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Random assignments on preference domains with a tier structure

Peng LIU
Singapore Management University, peng.liu.2013@phdecons.smu.edu.sg

Huaxia ZENG
Singapore Management University, huaxia.zeng.2011@phdecons.smu.edu.sg

Random Assignments on Preference Domains with a Tier Structure*

Peng Liu† and Huaxia Zeng‡

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Abstract

We address a standard random assignment problem (Bogomolnaia and Moulin (2001)) and search for *sd-strategy-proof*, *sd-efficient* and *sd-envy-free* (or *equal-treatment-of-equals*) rules. We introduce a class of restricted preference domains, *restricted tier domains*, and show that a rule is *sd-strategy-proof*, *sd-efficient* and *equal-treatment-of-equals* (or *sd-efficient* and *sd-envy-free*) if and only if it is *the Probabilistic Serial rule*. More importantly, we prove that a *restricted tier structure* is necessary for the existence of an *sd-strategy-proof*, *sd-efficient* and *sd-envy-free* (or *equal-treatment-of-equals*) rule, provided that the domain is *connected* (Monjardet (2009)).

Keywords: Probabilistic serial rule; sd-strategy-proofness; sd-efficiency; sd-envy-freeness; equal treatment of equals; restricted tier domains

JEL Classification: C78, D71.

1 Introduction

We consider the problem of allocating several indivisible objects to a group of agents, each of whom consumes at most one object. Classical examples include assigning college seats to applicants (Gale and Shapley (1962)), houses to residents (Shapley and Scarf (1974)), and jobs to workers (Hylland and Zeckhauser (1979)).

From the design point of view, the primary target is to identify rules that implement efficient allocations, and at the same time provide incentives for agents to truthfully reveal their preferences. Efficiency implies that no reallocation can be arranged to make all agents at least as well as before, and some agent strictly better off. Incentives, summarized by strategy-proofness of a rule, says that, in a revelation game associated to the rule, truth-telling is a weakly dominant

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†School of Economics, Singapore Management University, Singapore.

‡Lingnan (University) College, Sun Yat-sen University, Guangzhou, China.

strategy for each agent. The literature has introduced various classes of efficient and strategy-proof rules, e.g., serial dictatorship rules (Svensson (1999)), hierarchical exchange rules (Pápai (2000)), restricted endowment inheritance rules (Ehlers et al. (2002)) and trading cycles rules (Pycia and Ünver (2017)). However, none of these rules satisfies any fairness requirement. For instance, two agents reporting the same strict preference always receive distinct objects and hence are never treated equally. Consequently, one of them must envy the other. Instead of allocating deterministic objects, the literature has resorted to random assignment rules that assign to each agent a lottery on objects to restore ex ante fairness. Thus, agents representing the same preference may receive the same lottery, and the random assignment rule satisfies a classic fairness axiom: *equal treatment of equals*.

Since ordinal preferences on deterministic objects are collected to establish the random assignment, one needs to extend agents' preferences over deterministic objects to assess lotteries. A standard practice is to adopt the *stochastic dominance extension*.[1] A lottery is viewed at least as good as another if the former (first-order) stochastically dominates the latter according to the ordinal preference over objects. Equivalently, under the von-Neumann-Morgenstern hypothesis, a lottery (first-order) stochastically dominates another one if and only if it delivers an expected utility weakly higher than the expected utility delivered by the other lottery for *every* cardinal utility representing her ordinal preference on objects. By adopting the stochastic dominance extension, ex ante efficiency and strategy-proofness are defined and referred to as *sd-efficiency* and *sd-strategy-proofness*.[2] Beyond *equal treatment of equals*, ex ante fairness in random rules can be strengthened by *sd-envy-freeness* which requires that each agent always prefers her own lottery to any other's.

Two classic random assignment rules have been widely studied in the literature: the *Random Serial Dictatorship* (or RSD) *rule* (Abdulkadiroğlu and Sönmez (1998)) and the *Probabilistic Serial* (or PS) *rule* (Bogomolnaia and Moulin (2001)). Neither one of them resolves the conflict of *sd-strategy-proofness* and *sd-efficiency* with *sd-envy-freeness* or *equal treatment of equals*. Specifically, the PS rule is *sd-efficient* and *sd-envy-free* but fails *sd-strategy-proofness*, while the RSD rule is *sd-strategy-proof* and treats equals equally but is *sd-inefficient*. Moreover, an impossibility theorem is established by Bogomolnaia and Moulin (2001): when the numbers of objects and agents are identical and larger than four, and every preference in the universal domain is admissible[3], no random assignment rule satisfies *sd-strategy-proofness*, *sd-efficiency* and *equal treatment of equals*. Recently, this impossibility has been established on some restricted preference domains, e.g., single-peaked domains and single-dipped domains by Kasajima (2013), Chang and Chun (2016) and Altuntaş (2016). These results raise a natural question: Is there a reasonably restricted domain of preferences on which there exists an *sd-strategy-proof*, *sd-efficient* and *sd-envy-free* or *equal-treatment-of-equals* rule? Moreover, if the answer is affirmative, what is that rule?

This paper provides answers to the above questions by introducing a class of restricted

[1] For other preference extensions, please refer to Cho (2012) and Aziz et al. (2014).

