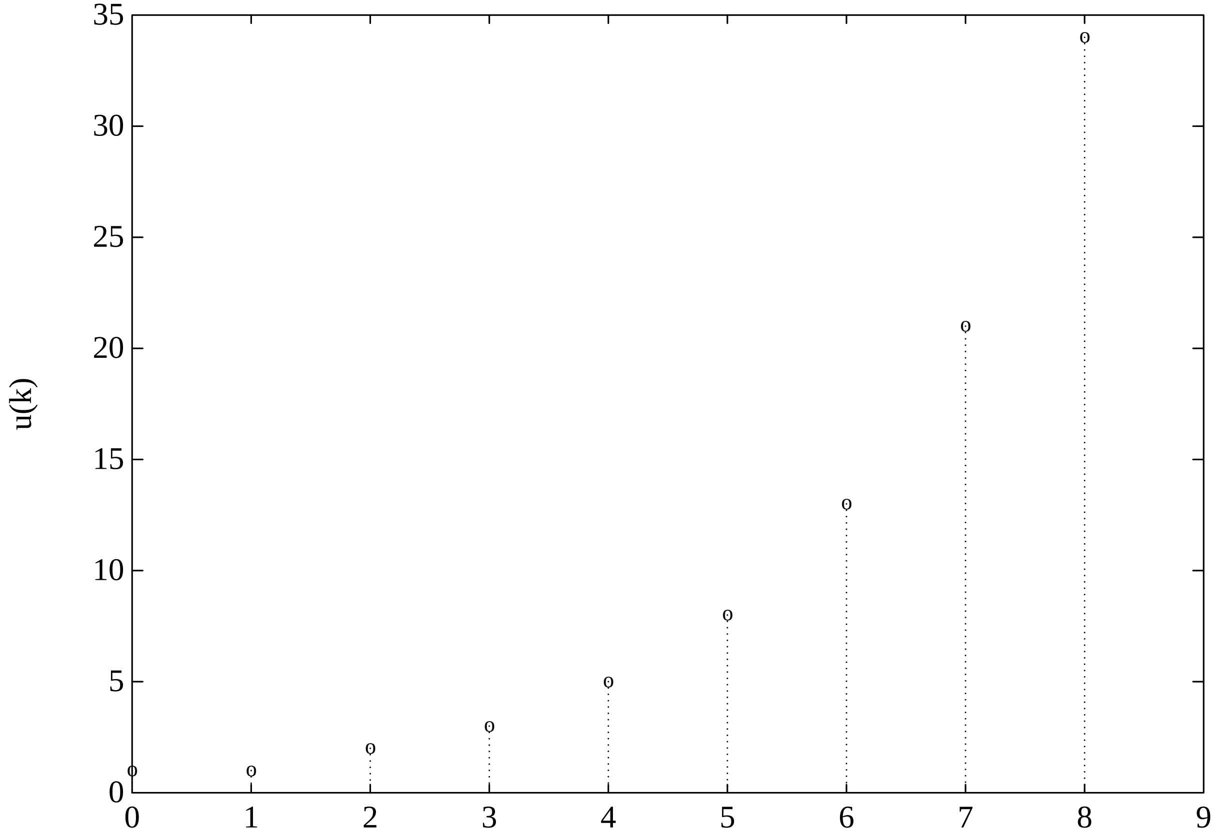
$$a_2 \nabla^2 u_k - (a_1 + 2a_2)\nabla u_k + (a_2 + a_1 + 1) u_k = b_0 e_k$$

# Difference equations

Solution of a difference equation with constant coefficients

$$u_k = u_{k-1} + u_{k-2} \qquad k \geq 2$$

$u_0 = u_1 = 1.$



# Difference equations

The elementary solution is in the form of $z^k$:

$$cz^k = cz^{k-1} + cz^{k-2}$$

$$z^2 - z - 1 = 0 \qquad\qquad z_{1,2} = (1 \pm \sqrt{5})/2$$

In general it holds:

$$u_k = c_1 z_1^k + c_2 z_2^k$$

with $c_1, c_2$ to be computed using initial conditions for $k = 0,\ 1$. In previous case:

$$u_k = \frac{1+\sqrt{5}}{2\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^k + \frac{-1+\sqrt{5}}{2\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^k$$

The trend is diverging, hence the system is unstable.

If **all** the roots of the characteristic equations are **within** the unitary circle, then the corresponding difference equation is **stable**, i.e., its solution will converge to zero in time for any finite initial condition.

# *The Z-Transform*

Given a sequence $x_k \in \mathbb{R}$, defined for $k = 0, 1, 2, \ldots$ and null for $k < 0$. La $\mathcal{Z}$-transform of $x_k$ is a function of the complex variable $z$ defined as

$$X(z) = \mathcal{Z}[x_k] \quad = \quad x_0 + x_1\, z^{-1} + \cdots + x_k\, z^{-k} + \cdots = \sum_{k=0}^{\infty} x_k z^{-k}$$

In the case of a sequence $x_k$ obtained by uniformly sampling a continuous signal $x(t)$, $t \geq 0$ with a sampling time $T$, then $x_k = x(kT)$:

$$X(z) = \sum_{k=0}^{\infty} x(k) z^{-k}$$

The extended equation

$$X(z) = x(0) + x(T)\, z^{-1} + x(2\,T)\, z^{-2} + \cdots + x(k\,T)\, z^{-k} + \cdots$$

implies the specification of the **sampling time** $T$, from which the samples depends (i.e., the coefficients of the series).

# The Z-transform

We write: $X(z) = \mathcal{Z}[X(s)]$ meaning $X(z) = \mathcal{Z}\left[\left\{\mathcal{L}^{-1}[X(s)]\big|_{t=kT}\right\}\right]$

In engineering applications, the function $X(z)$ assumes in general a rational fractional expression

$$X(z) = \frac{b_0\, z^m + b_1\, z^{m-1} + \cdots + b_m}{z^n + a_1\, z^{n-1} + \cdots + a_n}$$

that can also be expressed in powers of $z^{-1}$:

$$\begin{aligned} X(z) &= \frac{z^n\, (b_0\, z^{-(n-m)} + b_1\, z^{-(n-m+1)} + \cdots + b_m\, z^{-n})}{z^n\, (1 + a_1\, z^{-1} + \cdots + a_n z^{-n})} \\ &= \frac{b_0\, z^{-(n-m)} + b_1\, z^{-(n-m+1)} + \cdots + b_m\, z^{-n}}{1 + a_1\, z^{-1} + \cdots + a_n z^{-n}} \end{aligned}$$

Example:

$$X(z) = \frac{z(z+0.5)}{(z+1)(z+2)} = \frac{1 + 0.5\, z^{-1}}{(1+z^{-1})(1+2\, z^{-1})}$$

# Z-Transform of elementary terms

Unitary discrete impulse. Kronecker's function $\delta_0(t)$:

$$x(t) = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases}$$

$$X(z) = \mathcal{Z}[x(t)] = \sum_{k=0}^{\infty} x(kT) z^{-k} = 1 + 0z^{-1} + 0z^{-2} + 0z^{-3} + \cdots = 1$$

Unitary step:

$$x(t) = h(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \quad \text{i.e.} \quad h(k) = \begin{cases} 1 & k = 0, 1, 2, \ldots \\ 0 & k < 0 \end{cases}$$

$$\begin{aligned} H(z) &= \mathcal{Z}[h(t)] = \sum_{k=0}^{\infty} h(kT) z^{-k} = \sum_{k=0}^{\infty} z^{-k} = 1 + z^{-1} + z^{-2} + z^{-3} + \cdots \\ &= \frac{1}{1 - z^{-1}} = \frac{z}{z-1} \,. \end{aligned}$$

The series converges for $|z| > 1$ .

# Z-Transform of elementary terms

Unitary ramp:

$$x(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Since $x(kT) = kT,\ k = 0, 1, 2, \ldots$, the $\mathcal{Z}$-transform is

$$\begin{aligned} X(z) &= \mathcal{Z}[t] = \sum_{k=0}^{\infty} x(kT) z^{-k} = T \sum_{k=0}^{\infty} k z^{-k} \\ &= T(z^{-1} + 2z^{-2} + 3z^{-3} + \cdots) \\ &= Tz^{-1}(1 + 2z^{-1} + 3z^{-2} + \cdots) \\ &= T \frac{z^{-1}}{(1 - z^{-1})^2} = T \frac{z}{(z-1)^2} \end{aligned}$$

converging for $|z| > 1$.

# Z-Transform of elementary terms

Exponential function:

$$x(t) = \begin{cases} e^{-at} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

where $a$ is a real or complex constant. Since $x(kT) = e^{-akT}, k = 0, 1, 2, \ldots$, we have

$$\begin{aligned} X(z) &= \mathcal{Z}\left[e^{-at}\right] = \sum_{k=0}^{\infty} e^{-akT} z^{-k} \\ &= 1 + e^{-aT} z^{-1} + e^{-2aT} z^{-2} + e^{-3aT} z^{-3} + \cdots \\ &= \frac{1}{1 - e^{-aT} z^{-1}} = \frac{z}{z - e^{-aT}} \end{aligned}$$

that converges for $|z| > e^{-Re(a)T}$. Note that for $a = 0$ we move back to the unitary step.

# Z-Transform of elementary terms

Sinusoidal function:

$$x(t) = \begin{cases} \sin \omega t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

From Euler's equations

$$\sin \omega t = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$

$$\begin{aligned} X(z) &= \mathcal{Z}[\sin \omega t] = \frac{1}{2j}\left(\frac{1}{1 - e^{j\omega T} z^{-1}} - \frac{1}{1 - e^{-j\omega T} z^{-1}}\right) \\ &= \frac{1}{2j} \frac{(e^{j\omega T} - e^{-j\omega T})z^{-1}}{1 - (e^{j\omega T} + e^{-j\omega T})z^{-1} + z^{-2}} \\ &= \frac{z^{-1} \sin \omega T}{1 - 2z^{-1} \cos \omega T + z^{-2}} = \frac{z \sin \omega T}{z^2 - 2z \cos \omega T + 1} \end{aligned}$$

converging for $|z| > 1$.

# Z-Transform of elementary terms

Cosinusoidal function:

$$x(t) = \begin{cases} \cos\omega t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\begin{aligned} X(z) &= \mathcal{Z}[\cos\omega t] = \frac{1}{2}\left(\frac{1}{1-e^{j\omega T}\,z^{-1}} + \frac{1}{1-e^{-j\omega T}\,z^{-1}}\right) \\ &= \frac{1}{2}\frac{2-(e^{-j\omega T}+e^{j\omega T})z^{-1}}{1-(e^{j\omega T}+e^{-j\omega T})z^{-1}+z^{-2}} \\ &= \frac{1-z^{-1}\cos\omega T}{1-2z^{-1}\cos\omega T+z^{-2}} \\ &= \frac{z(z-\cos\omega T)}{z^2-2z\cos\omega T+1} \qquad |z|>1 \end{aligned}$$

# Z-Transform of elementary terms

Example:

$$X(s) = \frac{1}{s(s+1)}$$

First technique:

$$x(t) = 1 - e^{-t}$$

$$\begin{aligned} X(z) &= \mathcal{Z}\left[1 - e^{-t}\right] = \frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-T}z^{-1}} \\ &= \frac{(1 - e^{-T})z^{-1}}{(1 - z^{-1})(1 - e^{-T}z^{-1})} = \frac{(1 - e^{-T})z}{(z-1)(z - e^{-T})} \end{aligned}$$

Second technique:

$$X(s) = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{1+s}$$

$$X(z) = \frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-T}z^{-1}}$$

# The Z-transform

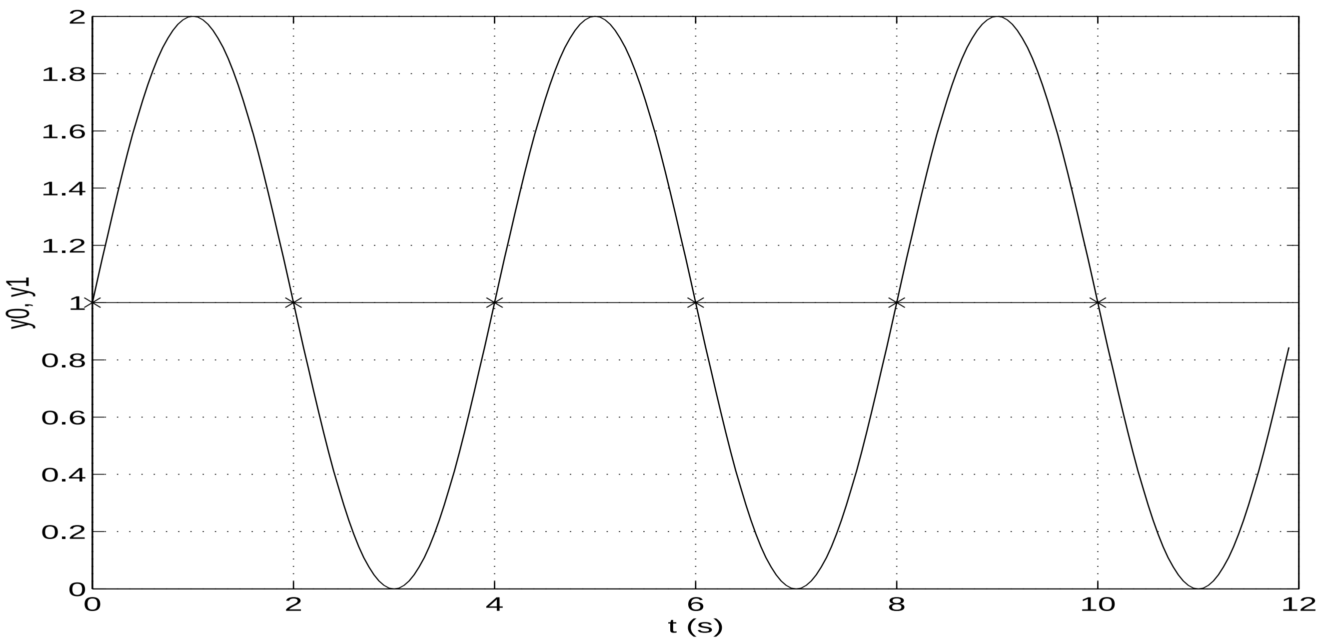
The $\mathcal{Z}$-transform $X(z)$ and the corresponding sequence $x(k)$ are in one-to-one correpondance

This is not true for the $\mathcal{Z}$-transform $X(z)$ and its inverse $x(t)$

From a $X(z)$ it is possible to obtain many $x(t)$

This ambiguity does not hold if restrictive conditions on the sampling time $T$ hold (**Shannon's theorem**)

Different continuous time functions can have the same samples: $x(k)$



# Properties of the Z-transform

♠ Linearity:

$$x(k) = af(k) + bg(k)$$

$$X(z) = aF(z) + bG(z)$$

♠ Multiplication for $a^k$:

Being $X(z)$ the $\mathcal{Z}$-transform of $x(t)$, $a$ a constant value.

$$\mathcal{Z}\left[a^k x(k)\right] = X(a^{-1}z)$$

$$\begin{aligned}\mathcal{Z}\left[a^k x(k)\right] &= \sum_{k=0}^{\infty} a^k x(k) z^{-k} = \sum_{k=0}^{\infty} x(k)(a^{-1}z)^{-k} \\ &= X(a^{-1}z)\end{aligned}$$

# Properties of the Z-transform

♠ Time shifting:

If $x(t) = 0, t < 0$, $X(z) = \mathcal{Z}[x(t)]$, and $n = 1, 2, \ldots$, then

$$\mathcal{Z}[x(t - nT)] = z^{-n} X(z) \qquad \text{(delay)}$$

$$\mathcal{Z}[x(t + nT)] = z^{n} \left[ X(z) - \sum_{k=0}^{n-1} x(kT) z^{-k} \right] \qquad \text{(anticipation)}$$

Note that:

$$z^{-1} x(k) = x(k-1)$$

$$z^{-2} x(k) = x(k-2)$$

$$z\ x(k) = x(k+1)$$

# Properties of the Z-transform

♠ Delay:

$$\begin{aligned}\mathcal{Z}[x(t-nT)] &= \sum_{k=0}^{\infty} x(kT-nT)z^{-k} \\ &= z^{-n}\sum_{k=0}^{\infty} x(kT-nT)z^{-(k-n)}\end{aligned}$$

defining $m = k - n$,

$$\mathcal{Z}[x(t-nT)] = z^{-n}\sum_{m=-n}^{\infty} x(mT)z^{-m}$$

Since $x(mT) = 0$ for $m < 0$, we can write

$$\mathcal{Z}[x(t-nT)] = z^{-n}\sum_{m=0}^{\infty} x(mT)z^{-m} = z^{-n}X(z)$$

# Properties of the Z-transform

♠ Anticipation:

$$\begin{aligned}
\mathcal{Z}[x(t+nT)] &= \sum_{k=0}^{\infty} x(kT+nT)z^{-k} = z^n \sum_{k=0}^{\infty} x(kT+nT)z^{-(k+n)} \\
&= z^n \left[ \sum_{k=0}^{\infty} x(kT+nT)z^{-(k+n)} + \sum_{k=0}^{n-1} x(kT)z^{-k} - \sum_{k=0}^{n-1} x(kT)z^{-k} \right] \\
&= z^n \left[ \sum_{k=0}^{\infty} x(kT)z^{-k} - \sum_{k=0}^{n-1} x(kT)z^{-k} \right] \\
&= z^n \left[ X(z) - \sum_{k=0}^{n-1} x(kT)z^{-k} \right]
\end{aligned}$$

# Properties of the Z-transform

♠ Initial value theorem: If $X(z)$ is the $\mathcal{Z}$-transform of $x(t)$ and if

$$\lim_{z\to\infty} X(z)$$

exists, then the initial value $x(0)$ of $x(t)$ is given by:

$$x(0) = \lim_{z\to\infty} X(z)$$

In fact, note that

$$X(z) = \sum_{k=0}^{\infty} x(k)z^{-k} = x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots$$

# Properties of the Z-transform

♠ Final value theorem: being all the poles of $X(z)$ within the unitary circle with at most a simple pole in $z = 1$.

$$\lim_{k\to\infty} x(k) = \lim_{z\to 1}\left[(1 - z^{-1})X(z)\right]$$

In fact:

$$\sum_{k=0}^{\infty} x(k)z^{-k} - \sum_{k=0}^{\infty} x(k-1)z^{-k} = X(z) - z^{-1}X(z)$$

$$\lim_{z\to 1}\left[\sum_{k=0}^{\infty} x(k)z^{-k} - \sum_{k=0}^{\infty} x(k-1)z^{-k}\right] =$$

$$\begin{aligned}
&= \sum_{k=0}^{\infty}[x(k) - x(k-1)] \\
&= [x(0) - x(-1)] + [x(1) - x(0)] + [x(2) - x(1)] + \cdots \\
&= \lim_{k\to\infty} x(k)
\end{aligned}$$

# Properties of the Z-transform

♠ Real convolution theorem:
given two functions $x_1(t)$ e $x_2(t)$, with $x_1(t) = x_2(t) = 0,\ t < 0$ and $\mathcal{Z}$-transform $X_1(z), X_2(z)$. Then

$$X_1(z)X_2(z) = \mathcal{Z}\left[\sum_{h=0}^{k} x_1(hT)x_2(kT - hT)\right]$$

Note that

$$\mathcal{Z}\left[\sum_{h=0}^{k} x_1(h)x_2(k-h)\right] = \sum_{k=0}^{\infty}\sum_{h=0}^{k} x_1(h)x_2(k-h)z^{-k} = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} x_1(h)x_2(k-h)z^{-k}$$

Since $x_2(k-h) = 0,\ h > k$. Defining $m = k - h$ we have

$$\mathcal{Z}\left[\sum_{h=0}^{k} x_1(h)x_2(k-h)\right] = \sum_{h=0}^{\infty} x_1(h)z^{-h}\sum_{m=0}^{\infty} x_2(m)z^{-m}$$

# The inverse Z-transform

Let to obtain a sequence $x_k$ (and possibly the continuous function $x(t)$ whose samples are $x_k$) from the $\mathcal{Z}$-transform $X(z)$.

$$X(z) \quad \longleftrightarrow \quad x(k) \quad \longleftarrow \quad x(t)$$

one-to-one    non one-to-one

If the **Shannon's theorem** on sampling holds, then the continuous time function $x(t)$ can be univocally derived from the sequence $x_k$.

# The inverse Z-transform: decomposition in simple fractions

$$X(z) = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_{m-1} z + b_m}{(z - p_1)(z - p_2) \cdots (z - p_n)}$$

Case 1: All the poles are simple

$$X(z) = \frac{c_1}{z - p_1} + \frac{c_2}{z - p_2} + \cdots + \frac{c_n}{z - p_n} = \sum_{i=1}^{n} \frac{c_i}{z - p_i}$$

residue $c_i$ are computed as: $c_i = \left[(z - p_i)X(z)\right]_{z=p_i}$ .

If $X(z)$ has a zero in the origine, the function $X(z)/z$ must be used

$$\frac{X(z)}{z} = \frac{c_1}{z - p_1} + \cdots + \frac{c_n}{z - p_n} \qquad c_i = \left[(z - p_i)\frac{X(z)}{z}\right]_{z=p_i}$$

When we have complex conjugated poles, also the coefficients $c_i$ are complex number.
In this case use Euler's equations to obtain trigonometry functions.
The mathematical expression of the inverse transform is

$$x(k) = \sum_{i=1}^{n} c_i p_i^k$$

# The inverse Z-transform: decomposition in simple fractions

Case 2: If $X(z)$, or $X(z)/z$, has **multiple poles**

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_{m-1} z + b_m}{(z-p_1)^{r_1}(z-p_2)^{r_2}\cdots(z-p_h)^{r_h}}$$

then

$$X(z) = \sum_{i=1}^{h} \sum_{k=1}^{r_i} \frac{c_{ik}}{(z-p_i)^{r_i-k+1}}$$

where residues can be computed as

$$c_{ik} = \left[\frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}(z-p_i)^{r_i}X(z)\right]_{z=p_i}$$

$$i = 1, \ldots, h;\;\; k = 1, \ldots, r_i$$

# The inverse Z-transform: decomposition in simple fractions

Example:

$$X(z) = \frac{1}{z^4 + 6z^3 + 13z^2 + 12z + 4} = \frac{1}{(z+2)^2(z+1)^2}$$

We have:

$$X(z) = \frac{c_{11}}{(z+2)^2} + \frac{c_{12}}{(z+2)} + \frac{c_{21}}{(z+1)^2} + \frac{c_{22}}{(z+1)}$$

$$\begin{aligned}
c_{11} &= \left[(z+2)^2 X(z)\right]\Big|_{z=-2} = 1 \\
c_{12} &= \left[\frac{d}{dz}(z+2)^2 X(z)\right]_{z=-2} = 2 \\
c_{21} &= \left[(z+1)^2 X(z)\right]_{z=-1} = 1 \\
c_{22} &= \left[\frac{d}{dz}(z+1)^2 X(z)\right]_{z=-1} = -2
\end{aligned}$$

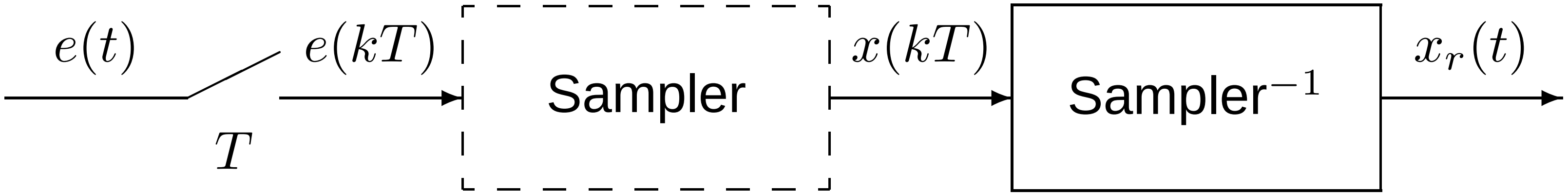
# SAMPLING AND INVERSE SAMPLING

# Sampling and inverse sampling

Digital feedback systems are characterized by a **continuous time part** (the plant) and a **discrete time part** (the digital controller)

Hence both **continuous time variables** and **discrete time variables** coexist

Interface devices are the **sampler** and the **inverse sampler**




Zero order hold (hold inverse sampling):

$$x_r(t) = \sum_{k=0}^{\infty} x(kT)[h(t-kT) - h(t-(k+1)T)]$$

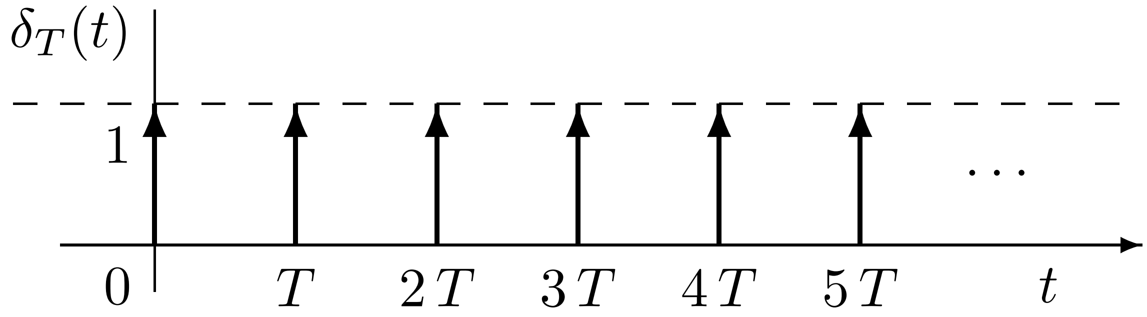
$$X_r(s) = \sum_{k=0}^{\infty} x(kT) \left[\frac{e^{-kTs} - e^{-(k+1)Ts}}{s}\right] = \frac{1-e^{-Ts}}{s} \sum_{k=0}^{\infty} x(kT) e^{-kTs}$$

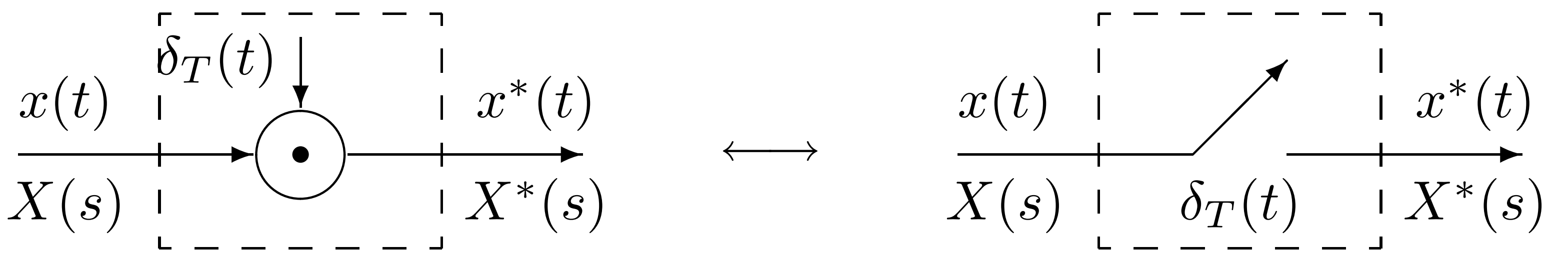
# Impulsive sampling

$$H_0(s) = \frac{1 - e^{-Ts}}{s} \qquad X^*(s) = \sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

$$x^*(t) = \mathcal{L}^{-1}[X^*(s)] = \sum_{k=0}^{\infty} x(kT)\delta(t - kT)$$

$$\delta_T(t) = \sum_{k=0}^{\infty} \delta(t - kT)$$



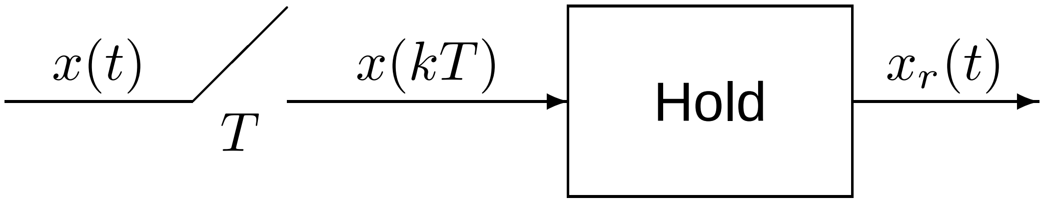



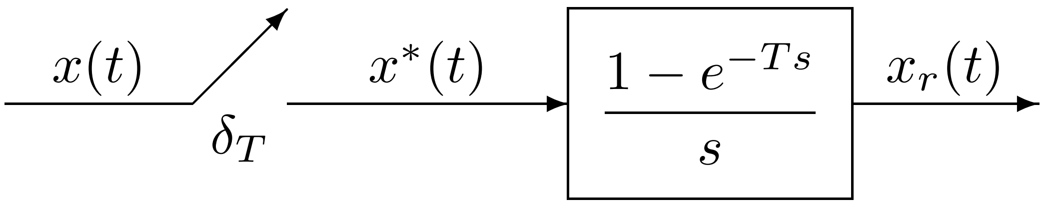
# *Impulsive sampling*

The **impulsive sampler** is an ideal model of the real sampler (A/D converter) used to analyze and design digital control systems

The output of the zero oder hold is:

$$X_r(s) = H_0(s)\,X^*(s) = \frac{1-e^{-Ts}}{s}\,X^*(s)$$




$$X^*(s) = \sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

$$z = e^{sT} \qquad \Longleftrightarrow \qquad s = \frac{1}{T}\ln z \qquad \Rightarrow \qquad X^*(s)\Big|_{s\,=\,\frac{1}{T}\ln z} = \sum_{k=0}^{\infty} x(kT)\,z^{-k}$$

# *Impulsive sampling*

Compute the Laplace transform of the sampled signal $x^*(t)$:

$$x^*(t) = x(t)\,\delta_T(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT)$$

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{j\,n\omega_s t} \quad \text{con} \quad c_n = \frac{1}{T}\int_0^T \delta_T(t)\, e^{-j\,n\omega_s t}dt = \frac{1}{T}$$

which is

$$x^*(t) = x(t)\,\frac{1}{T}\sum_{n=-\infty}^{\infty} e^{j\,n\omega_s t} = \frac{1}{T}\sum_{n=-\infty}^{\infty} x(t)\, e^{j\,n\omega_s t}$$

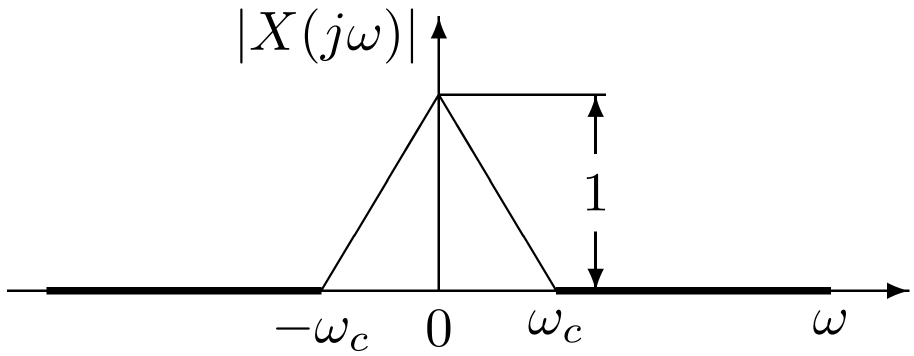
$$X^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty}\mathcal{L}\left[x(t)\, e^{j\,n\omega_s t}\right] = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(s - j\,n\omega_s)$$

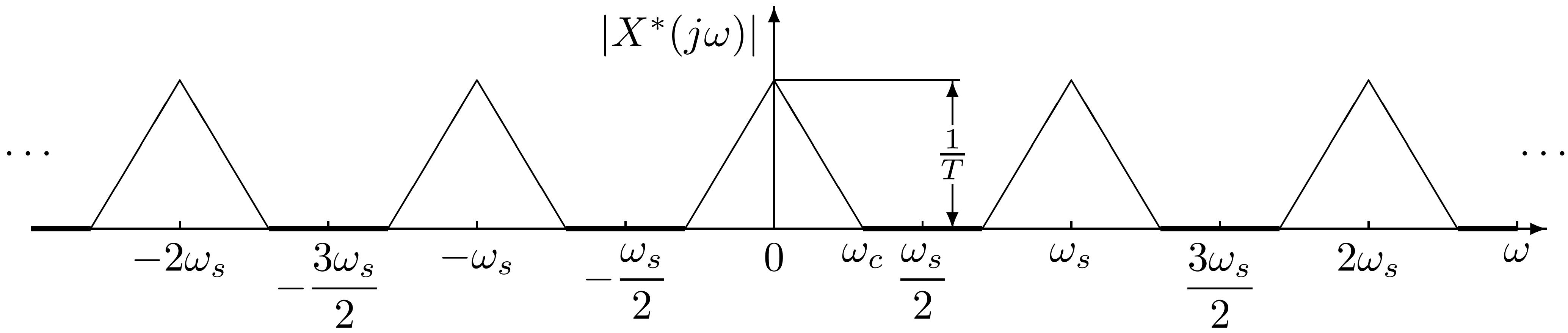
Disregarding the gain $1/T$, the Laplace transform of the sampled signal $X^*(s)$ is the sum of infinite terms , $X(s-j\,n\omega_s)$, each of them obtained by a $j\,n\omega_s$ shifting of $X(s)$.

# *Impulsive sampling*

Hence the spectrum of the sampled signal is:

$$X^*(j\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(j\omega - j\,n\omega_s)$$

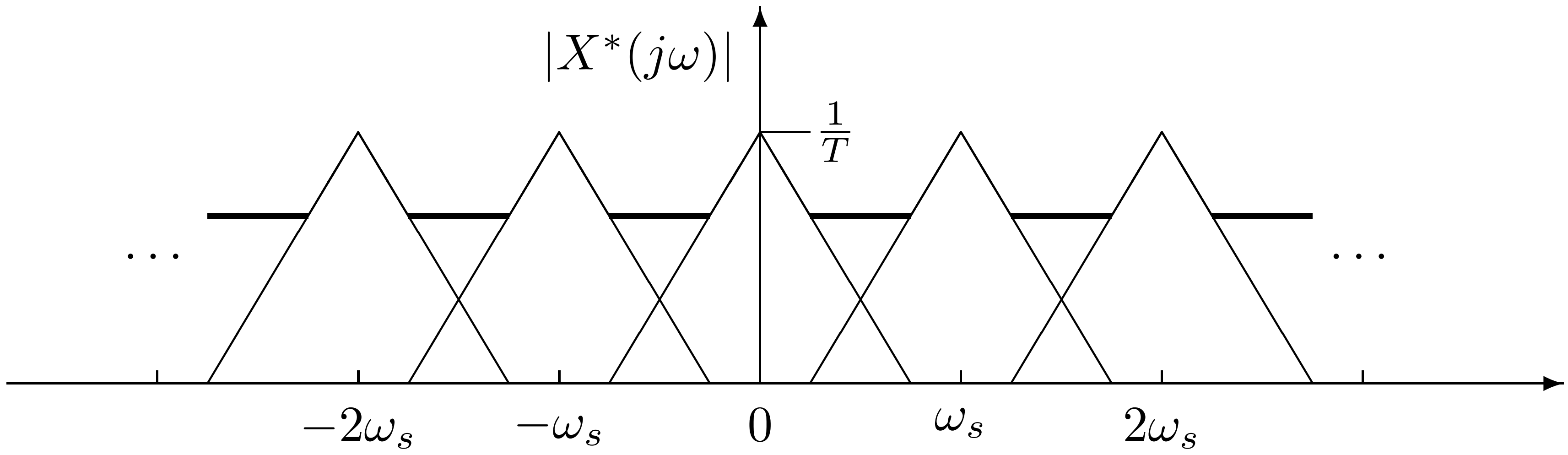


# Sampling and inverse sampling

The condition $\omega_s > 2\omega_c$ let the spectrum of the main spectral component be divided by the repetitions. Hence, through a filtering operation, it is possible to exactly recovery the original signal $x(t)$ from the sampled signal $x^*(t)$.

In case the condition $\omega_s > 2\omega_c$ does not hold:




The main spectral component is partially superimposed to its repetitions, hence it is not possible to isolate it recovering the original signal.

**Shannon's theorem**

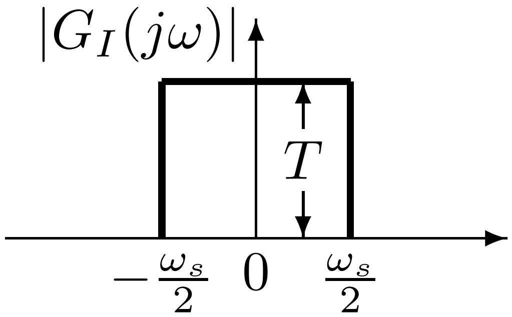
Let $\omega_s = \frac{2\pi}{T}$ be the sampling pulse($T$ is the sampling tim), and let $\omega_c$ be the higher spectral component of the continuous time signal $x(t)$.
Signal $x(t)$ can be recovered starting from the sampled signal $x^*(t)$ if and only if:

$$\omega_s > 2\omega_c$$

■

The perfect recovery can be pursued using the ideal filter:

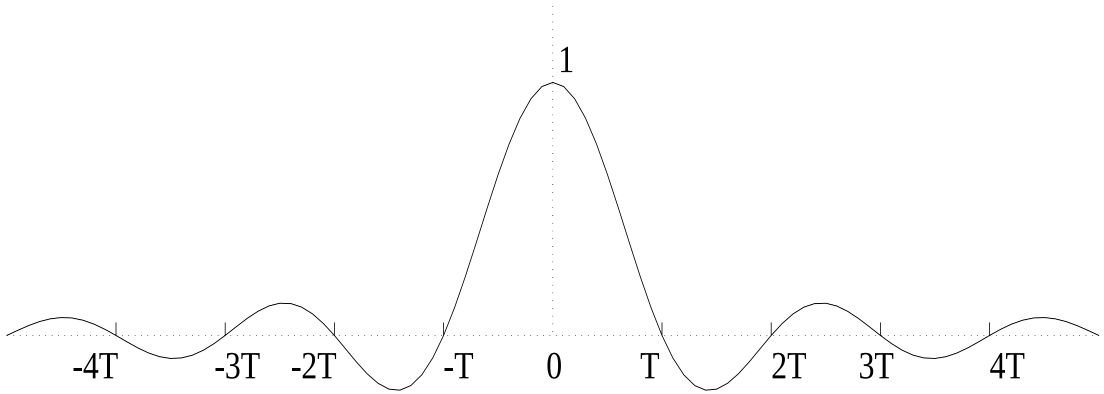
$$G_I(j\omega) = \begin{cases} T & -\frac{\omega_s}{2} \leq \omega \leq \frac{\omega_s}{2} \\ 0 & \text{elsewhere} \end{cases}$$



# Sampling and inverse sampling

The ideal filter $G_I(j\omega)$ is not feasible; in fact, its impulsive response is:

$$g_I(t) = \frac{\sin(\omega_s t/2)}{\omega_s t/2}$$




This means that the recovered signal is

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} x^*(\tau)\, g_I(t-\tau)\, d\tau = \sum_{k=-\infty}^{\infty} x(kT) \int_{-\infty}^{\infty} \delta(\tau - kT) \frac{sin(\omega_s(t-\tau)/2)}{\omega_s(t-\tau)/2}\, d\tau \\ &= \sum_{k=-\infty}^{\infty} x(kT) \frac{sin(\omega_s(t-kT)/2)}{\omega_s(t-kT)/2} \end{aligned}$$

We need all the past and future samples $x(kT)$!!

We will use approximated feasible inverse sampler.

# The aliasing phenomenon

With the term **aliasing** we intend the generation of new spectral components, due to the sampling operation, partially superimposed to the main component. These new components do not allow the exact recovery of the original signal.
The aliasing appears just if the Shannon's theorem condition $\omega_s > 2\omega_c$ is not met.
Example: consider

$$\begin{cases} x(t) & = & \sin(\omega_2 t + \theta) \\ y(t) & = & \sin((\omega_2 + n\omega_s)t + \theta) \end{cases}$$

having the same phase $\theta$ and pulses that differ for a integer multiple of $\omega_s$.
If the signals are sampled

$$\begin{cases} x(kT) & = & \sin(\omega_2 kT + \theta) \\ y(kT) & = & \sin((\omega_2 + n\omega_s)kT + \theta) \\ & = & \sin(\omega_2 kT + 2k\pi n + \theta) \\ & = & \sin(\omega_2 kT + \theta) \end{cases}$$

samples are the same: $x(kT) = y(kT)$

# The aliasing phenomenon

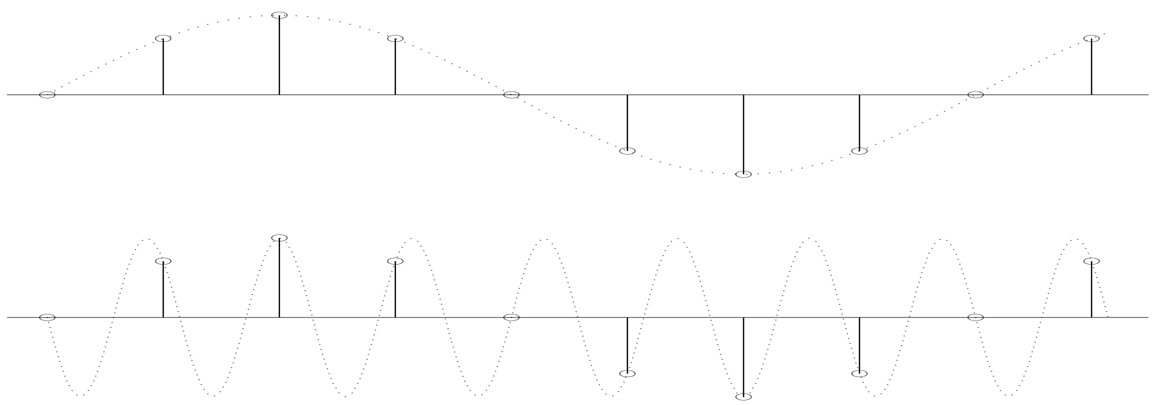
Example: $\omega_2 + \omega_1 = n\omega_s$

Being $\omega_1 = \frac{1}{8}\omega_s$ e $\omega_2 = \omega_s - \omega_1 = \frac{7}{8}\omega_s$

$$\begin{cases} x(t) &= \sin(\omega_1 t) = \sin(\omega_s t/8) \\ y(t) &= \sin(\omega_2 t) = \sin(7\omega_s t/8 + \pi) \end{cases}$$

Sampling we have

$$\begin{cases} x(kT) &= \sin(\omega_s k\,T/8) = \sin(2k\pi/8) = \sin(k\pi/4) \\ y(kT) &= \sin(7\omega_s kT/8 + \pi) = \sin(7k\pi/4 + \pi) = \sin(k\pi/4) \end{cases}$$



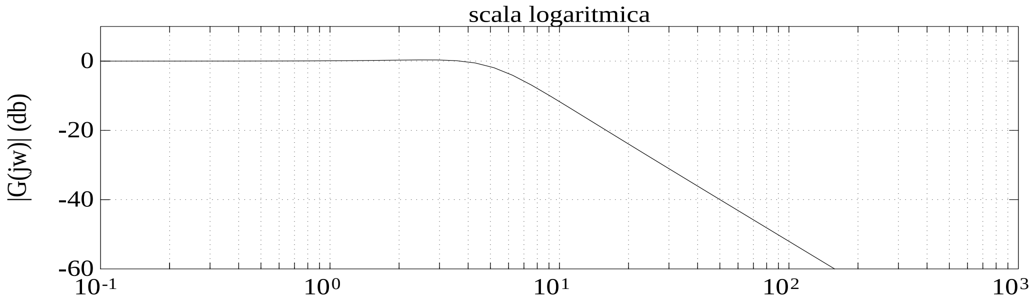
To avoid aliasing it is important to opportunely filter the signal **before** the sample: **anti aliasing filters**.

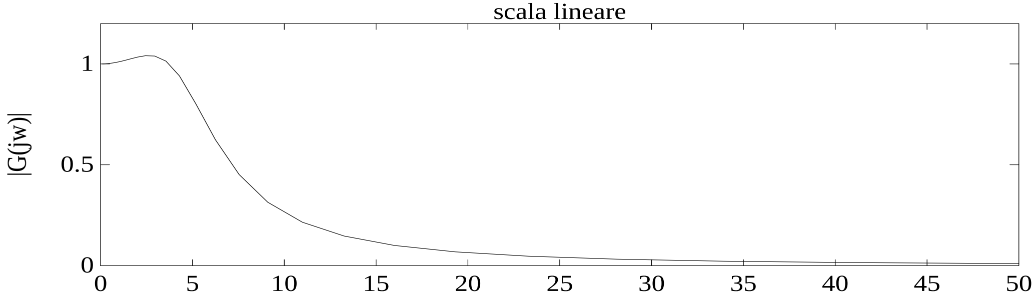
# Sampling example

Sampling of the impulsive response of the second order system

$$G(s) = \frac{25}{s^2 + 6\,s + 25}$$

Unitary DC gain, complex conjugated poles $p_{1,2} = -3 \pm j4$, natural pulse $\omega_n = 5\ rad/s$ and damping coefficient $\delta = 3/5$.
Amplitude Bode diagram of $G(j\omega)$:




For $\omega > 10\omega_n = 50\ rad/s = \bar{\omega}$,the module of $G(j\omega)$ is below 1/100 (-40 db) of the DC gain.

# Sampling example

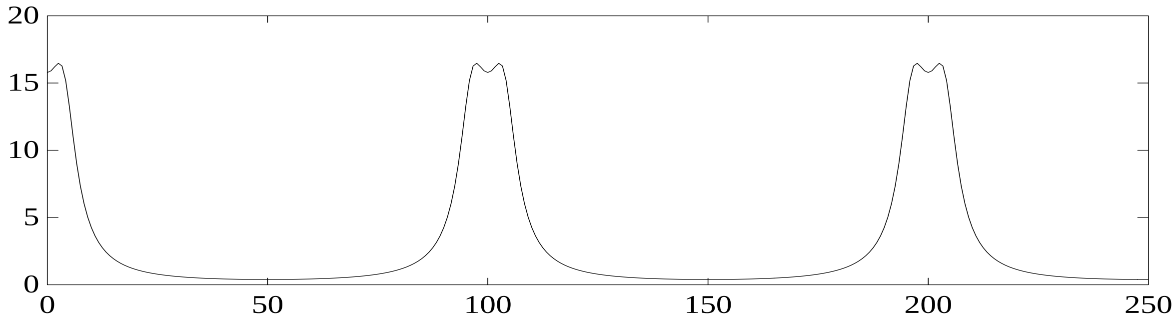
The spectrum, even if ideally has spectral components until unbounded frequency, can be neglected for $\omega \geq \bar{\omega} = 50\ rad/s$

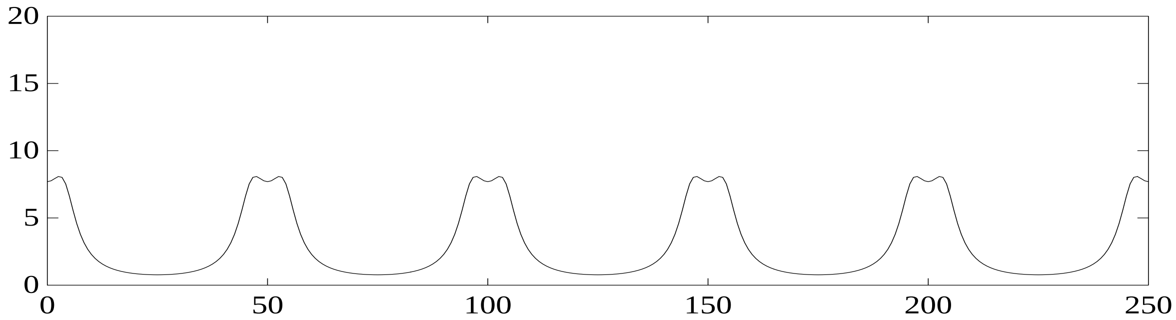
Applying the $\mathcal{Z}$-transform we have

$$G(z) = \frac{25}{4} \frac{e^{-3T} \sin(4T)\ z}{z^2 - 2e^{-3T} \cos(4T)\ z + e^{-6T}}$$

We can can compute the spectral behavior as $G^*(j\omega) = G(z)|_{z=e^{j\omega T}} \qquad (0 \leq \omega \leq \frac{\pi}{T})$

Different trends for $T = \frac{\pi}{50}$ and $T = \frac{\pi}{25}$:



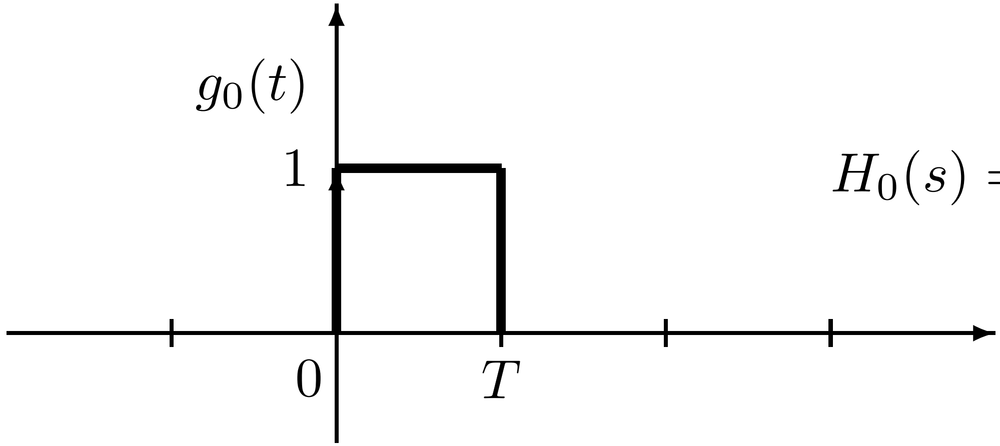
# Typical inverse samplers:


$$x(t) = x(kT) + \left.\frac{d\,x(t)}{dt}\right|_{t=kT} (t - kT) + \left.\frac{d^2\,x(t)}{dt^2}\right|_{t=kT} \frac{(t-kT)^2}{2!} + \cdots$$

$$\left.\frac{d\,x(t)}{dt}\right|_{t=kT} \simeq \frac{x(kT) - x((k-1)T)}{T} \qquad \cdots$$

Zero order hold:

$$x_0(t) = x(kT) \qquad kT \leq t < (k+1)T$$




$$H_0(s) = \frac{1 - e^{-sT}}{s}$$

# Zero order hold

The frequential response of the zero order hold is:

$$H_0(j\omega) = \frac{1 - e^{-j\omega T}}{j\omega} = \frac{2\, e^{-j\omega T/2}}{\omega} \; \frac{e^{j\omega T/2} - e^{-j\omega T/2}}{2j}$$

$$= T \frac{\sin(\omega T/2)}{\omega T/2}\, e^{-j\omega T/2}$$

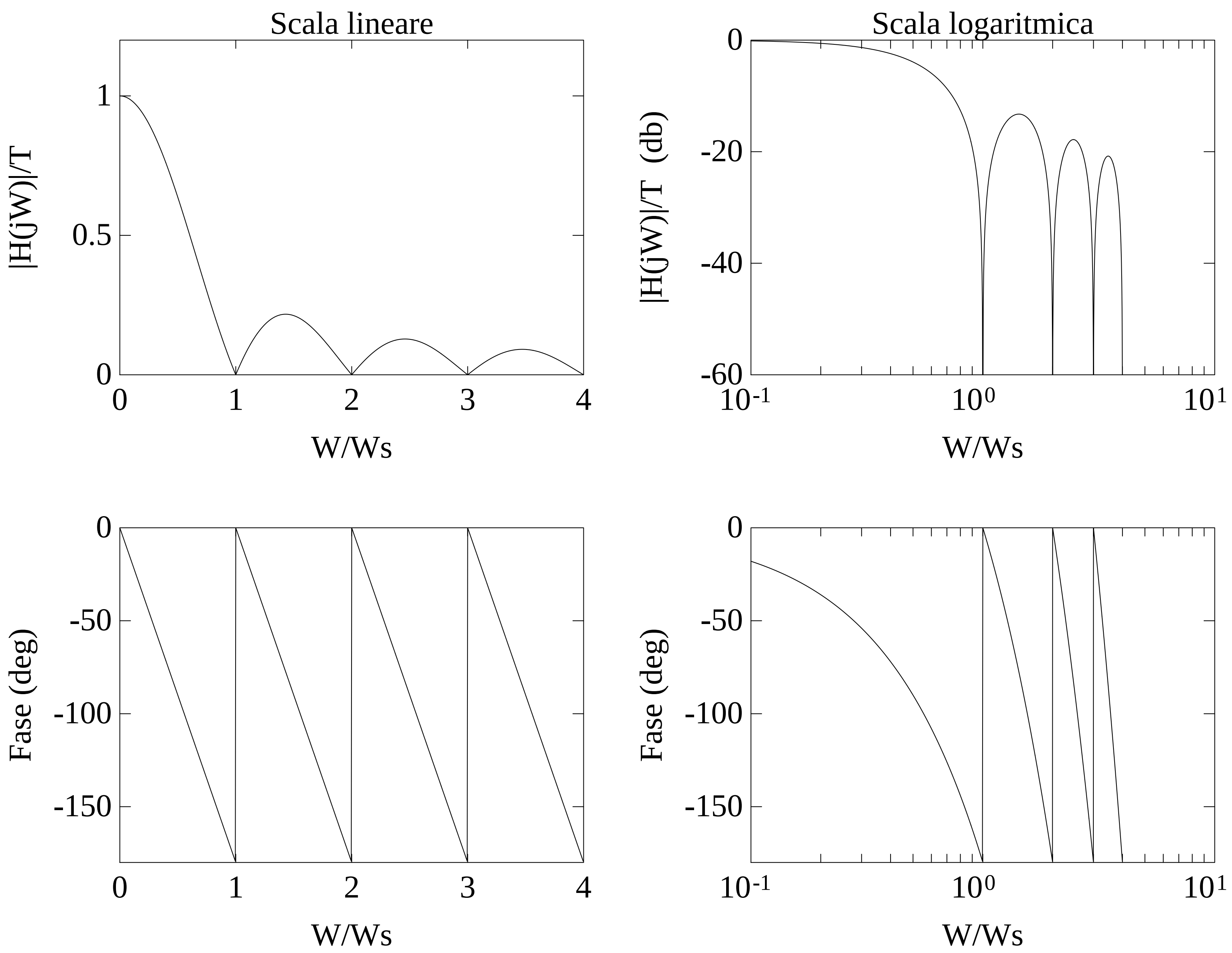
Module

$$|H_0(j\omega)| = T \left| \frac{\sin(\omega T/2)}{\omega T/2} \right| \approx T \quad \text{for } \omega \ll \omega_s = 2\pi/T$$

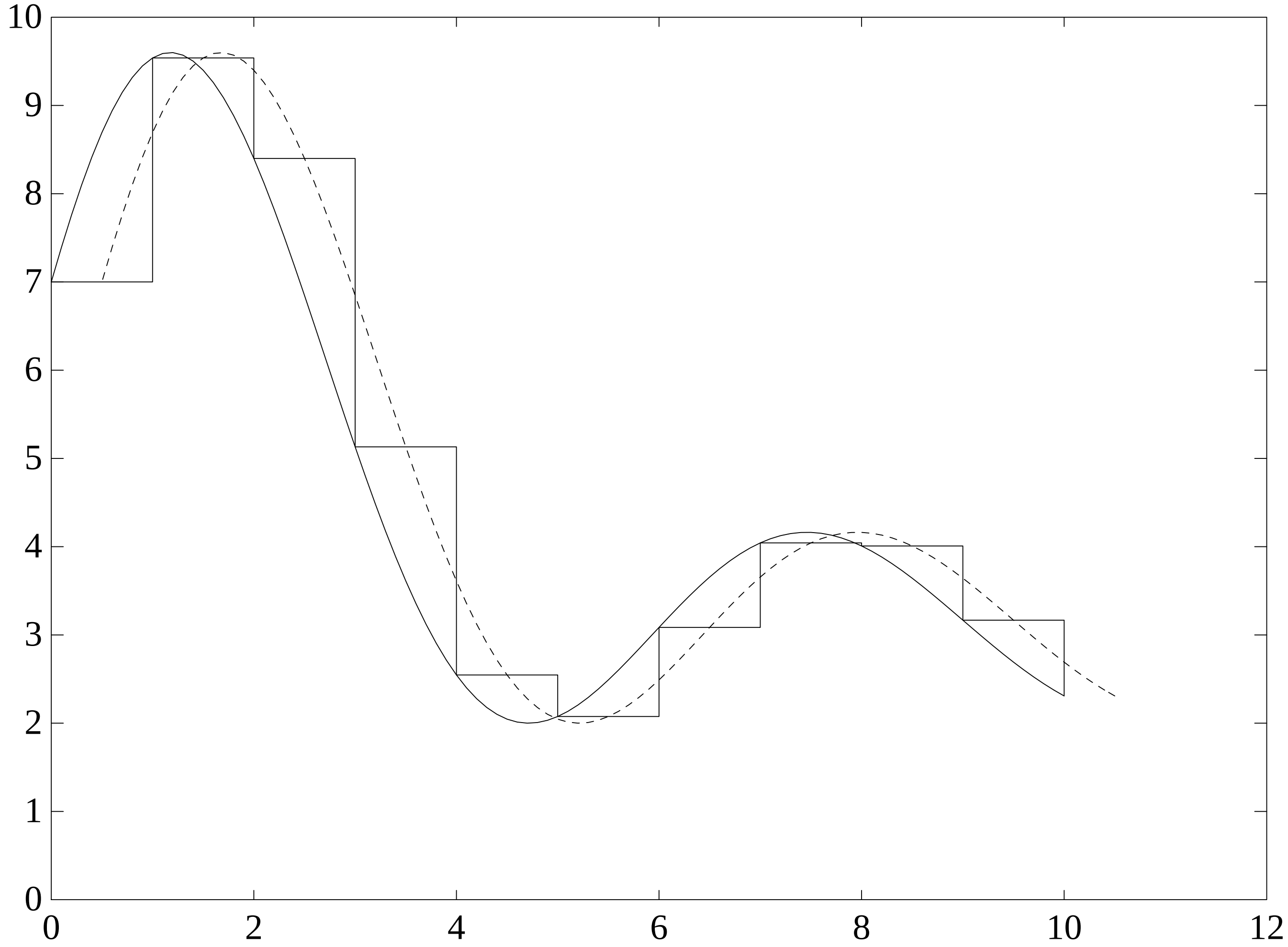
Phase

$$Arg\,[H_0(j\omega)] = Arg\left[\sin\frac{\omega T}{2}\right] - \frac{\omega T}{2} \approx -\frac{\omega T}{2} \quad \text{for } \omega \ll \omega_s = 2\pi/T$$

# Zero order hold



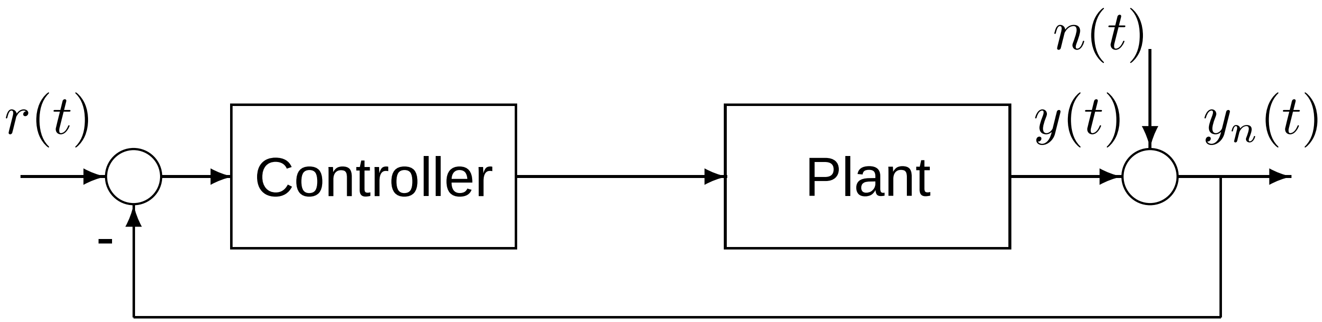
# Zero order hold



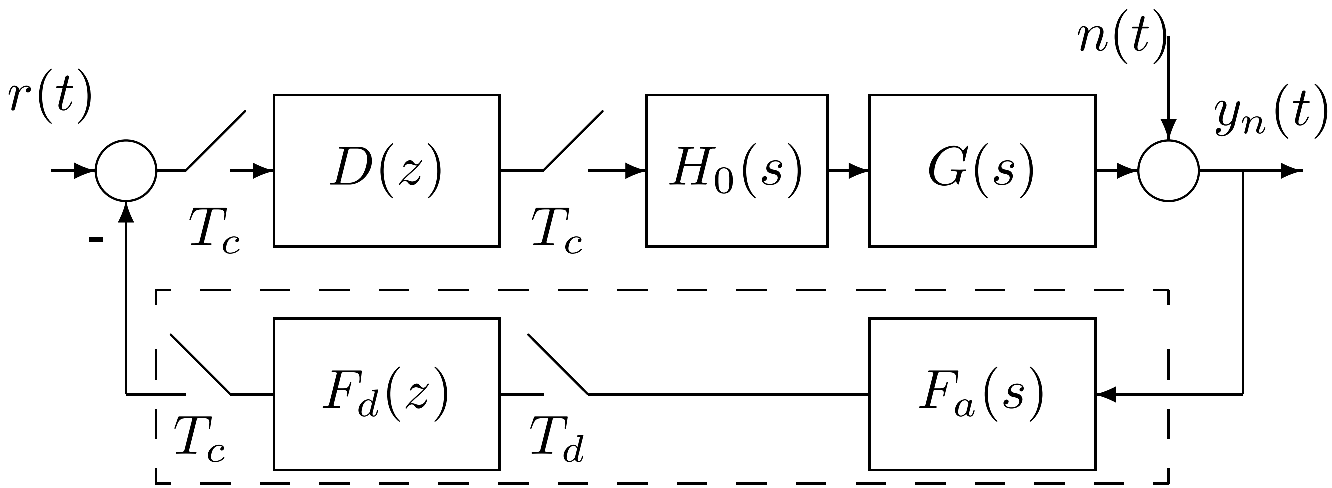

$$H_0(j\omega) \simeq T\, e^{-j\omega T/2}$$

# Antialiasing filters

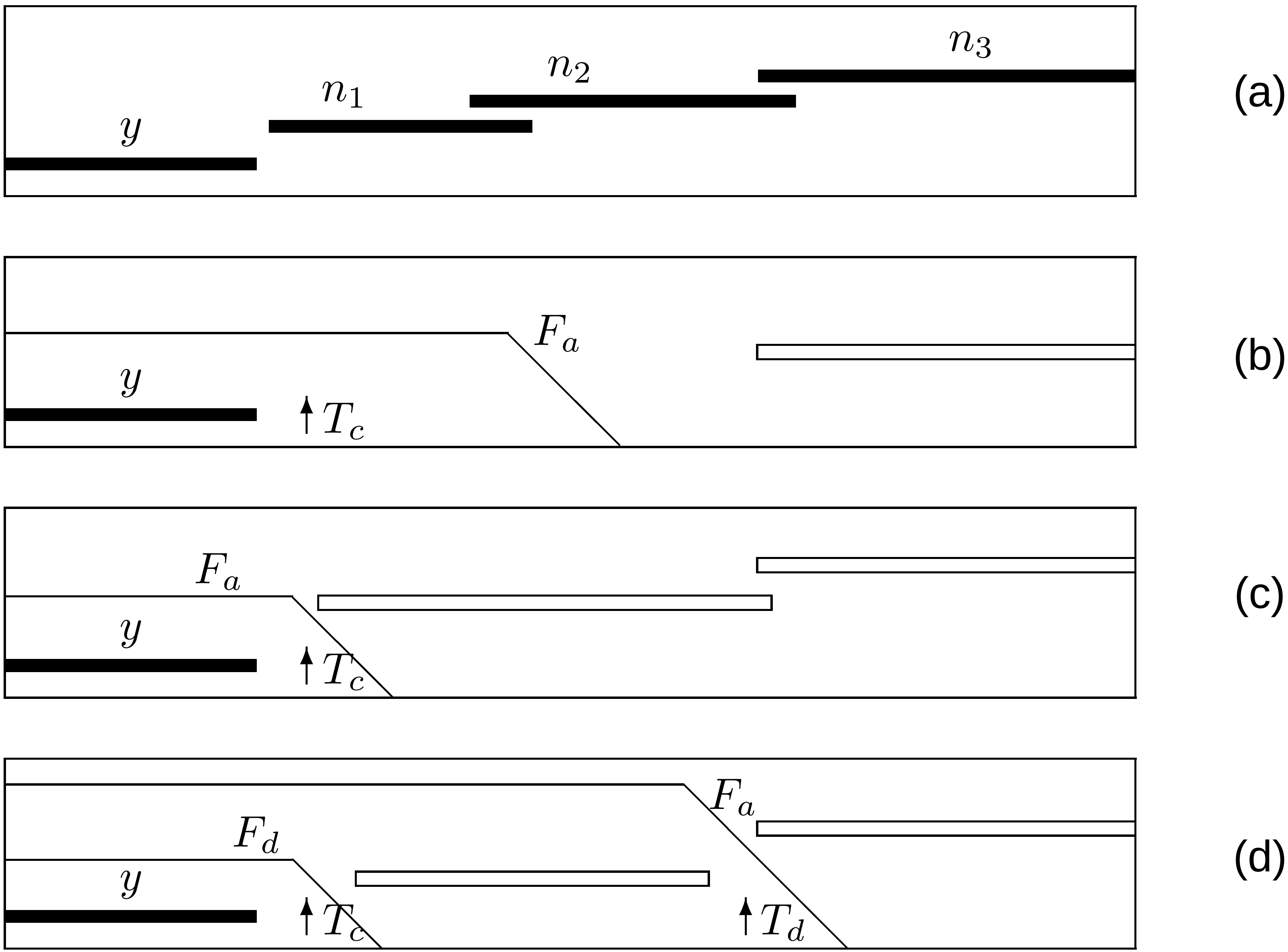
The aliasing produced by the sampling introduces undesired harmonic components within the bandwidth of the closed loop system.