[2] Henceforth, we add prefix "sd-" to emphasize that the corresponding axiom is established with respect to the stochastic dominance extension.

[3] The universal domain is referred to as the collection of all strict preferences. Throughout this paper, we assume that the preference is strict.

domains of preferences: *restricted tier domains*, and characterizing all *sd-strategy-proof*, *sd-efficient* and *equal-treatment-of-equals* rules. To construct a *restricted tier domain*, objects are first partitioned into several tiers each of which contains one or two objects; and all preferences are required to respect a common ranking of these tiers while the relative rankings of objects within a tier may vary arbitrarily. Such a common ranking of 1-or-2-object tiers is referred to as a *restricted tier structure*. As an example, consider a skyscraper with two apartments on each floor. A restricted tier structure can be generated according to floors (for instance, from the top down to the bottom), i.e., all agents prefer higher apartments to lower ones. Between two apartments on the same floor, however, the preferences may be arbitrary across agents. For another example, consider a road from the downtown to the suburb along which houses of similar quality are located on both sides. A restricted tier structure can be generated according to the distance away from the downtown.

Our first theorem shows that a rule on a *restricted tier domain* is *sd-strategy-proof*, *sd-efficient* and *equal-treatment-of-equals* if and only if it is the PS rule. Recall that the PS rule is manipulable on the universal domain since the lotteries prescribed by the PS rule are sensitive to unilateral deviations. Intuitively, the *restricted tier structure* embedded in a *restricted tier domain* reduces such sensitivity, and therefore restores appropriate incentive property on the PS rule.[4] At every preference profile of a *restricted tier domain*, according to the PS rule, all agents first equally share each tier of object(s), and moreover, within a tier with two objects, say a and b, each agent in the (weak) majority group (e.g., the group of agents with cardinality $l \geqslant \frac{n}{2}$ who prefer a to b, provided that n is the total number of agents) consumes $\frac{1}{l}$ of her preferred object a and obtains $\frac{2}{n} - \frac{1}{l}$ of her less preferred object b, while each agent in the minority group (i.e., the complementary group of the (weak) majority group) merely consumes $\frac{2}{n}$ of her preferred object b. Consequently, any individual preference misrepresentation does not affect the manipulator's share on each tier, and cannot increase the consumption of her sincerely preferred object in each 2-object tier. Therefore, we restore *sd-strategy-proofness* of the PS rule on a *restricted tier domain*. Moreover, in the verification of this characterization, we find that *sd-envy-freeness* is endogenized in an *sd-strategy-proof*, *sd-efficient* and *equal-treatment-of-equals* rule, and essentially *sd-efficiency* and *sd-envy-freeness* pin down all random assignments induced by the PS rule. Therefore, the PS rule is also uniquely characterized by *sd-efficiency* and *sd-envy-freeness* on a *restricted tier domain*.

As the *restricted tier structure* helps to restore *sd-strategy-proofness* of the PS rule, it provides one *particular* sufficient condition for the existence of an *sd-strategy-proof*, *sd-efficient* and *sd-envy-free* or *equal-treatment-of-equals* rule. More importantly, we characterize *restricted tier domains* for the existence of such an admissible rule. We restrict attention to the class of *connected domains* which have been widely studied in the voting literature (e.g., Monjardet (2009), Sato (2013), Chatterji et al. (2013) and Chatterji et al. (2016)), and recently have been adopted for characterizing random assignment rules in Cho (2012) and Cho (2016a). A pair of preferences is said to be *adjacent* if they are identical up to a switch of two consecutively ranked objects. Given a domain, a undirected graph is constructed such that the vertex set is the prefer-

[4] Kojima and Manea (2010) restrict such sensitivity by increasing the copies of objects, and hence restore *sd-strategy-proofness* of the PS rule.

ences in the domain and an edge is drawn between two adjacent preferences. Correspondingly, a domain is said to be *connected* if this graph is connected. Our second theorem proves that if a connected domain admits an *sd-strategy-proof*, *sd-efficient* and *sd-envy-free* rule, it must be a *restricted tier domain*. This axiomatically justifies the necessity of the *restricted tier structure*, and clearly specifies a boundary between the impossibility and possibility for designing desirable strategy-proof random assignment rules. Furthermore, when we weaken the fairness axiom from *sd-envy-freeness* to *equal treatment of equals*, the domain characterized is, surprisingly, not enlarged at all (see Theorem 3).

Last, we extend our preference restriction to a generalized model where each agent has an outside option, and the number of agents may differ from the number of objects. This domain strictly nests the one investigated by Bogomolnaia and Moulin (2002). On this domain, we analogously characterize the PS rule by either *sd-strategy-proofness*, *sd-efficiency* and *equal treatment of equals* or *sd-efficiency* and *sd-envy-freeness*. Hence, our result strengthens the characterization in Bogomolnaia and Moulin (2002).

The rest of the paper is organized as follows. The remainder of the Introduction explains the detailed relation of this paper to the literature. Section 2 specifies the model and axioms. Section 3 first introduces our preference restriction, characterizes the PS rule and shows the necessity of our preference restriction. Section 4 discusses the generalized model with outside options while Section 5 concludes. An appendix gathers the omitted proofs.