We need to introduce filters that attenuate the more as possible the noise.



# Antialiasing filters



# Antialiasing filters

Analogical filters: passive or actives

- RC filters (first order)
- Slope $20\ db$ per decade
- The filter should not perturb the frequencies at which system works

Digital filters

- Sampling time lower then the sampling time of the controller
- Mean filter:

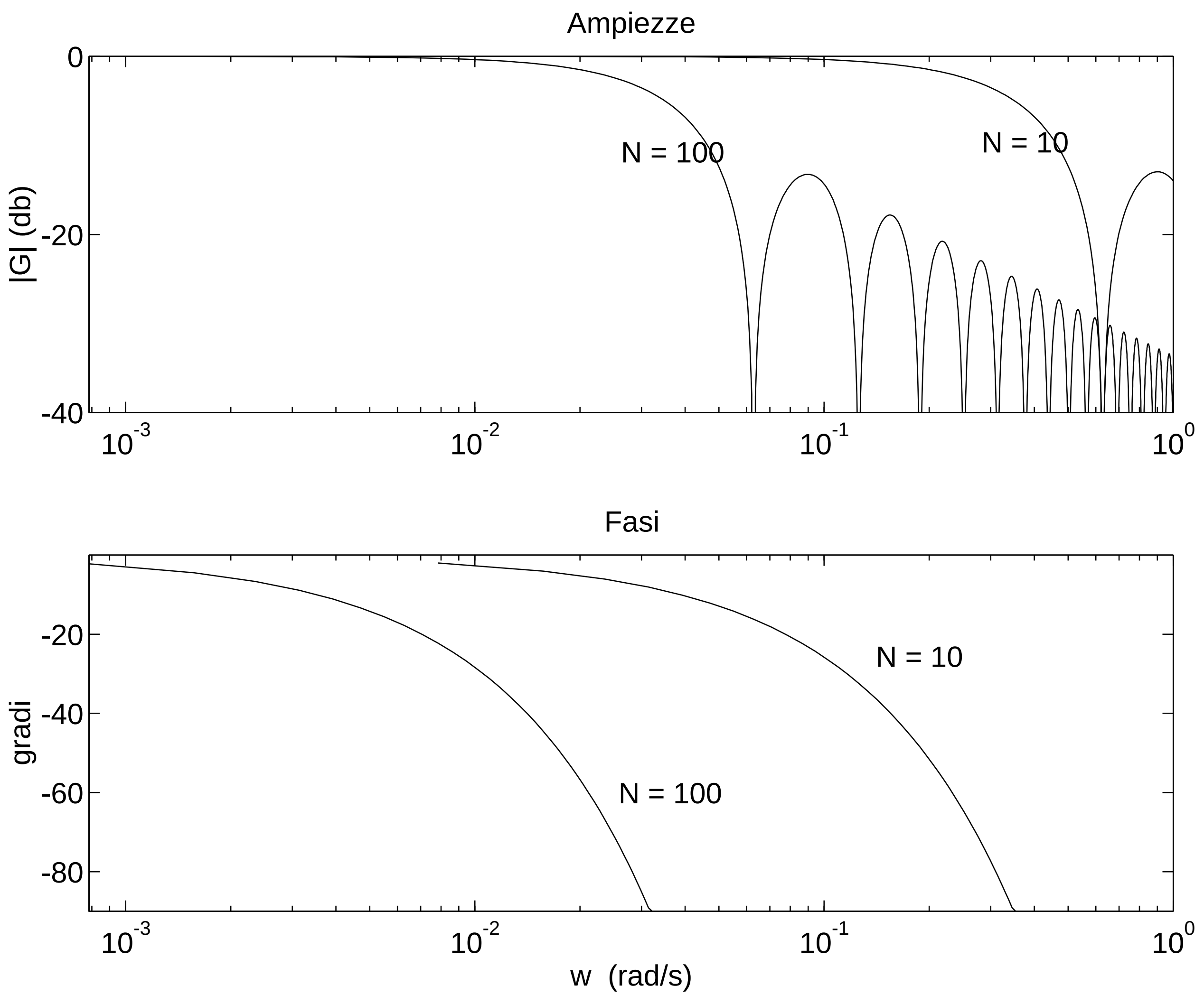
$$y(k) = \frac{1}{N} \sum_{i=0}^{N-1} u(k-i)$$

with sampling time

$$T_d = \frac{T_c}{N}$$

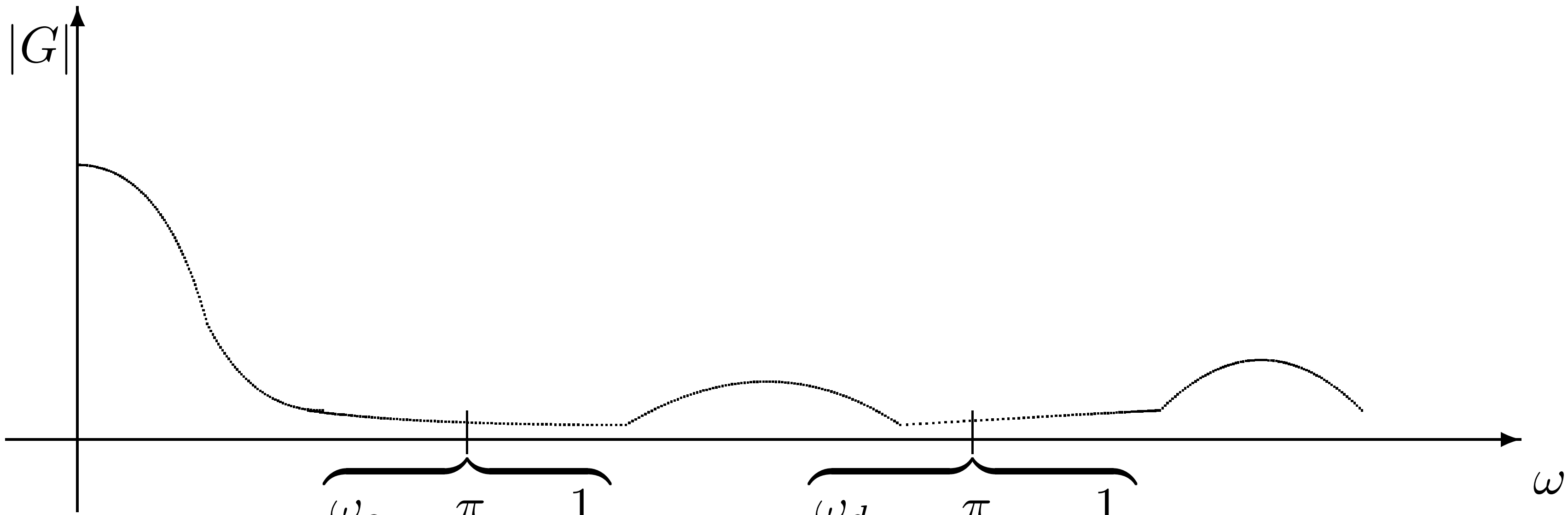
# Antialiasing filters

Bode diagrams of Bode filters



# Antialiasing filters

How to chose time constants for the analog and digital filters




$$\frac{\omega_c}{2} = \frac{\pi}{T_c} = \frac{1}{\tau_d} \qquad \frac{\omega_d}{2} = \frac{\pi}{T_d} = \frac{1}{\tau_a}$$

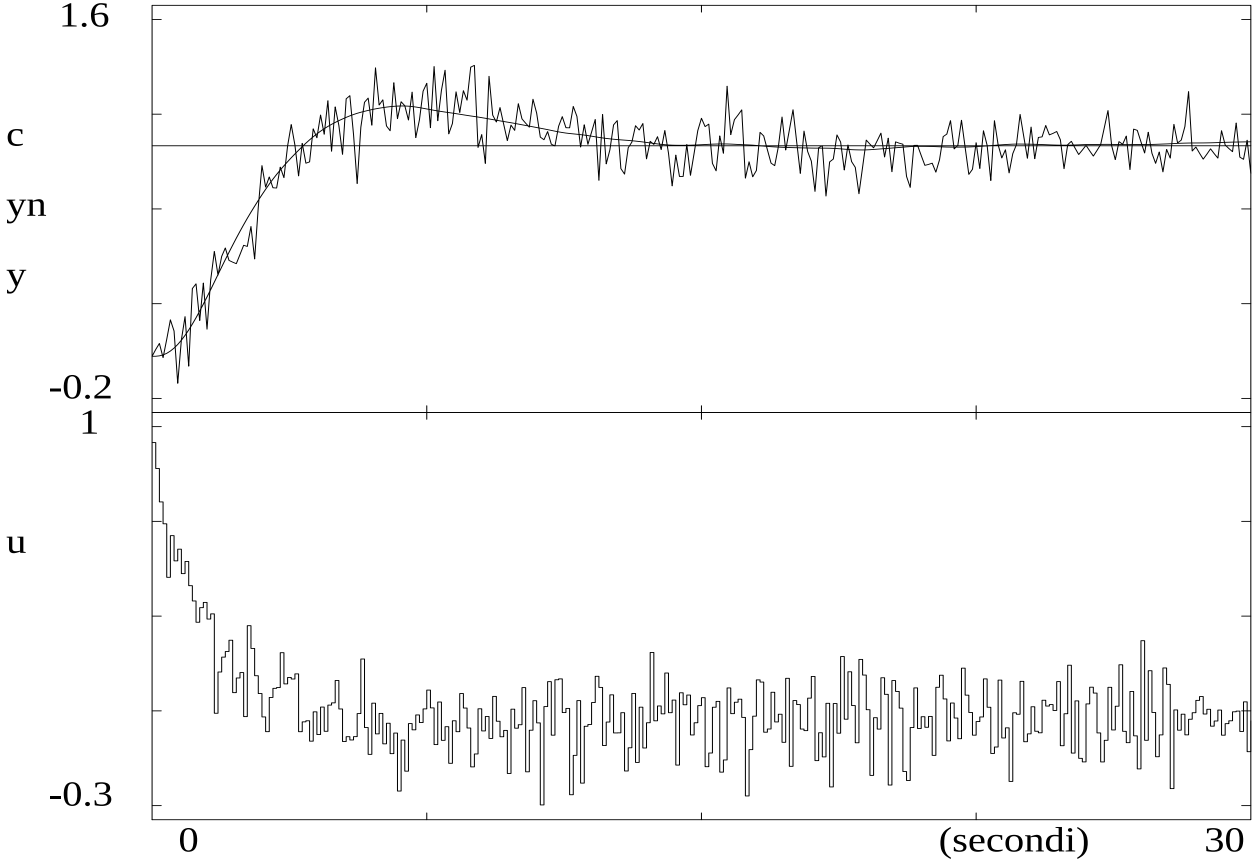
$$\omega_a = \frac{\omega_c}{2} = \frac{\pi}{T_c}, \qquad \omega_a \approx 2/T_c, \qquad \tau_a = 1/\omega_a \approx T_c/2$$

$\tau_d \approx T_c/2$, while the time constant of the analog filter $\tau_a$ must be computed according to the sampling time of the filter: $\tau_a \approx T_d/2$

# Antialiasing filters: example

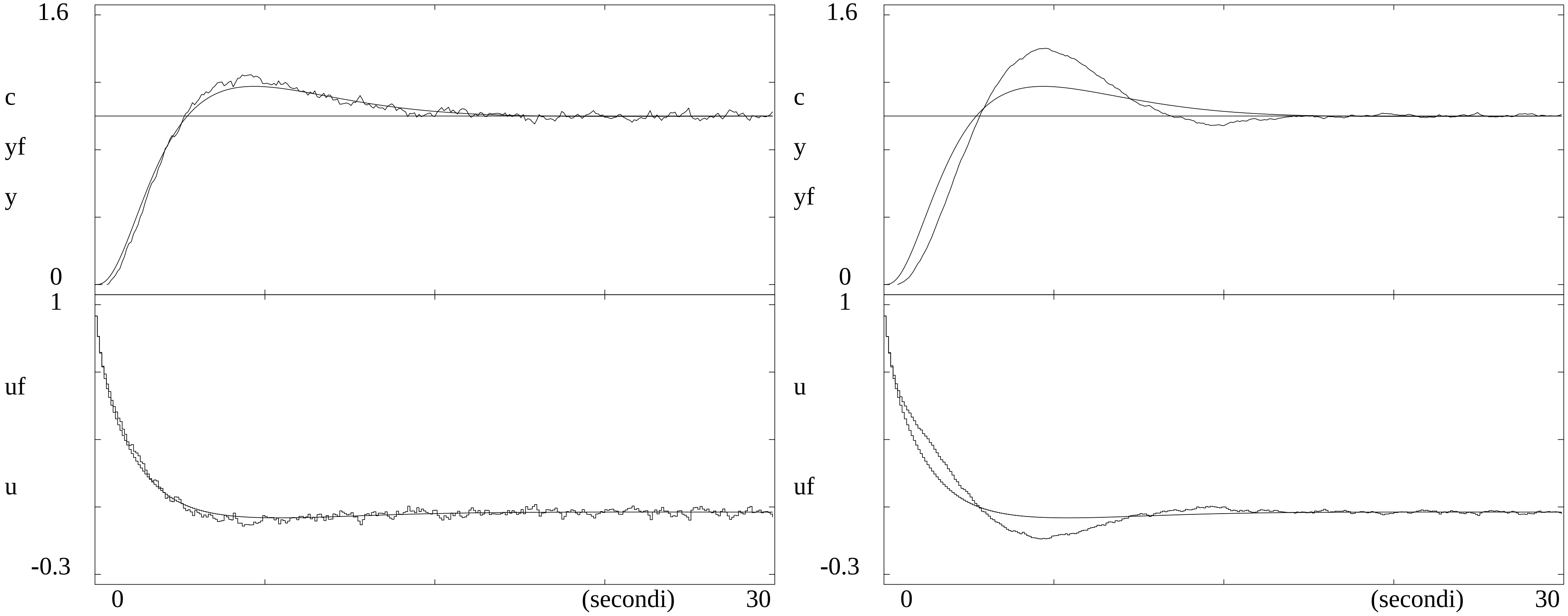
$$G(s) = \frac{2}{s(s+1)(s+2)} \qquad D(z) = \frac{0.94527(z-0.97884)(z-0.92433)}{(z-0.80687)(z-0.99216)}$$

Response without antialiasing filtering. Gaussian noise with variance 0.1



# Antialiasing filters: example

Antialiasing first order filter with $1/\tau = 5\ rad/s$ ($1/\tau = 2\ rad/s$)



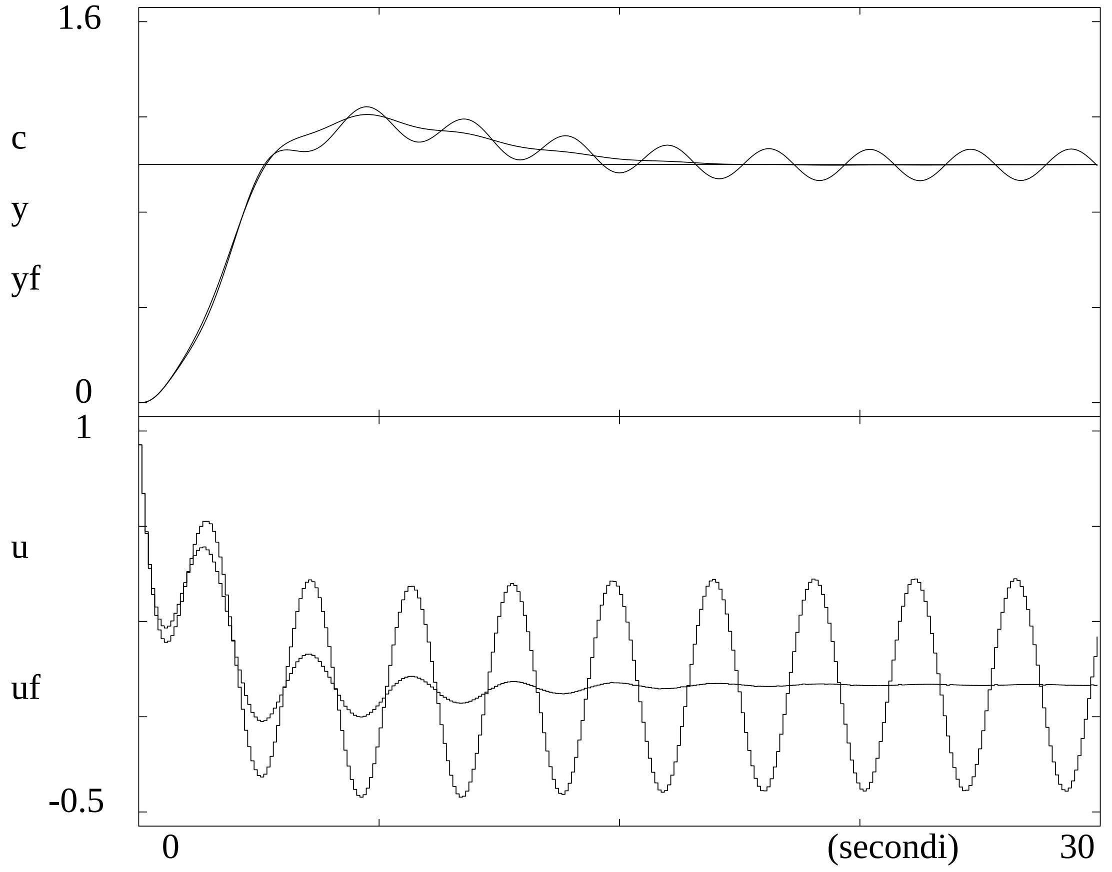

Drawback: system slows down and overshoot increase

# Antialiasing filters: example

Sinusoidal noise with amplitude 0.5 and frequency $f = 0.3183\ Hz = 2\ rad/s$

Selective filter (notch): $\omega = 2\ rad/s$

$$F(s) = \frac{s^2 + 4}{s^2 + 0.4s + 4}$$



# Plane $s$ and plane $z$ correspondence

Since

$$X^*(s) = X(z)|_{z=e^{sT}}$$

variables $s$ and $z$ are linked by the relation $z = e^{sT}$
Being $s = \sigma + j\omega$ we have

$$z = e^{T(\sigma+j\omega)} = e^{T\sigma}e^{jT\omega} = e^{T\sigma}e^{jT(\omega+\frac{2k\pi}{T})}$$

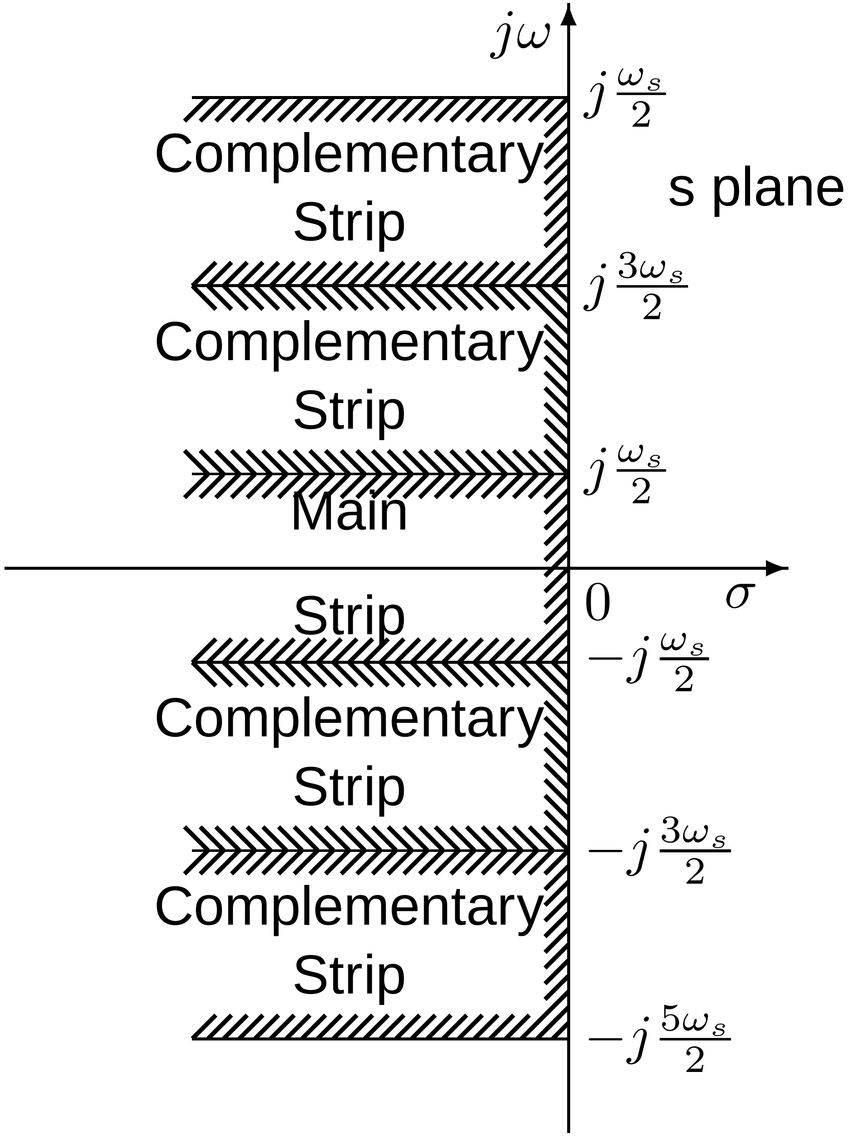
**Important**: any point in $z$ is in correspondence with infinite points if the plane $s$.
The points in $s$ with negative real part ($\sigma < 0$) are in correspondence with the points in $z$ within the unitary circle

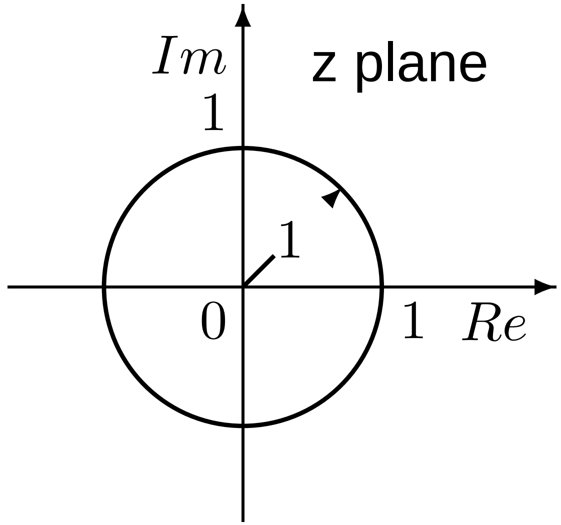
$$|z| = e^{T\sigma} < 1$$

The points in $s$ on the imaginary axis ($\sigma = 0$) are mapped on the unitary circle $(|z| = 1)$, while those with positive real part ($\sigma > 0$) are mapped outside the unitary circle $(|z| > 1)$.
The strip of the $s$ plane bounded by the horizontal lines $s = j\omega_s/2$ and $s = -j\omega_s/2$ is called **main strip**.

# Plane $s$ and plane $z$ correspondence

Main strip and complementary strips



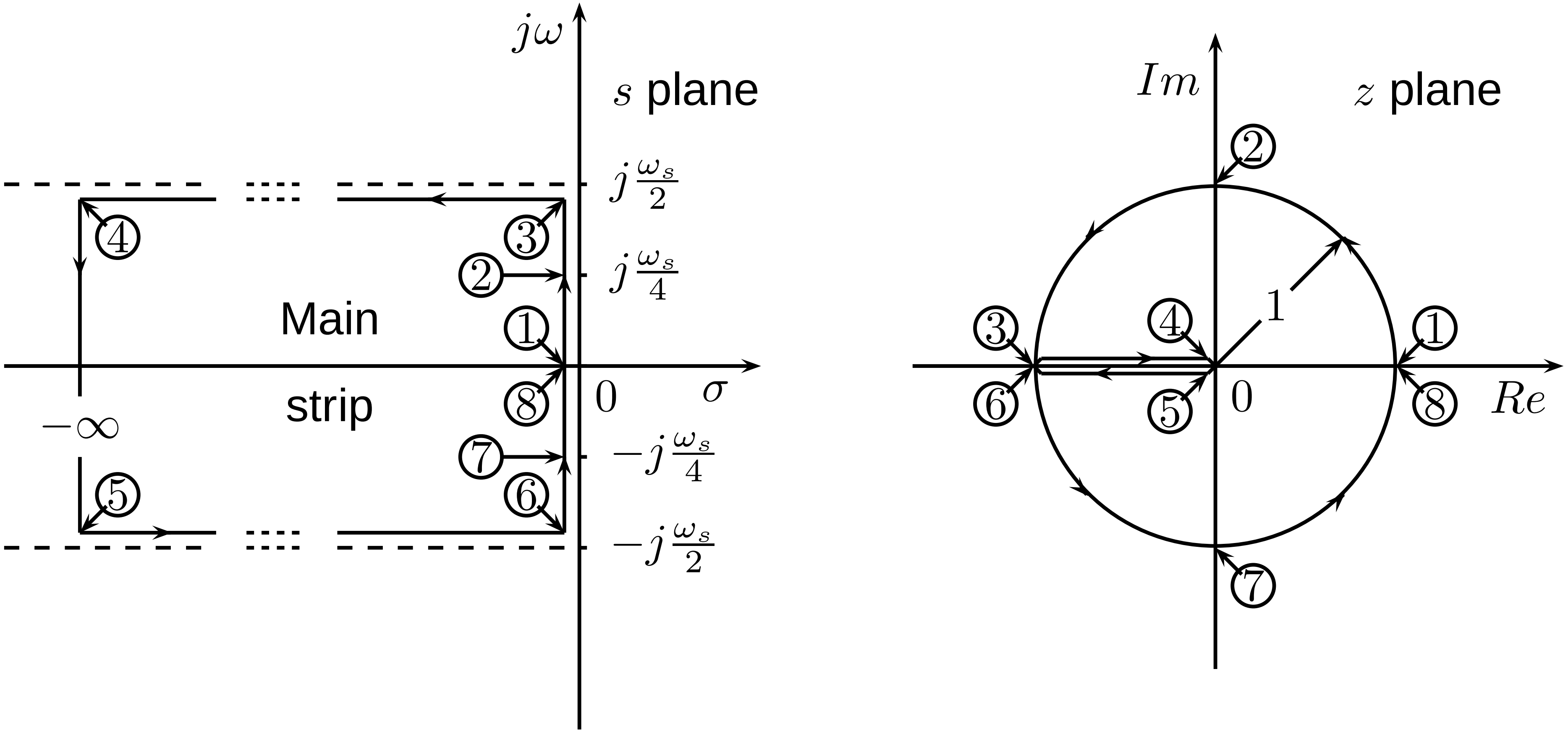



# Plane $s$ and plane $z$ correspondence

Complex variables $s$ and $z$ are linked by the relation $z = e^{sT}$
Being $s = \sigma + j\omega$ we have

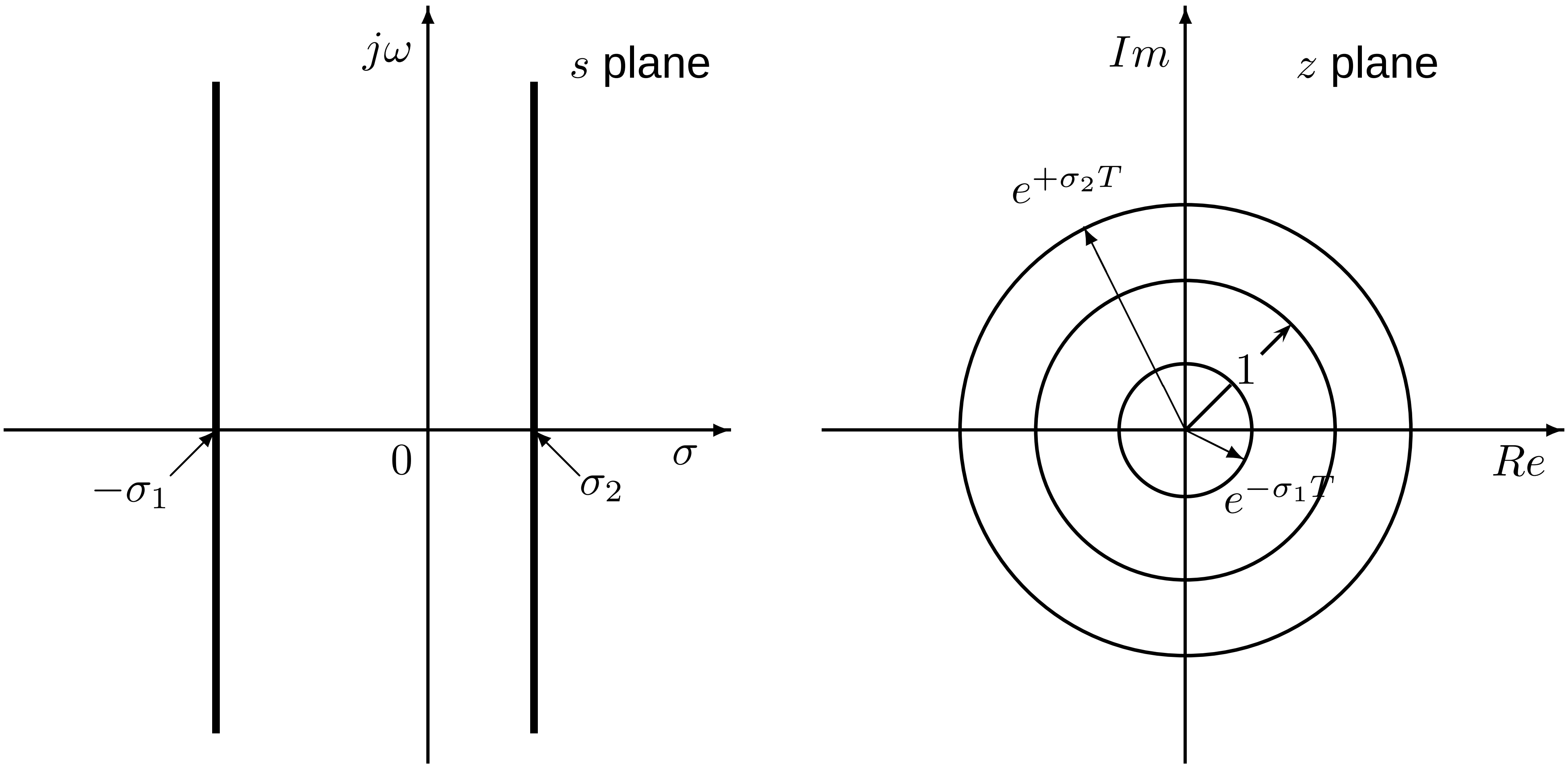
$$z = e^{T(\sigma+j\omega)} = e^{T\sigma} e^{jT\omega} \qquad \left(0 \leq \omega \leq \frac{\omega_s}{2} = \frac{\pi}{T}\right)$$

Mapping between main strip and $z$ plane



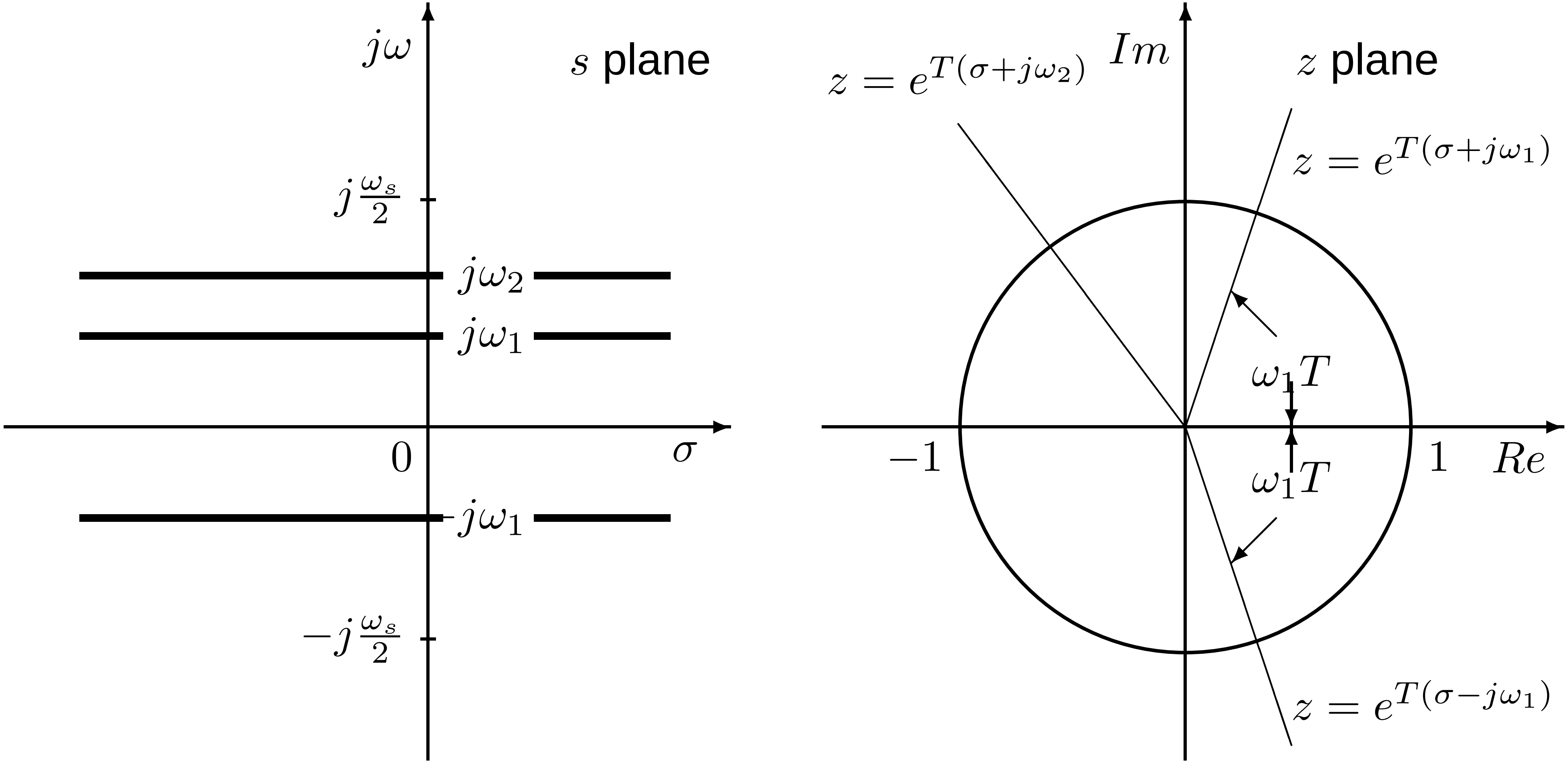
# Plane $s$ and plane $z$ correspondence

Constant exponential decay loci ($\sigma =$const)



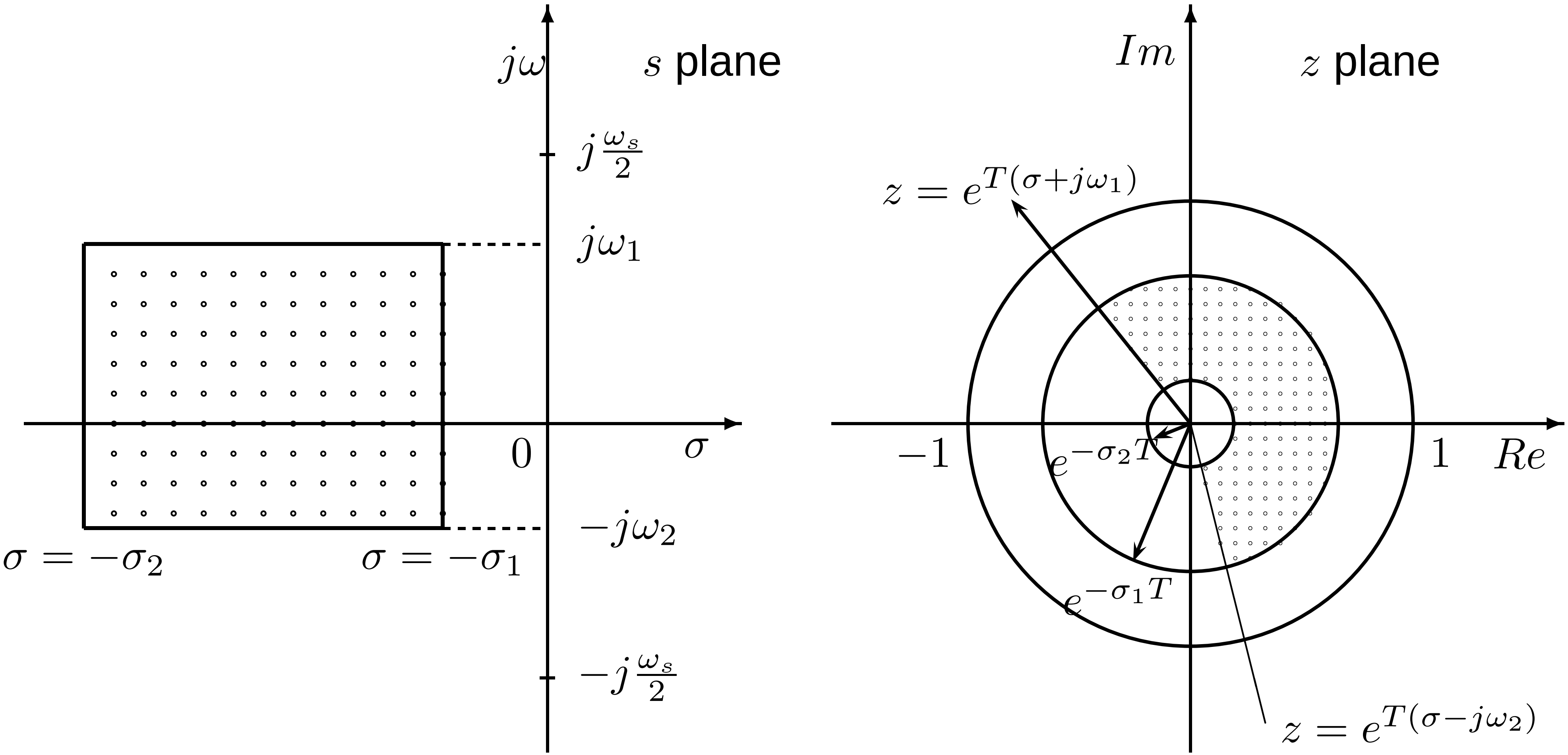
# Plane $s$ and plane $z$ correspondence

Constant pulse loci ($\omega$ =const)



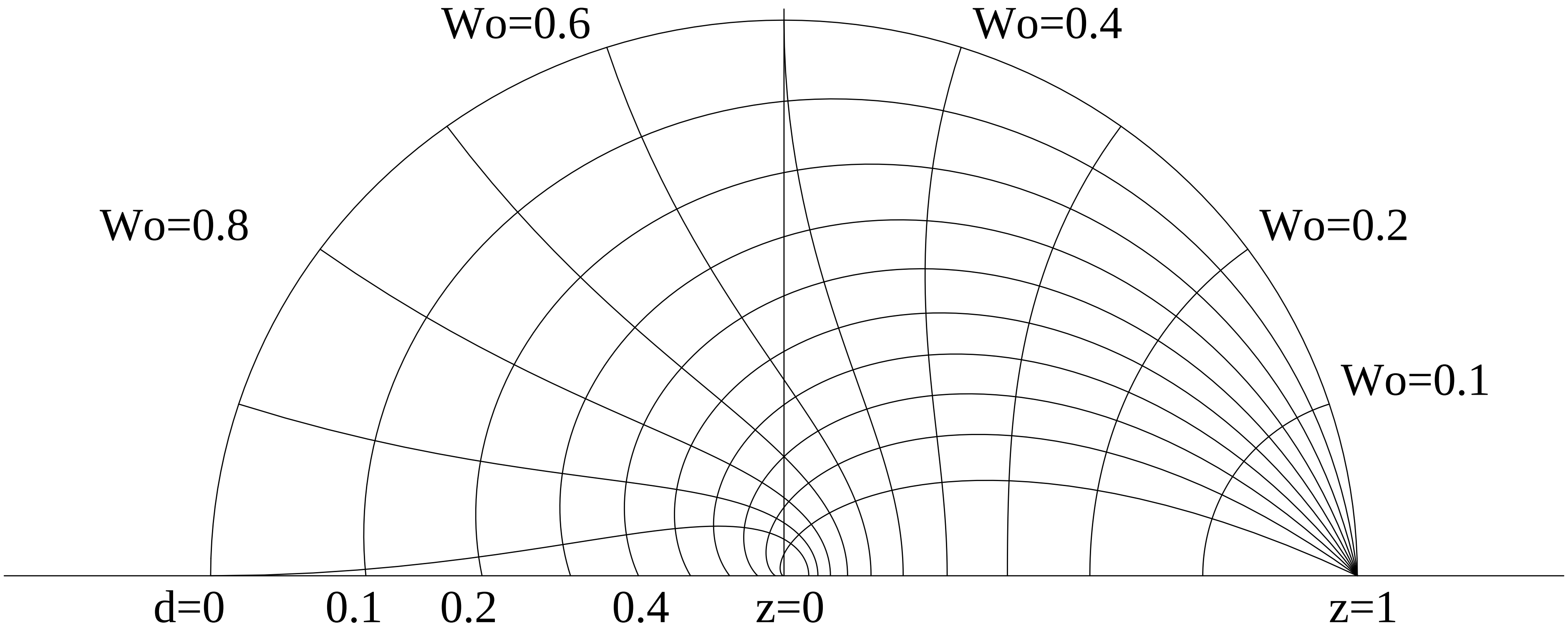
# *Plane $s$ and plane $z$ correspondence*

Example:



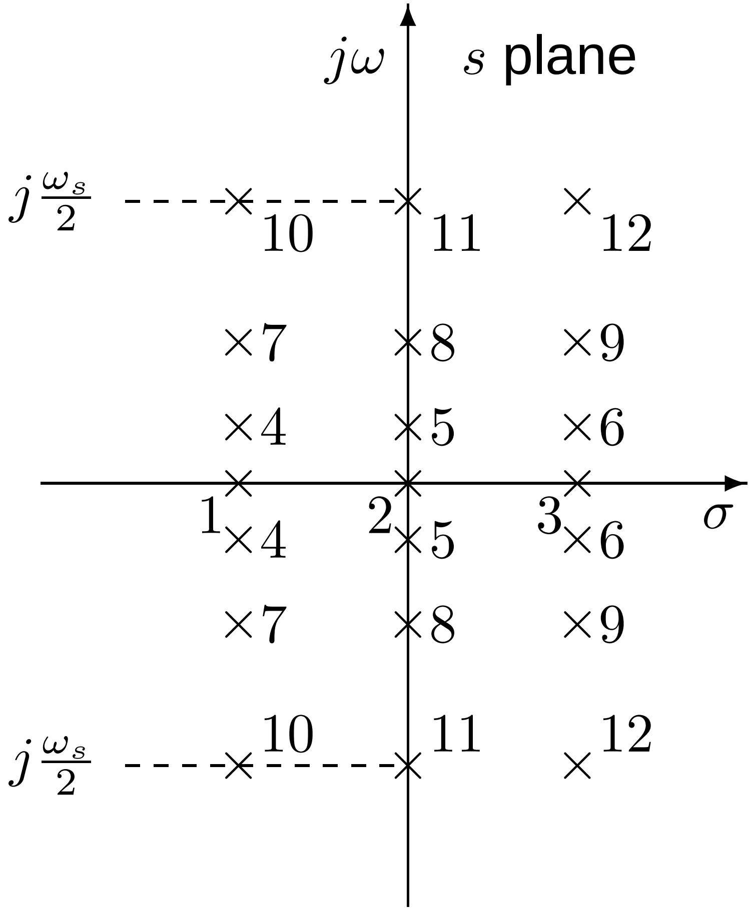
# Plane $s$ and plane $z$ correspondence

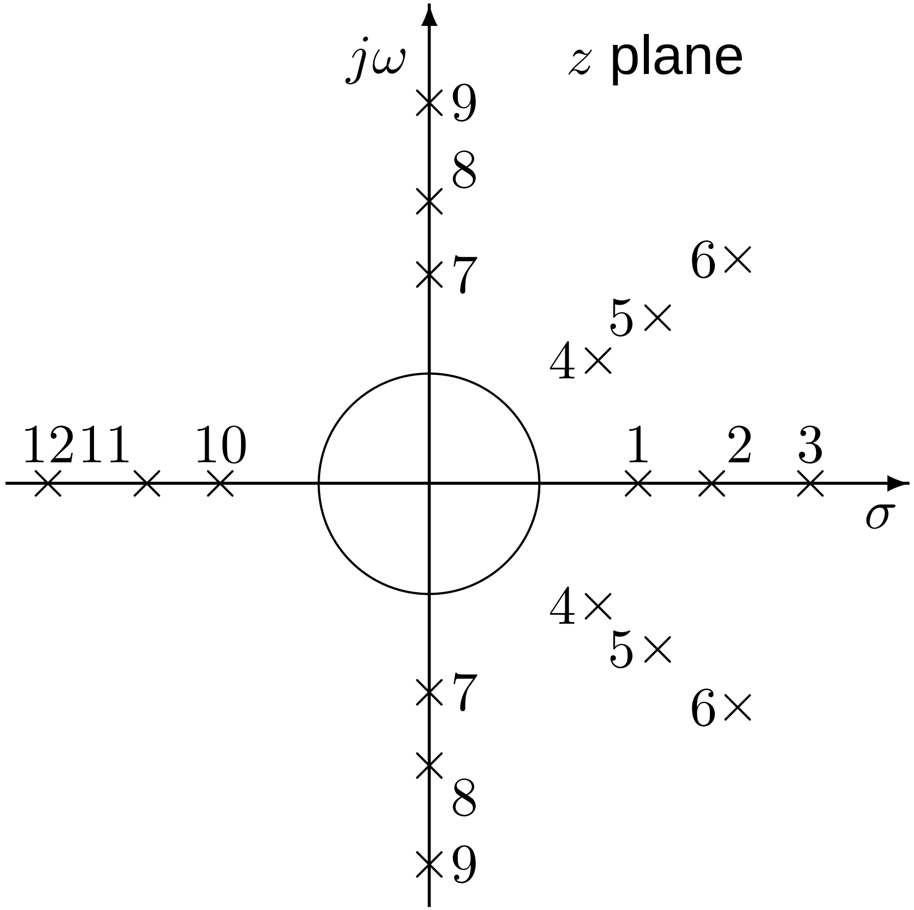
Constant damping coefficient $\delta$ and natural pulse $\omega_n$ loci in $z$ plane



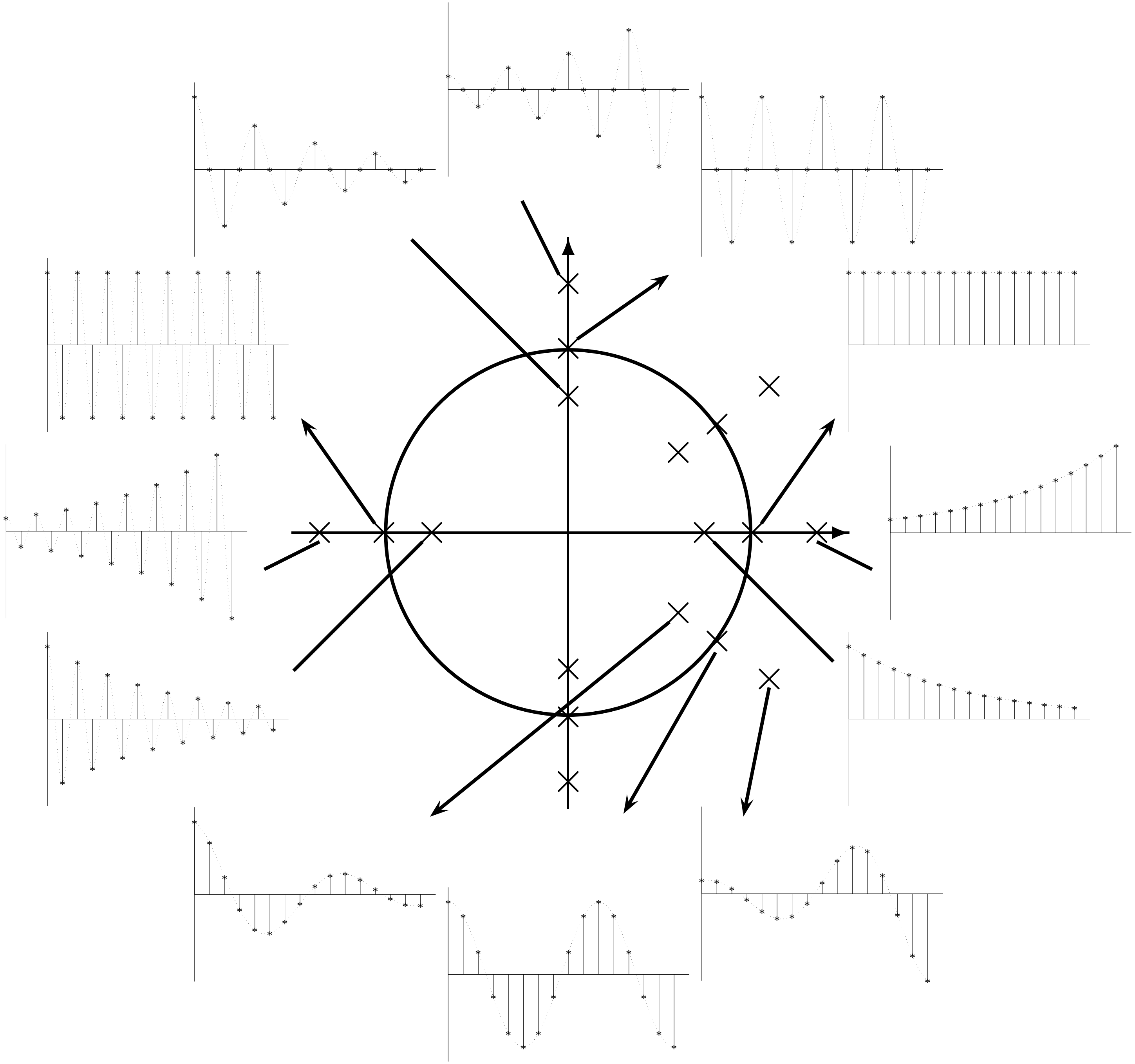
# Plane $s$ and plane $z$ correspondence

The points in $s$ and in $z$, such that $z = e^{sT}$, can be considered as the poles of the corresponding Laplace transform $F(s)$ and $\mathcal{Z}$-transform $F(z)$, with $F(z)$ obtained by sampling $F(s)$.



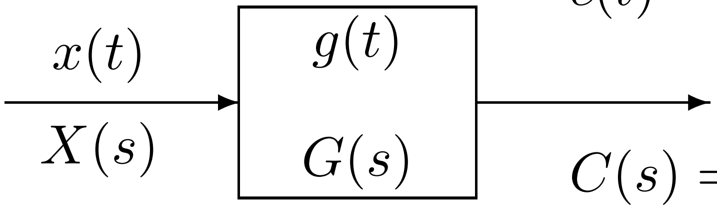



# Transient responses in $z$ plane



# Discrete Time Systems

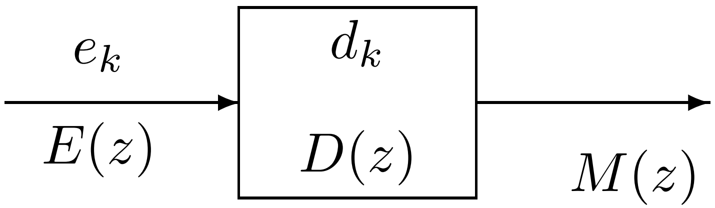
## Continuous time systems




$$c(t) = \int_0^t g(\tau)x(t-\tau)\,d\tau$$

$$C(s) = G(s)\,X(s)$$

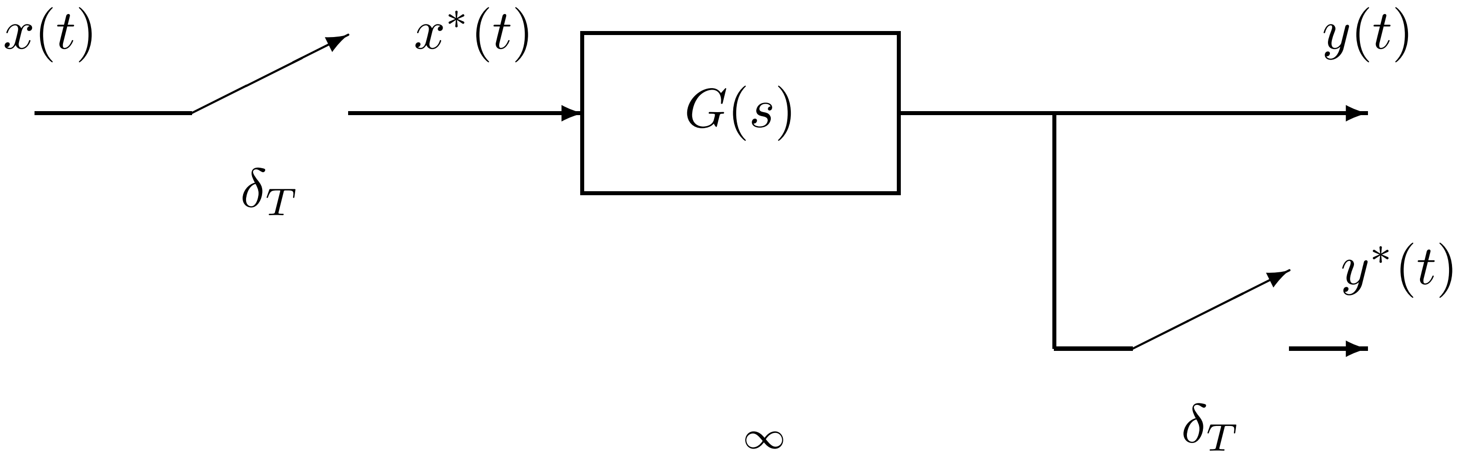
## Discrete time systems




$$m_k = \sum_{i=0}^{k} d_i\,e_{k-i} = \sum_{i=0}^{k} e_i\,d_{k-i}$$

$$M(z) = D(z)\,E(z)$$

## Discrete convolution




$$x^*(t) = \sum_{k=0}^{\infty} x(kT)\delta(t - kT)$$

$$y(t) = \begin{cases} g(t)x(0) & 0 \le t < T \\ g(t)x(0) + g(t-T)x(T) & T \le t < 2T \\ g(t)x(0) + g(t-T)x(T) + g(t-2T)x(2T) & \dots \\ \vdots & \\ g(t)x(0) + g(t-T)x(T) + \dots + g(t-kT)x(kT) & \end{cases}$$

# Discrete time systems

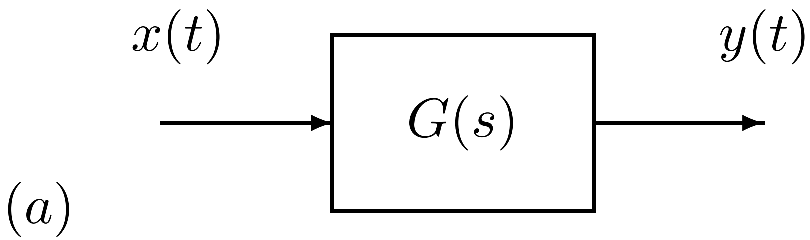
Since $g(t) = 0, t < 0$, we have

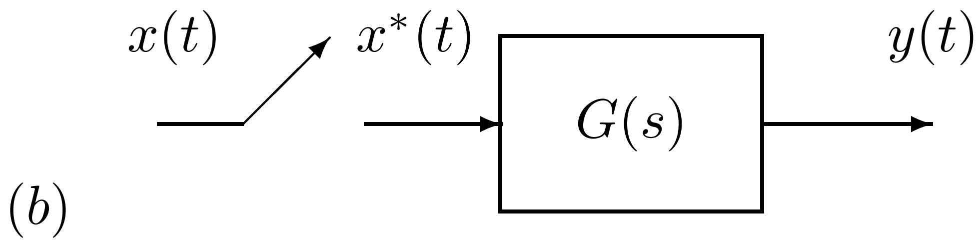
$$\begin{aligned} y(t) &= g(t)x(0) + g(t-T)x(T) + \ldots + g(t-kT)x(kT) \\ &= \sum_{h=0}^{k} g(t-hT)x(hT) \quad 0 \leq t < (k+1)T \end{aligned}$$

Considering the samples of $y(t)$ $t = kT$, $k = 0, 1, 2, \ldots,$ we have

$$\begin{aligned} y(kT) &= \sum_{h=0}^{k} g(kT-hT)x(hT) \\ &= \sum_{h=0}^{k} x(kT-hT)g(hT) \end{aligned}$$