1.1 Relation to the literature

In the literature, there are two main strains on the axiomatic characterizations of the PS rule. The first strain focuses on identifying axioms that characterize the PS rule on the universal domain.[5] For instance, Bogomolnaia and Heo (2012) proposes the axiom *bounded invariance*, and characterizes the PS rule along with *sd-efficiency* and *sd-envy-freeness*.[6] Recently, Bogomolnaia (2015) adopts a different preference extension: *lexicographic preference extension* to establish a weaker incentive notion, and then shows that the PS rule is unique for *sd-efficiency*, *sd-envy-freeness* and strategy-proof with respect to the lexicographic preference extension.[7] Alternatively, Hashimoto et al. (2014) neglect the incentive issue in random assignment rules,

[5]We discuss here only the characterizations in the model where ordinal preferences are strict; each agent receives exactly one object; and each object has one unit. There are also interesting characterizations of the PS rule in other environments, e.g., Heo and Yılmaz (2015) add indifferences in preferences; Heo (2014b) allows each agents to consume more than one object; Liu and Pycia (2011a) increase the copies of each object to infinity; while both infinitely many copies and multiple-unit consumption are allowed in Liu and Pycia (2011b).

[6]Bounded invariance requires that whenever an agent's unilateral deviation does not involve her top k ranked objects, the allocation of each of these k objects remains unchanged. Hashimoto et al. (2014) weaken bounded invariance and characterize the PS rule accordingly.

[7]Fix a preference and two distinct lotteries over objects. We rearrange each lottery according to the preference from the worst object up to the best object. One lottery is evaluated better than the other according to the lexicographic preference extension, if we can find one object which has strictly higher probability in the former lottery than that in the latter one while for any less preferred object, the probabilities in both lotteries are identical. The lexicographic preference extension induce a linear order over all lotteries while stochastic-dominance preference extension only produces a partial order over all lotteries.

and characterize the PS rule with a new axiom *ordinal fairness* which in fact strengthens *sd-efficiency* and *sd-envy-freeness* combined.[8]

In the second strain, restrictions are imposed on either preference domains or preference profiles, and the PS rule is characterized by canonical axioms, i.e., the axioms studied in our paper. Bogomolnaia and Moulin (2002) introduce a model with preference restrictions which can be described by the following realistic application. Consider a public service center which is able to serve only one agent in each time slot. All agents want to be served earlier and differ only on their deadlines of services beyond which the services are perceived of no value. The deadlines are private information of agents, and the planner wants to truthfully elicit them and then schedule an efficient and fair service plan. Then, they characterize the PS rule by either *sd-strategy-proofness*, *sd-efficiency* and *equal treatment of equals* or *sd-efficiency* and *sd-envy-freeness*. As we mentioned above, the model studied in Bogomolnaia and Moulin (2002) is nested in our generalized model with outside options, and the same characterization results are established. As to the scheduling problem of the service center mentioned above, our generalization can be interpreted as follows. On each day, there is one service slot in the morning and one in the afternoon. All agents want to be served on an earlier date. However, given a service date, some agents may prefer to be served in the morning while some others the afternoon. In other words, our generalization suggests the use of the PS rule in these scheduling problems even when agents' preferences endows a slight perturbation.

Alternatively, various restrictions are introduced on preference profiles, and the PS assignments are characterized via *sd-efficiency* and *sd-envy-freeness*, e.g., the *full support* requirement in Liu and Pycia (2011a), *rich support on a partition* in Heo (2014a) and *rich preferences* in Cho (2016b).[9]

Our paper lies in the same vein of the second strain. We focus on the incentive property of rules on restricted preference domains (provided that each agent's domain is assumed to be identical), and canonical characterizations of the PS rule on these domains. More importantly, we axiomatically justify the necessity of our domain restriction for the existence of *sd-strategy-proof*, *sd-efficient* and *sd-envy-free* or *equal-treatment-of-equals* rules. Our characterization result implies that the PS assignments are unique for *sd-efficiency* and *sd-envy-freeness* on profiles of restricted tier preferences.

In the verification of our domain characterization theorems (Theorems 2 and 3), we intro-

[8]Ordinal fairness requires that whenever an agent is assigned an object a with strictly positive probability, the probability of this agent receiving an object better than a is no greater than the probability of any other agent getting an object better than a.

[9]The *full support* requires that in a preference profile, each preference in the universal domain is adopted by some agent. In a preference profile which is *rich support on a partition*, we first observe that all agents preference share a common ranking on a partition of objects where some block of the partition may contain more than 2 objects. Moreover, in each block of the partition, the full support requirement holds. In a *rich preference* profile, for each agent j and each object a, we can find an agent k (either $k = j$ or $k \neq j$) who prefers a the most and moreover, after partitioning all objects into two blocks according to agent j's preference: objects better than or identical to a and objects worse than a, we note that agent k also prefers the first block to the second one. Cho (2016b) studies economies with random assignments, and shows that if an economy is able to be decomposed into several feasible sub-economies via his *recursive decomposability* condition, then the PS assignment is the unique one satisfying *sd-efficiency* and *sd-envy-freeness* in the economy. For more detailed relation of our paper to Heo (2014a) and Cho (2016b), please refer to Remark 4.

duce an important notion called *the elevating property* which is a sufficient domain condition for the incompatibility of *sd-strategy-proofness*, *sd-efficiency* and *sd-envy-freeness* or *equal treatment of equals*. To the best of our knowledge, the *elevating property* covers all existing literature related to the study of impossibility on the existence of *sd-strategy-proof*, *sd-efficient* and *sd-envy-free* or *equal-treatment-of-equals* rules (e.g., Bogomolnaia and Moulin (2001), Kasajima (2013), Chang and Chun (2016) and Altuntaş (2016)). More importantly, in contrast to these literature which proposes some domain conditions and establishes negative results, we formulate the elevating property in a more general sense so that the avoidance of the *elevating property* becomes a critical and informative condition which is then adopted to characterize the *restricted tier domains*.

2 The model

Let $A \equiv \{a, b, \dots\}$ be a finite set of objects and $I \equiv \{1, 2, \dots, n\}$, $n \geqslant 4$, a finite set of agents. As a benchmark, we assume $|A| = |I| = n$. Each agent i is equipped with a complete, transitive and antisymmetric binary relation P_i over A, i.e., a linear order. Let $\mathbb{P}$ denote the set consisting of *all* strict preferences over A. The set of admissible preferences is a set $\mathbb{D} \subseteq \mathbb{P}$, referred to as the *preference domain*. Thus, $\mathbb{P}$ is referred to as the universal domain. Given $P_i \in \mathbb{D}$ and $a \in A$, let $r_k(P_i)$, $k = 1, \dots, n$, denote the kth ranked object according to P_i, and $B(P_i, a) = \{x \in A | x \quad P_i \quad a\}$ denote the (strict) upper contour set of a in P_i. A *preference profile* $P \equiv (P_1, \dots, P_n) \equiv (P_i, P_{-i}) \in \mathbb{D}^n$ is an n-tuple of admissible preferences.

Let $\Delta(A)$ denote the set of lotteries, or probability distributions, over A. Given $\lambda \in \Delta(A)$, λ_a denotes the probability assigned to object a. A (random) **assignment** is a bi-stochastic matrix $L \equiv [L_{ia}]_{i \in I, a \in A}$, namely a non-negative square matrix whose elements in each row and each column sum to unity, i.e., $L_{ia} \geqslant 0$ for all $i \in I$ and $a \in A$, $\sum_{a \in A} L_{ia} = 1$ for all $i \in I$, and $\sum_{i \in I} L_{ia} = 1$ for all $a \in A$. Evidently, in a bi-stochastic matrix L, each row is a lottery, i.e., $L_i \in \Delta(A)$ for all $i \in I$. Let $\mathcal{L}$ denote the set of all bi-stochastic matrices. Agents assess lotteries according to (first-order) stochastic dominance. Given $P_i \in \mathbb{D}$ and lotteries $\lambda, \lambda' \in \Delta(A)$, λ *stochastically dominates* λ' according to P_i, denoted $\lambda \quad P_i^{sd} \quad \lambda'$, if $\sum_{l=1}^{k} \lambda_{r_l(P_i)} \geqslant \sum_{l=1}^{k} \lambda'_{r_l(P_i)}$ for all $1 \leqslant k \leqslant n$. Analogously, given $P \in \mathbb{D}^n$, we say an assignment L stochastically dominates L' according to P, denoted $L \quad P^{sd} \quad L'$, if $L_i \quad P_i^{sd} \quad L'_i$ for all $i \in I$.

A *rule* is a mapping $\varphi : \mathbb{D}^n \to \mathcal{L}$. Given $P \in \mathbb{D}^n$, $\varphi_{ia}(P)$ denotes the probability of agent i receiving object a, and thus $\varphi_i(P)$ denotes the lottery assigned to agent i.

Given $P \in \mathbb{D}^n$, an assignment L is *sd-efficient* if it is not stochastically dominated by any another assignment L', i.e., $[L' \quad P^{sd} \quad L] \Rightarrow [L' = L]$. Accordingly, a rule $\varphi : \mathbb{D}^n \to \mathcal{L}$ is **sd-efficient** if the assignment $\varphi(P)$ is *sd-efficient* for all $P \in \mathbb{D}^n$.

Next, a rule is *sd-strategy-proof* if for every agent, her lottery under truthtelling always stochastically dominates her lottery induced by any misrepresentation, according to her true preference. Formally, a rule $\varphi : \mathbb{D}^n \to \mathcal{L}$ is **sd-strategy-proof** if for all $i \in I$, $P_i, P'_i \in \mathbb{D}$, and $P_{-i} \in \mathbb{D}^{n-1}$, $\varphi_i(P_i, P_{-i}) \quad P_i^{sd} \quad \varphi_i(P'_i, P_{-i})$.

Last, we require that every agent weakly prefer her own lottery to any other's. Given $P \in$

$\mathbb{D}^n$, an assignment L is *sd-envy-free* if $L_i \;\; P_i^{sd} \;\; L_j$ for all $i, j \in I$. Accordingly, a rule φ : $\mathbb{D}^n \rightarrow \mathcal{L}$ is **sd-envy-free** if $\varphi(P)$ is *sd-envy-free* for all $P \in \mathbb{D}^n$. As a weaker notion of fairness, we say that an assignment $L \in \mathcal{L}$ satisfies *equal treatment of equals* if for all $i, j \in I$, $[P_i = P_j] \Rightarrow [L_i = L_j]$. Similarly, a rule $\varphi : \mathbb{D}^n \rightarrow \mathcal{L}$ satisfies **equal treatment of equals** if $\varphi(P)$ satisfies *equal treatment of equals* for all $P \in \mathbb{D}^n$. Evidently, *sd-envy-freeness* implies *equal treatment of equals*.