Example: $x(t) = e^{-t}$ $\qquad G(s) = \dfrac{1}{1+s}$ $\qquad T = 1$



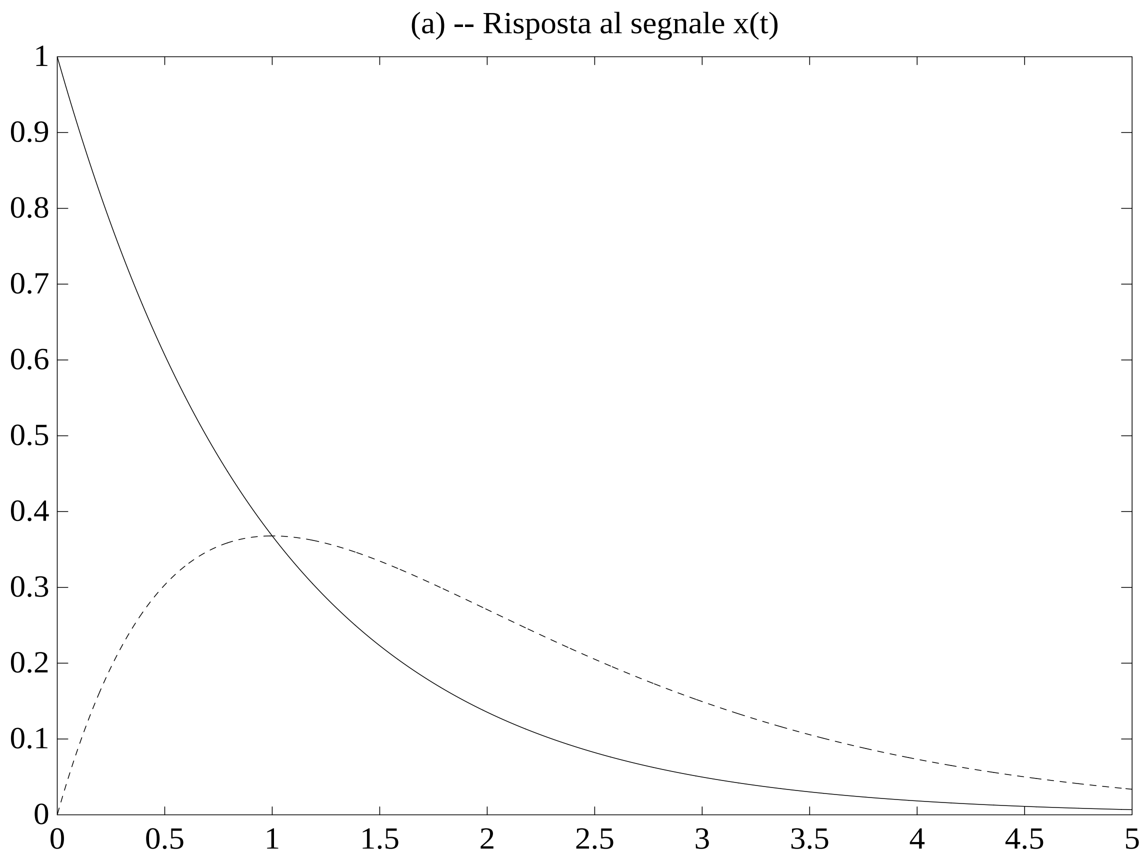



Example (cont'd)

Case a)

$$Y_a(s) = G(s)X(s) = \frac{1}{s+1}\frac{1}{s+1} = \frac{1}{(s+1)^2}$$

$$y_a(kT) = kTe^{-kT}$$



Example (cont'd)
Case b)

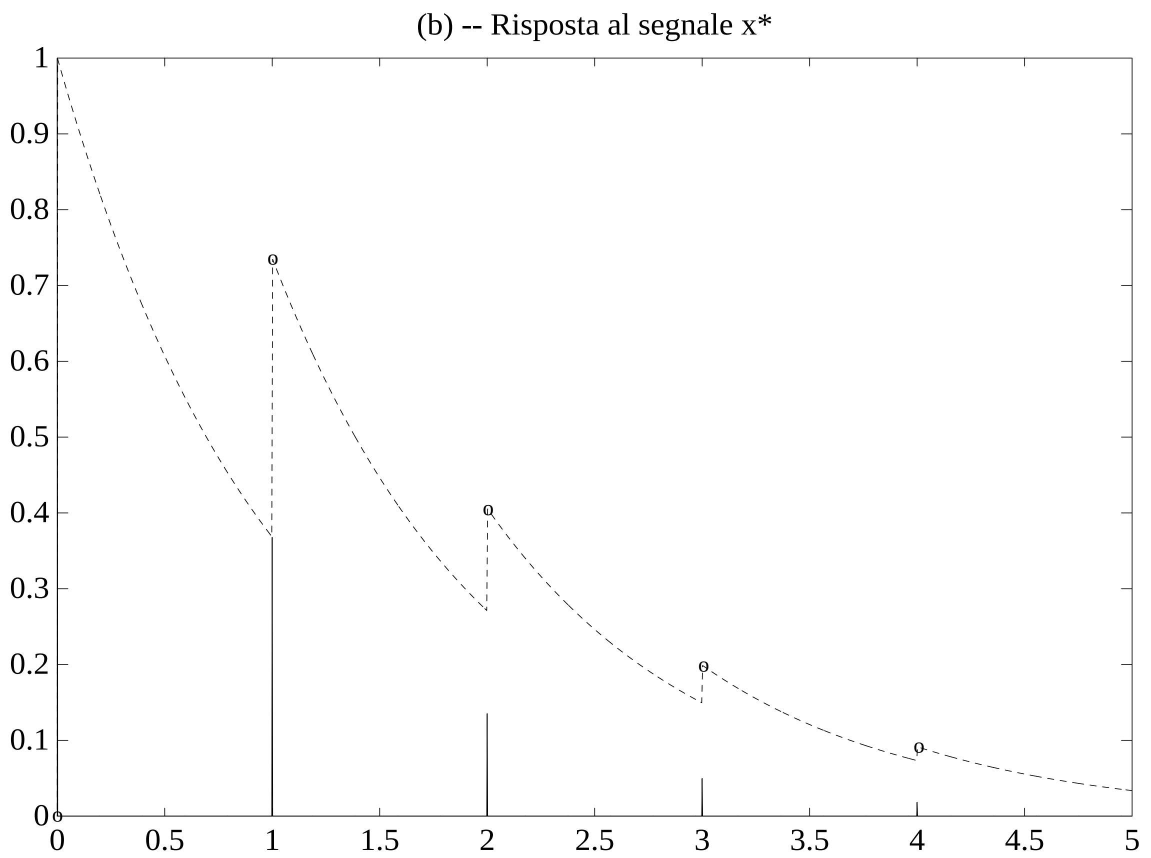
$$y_b(t) = \begin{cases} g(t)x(0) \\ g(t)x(0) + g(t-T)x(T) \\ \vdots \\ g(t)x(0) + g(t-T)x(T) + \ldots + g(t-kT)x(kT) \end{cases}$$

In this case $g(t) = e^{-t}$ (inverse transform of $G(s)$), hence:

$$y_b(t) = \begin{cases} e^{-t} \\ e^{-t} + e^{-(t-T)}e^{-T} = 2e^{-t} \\ \vdots \\ e^{-t} + \ldots + e^{-(t-kT)}e^{-kT} = (k+1)e^{-t} \end{cases}$$

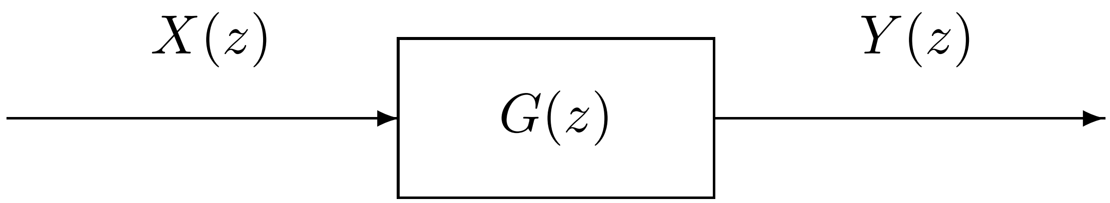
$$y_b(kT) = (k+1)e^{-kT}$$

Example (cont’d)
Case b)



# Discrete transfer function

$$y(kT) = \sum_{h=0}^{\infty} g(kT - hT)x(hT)$$




$$X(z) = \mathcal{Z}[x(kT)] = 1 \qquad \rightarrow \qquad Y(z) = G(z)$$

Discrete harmonic response function

$$G(e^{j\omega T}), \qquad 0 \leq \omega \leq \frac{\pi}{T}$$

$$G(e^{j(\omega + k\omega_s)T}) = G(e^{j\omega T}), \qquad G(e^{j(-\omega)T}) = G^*(e^{j\omega T})$$

# Discrete transfer function

The response of an asymptotically stable system $G(z)$ wrt a sinusoidal input $\sin(\omega kT)$ is, in steady state, a sinusoidal function $A\sin(\omega kT+\varphi)$ whose amplitude $A$ and phase $\varphi$ are respectively given by the module and the phase of the vector $G(e^{j\omega T})$:

$$A = |G(e^{j\omega T})| \qquad\qquad \varphi = Arg[G(e^{j\omega T})]$$

$\mathcal{Z}$-transform of the sinusoidal signal:

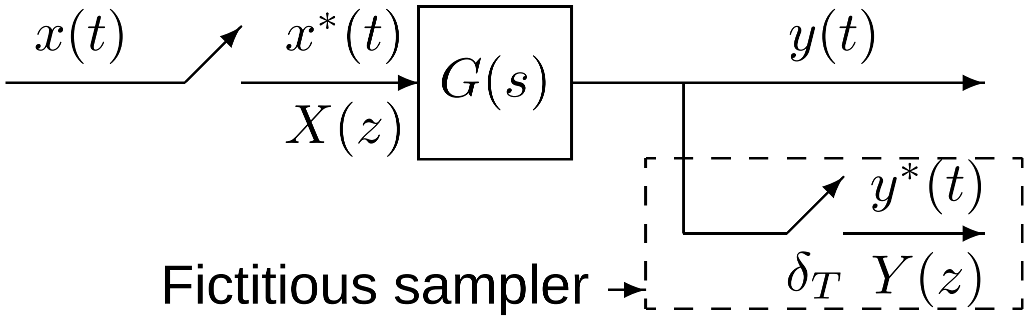
$$\begin{aligned} X(z) &= \mathcal{Z}[\sin(\omega t)] = \frac{z\sin\omega T}{z^2-(2\cos\omega T)z+1} \\ &= \frac{1}{2j}\left(\frac{z}{z-e^{j\omega T}} - \frac{z}{z-e^{j\omega T}}\right) \end{aligned}$$

$$Y(z) = G(z)\,X(z) = Y_0(z) + \frac{|G(e^{j\omega T})|}{2j}\left(\frac{e^{j\varphi}z}{z-e^{j\omega T}} - \frac{e^{j\varphi}z}{z-e^{-j\omega T}}\right)$$

is the sum of the asymptotically vanishing term $Y_0(z)$, corresponding to stable poles of $G(z)$, and a sinusoidal term with amplitude and phase equal to $|G(e^{j\omega T})|$ and $\varphi = Arg[G(e^{j\omega T})]$ respectivey.

# Composing discrete transfer functions

Case a)

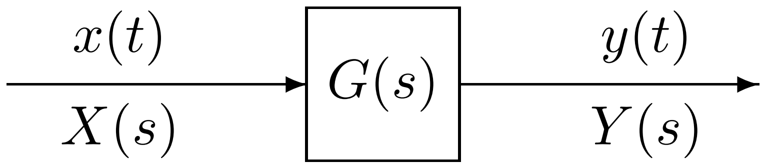



$$Y(s) = G(s)\,X^*(s)$$

$$Y^*(s) = [G(s)\,X^*(s)]^* = G(s)^*\,X^*(s)$$

$$Y(z) = G(z)\,X(z)$$

Case b)



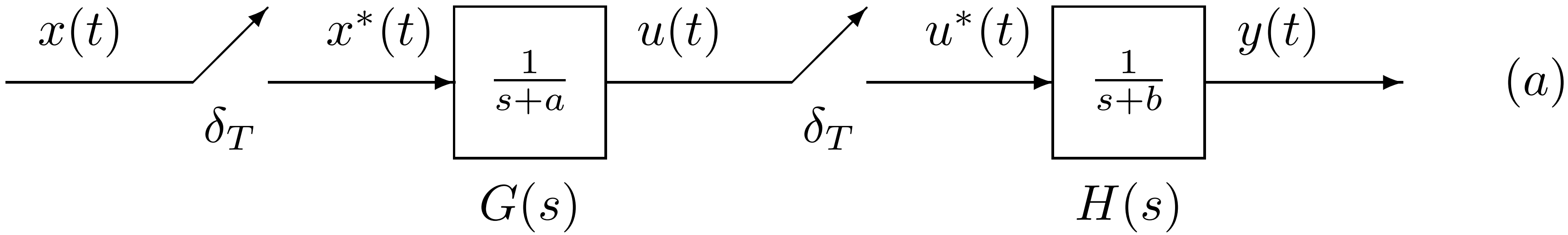

$$Y(s) = G(s)\,X(s)$$

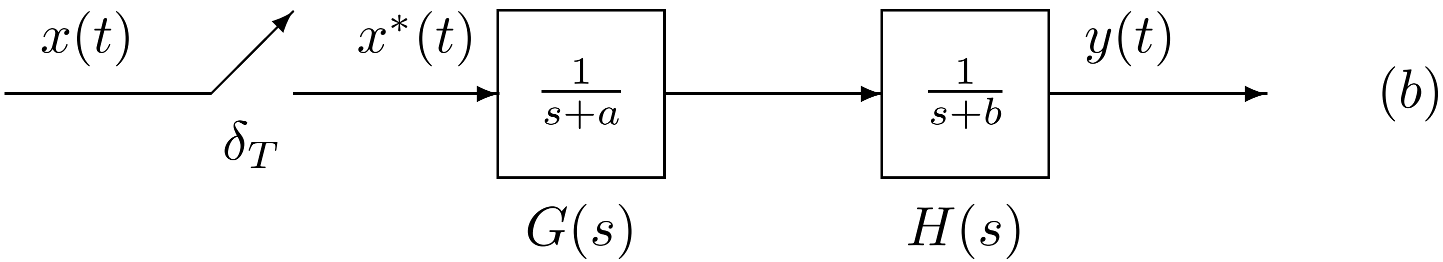
$$Y^*(s) = [G(s)\,X(s)]^*$$

$$Y(z) = \mathcal{Z}[G(s)\,X(s)] = GX(z) \neq G(z)\,X(z)$$

# Composing discrete transfer functions

Example



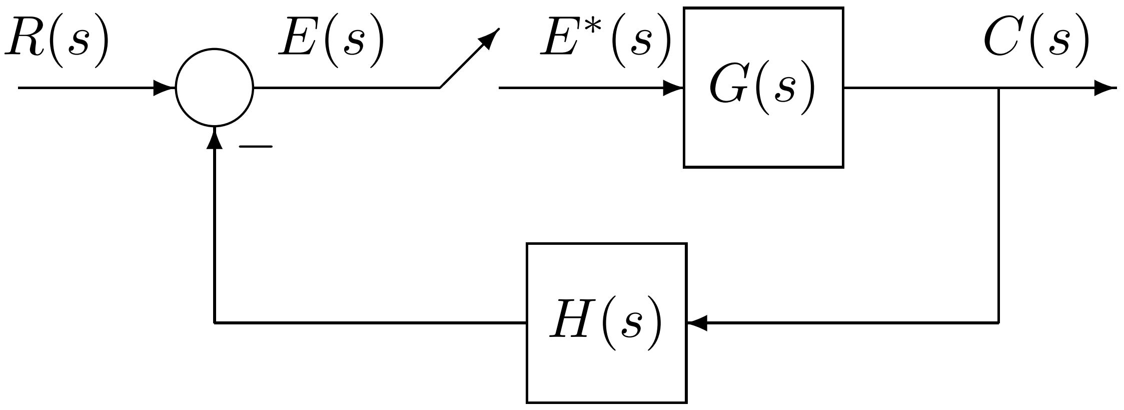




Case a) $$\frac{Y(z)}{X(z)} = G(z)\,H(z) = \mathcal{Z}\left[\frac{1}{s+a}\right]\mathcal{Z}\left[\frac{1}{s+b}\right]$$

Case b) $$\frac{Y(z)}{X(z)} = \mathcal{Z}[G(s)\,H(s)] = \mathcal{Z}\left[\frac{1}{s+a}\,\frac{1}{s+b}\right]$$

# Composing discrete transfer functions

Feedback interconnection:




$$E(s) = R(s) - H(s)\,C(s) \qquad C(s) = G(s)\,E^*(s)$$
$$E(s) = R(s) - H(s)\,G(s)\,E^*(s)$$

Sampling: $$\begin{cases} E^*(s) = R^*(s) - GH^*(s)\,E^*(s) \\ C^*(s) = G^*(s)\,E^*(s) \end{cases}$$

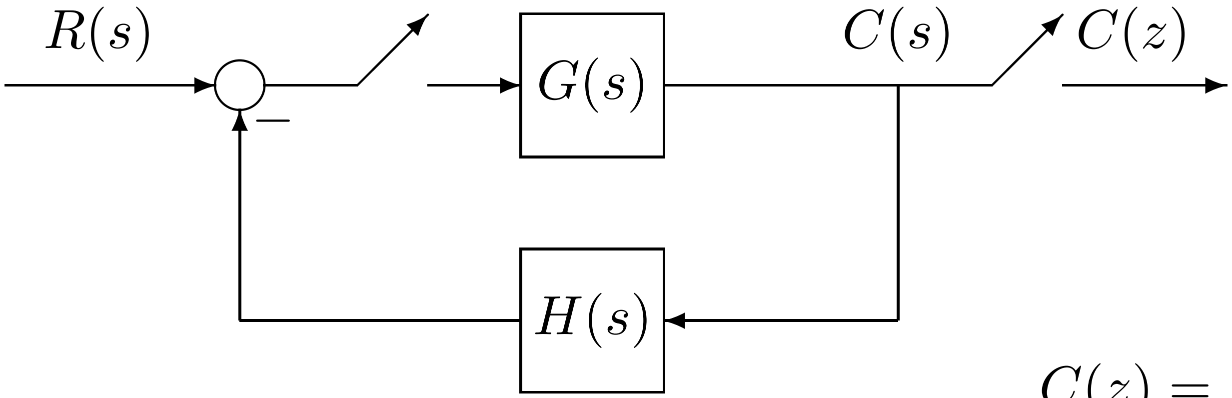
$$C^*(s) = \frac{G^*(s)\,R^*(s)}{1 + GH^*(s)} \qquad \rightarrow \qquad C(z) = \frac{G(z)\,R(z)}{1 + GH(z)}$$

The discrete transfer function of the sampled system is:

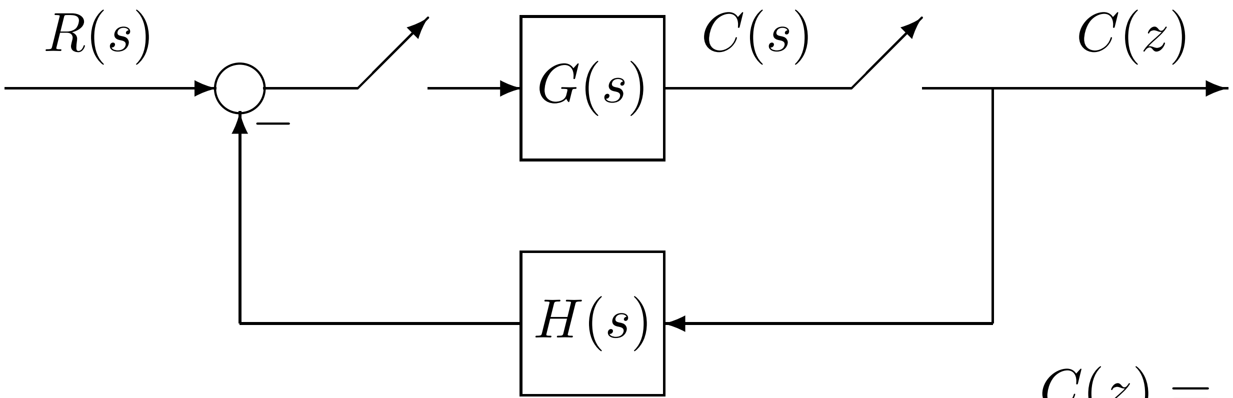
$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + GH(z)}$$

# Composing discrete transfer functions

Typical feedback interconnections

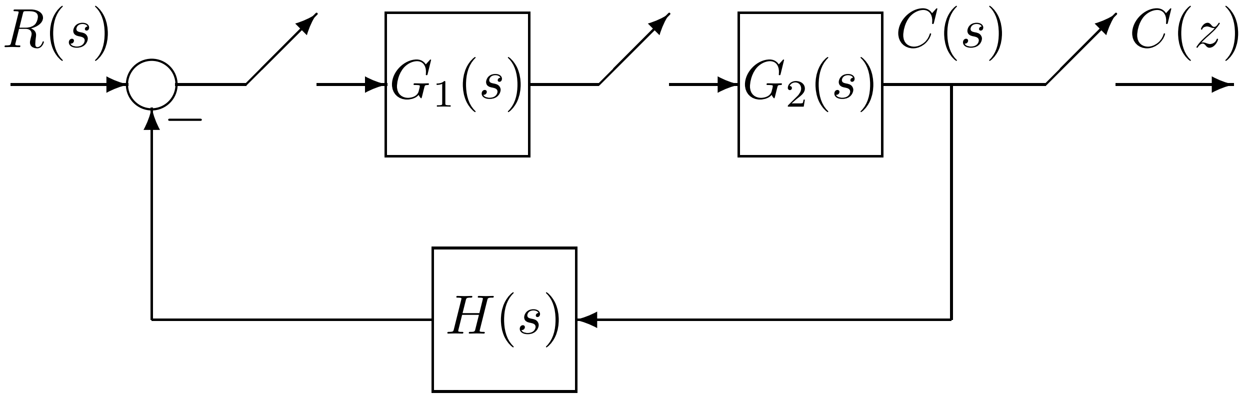



$$C(z) = \frac{G(z)\,R(z)}{1 + GH(z)}$$

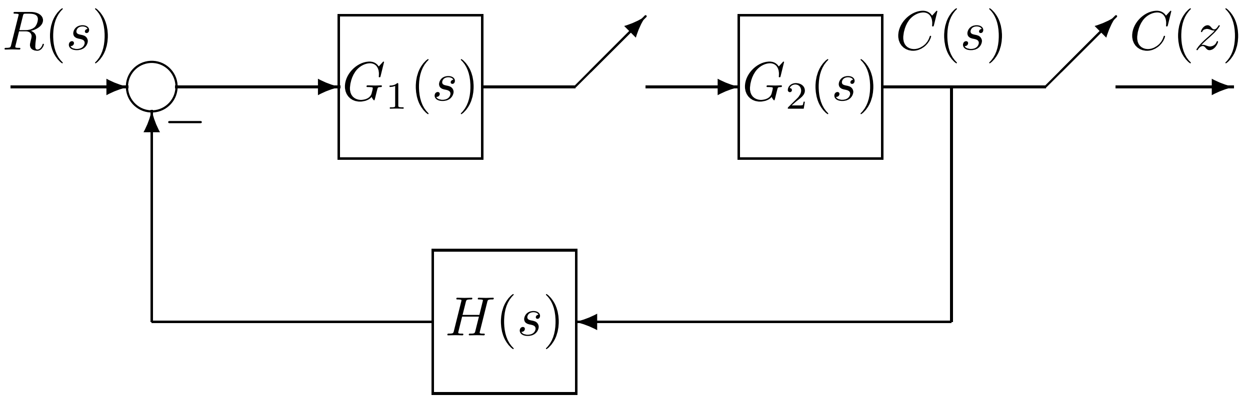



$$C(z) = \frac{G(z)\,R(z)}{1 + GH(z)}$$

# Composing discrete transfer functions

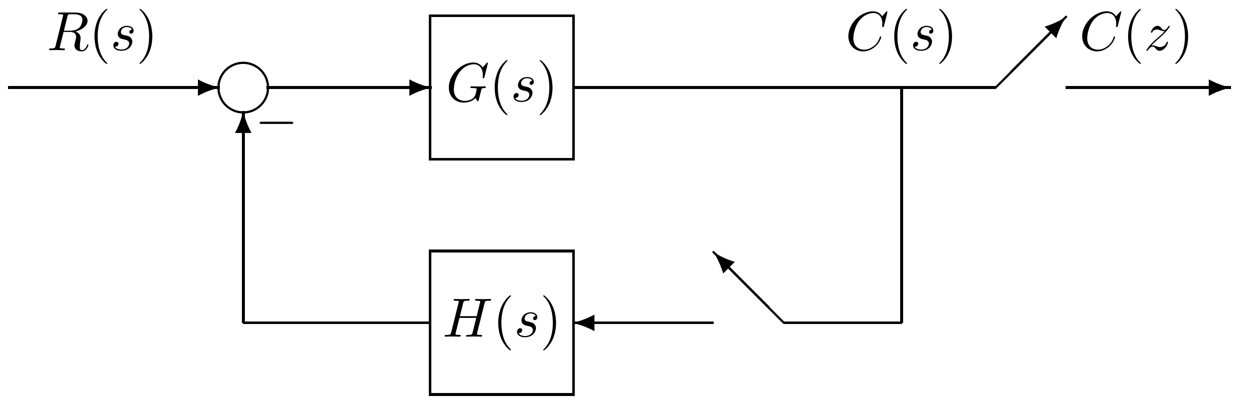



$$C(z) = \frac{G_1(z)\, G_2(z)\, R(z)}{1 + G_1(z)\, G_2H(z)}$$




$$C(z) = \frac{G_2(z)\, G_1R(z)}{1 + G_1G_2H(z)}$$

# Composing discrete transfer functions




$$C(z) = \frac{GR(z)}{1 + GH(z)}$$

# *Digital controllers design methods*

1. **Indirect method**: discretization of an analog controller
2. **Direct method**: using analytical methods in discrete time domain
3. **Standard controllers**: PID

# DISCRETIZATION OF ANALOG CONTROLLERS

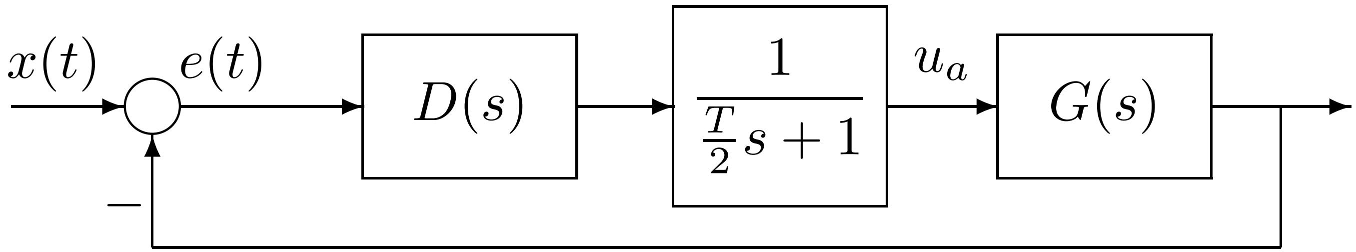
# Design via discretization

Three steps

1. Definition of sampling time $T$ and verification of the phase margin of the system

$$H_0(s) = \frac{1 - e^{-sT}}{s} \approx \frac{T}{\frac{T}{2}s + 1}$$

$$H_0(s) \approx e^{-sT/2}$$




2. Discretization of the analog controller $D(s)$
3. A posteriori simulative verification

# Backward difference method

$$D(z) = D(s)|_{s = \frac{1 - z^{-1}}{T}}$$

Example:

$$\frac{d\,y(t)}{d\,t} + ay(t) = ax(t)$$

$$\int_0^t \frac{d\,y(t)}{d\,t} dt = -a \int_0^t y(t)dt + a \int_0^t x(t)dt$$

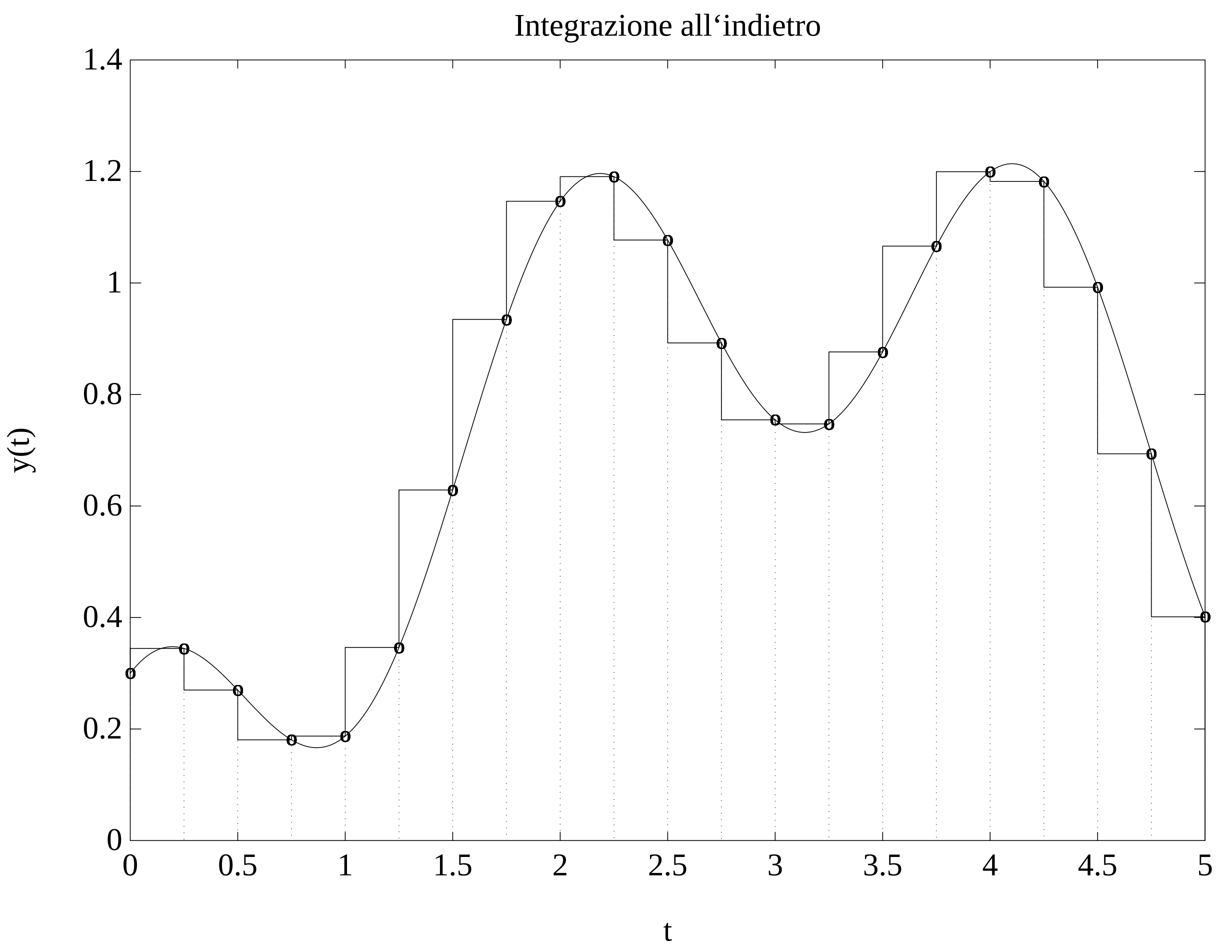
evaluating for $t = kT$, and for $t = (k-1)T$ and subtracting we obtain

$$y(kT) - y((k-1)T) = -a \int_{(k-1)T}^{kT} y(t)dt + a \int_{(k-1)T}^{kT} x(t)dt \simeq -aT\left[y(kT) - x(kT)\right]$$

$$Y(z) = z^{-1}Y(z) - aT\left[Y(z) - X(z)\right]$$

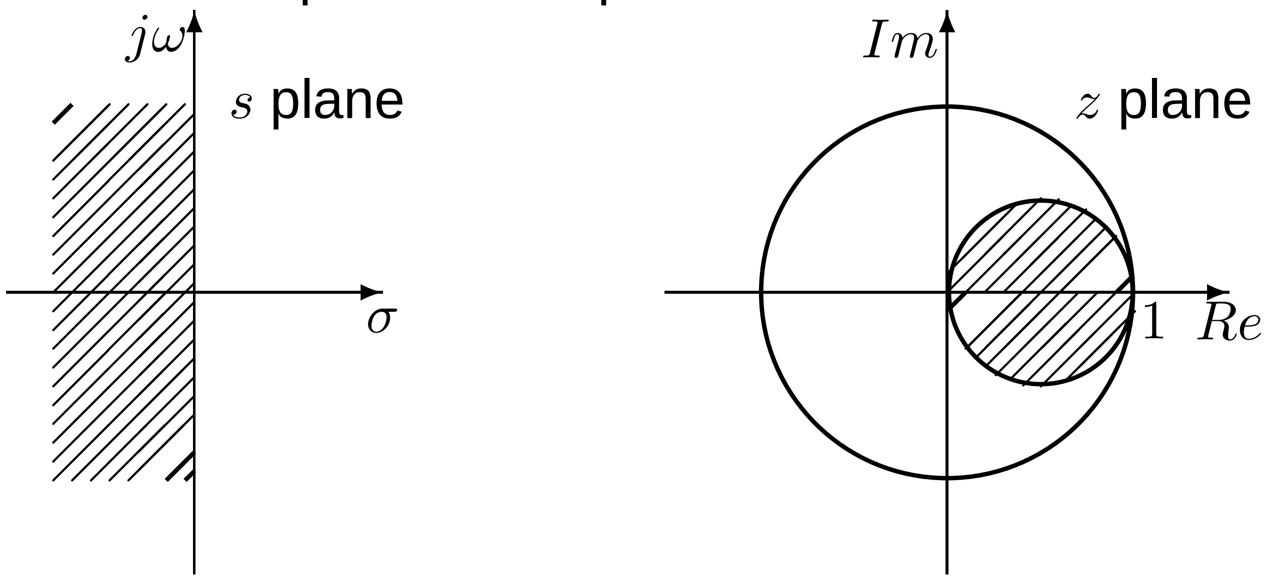
$$\frac{Y(z)}{X(z)} = G(z) = \frac{aT}{1 - z^{-1} + aT} = \frac{a}{\frac{1-z^{-1}}{T} + a}$$

# Backward difference method



# Metodo delle differenze all’indietro

Correspondence between plane $s$ and plane $z$:




If $z = \sigma + j\omega$

$$Re\left(\frac{\sigma + j\omega - 1}{\sigma + j\omega}\right) = Re\left(\frac{(\sigma + j\omega - 1)(\sigma - j\omega)}{\sigma^2 + \omega^2}\right) = \frac{\sigma^2 - \sigma + \omega}{\sigma^2 + \omega^2} < 0$$

$$(\sigma - \frac{1}{2})^2 + \omega^2 < \left(\frac{1}{2}\right)^2$$

Stable controllers $D(s)$ are mapped in discrete time stable controllers $D(z)$.