2.1 Random serial dictatorship rule and probabilistic serial rule

There are essentially two random assignment rules in the literature: the *Random Serial Dictatorship* (or RSD) *rule* (Abdulkadiroğlu and Sönmez (1998)) and the *Probabilistic Serial (or PS) rule* (Bogomolnaia and Moulin (2001)). In deterministic assignment models, serial dictatorship rules are known to be (ex-post) strategy-proof and efficient (Svensson (1999)). As a uniform randomization among all serial dictatorship rules, the RSD rule treats equals equally and inherits ex ante incentive property, i.e., *sd-strategy-proofness* from ex-post strategy-proofness of serial dictatorship rules. However, the RSD rule fails *sd-efficiency*, for which Abdulkadiroğlu and Sönmez (2003) and Kesten (2009) provide extensive explanations.

The PS rule is initially introduced by Crès and Moulin (2001) to deal with the scheduling problem and later introduced to the standard random assignment problem by Bogomolnaia and Moulin (2001). The PS rule is fundamentally different from the RSD rule as it specifies directly a random assignment for each preference profile, rather than using a mixture of some deterministic assignments to determine the random assignment. The PS rule treats the objects as infinitely divisible and agents consume the objects as time flows. When time starts, each agent consumes her favorite object at the uniform speed, until some object(s) are exhausted. Then agents reformulate their preferences by removing the exhausted object(s), and resume consuming their favorite objects in the remaining ones at the uniform speed. Such procedure proceeds until all the objects are exhausted. Finally, the share of an object consumed by an agent is interpreted as the probability she receives this object. The axiomatic performance of the PS rule is very different from the RSD rule. It is *sd-efficient* and *sd-envy-free*, since at each time point each agent is consuming her favorite available object. However, the major drawback of the PS rule is that it is manipulable, i.e., not *sd-strategy-proof*. This happens because the consumption procedure is sensitive to unilateral deviations, which will be elaborated by Example 1 in the next section.

From the above discussion of the RSD rule and the PS rule, there seems to be a fundamental conflict between *sd-strategy-proofness* and *sd-efficiency* given fairness present. Such conflict is formally established in the following impossibility result.

Proposition 1 (Bogomolnaia and Moulin (2001)) *There exists no sd-strategy-proof, sd-efficient and equal-treatment-of-equals rule on the universal domain.*

3 Results

As shown in Proposition 1, on the domain of unrestricted preferences, *sd-strategy-proofness*, *sd-efficiency* and *equal-treatment-of-equals* (or *sd-envy-freeness*) are incompatible. In this paper, we ask what appropriate preference restriction is sufficient for the existence of an *sd-strategy-proof*, *sd-efficient* and *equal-treatment-of-equals* (or *sd-envy-free*) rule. As a simpler starting point, we investigate on what restricted preference domains, we can restore *sd-strategy-proofness* of the PS rule. We start our investigation from a heuristic example.

Example 1 Let $A = \{a, b, c, d\}$. Let $P \equiv (P_{-4}, P_4)$ and $P' \equiv (P_{-4}, P_4')$ be two preference profiles below, which specify an profitable manipulation of agent 4. According to the PS rule, the corresponding consumption procedures are depicted in Figure 1.

$$P = \begin{pmatrix} P_1 : a \succ c \succ b \succ d \\ P_2 : a \succ b \succ c \succ d \\ P_3 : b \succ a \succ c \succ d \\ P_4 : b \succ a \succ c \succ d \end{pmatrix} \qquad P' = \begin{pmatrix} P_1 : a \succ c \succ b \succ d \\ P_2 : a \succ b \succ c \succ d \\ P_3 : b \succ a \succ c \succ d \\ P_4' : a \succ b \succ c \succ d \end{pmatrix}$$

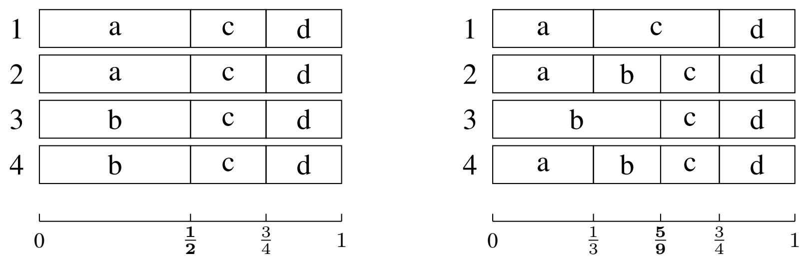

Figure 1: Consumption procedures under P and P' in the PS rule

Observe that $\varphi_{4a}(P) + \varphi_{4b}(P) = \frac{1}{2} < \frac{5}{9} = \varphi_{4a}(P') + \varphi_{4b}(P')$. Thus, agent 4 can profitably manipulate at profile P via P_4'. This indicates that the PS rule is vulnerable to small manipulations like P_4 and P_4' which differ on the relative rankings of exactly one pair of objects. Note that in profile P, each of a and b is most preferred by two agents. Therefore, as shown in Figure 1, objects a and b are exhausted simultaneously at time $\frac{1}{2}$, and all agents turn to objects in $\{c, d\}$ at the same time. However, in profile P', a is top ranked in the preferences of agents 1, 2 and 4, and therefore is exhausted in a shorter time: $\frac{1}{3}$. This indicates that agent 3, who prefers b the most, only consumes $\frac{1}{3}$ of b while all others exhaust a. Furthermore, since c is the second best in agent 1's preference while a and b occupy the top two positions in all others' preferences, after time $\frac{1}{3}$, agent 1 starts to consume c while agents 2, 3 and 4 are going to equally share the rest of object b. Consequently, agent 4 obtains $\frac{2}{9}$ of b, and therefore has $\frac{5}{9}$ of a and b combined which is more desirable than that under profile P.

This manipulation is made possible by the following two facts. First, according to the PS rule, agents are myopic and greedy: every agent consumes her favorite object among what are not exhausted at every time point. Second, more specifically, agent 1's preference differs to the others in both profiles P and P' in the sense that c is ranked in between a and b in P_1 while all others rank both a and b above c.

Observe that in preferences P_1, P_2 and P_3, object b occupies three distinct ranking positions, and more specifically, is *elevated* successively from the third position in P_1 to the second in P_2, and to the top in P_3. Now, we impose an additional restriction on *all* agents' preferences to avoid such 3-position elevating phenomenon: both a and b occupy the top two positions, and c and d obtain the other two positions. Thus, preference P_1 is no longer admissible, and more importantly, all preferences preserve a *common tier structure*: both a and b are ranked above c and d. Accordingly, let $\bar{P} \equiv (\bar{P}_1, P_2, P_3, P_4)$ and $\bar{P}' \equiv (\bar{P}_1, P_2, P_3, P_4')$ where for instance, $\bar{P}_1 = P_2$. The consumption procedure at $\bar{P}$ (specified in Figure 2 below) remains identical to that in Figure 1, while the consumption procedure at $\bar{P}'$ becomes significantly simpler than that at profile P', and is depicted in Figure 2 below.

$$\bar{P} = \begin{pmatrix} \bar{P}_1 : a \succ b \succ c \succ d \\ P_2 : a \succ b \succ c \succ d \\ P_3 : b \succ a \succ c \succ d \\ P_4 : b \succ a \succ c \succ d \end{pmatrix} \qquad \bar{P}' = \begin{pmatrix} \bar{P}_1 : a \succ b \succ c \succ d \\ P_2 : a \succ b \succ c \succ d \\ P_3 : b \succ a \succ c \succ d \\ P_4' : a \succ b \succ c \succ d \end{pmatrix}$$

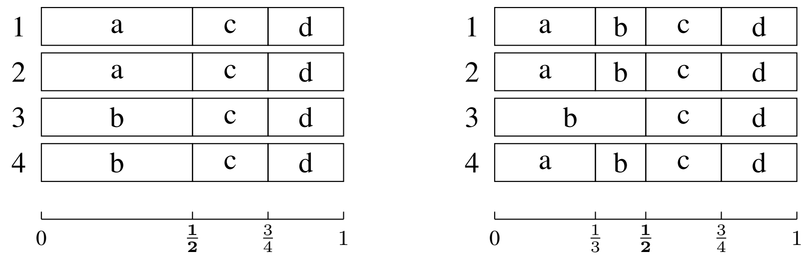

Figure 2: Consumption procedures under $\bar{P}$ and $\bar{P}'$ in the PS rule

Now, the manipulation of agent 4 at $\bar{P}$ via P_4' is non-profitable, i.e., the lottery assigned to agent 4 at $\bar{P}$ first-order stochastically dominates that at $\bar{P}'$ according to her sincere preference P_4. First, it is evident that agent 4 obtains identical shares of objects c and d across profiles $\bar{P}$ and $\bar{P}'$. Second, more importantly, due to the common tier structure, the combined share of a and b assigned to agent 4 at $\bar{P}'$ is fixed to $\frac{1}{2}$ which is identical to that at profile $\bar{P}$. Last, the switch of a and b in $\bar{P}_1$ and $\bar{P}_1'$ makes agent 4 worse off as she consume less of b at $\bar{P}'$ than that at $\bar{P}$, i.e., agent 4 gets $\frac{1}{6}$ of b at $\bar{P}'$, and $\frac{1}{2}$ at $\bar{P}$. $\square$

3.1 Restricted tier domains: A possibility result

We now formally introduce the preference restriction suggested by Example 1: all object are partitioned into tiers; each tier consists of one or two objects; and all admissible preferences respect a common ranking of tiers.

Let $\mathcal{P} \equiv (A_k)_{k=1}^T$ denote a **tier structure**, i.e., (i) tier $A_k \subseteq A$ is not empty, $k = 1, \ldots, T$, (ii) $A_k \cap A_{k'} = \emptyset$ for all $k \neq k'$, (iii) $\cup_{k=1}^T A_k = A$. According to an arbitrary tier structure, we have a *tier domain* where the relative rankings over tiers in *every* preference are identical. Moreover, we impose an additional restriction: every tier contains *at most* two objects, and then construct a *restricted tier domain*.