# Foreward difference method

$$D(z) = D(s)\Big|_{s=\frac{z-1}{T}}$$

Example:

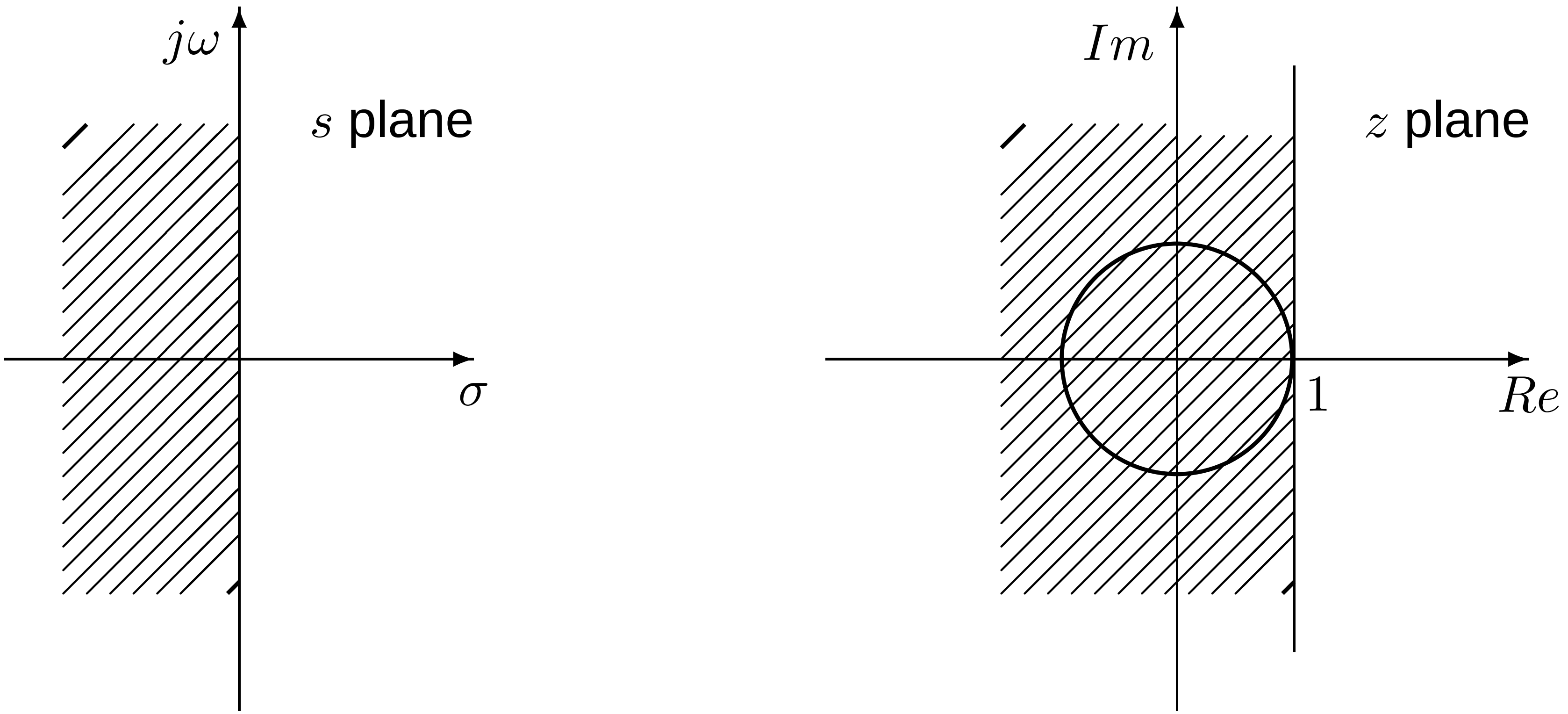
$$\int_{(k-1)T}^{kT} y(t)dt \approx Ty((k-1)T), \qquad \int_{(k-1)T}^{kT} x(t)dt \approx Tx((k-1)T)$$

$$y(kT) = y((k-1)T) - aT\left[y((k-1)T) - x((k-1)T)\right]$$

$$\frac{Y(z)}{X(z)} = G(z) = \frac{aTz^{-1}}{1-(1-aT)z^{-1}} = \frac{a}{\frac{1-z^{-1}}{Tz^{-1}} + a}$$

# Foreward difference method

$$Re(s) = Re\left(\frac{z-1}{T}\right) < 0 \qquad \rightarrow \qquad Re(z) < 1$$




Stable controllers $D(s)$ may be mapped in discrete time **unstable** controllers $D(z)$!!

# Bilinear transformation

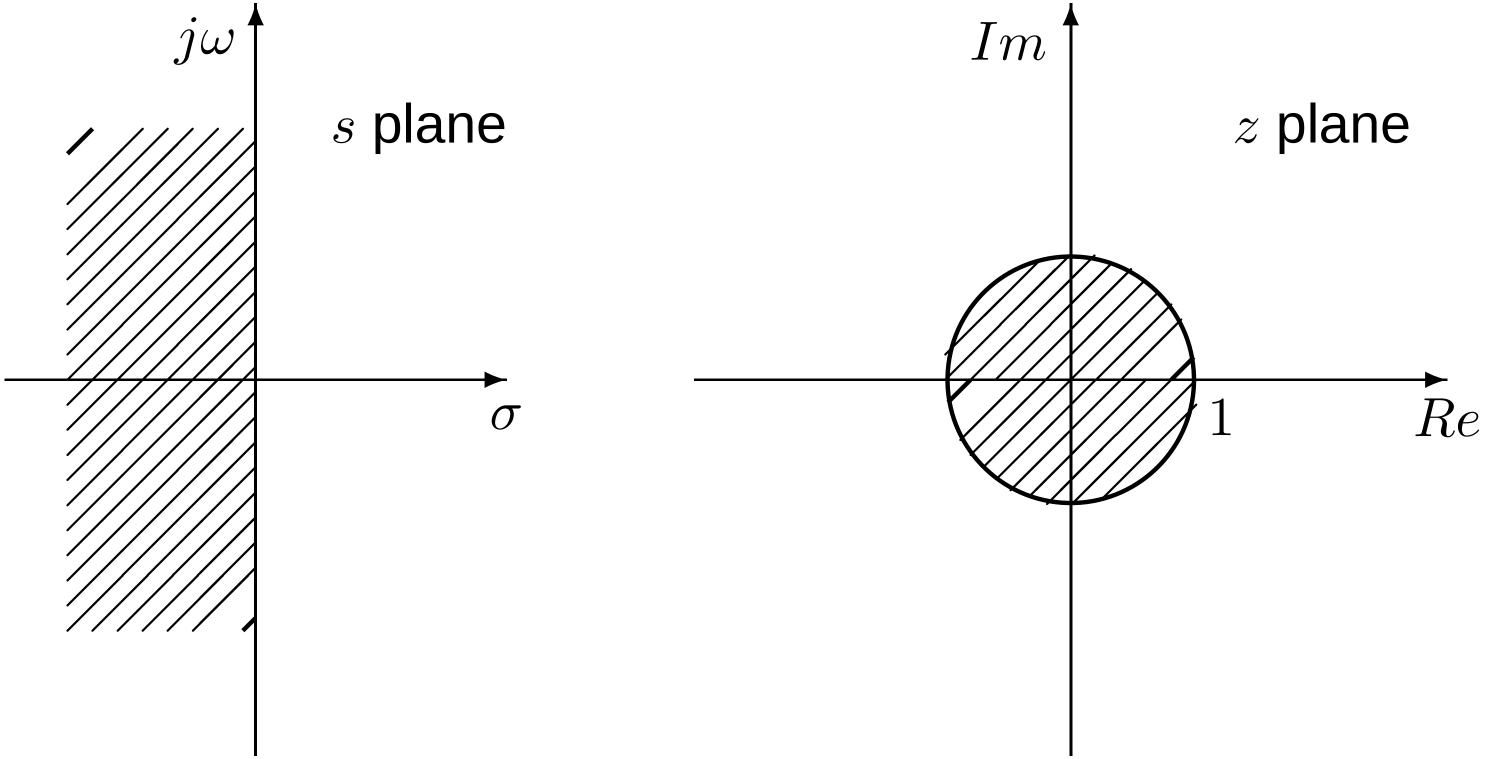
$$D(z) = D(s)|_{s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

also called **trapezoidal integration** (or Tustin's transformation)

$$\int_{(k-1)T}^{kT} y(t)dt \quad \approx \quad \frac{[y(kT) + y((k-1)T)]T}{2}$$

$$\int_{(k-1)T}^{kT} x(t)dt \quad \approx \quad \frac{[x(kT) + x((k-1)T)]T}{2}$$

# Bilinear transformation



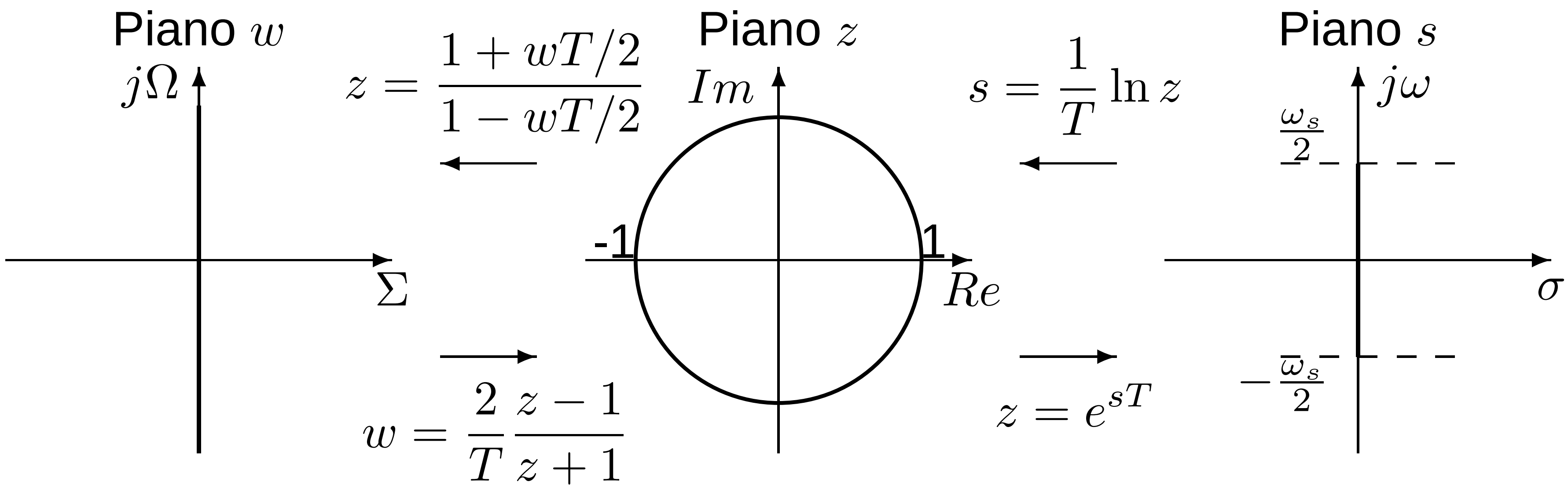

$$Re\left(\frac{z-1}{z+1}\right) < 0$$

$$Re\left(\frac{\sigma + j\omega - 1}{\sigma + j\omega + 1}\right) = Re\left[\frac{\sigma^2 - 1 + \omega^2 + j2\omega}{(\sigma+1)^2 + \omega^2}\right] < 0$$

$$\sigma^2 + \omega^2 < 1$$

# Bilinear transformation

Frequential relation between $w$ plane, $z$ plane and $s$ plane




The transformation does not generate frequential overlapping but introduce distortions!!

$$j\Omega = \frac{2}{T}\frac{1-e^{-j\omega T}}{1+e^{-j\omega T}} = \frac{2}{T}\frac{e^{j\omega T/2}-e^{-j\omega T/2}}{e^{j\omega T/2}+e^{-j\omega T/2}}$$

$$= \frac{2}{T}\frac{2j\sin\omega T/2}{2\cos\omega T/2} = j\frac{2}{T}\tan\frac{\omega T}{2}$$

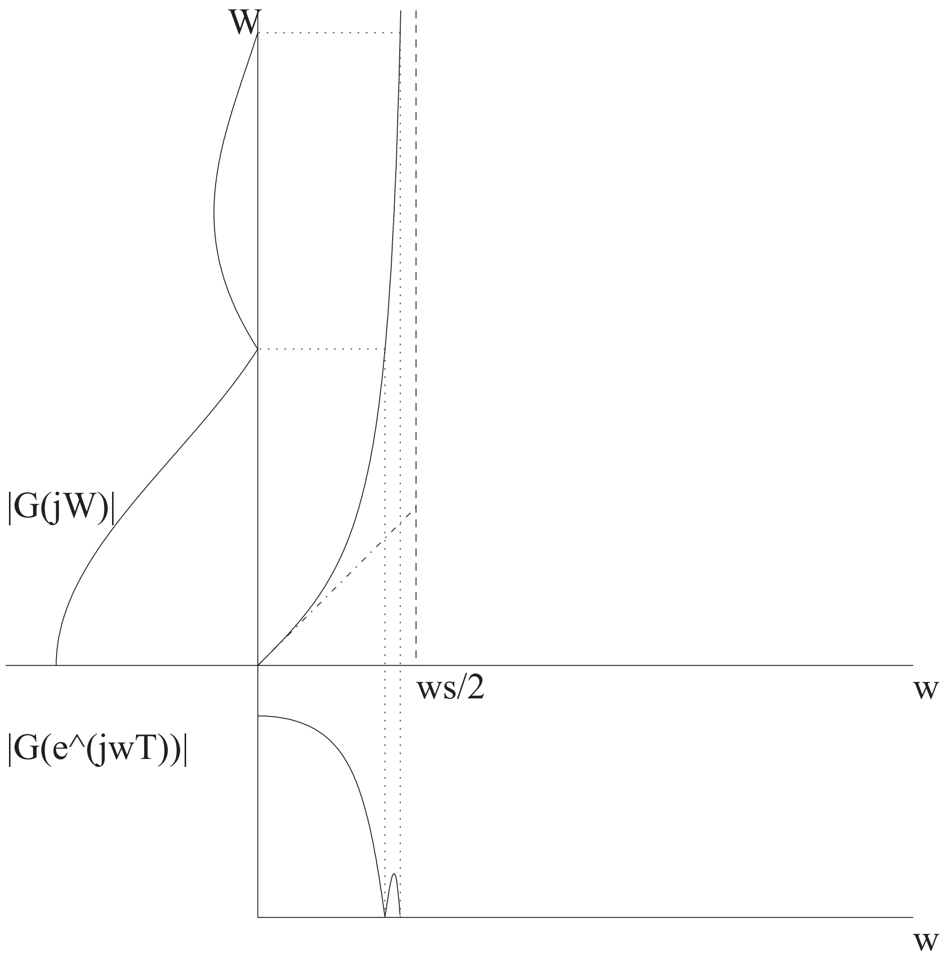
# Bilinear transformation

$$j\Omega = j\frac{2}{T}\tan\frac{\omega T}{2}$$

$$D_c(j\Omega) = D_d(e^{j\omega T})$$

for

$$\Omega = \frac{2}{T}\tan\frac{\omega T}{2}$$



# Bilinear transformation with prewarping

$$s = \frac{\omega_1}{\tan \frac{\omega_1 T}{2}} \frac{1 - z^{-1}}{1 + z^{-1}} = \frac{\omega_1}{\tan \frac{\omega_1 T}{2}} \frac{z - 1}{z + 1}$$

For $\Omega = \omega_1$ we have $\omega = \omega_1$

Example

$$G(s) = \frac{a}{s + a}$$

Prewarping at $\omega = a$

$$s = \frac{a}{\tan \frac{aT}{2}} \frac{1 - z^{-1}}{1 + z^{-1}}$$

$$G_d(z) = \frac{\tan \frac{aT}{2}(1 + z^{-1})}{(\tan \frac{aT}{2} - 1)z^{-1} + (\tan \frac{aT}{2} + 1)}$$

# Bilinear transformation with prewarping

Example

Design a discrete time low pass filter that approximate the frequential behavior for $\omega \in [0,\ 10] rad/s$ del filtro analogico

$$G(s) = \frac{10}{s+10} \qquad \text{with} \qquad T = 0.2\ s$$

$$G_d(z) = \frac{10}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} + 10} = \frac{1+z^{-1}}{2}$$

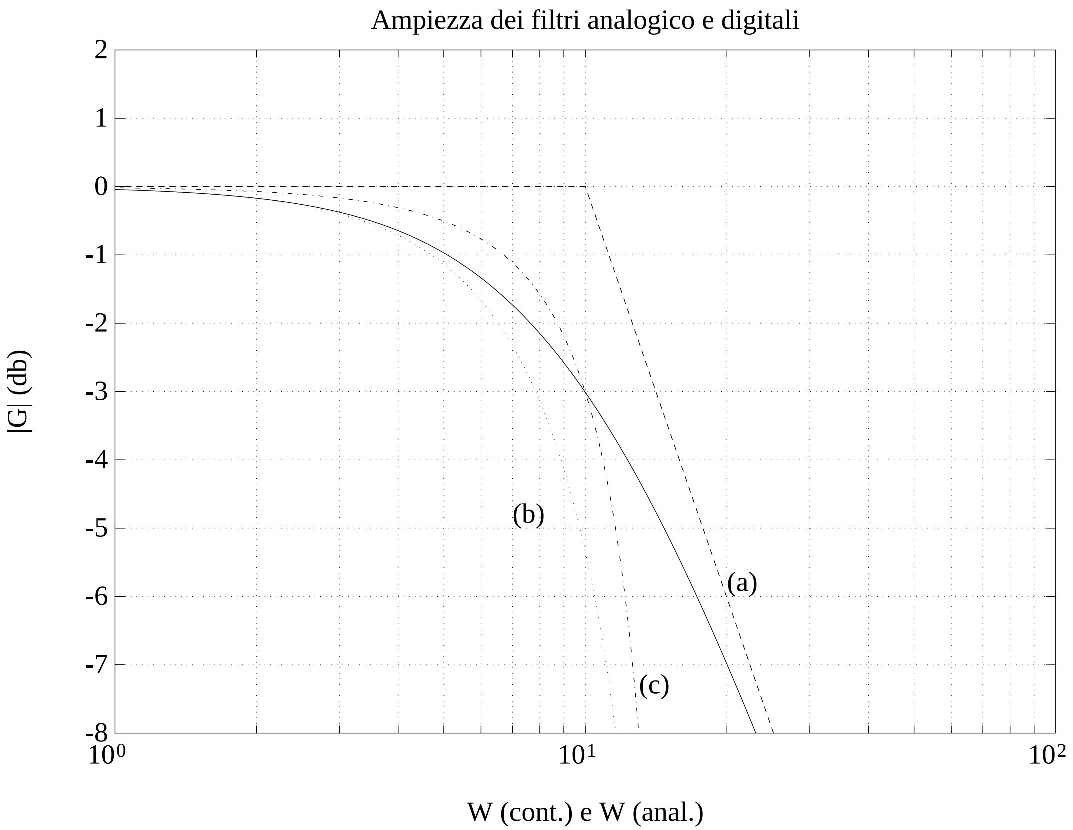
$$G_d(e^{j\omega T}) = \frac{10}{j\frac{2}{T}\tan\frac{\omega T}{2} + 10} = \frac{1}{j\tan 0.1\omega + 1}$$

# Bilinear transformation with prewarping

Example (cont'd)

Using prewarping at $\omega = 10\ rad/s$, we get

$$G_d(z) = \frac{10}{\frac{10}{\tan\frac{10T}{2}}\frac{1-z^{-1}}{1+z^{-1}} + 10} = \frac{0.609(1+z^{-1})}{1+0.218z^{-1}}$$



# Zeros/Poles matching

Write $D(s)$ enlightening poles and zeros factors.
Transform each pole and zero as

$$(s+a) \quad \rightarrow \quad (1 - e^{-aT}z^{-1})$$

$$(s+a \pm jb) \quad \rightarrow \quad (1 - 2e^{-aT}\cos bTz^{-1} + e^{-2aT}z^{-2})$$

Introduce as many zeros in $z = -1$ as the relative degree
Adjust the low frequencies gain ($z = 1$) or the high frequency gain ($z = -1$)

Example

$$D(s) = \frac{s+b}{s+a}$$

$$D(z) = k\frac{z - e^{-bT}}{z - e^{-aT}}$$

$$D(z=1) = k\frac{1 - e^{-bT}}{1 - e^{-aT}} = D(s=0) = \frac{b}{a} \qquad \left(k = \frac{b}{a}\frac{1 - e^{-aT}}{1 - e^{-bT}}\right)$$

# Zeros/Poles matching

Example: High pass filter

$$D(s) = \frac{s}{s+a}$$

$$D(z) = k\frac{z-1}{z-e^{-aT}} \qquad k = \frac{1+e^{-aT}}{2}$$

Example:

$$D(s) = \frac{1}{(s+a)^2+b^2} = \frac{1}{(s+a+jb)(s+a-jb)}$$

$\Rightarrow$ relative degree equal to 2

$$D(z) = k\frac{(z+1)^2}{z^2-2ze^{-aT}\cos bT+e^{-2aT}}$$

$$k = \frac{1-2e^{-aT}\cos bT+e^{-2aT}}{4(a^2+b^2)}$$

# Discretization design example

Plant:

$$G(s) = \frac{1}{s(s+2)}$$

Feedback specification: $\delta = 0.5$ ($S = 16.3\%$) e $T_a \leq 2\ s$ (al $2\%$)

$$\frac{4}{\delta\omega_n} = 2 \qquad \Rightarrow \qquad \omega_n = 4\ rad/s$$

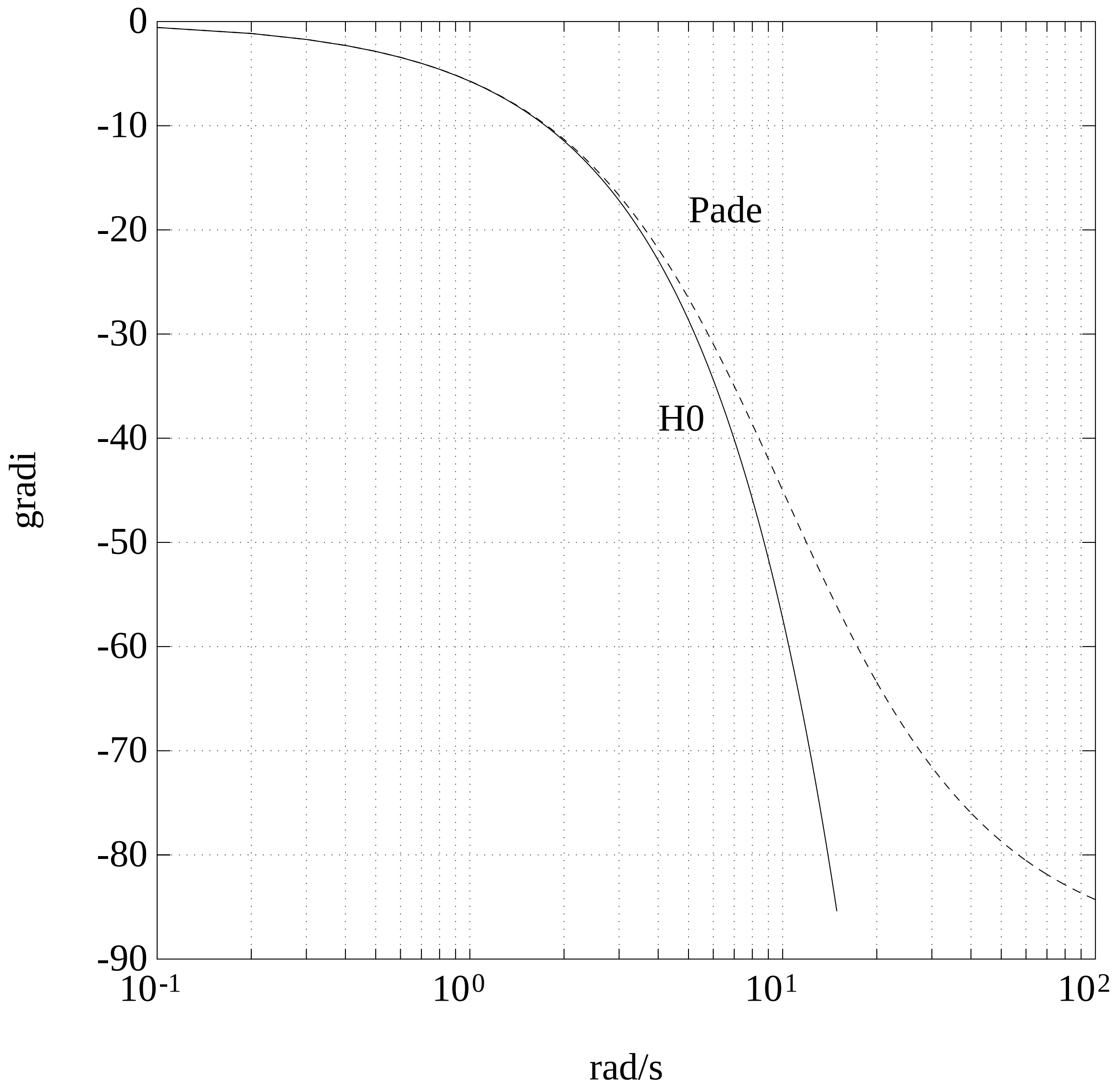
Sampling time $T$:

- damped oscillations with period $2\pi/(\omega_n\sqrt{1-\delta^2}) = 1.814\ s$
- we want 8-10 samples per period
- $T = 0.2\,s$

Effect of the zero order hold

$$H_0(s) = \frac{1-e^{-sT}}{s} \quad \approx \quad G_h(s) = \frac{1}{Ts/2+1} = \frac{10}{s+10}$$

# Discretization design example (cont'd)



# Discretization design example (cont'd)

An analog controller $D(s)$ that meets the specifications for the modified plant $G_m$ is

$$G_m(s) = G_h(s)G(s) \qquad \Rightarrow \qquad D(s) = 20.25\frac{s+2}{s+6.667}$$

Being $G_a(s) = D(s)G_h(s)G(s)$, the feedback transfer function

$$G_0(s) = \frac{G_a(s)}{1+G_a(s)} = \frac{202.5}{s^3 + 16.667s^2 + 66.67s + 202.5}$$

has poles in $s = -12.665$ e in $s = -2 \pm j3.462$ ($\delta = 0.5$, $\delta\omega_n = 2 \Rightarrow T_a = 2\ s$)
Discretizing the controller using the zeros/poles matching method:

$$z = e^{-6.667T} = 0.2644, \qquad z = e^{-2T} = 0.6703$$

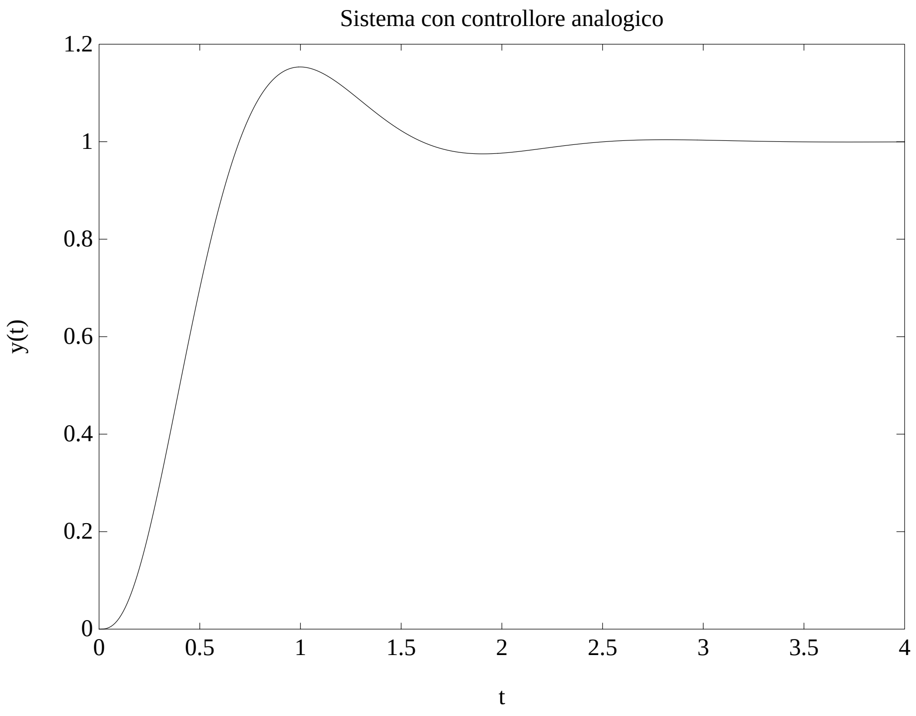
$$D(z) = k\frac{z-0.6703}{z-0.2644}$$

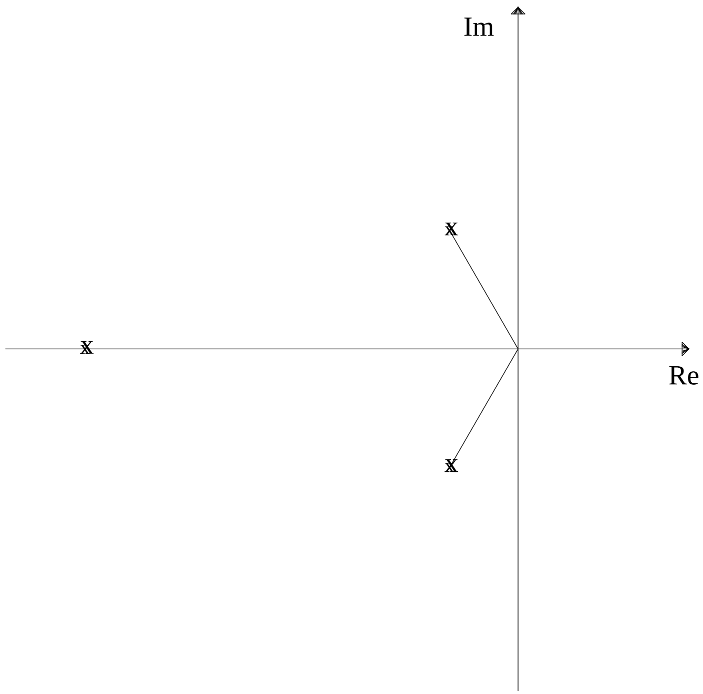
$$D(z=1) = D(s=0) \qquad \Rightarrow \qquad k = 13.57$$

$$D(z) = 13.57\frac{z-0.6703}{z-0.2644}$$

# Discretization design example (cont'd)

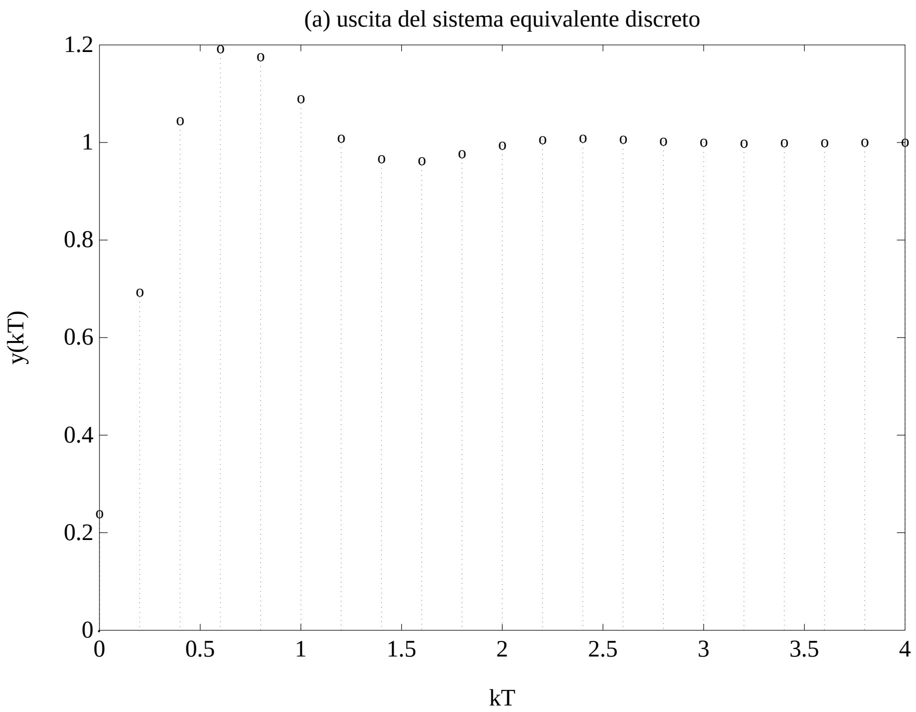
Step response of the feedback continuous time system (left) and feedback poles position (right)

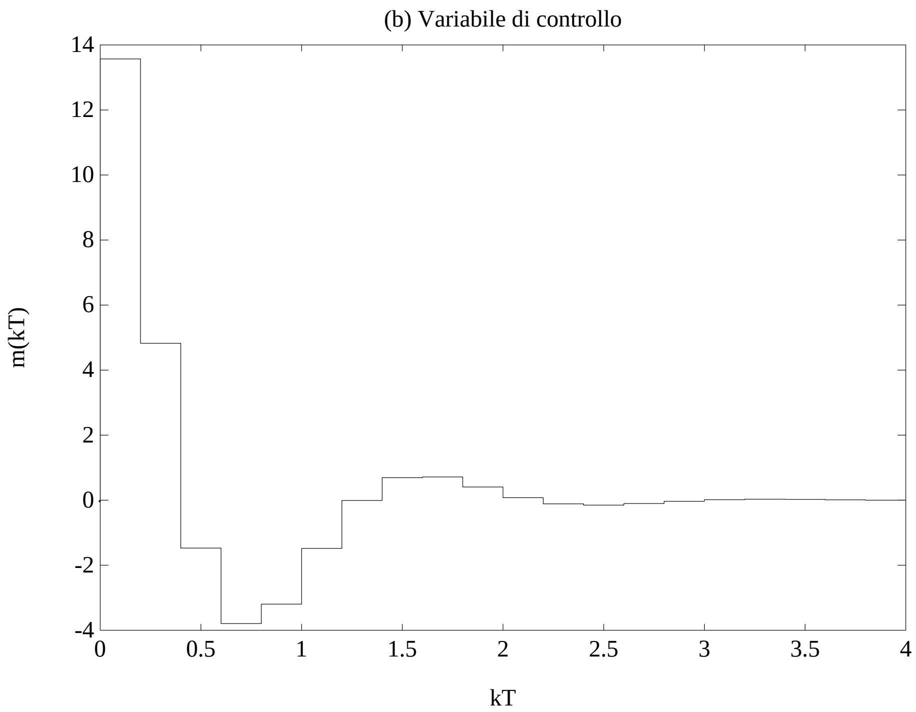


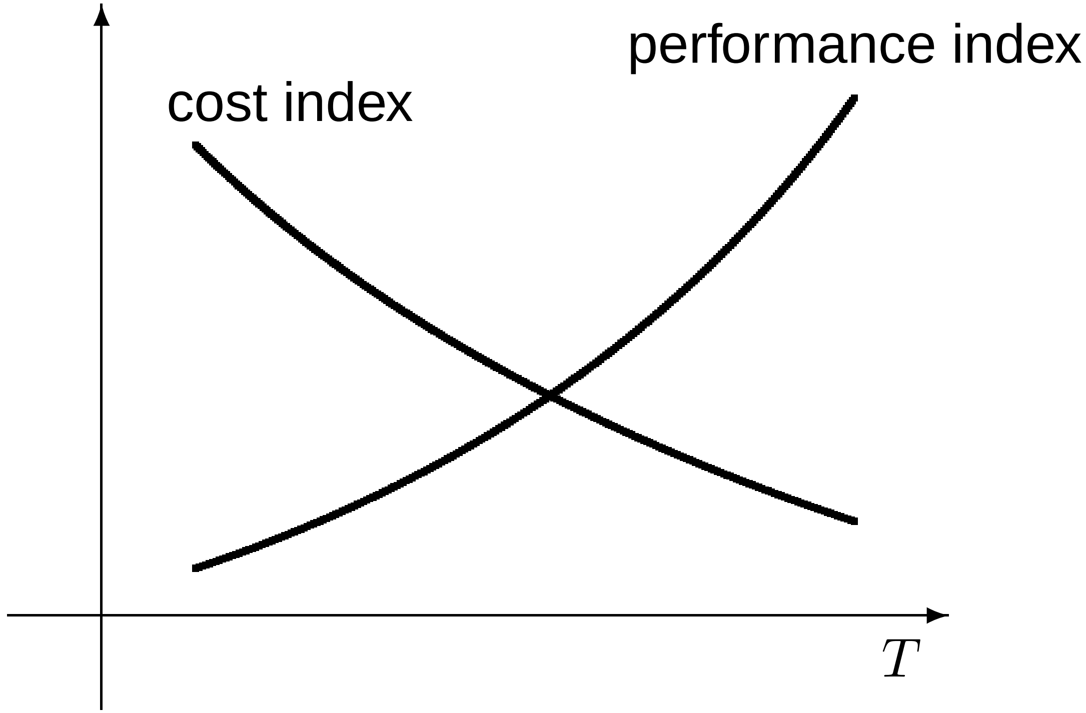
# Discretization design example (cont’d)

Step response of the feedback system with discrete time controller (a) and corresponding control action (b)



# *How to choose sample time*




Performances

- disturbance rejection
- set-point tracking
- control effort
- delays and stability
- robustness

Costs

- computational burden
- speed of computation
- precision

# How to choose sample time

The effects of $T$ on performances are:

- effects of destabilization grow when $T$ grows;
- information loss grows when $T$ grows;
- discretization accuracy grows when $T$ decrease;

The best choice is the higher value of $T$ that guarantees good performances in terms of:

1) Loss of information: $\omega_s > 2\omega_b$
2) Smooth dynamics without delays: $$6 < \frac{\omega_s}{\omega_b} < 20$$
3) Disturbance rejection efficacy: $\omega_s > 2\omega_r$
4) Antialiasing filter efficacy: $$\frac{\omega_s}{\omega_b} \geq 20$$

# How to choose sample time

Some practical rules:

a)

$$T \leq \frac{\tau_{dom}}{10}$$

b)

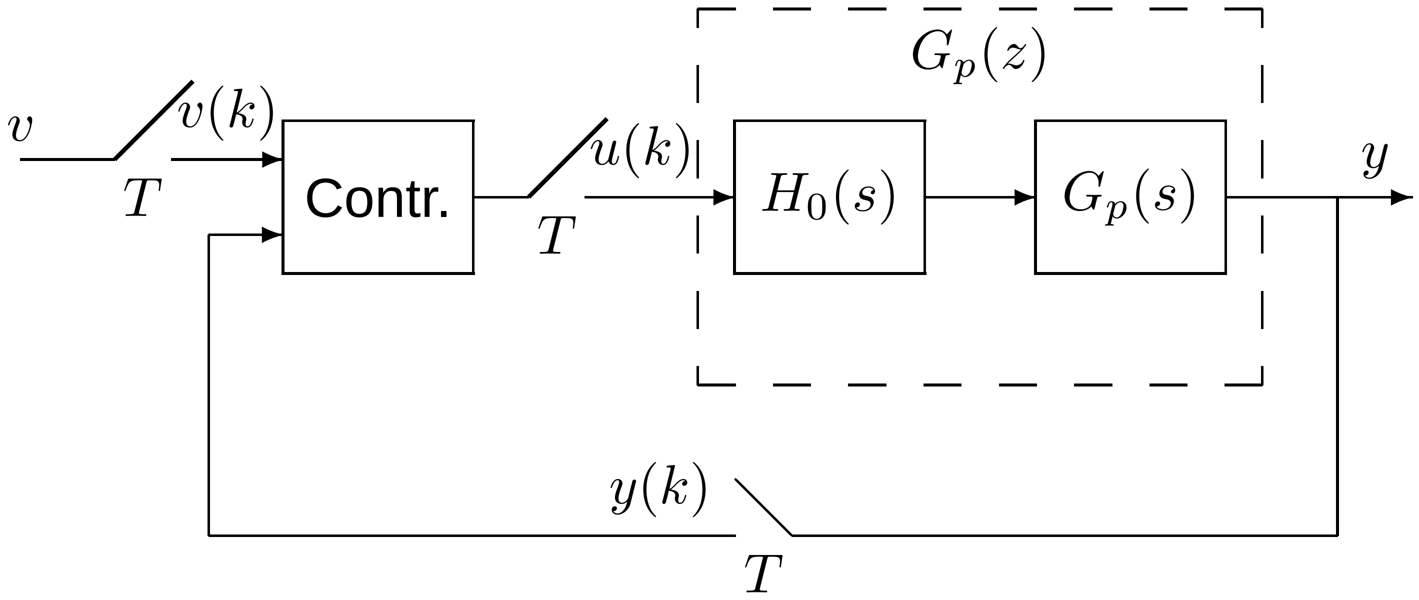
$$T \leq \frac{\theta}{4}$$

c)

$$T < \frac{T_a}{10} \qquad \omega_s > 10\ \omega_n$$

# Analytical Design of Digital Controllers

# Analytical design: poles and zeros assignment




$$G_p(z) = \frac{Y(z)}{U(z)} = \frac{B(z)}{A(z)}$$

where $B(z)$ and $A(z)$ have no common factors and have degree equal to $m$ and $n$ respectively with $n \geq m$

Controller:

$$R(z)\,U(z) = T(z)\,V(z) - S(z)\,Y(z)$$

# Analytical design: poles and zeros assignment

Control action combine a feedforward action

$$H_{ff}(z) = \frac{T(z)}{R(z)}$$

and a feedback action

$$H_{fb}(z) = \frac{S(z)}{R(z)}$$

Causality implies that

$$grado(R) \geq grado(T), \qquad grado(R) \geq grado(S)$$

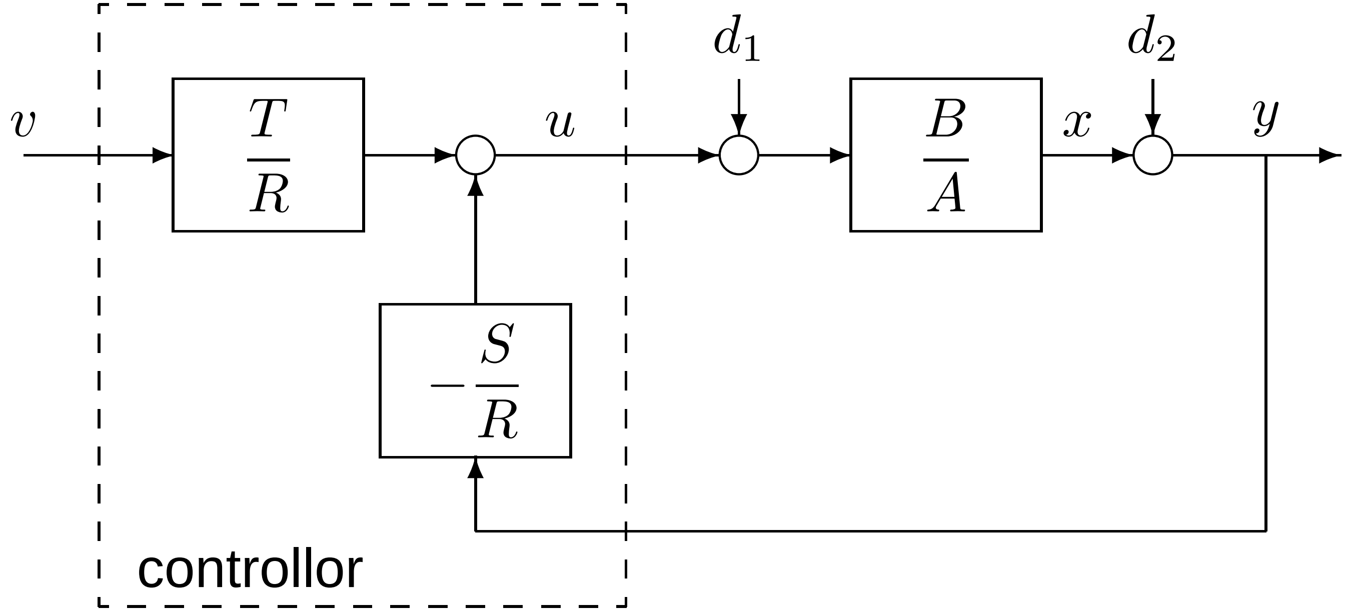
In pratice

$$grado(R) = grado(T) = grado(S)$$

or

$$grado(R) = 1 + grado(T) = 1 + grado(S)$$

# Analytical design: poles and zeros assignment




Closed loop system: $$\frac{Y(z)}{V(z)} = \frac{B\,T}{A\,R + B\,S}$$

Specs: $$G_m(z) = \frac{B_m(z)}{A_m(z)}$$

Design equation: $$\frac{B\,T}{A\,R + B\,S} = \frac{B_m}{A_m}$$ where

$$grado(A_m) - grado(B_m) \geq grado(A) - grado(B)$$

# Analytical design: poles and zeros assignment

Considering internal stable dynamics (“observer dynamics”) we define

$$G_m(z) = \frac{A_0(z)\,B_m(z)}{A_0(z)\,A_m(z)}$$

In order to have small errors for low frequency disturbances, the gain function

$$\left.\frac{B(z)\,S(z)}{A(z)\,R(z)}\right|_{z=e^{j\omega T}} \quad \text{must be high for } \omega \to 0.$$

We can use integral actions: $R(z) = (z-1)^q\,R_1(z)$

**Problem**: design $R$, $S$ and $T$
The cancellation among zeros of $B$ and zeros of $A\,R + B\,S$, i.e., poles of the closed loop system, must be limited to stable zeros

$$B = B^+B^-$$

$$B_m = B^-B'_m$$

# Analytical design: poles and zeros assignment

The presence of non minimum phase zeros in $G_p(z)$ depends on $T$. In fact for $T \to 0$

$$G_p(z) = \mathcal{Z}\left[\frac{1-e^{-Ts}}{s} G(s)\right] \simeq \mathcal{Z}\left[\frac{1-e^{-Ts}}{s} \, \frac{1}{s^l}\right]$$

For $l \geq 3$ we always have unstable zeros. If $l = 3$ (and for $T \to 0$)

$$G_p(z) \simeq \frac{T^3}{3!} \frac{z^{-1}(1+4\,z^{-1}+z^{-2})}{(1-z^{-1})^3}$$

with a zero in $z = -3.73$.

The stable factor $B^+$ can be canceled by chosing $R = B^+ R'$
Rewriting the design equation:

$$\frac{B^+ B^- T}{B^+(A\,R' + B^- S)} = \frac{B^- B'_m}{A_m} \quad \text{which is}$$

$$\frac{T}{A\,R' + B^- S} = \frac{B'_m}{A_m}$$

# Analytical design: poles and zeros assignment

Considering also the observer dynamics $A_0$, the two design equations became

$$A\,R' + B^- S = A_0 A_m$$

$$T = A_0 B'_m$$

The characteristic equation of the feedback loop is

$$A\,R + B\,S = B^+ A_0 A_m$$

whose roots are

- stable zeros of the plant $(B^+)$
- specification poles $(A_m)$
- poles of the "observer dynamics" $(A_0)$

# The Diophantine equation

$$A\,X + B\,Y = C$$

Necessary and sufficient condition for the existence of a solution $(X, Y)$ is that the maximum common divider of $A$ and $B$ is a factor of $C$.
This is satisfied if $A$ and $B$ do not have common factors.
If $(X_0, Y_0)$ then exist infinite solutions

$$\begin{aligned} X &= X_0 + Q\,B \\ Y &= Y_0 - Q\,A \end{aligned}$$

Example

$$3\,x + 4\,y = 7$$

with $x$ and $y$ integers
Particular solution: $x_0 = y_0 = 1$
General solution ($n$ integer number):

$$\begin{cases} x = x_0 + 4\,n \\ y = y_0 - 3\,n \end{cases}$$

# The Diophantine equation

Exists a unique solution if

$$degree(X) < degree(B)$$

or

$$degree(Y) < degree(A)$$

The solution of the Diophantine equation can be obtained by solving the linear equations system

$$\begin{aligned} A(z) &= z^m + a_1 z^{m-1} + a_2 z^{m-2} + \cdots + a_m \\ B(z) &= b_0 z^n + b_1 z^{n-1} + b_2 z^{n-2} + \cdots + b_n \\ C(z) &= c_0 z^p + c_1 z^{p-1} + c_2 z^{p-2} + \cdots + c_p \end{aligned}$$

For $degree(Y) = m - 1$ and $degree(X) = p - m$

$$\begin{aligned} Y(z) &= y_0 z^{m-1} + y_1 z^{m-2} + \cdots + y_{m-1} \\ X(z) &= x_0 z^{p-m} + x_1 z^{p-m-1} + \cdots + x_{p-m} \end{aligned}$$

the system is squared and with a degree equal to $p + 1$

# The Diophantine equation

$$
\begin{bmatrix}
1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
a_1 & 1 & \ddots & \vdots & b_0 & 0 & \ddots & \vdots \\
a_2 & a_1 & \ddots & 0 & b_1 & b_0 & \ddots & 0 \\
\vdots & \vdots & \ddots & 1 & \vdots & \vdots & \ddots & 0 \\
a_m & \vdots & & a_1 & b_n & \vdots & & b_0 \\
0 & a_m & & \vdots & 0 & b_n & & \vdots \\
\vdots & \ddots & \ddots & & \vdots & \ddots & \ddots & \\
0 & \dots & 0 & a_m & 0 & \dots & 0 & b_n
\end{bmatrix}
\begin{bmatrix}
x_0 \\ x_1 \\ \vdots \\ x_{p-m} \\ y_0 \\ y_1 \\ \vdots \\ y_{m-1}
\end{bmatrix}
=
\begin{bmatrix}
c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{p-2} \\ c_{p-1} \\ c_p
\end{bmatrix}
$$

# The Diophantine equation

Being $x_0 = c_0$, we have a reduced order system $p$

$$\begin{bmatrix} 1 & 0 & \dots & 0 & b_0 & 0 & \dots & 0 \\ a_1 & 1 & \ddots & \vdots & b_1 & b_0 & \ddots & \vdots \\ a_2 & a_1 & \ddots & 0 & b_2 & b_1 & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 & \vdots & \vdots & \ddots & b_0 \\ a_m & \vdots & & a_1 & b_n & \vdots & & b_1 \\ 0 & a_m & & \vdots & 0 & b_n & & \vdots \\ \vdots & \ddots & \ddots & & \vdots & \ddots & \ddots & \\ 0 & \dots & 0 & a_m & 0 & \dots & 0 & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{p-m} \\ y_0 \\ y_1 \\ \vdots \\ y_{m-1} \end{bmatrix} = \begin{bmatrix} c_1 - c_0 a_1 \\ c_2 - c_0 a_2 \\ \vdots \\ c_m - c_0 a_m \\ c_{m+1} \\ \vdots \\ c_p \end{bmatrix}$$

$\Rightarrow$ Sylvester's matrix

In our application, in order to have a unique solution $degree(S) = degree(A) - 1$ and for causality

$$degree(A_m) - degree(B_m) \geq degree(A) - degree(B)$$

$$degree(A_0) \geq 2\, degree(A) - degree(A_m) - degree(B^+) - 1$$

# Poles and zeros assignment: design procedure

1. Inputs $G_p = B/A$, $A_0$ and $G_m = B_m/A_m$
2. Decompose $B$

$$B = B^- B^+ \qquad\qquad B_m = B^- B'_m$$

   where $B^+$ ismonic
3. Solve

$$(z-1)^q A\, R'_1 + B^- \, S = A_0 A_m$$

   with

$$grado(S) = grado(A) + q - 1$$

$$grado(R'_1) = grado(A_0) + grado(A_m) - grado(A) - q$$

4. write the control law

$$R\, u = T\, v - S\, y \qquad\qquad \text{con}$$

$$R = B^+ R', \qquad T = B'_m A_0, \qquad R' = (z-1)^q R'_1$$

# Poles and zeros assignment: design procedure

Chose $G_m = B_m/A_m$ as

$$G_m(z) = \frac{Q(1)\,B^-(z)}{B^-(1)\,z^k Q(z)}$$

where: $Q(z) = z^2 + p_1 z + p_2$

$$\begin{aligned} p_1 &= -2e^{-\delta\omega_n T}\cos(\omega_n T\sqrt{1-\delta^2}) \\ p_2 &= e^{-2\delta\omega_n T} \end{aligned}$$

or: $Q(z) = z - a$ $\qquad a = e^{-T/\tau}$

# Poles and zeros assignment: example

Example: $$G_p(s) = \frac{1}{s(s+1)}$$

$$G_p(z) = \mathcal{Z}\left[\frac{1-e^{-sT}}{s}\frac{1}{s(s+1)}\right] = \frac{K(z-b)}{(z-1)(z-a)}$$

where

$$a = e^{-T}, \qquad K = a + T - 1, \qquad b = 1 - \frac{T(1-a)}{K}$$

Feedback system specification

$$G_m(z) = \frac{z(1+p_1+p_2)}{z^2+p_1 z+p_2}$$

$G_p(z)$ has a zero in $z = b$ which is not present in $G_m(z)$ hence

$$B = B^+B^-, \qquad B^+ = z - b, \qquad B^- = K$$

# Poles and zeros assignment: example

Example (cont'd) It must hold :

- $B'_m = \dfrac{B_m}{K} = \dfrac{z(1+p_1+p_2)}{K}$
- $degree(A_0) \geq 0$ and we choose $A_0 = 1$
- $degree(R') = degree(A_0) + degree(A_m) - degree(A) = 0$
- $degree(S) = degree(A) - 1 = 1$

hence $R' = r_0$ and $S = (s_0 z + s_1)$
The design equation is

$$(z-1)(z-a)\,r_0 + K\,(s_0 z + s_1) = z^2 + p_1 z + p_2$$

from which

$$r_0 = 1, \qquad s_0 = \frac{1+a+p_1}{K}, \qquad s_1 = \frac{p_2 - a}{K}$$

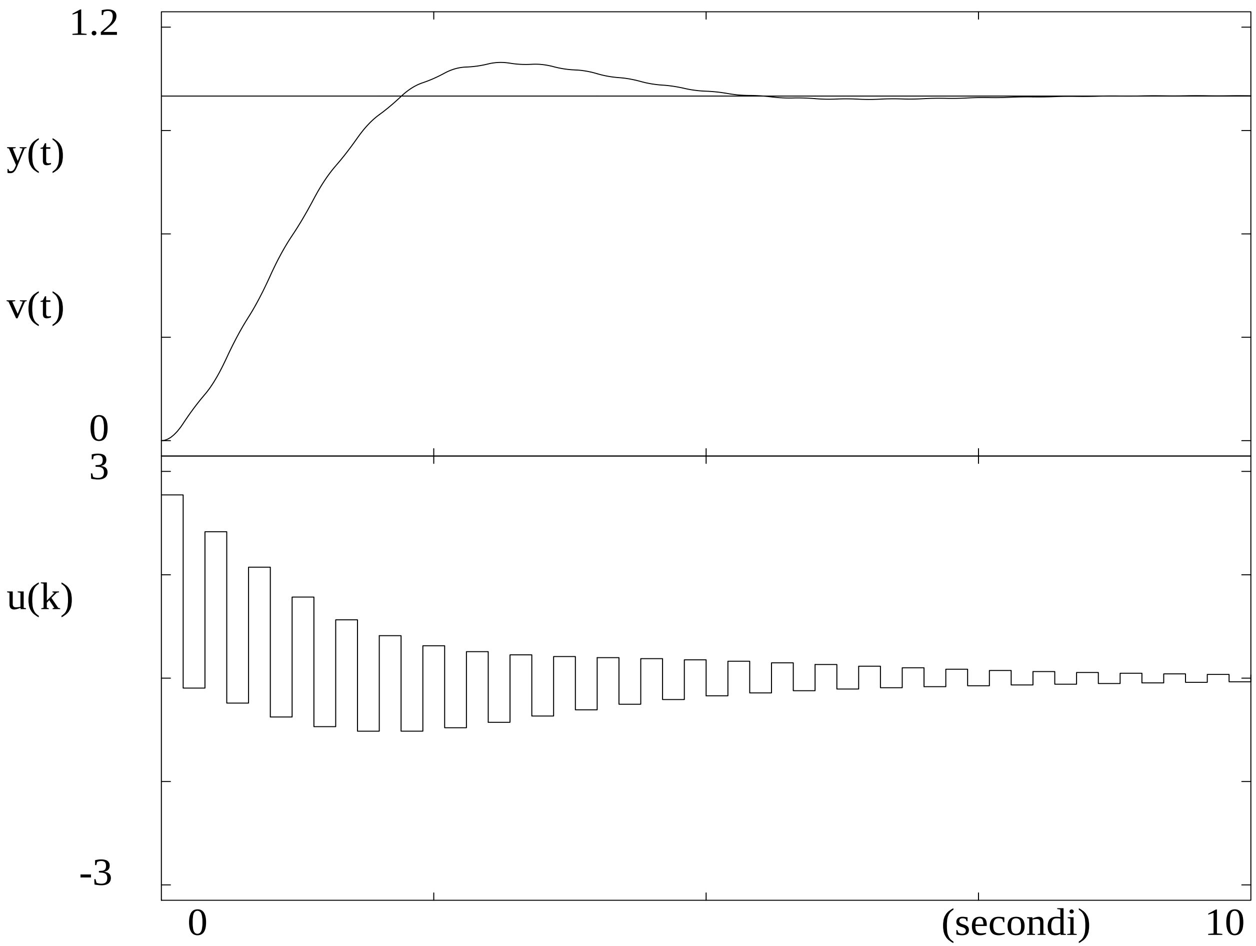
moreover

$$T(z) = A_0 B'_m = \frac{z(1+p_1+p_2)}{K} = t_0 z$$

# Poles and zeros assignment: example

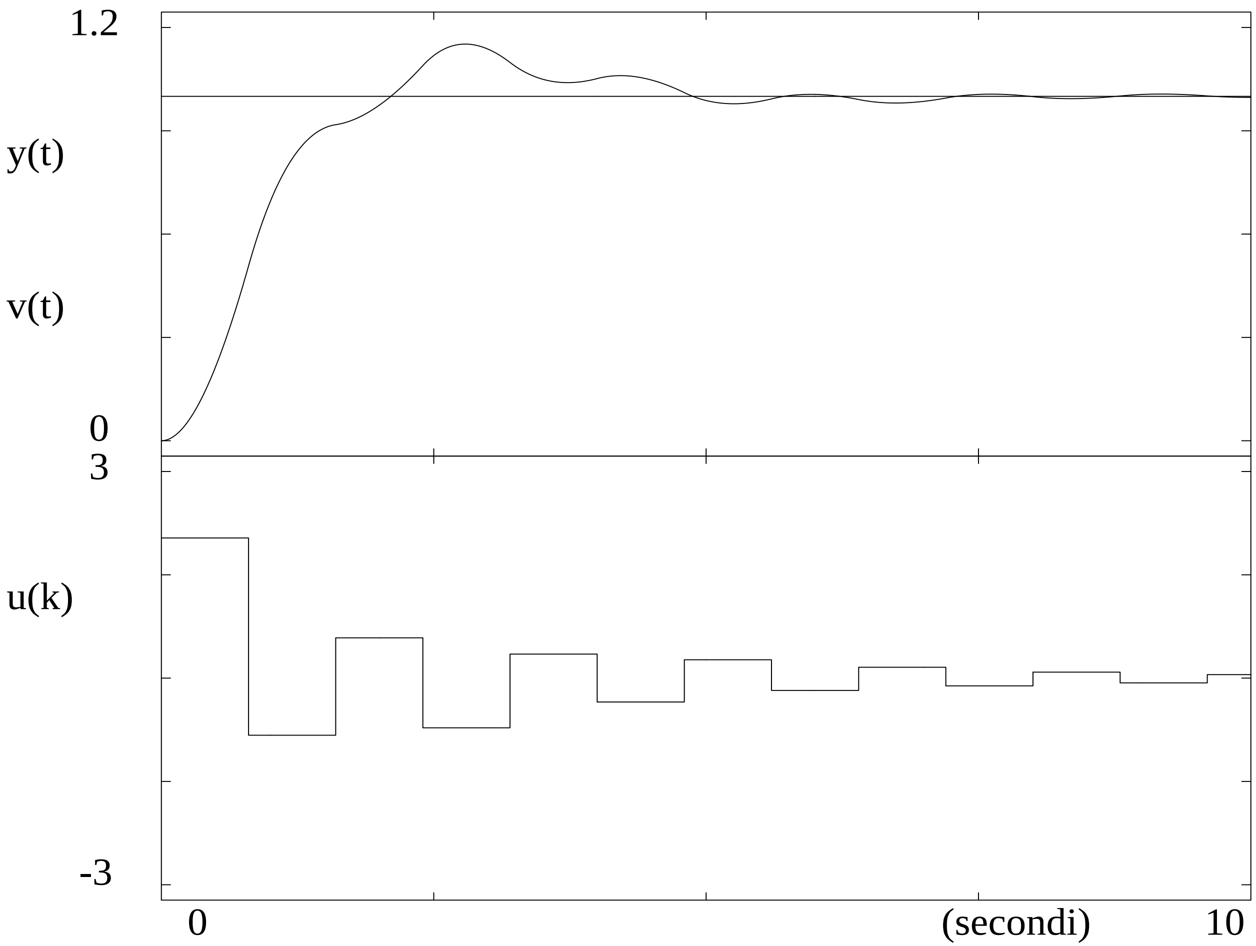
Example (cont'd) The control law $Ru = Tv - Sy$ is

$$u(k) = bu(k-1) + t_0 v(k) - s_0 y(k) - s_1 y(k-1)$$




Output $y(t)$ and control action $u(k)$ in case $\delta = 0.6$, $\omega_n = 1.2$ and $T = 0.2$