Definition 1 *A domain $\mathbb{D}$ is a **restricted tier domain** if there exists a restricted tier structure $\mathcal{P} \equiv (A_k)_{k=1}^T$ such that*

1. *For all $1 \leqslant k \leqslant T$, $|A_k| \leqslant 2$;*
2. *Given $P_i \in \mathbb{D}$ and $a, b \in A$, $[a \in A_k, b \in A_{k'}$ and $k < k'] \Rightarrow [a \;\; P_i \;\; b]$.*

Let $\mathbb{D}(\mathcal{P})$ denote the restricted tier domain containing all admissible preferences.

Remark 1 Given a tier structure where some tier contains more than two objects, let $\mathbb{D}$ be the tier domain containing *all* admissible preferences. Then, there are three preferences analogous to P_1, P_2 and P_3 in Example 1, and consequently, by a similar argument in Example 1, the PS rule fails *sd-strategy-proofness*.

Remark 2 In an auction model, Bikhchandani et al. (2006) study a particular class of tiered domains, named "order-based domains" where all (quasi linear) cardinal preferences they examine induce an identical ordinal preference on objects at each payment level. More recently, tiered domains are examined in two-sided matching (Akahoshi (2014) and Kandori et al. (2010)), school choice (Kesten (2010) and Kesten and Kurino (2013)), and spectrum license auctions (Serizawa and Zhou (2016)).

Remark 3 Let $\mathcal{P} \equiv (A_k)_{k=1}^T$ be a tier structure with $|A_k| = 2$ for all $1 \leqslant k \leqslant T$. The cardinality of the *restricted tier domain* $\mathbb{D}(\mathcal{P})$ is 2^T.

On a *restricted tier domain*, we can escape the impossibility in Proposition 1 by restoring *sd-strategy-proofness* of the PS rule. Moreover, Theorem 1 below shows that the PS rule is the unique one on a *restricted tier domain* satisfying *sd-strategy-proofness*, *sd-efficiency* and *equal treatment of equals*.

Theorem 1 *On a restricted tier domain, a rule is sd-strategy-proof, sd-efficient and equal-treatment-of-equals if and only if it is the PS rule.*

Proof: Given $\mathcal{P} \equiv (A_k)_{k=1}^T$, let $\mathbb{D} \subseteq \mathbb{D}(\mathcal{P})$ be a *restricted tier domain*.

Due to the restricted tier structure embedded in $\mathbb{D}$, at each preference profile, we can clearly specify the random assignment induced by the PS rule as shown in Fact 1 below.

Fact 1 *Given $P \in \mathbb{D}^n$, the random assignment specified by the PS rule, $L \equiv PS(P)$, is the one that satisfies the following two conditions: for each $1 \leqslant k \leqslant T$,*

1. *$L_{iA_k} \equiv \sum_{x \in A_k} L_{ix} = \frac{|A_k|}{n}$ for all $i \in I$.*
2. *Assume $A_k = \{a, b\}$. Let $I_k \equiv \{i \in I | a \;\; P_i \;\; b\}$ and $l \equiv |I_k|$.*
   (i) *If $\frac{n}{2} \leqslant l \leqslant n$, then*
      - *$L_{ia} = \frac{1}{l}$ and $L_{ib} = \frac{2}{n} - \frac{1}{l}$ for all $i \in I_k$;*
      - *$L_{ja} = 0$ and $L_{jb} = \frac{2}{n}$ for all $j \in I \backslash I_k$.*

(ii) *If* $0 \leqslant l < \frac{n}{2}$, *then*

- $L_{ia} = \frac{2}{n}$ *and* $L_{ib} = 0$ *for all* $i \in I_k$;
- $L_{ja} = \frac{2}{n} - \frac{1}{n-l}$ *and* $L_{jb} = \frac{1}{n-l}$ *for all* $j \in I \backslash I_k$.

The verification of Fact 1 is routine, and we hence omit it. We first intuitively explain two conditions in Fact 1. In the assignment L , all agents first equally share every tier. Next, in a particular tier with two objects, say $A_k = \{a, b\}$, the set of agents who prefer a to b, i.e., $I_k \equiv \{i \in I | a \;\; P_i \;\; b\}$, is either a (weak) majority, i.e., $\frac{n}{2} \leqslant |I_k| \leqslant n$, or a (strict) minority, i.e., $0 \leqslant |I_k| < \frac{n}{2}$. If I_k is a (weak) majority, then all agents in I_k share a equally and exclusively, and hence each receives the share $\frac{1}{|I_k|}$ of a; $I \backslash I_k$ only consume b, and each of them receives the share $\frac{2}{n}$ of b. Moreover, all agents in I_k split what remains of b, and hence each obtains the share $\frac{1-(n-|I_k|)\times\frac{2}{n}}{|I_k|} = \frac{2}{n} - \frac{1}{|I_k|}$ of b. If I_k is a (strict) minority, then $I \backslash I_k$ is a (strict) majority, i.e., $\frac{n}{2} < |I \backslash I_k| \leqslant n$, and objects a and b are shared in an opposite symmetric way.