# Poles and zeros assignment: example




Output $y(t)$ and control action $u(k)$ in case $\delta = 0.6$, $\omega_n = 1.2$ and $T = 0.8$

# Poles and zeros assignment: example

Example (cont'd) To eliminate the "ringing" problem, let's modify $G_m(z)$ as

$$G_m(z) = \frac{1+p_1+p_2}{1-b}\,\frac{z-b}{z^2+p_1z+p_2}$$

from which: $B^+ = 1,$ $B^- = K(z-b)$

$$B'_m = \frac{1+p_1+p_2}{K(1-b)}$$

Since

$$degree(A_0) \geq 2degree(A) - degree(A_m) - degree(B^+) - 1 = 1$$

we chose $A_0(z) = z$. Moreover:

$$\begin{aligned} degree(R) &= degree(A_m) + degree(A_0) - degree(A_m) = 1 \\ degree(S) &= degree(A) - 1 = 1 \end{aligned}$$

# Poles and zeros assignment: example

Example (cont'd) The design equation is

$$(z-1)(z-a)(z+r_1)+K(z-b)(s_0z+s_1)=z^3+p_1z^2+p_2z$$

from which

$$r_1=-b+\frac{b(b^2+p_1b+p_2)}{(b-1)(b-a)}$$

$$\begin{aligned} K(1-b)(s_0+s_1) &= 1+p_1+p_2 \\ K(a-b)(s_0a+s_1) &= a^3+p_1a^2+p_2a \end{aligned}$$

Solving

$$s_0=\frac{\alpha_1-\alpha_2}{1-a} \qquad s_1=\frac{\alpha_2-\alpha_1 a}{1-a}$$

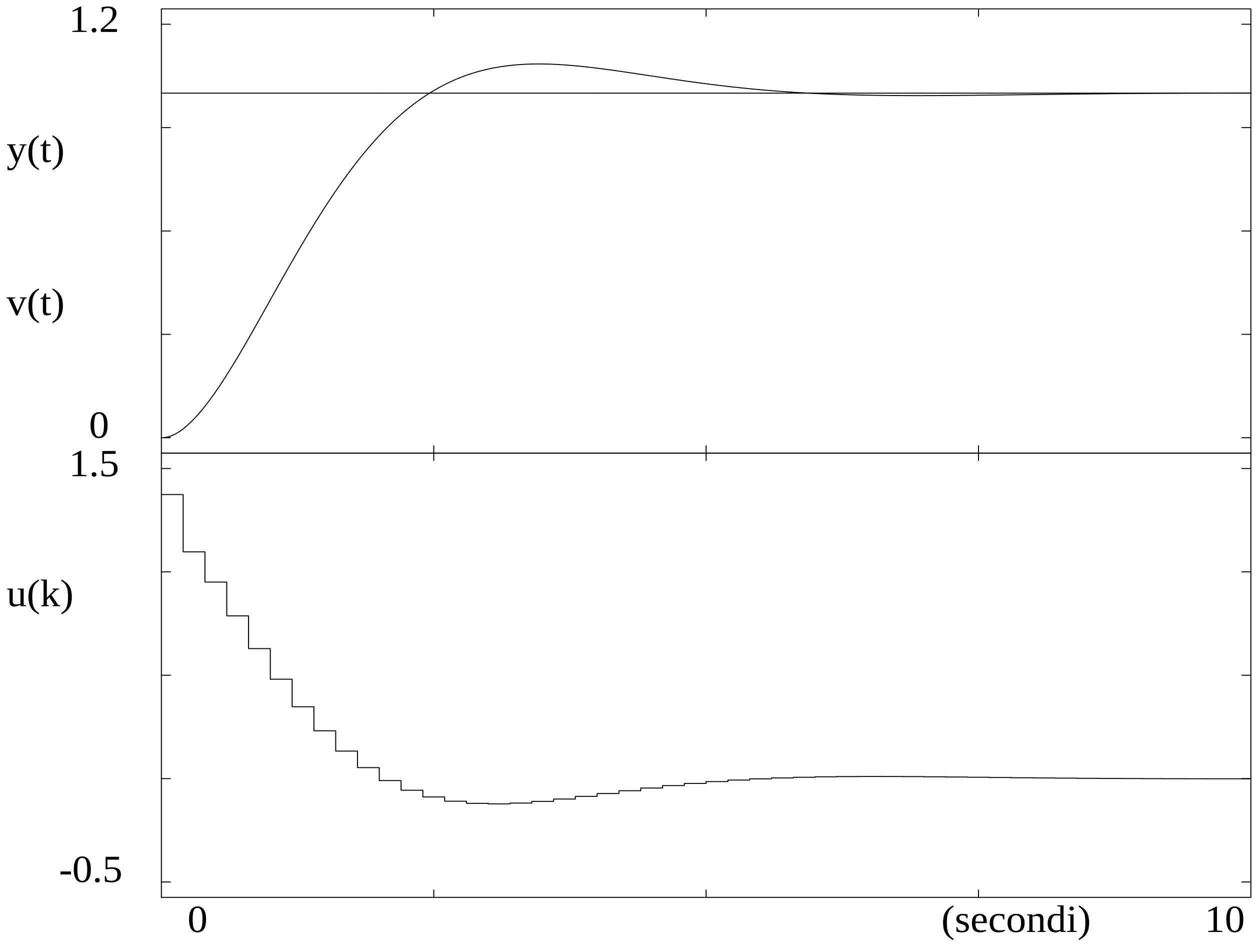
$$\alpha_1=\frac{1+p_1+p_2}{K(1-b)} \qquad \alpha_2=\frac{a^3+p_1a^2+p_2a}{K(a-b)}$$

Since: $T(z)=A_0B'_m=z\dfrac{1+p_1+p_2}{k(1-b)}=t_0z$, the resulting control law is:

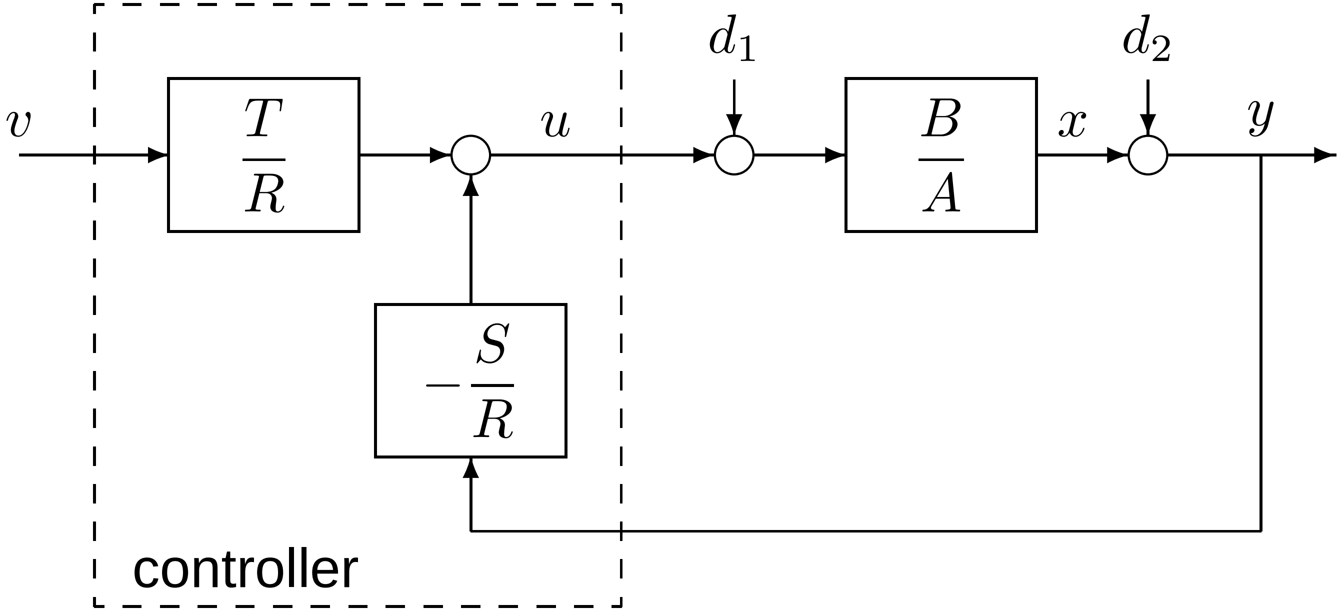
$$u(k)=-r_1u(k-1)+t_0v(k)-s_0y(k)-s_1y(k-1)$$

# Poles and zeros assignment: example

Example (cont'd) If $\delta = 0.6$, $\omega_n = 1.2$ and $T = 0.2$ we obtain



# Poles and zeros assignment: dealing with noises




$$
\begin{aligned}
x &= \frac{\frac{T}{R}\frac{B}{A}}{1+\frac{S}{R}\frac{B}{A}} v + \frac{\frac{B}{A}}{1+\frac{S}{R}\frac{B}{A}} d_1 - \frac{\frac{S}{R}\frac{B}{A}}{1+\frac{S}{R}\frac{B}{A}} d_2 \\
&= \frac{TB}{RA+BS} v + \frac{RB}{RA+BS} d_1 - \frac{SB}{RA+BS} d_2
\end{aligned}
$$

# Poles and zeros assignment: dealing with noises

Defining:

$$H_{fb} = \frac{S}{R} \qquad \text{feedback gain}$$

$$H_a = \frac{B}{A}\frac{S}{R} \qquad \text{loop gain}$$

we obtain

$$x = \frac{B_m}{A_m} v + \frac{H_a}{1+H_a}\frac{1}{H_{fb}} d_1 - \frac{H_a}{1+H_a} d_2$$

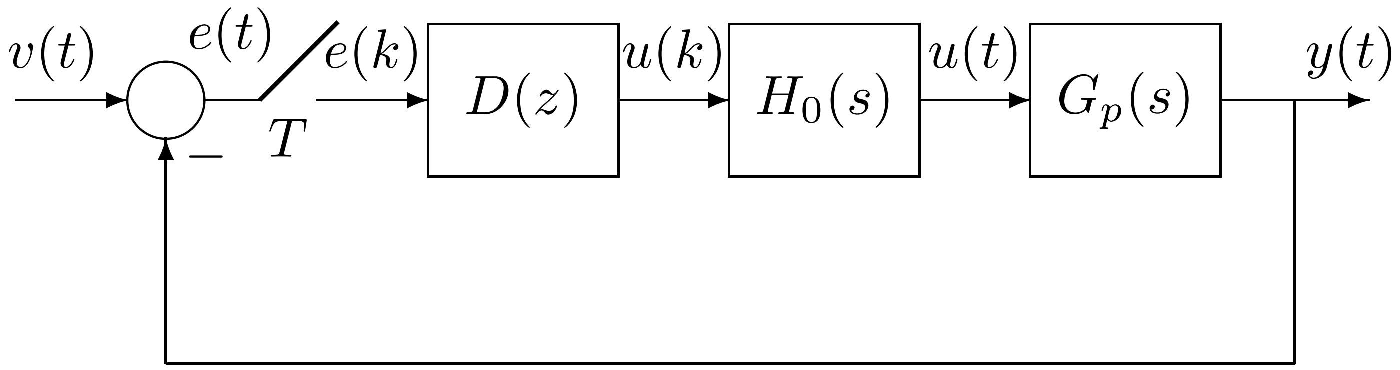
By substituting

$$\begin{aligned} x &= \frac{B_m}{A_m} v + \frac{R\,B}{B^+\,A_0\,A_m} d_1 - \frac{S\,B}{B^+\,A_0\,A_m} d_2 \\ &= \frac{B_m}{A_m} v + \frac{R\,B^-}{A_0\,A_m} d_1 - \frac{S B^-}{A_0\,A_m} d_2 \end{aligned}$$

# Analytical design: deadbeat controller

The controller has just one degree of freedom (feedback control)

$$\frac{U(z)}{E(z)} = \frac{S(z)}{R(z)} = D(z)$$




The deadbeat specifications, in case of step references, are

a) the output must reach its final value in minimum time

b) steady state error must be zero

c) no oscillations between samples

# Analytical design: deadbeat controller

To satisfy the deadbeat specs we impose

$$G_m(z) = \frac{a_0 z^N + a_1 z^{N-1} + \cdots + a_N}{z^N}$$

i.e.,

$$G_m(z) = a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}$$

with $N \geq n$, $n$ degree of the denominator of $G_p(z)$

From

$$\frac{D(z)G_p(z)}{1 + D(z)G_p(z)} = G_m(z)$$

we obtain

$$D(z) = \frac{G_m(z)}{G_p(z)[1 - G_m(z)]}$$

# Analytical design: deadbeat controller

Causality conditions:

1. $D(z)$ with positive relative degree
2. If $G_p(z)$ has a factor $z^{-k}$, $G_m(z)$ must have a factor $z^{-h}$ with $h \geq k$

Stability conditions:

1. All unstable poles of $G_p(z)$ must be zeros of$1 - G_m(z)$
2. All unstable zeros of $G_p(z)$ must be zeros of $G_m(z)$

We refer to reference signals

$$V(z) = \frac{P(z)}{(1 - z^{-1})^{q+1}}$$

- if $P(z) = 1$, $q = 0$ we have the unitary step
- if $P(z) = Tz^{-1}$, $q = 1$ we have the unitary ramp
- if $P(z) = \frac{1}{2}T^2 z^{-1}(1 + z^{-1})$, $q = 2$ we have the parable $v(t) = \frac{1}{2}t^2$

# Analytical design: deadbeat controller

Since

$$\begin{aligned} E(z) &= V(z) - Y(z) = V(z)\left[1 - G_m(z)\right] \\ &= \frac{P(z)\left[1 - G_m(z)\right]}{(1 - z^{-1})^{q+1}} \end{aligned}$$

the error goes to zero in finite time and remains null if

$$1 - G_m(z) = (1 - z^{-1})^{q+1} N(z)$$

which is

$$E(z) = P(z) N(z)$$

Hence the controller is given by

$$D(z) = \frac{G_m(z)}{G_p(z)(1 - z^{-1})^{q+1} N(z)}$$

# Analytical design: deadbeat controller

If $G_p(s)$ is stable, in order to avoid oscillations between samples ("ripple"), we aske for $t \geq nT$

$$y(t) = \text{const. for step}$$

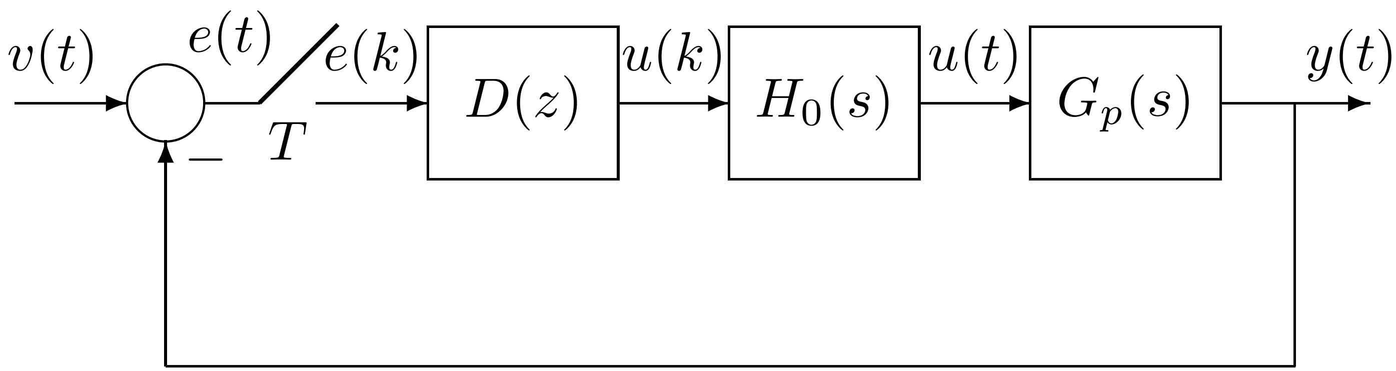
$$\dot{y}(t) = \text{const. for ramp}$$

$$\ddot{y}(t) = \text{const. for parable}$$

These must be translated in conditions on the control.

For example, in case of step input, the control $u(t)$ must be constant in steady state.

# Deadbeat controller: example




$$G_p(s) = \frac{1}{s(s+1)}$$

Design $D(z)$ so that the closed loop system has a deadbeat step response.
We choose $T = 0.8\ s$

$$G_p(z) = \mathcal{Z}\left[\frac{1-e^{-sT}}{s}\frac{1}{s(s+1)}\right] = \frac{K(z-b)}{(z-1)(z-a)} = \frac{K(1-bz^{-1})z^{-1}}{(1-z^{-1})(1-az^{-1})}$$

$$= \frac{0.2493(1+0.7669z^{-1})z^{-1}}{(1-z^{-1})(1-0.4493z^{-1})}$$

# Deadbeat controller: example

Since $G_p(z)$ has a delay $z^{-1}$ and $n = 2$, we choose

$$G_m(z) = a_1 z^{-1} + a_2 z^{-2}$$

Since the input is a step:

$$1 - G_m(z) = (1 - z^{-1})N(z)$$

this let also to avoid the cancellation of the critical pole of $G_p(z)$ in $z = 1$.

To avoid ripple we impose that $c(t) = cost$ for $t \geq 2T$, which is guranteed by $u(t) = cost$ for $t \geq 2T$, i.e.

$$U(z) = b_0 + b_1 z^{-1} + b(z^{-2} + z^{-3} + \dots)$$

where $b = 0$ since $G_p(s)$ has an integral action.

# Deadbeat controller: example

Therefore

$$U(z) = b_0 + b_1 z^{-1}$$

Moreover

$$\begin{aligned}
U(z) &= \frac{Y(z)}{G_p(z)} = \frac{Y(z)V(z)}{V(z)G_p(z)} = G_m(z)\frac{V(z)}{G_p(z)} \\
&= G_m(z)\frac{1}{(1-z^{-1})}\frac{(1-z^{-1})(1-0.4493z^{-1})}{0.2493(1+0.7669z^{-1})z^{-1}} \\
&= G_m(z)\frac{(1-0.4493z^{-1})}{0.2493(1+0.7669z^{-1})z^{-1}}
\end{aligned}$$

Making equal

$$\begin{aligned}
G_m(z) &= (1+0.7669z^{-1})z^{-1}G_1 \\
U(z) &= 4.01(1-0.4493z^{-1})G_1
\end{aligned}$$

with $G_1 = cost.$

# Deadbeat controller: example

From

$$1 - a_1 z^{-1} - a_2 z^{-2} = (1 - z^{-1})N(z)$$

we have

$$N(z) = 1 + (1 - a_1)z^{-1} \qquad\qquad 1 - a_1 - a_2 = 0$$

Making equal we obtain

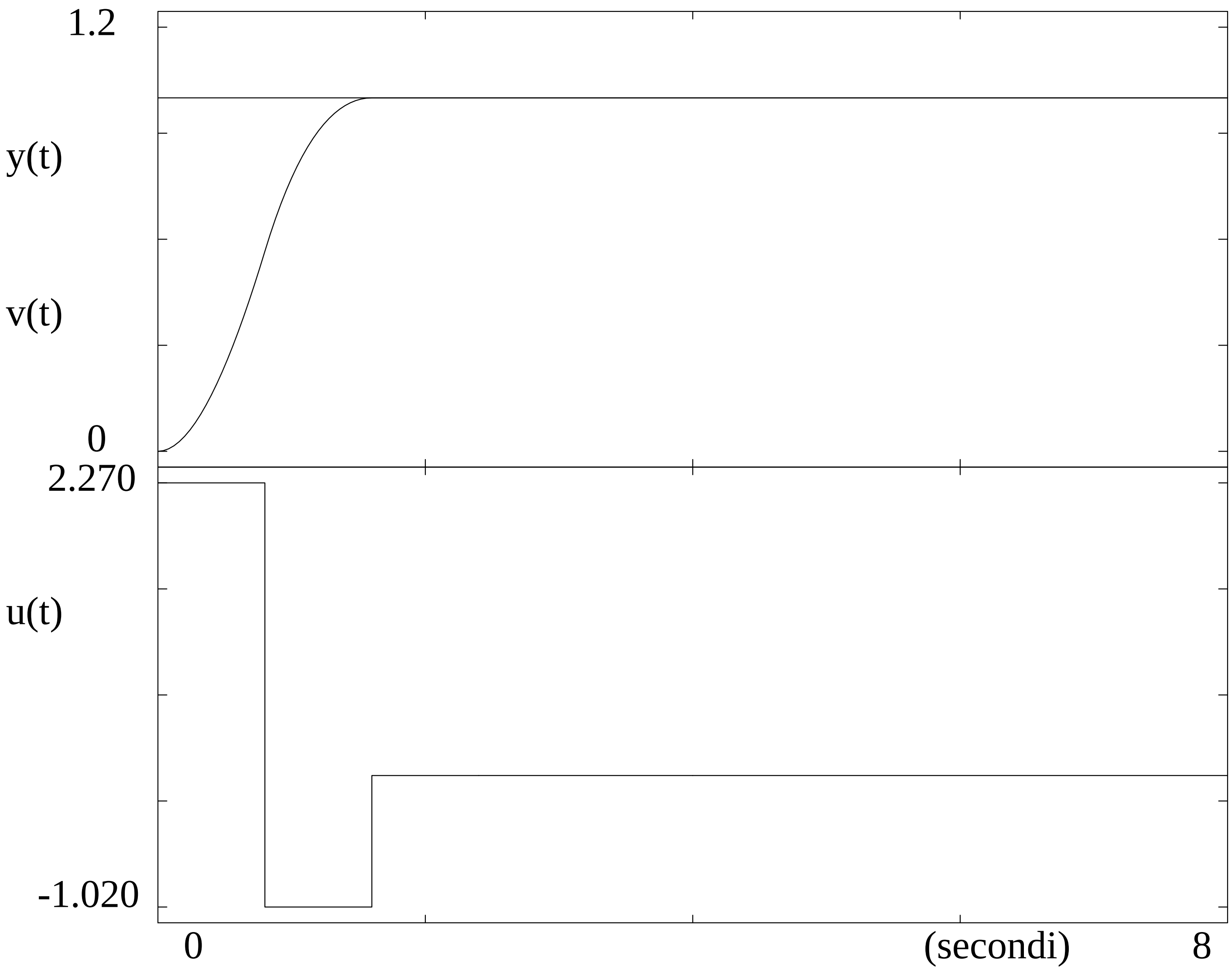
$$G_1 = a_1, \qquad\qquad a_2 - 0.7669a_1 = 0$$

$$a_1 = 0.566, \qquad\qquad a_2 = 0.434$$

And finally

$$\begin{aligned} G_m(z) &= 0.566z^{-1} + 0.434z^{-2} \\ N(z) &= 1 + 0.434z^{-1} \\ D(z) &= \frac{G_m(z)}{G_p(z)(1 - z^{-1})N(z)} = \frac{2.27 - 1.02z^{-1}}{1 + 0.434z^{-1}} \end{aligned}$$

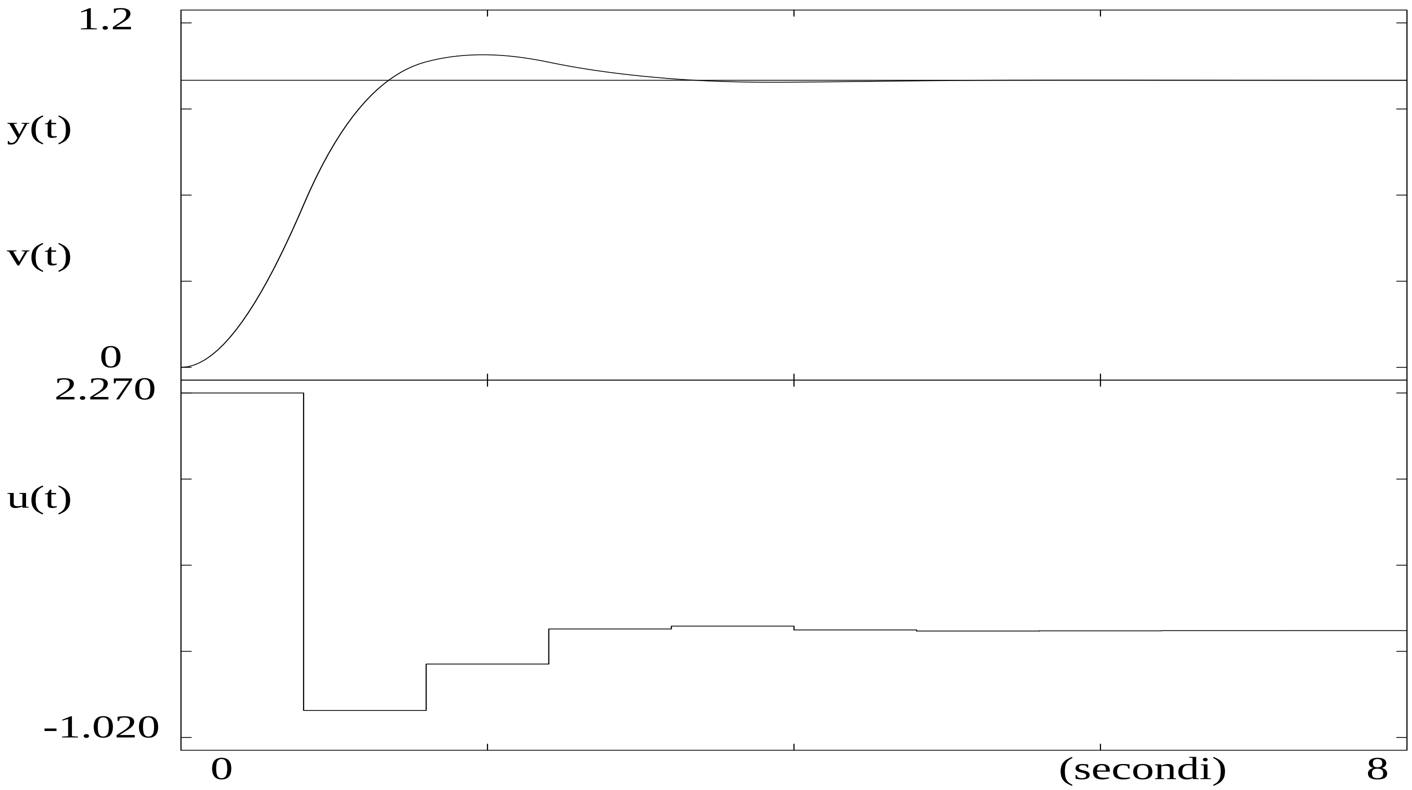
# Deadbeat controller: example



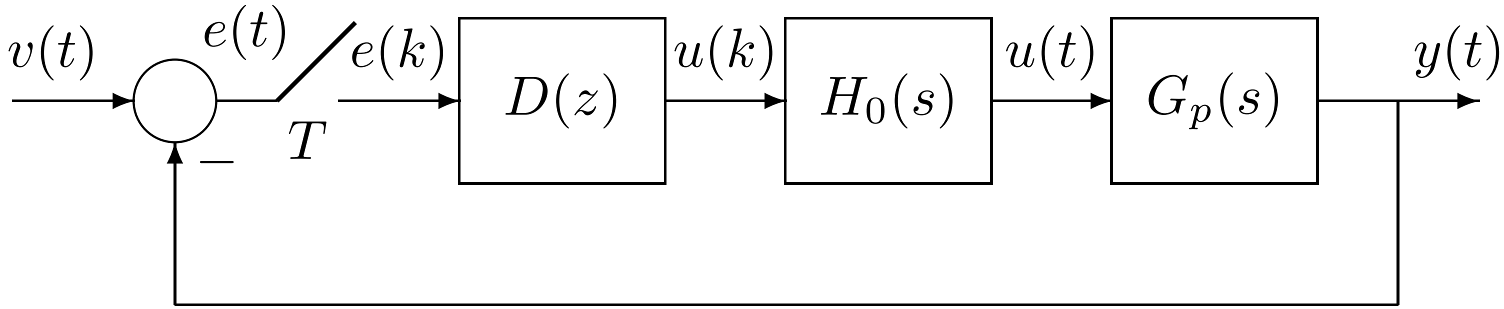
# Deadbeat controller: example

In case the plant has an unmodelled dynamics, i.e.,

$$G_p(s) = \frac{10}{s(s+1)(s+10)}$$



# Simplified deadbeat controller




- Just for step input
- $G_p(z)$ is stable and minimum phase
- Controller cancels all system dynamics
- Simplified specs: $G_m(z) = z^{-k}$ with $k$ greater or equal to the intrinsic delay of $G_p(z)$
- The controller $D(z)$ results:

$$D(z) = \frac{1}{G_p(z)} \frac{z^{-k}}{1 - z^{-k}}$$