It is evident that the PS rule is always *sd-efficient* and *equal-treatment-of-equals*. We verify that the PS rule is *sd-strategy-proof* on $\mathbb{D}$. Given $i \in I$, $P_i, P'_i \in \mathbb{D}$ and $P_{-i} \in \mathbb{D}^{n-1}$, let L and L' be two random assignments induced by the PS rule at profile $P \equiv (P_i, P_{-i})$ and $P' \equiv (P'_i, P_{-i})$ respectively. We show $L_i \; P_i^{sd} \; L'_i$.

According to condition 1 above, we know that for every $1 \leqslant k \leqslant T$, $\sum_{t=1}^{k} L_{iA_t} = \sum_{t=1}^{k} L'_{iA_t}$. Therefore, to complete the verification, it suffices to show that given $1 \leqslant k \leqslant T$, assuming $A_k = \{a, b\}$ and $a \; P_i \; b$, we have $L_{ia} \geqslant L'_{ia}$. If $a \; P'_i \; b$, condition 2 above implies $L_{ia} = L'_{ia}$. Next, assume $b \; P'_i \; a$. Let l be the number of agents who prefer a to b at P, i.e., $l \equiv |\{j \in I | a \;\; P_j \;\; b\}|$. Thus, $1 \leqslant l \leqslant n$ and the number of agents who prefer a to b at P' must be $l-1$. If $\frac{n}{2} < l \leqslant n$, condition 2(i) implies $L_{ia} = \frac{1}{l} > 0 = L'_{ia}$. If $1 \leqslant l \leqslant \frac{n}{2}$, condition 2(i) (if $l = \frac{n}{2}$) or condition 2(ii) (if $1 \leqslant l < \frac{n}{2}$) implies $L_{ia} = \frac{2}{n}$. Moreover, since $L'_{ia} \leqslant \frac{2}{n}$ by condition 1, we have $L_{ia} \geqslant L'_{ia}$. Therefore, $L_{ia} \geqslant L'_{ia}$ as required and hence $L_i \; P_i^{sd} \; L'_i$. In conclusion, the PS rule is *sd-strategy-proof* on domain $\mathbb{D}$. This completes the verification of the sufficiency part of Theorem 1.

Henceforth, we prove the necessity part of Theorem 1. Let $\varphi : \mathbb{D}^n \to \mathcal{L}$ a rule which satisfies all three axioms. Fix $P \equiv (P_1, \ldots, P_n) \in \mathbb{D}^n$ and $L \equiv \varphi(P)$ for the verifications below. Specifically, we show that L satisfies conditions 1 and 2 of Fact 1.

Lemma 1 *For all* $k \in \{1, \ldots, T\}$ *and* $i \in I$, $L_{iA_k} = \frac{|A_k|}{n}$.

Proof: Let $\bar{P} \in \mathbb{D}^n$ be such that $\bar{P}_i = \bar{P}_j$ for all $i, j \in I$. Then *equal treatment of equals* implies $\varphi_{ia}(\bar{P}) = \frac{1}{n}$ for all $i \in I$ and all $a \in A$. Hence, $\varphi_{iA_k}(\bar{P}) = \frac{|A_k|}{n}$ for all $i \in I$ and all $k \in \{1, \ldots, T\}$. According to P and $\bar{P}$, we can separate all agents into two groups: $\hat{I} = \{i \in I | P_i \neq \bar{P}_i\}$ and $I \backslash \hat{I} = \{i \in I | P_i = \bar{P}_i\}$. Given $S \subseteq \hat{I}$, let $\bar{P}^S \in \mathbb{D}^n$ be such that $\bar{P}_i^S = P_i$ for all $i \in S$ and $\bar{P}_i^S = \bar{P}_i$ for all $i \notin S$. Thus, $\bar{P}^S = (P_S, \bar{P}_{-S})$. Evidently, $\bar{P}^{\emptyset} = \bar{P}$ and $\bar{P}^{\hat{I}} = P$.

Now, given $i \in \hat{I}$, *sd-strategy-proofness* implies $\sum_{t=1}^{k} \varphi_{iA_t}(\bar{P}^{\{i\}}) = \sum_{t=1}^{k} \varphi_{iA_t}(\bar{P}^{\emptyset})$ for all $k \in \{1, \ldots, T\}$, which in turn implies $\varphi_{iA_k}(\bar{P}^{\{i\}}) = \frac{|A_k|}{n}$ for all $k \in \{1, \ldots, T\}$. Furthermore, *equal treatment of equals* implies $\varphi_{jA_k}(\bar{P}^{\{i\}}) = \frac{|A_k|}{n}$ for all $j \neq i$ and all $k \in \{1, \ldots, T\}$.

Therefore, given $i \in \hat{I}$, $\varphi_{jA_k}(\bar{P}^{\{i\}}) = \frac{|A_k|}{n}$ for all $j \in I$ and all $k \in \{1, \ldots, T\}$. We continue with an induction argument on $S \subseteq \hat{I}$.

Induction Hypothesis: Given $1 < l \leqslant |\hat{I}|$, for all $S \subseteq \hat{I}$ with $1 \leqslant |S| \leqslant l-1$, we have $\varphi_{iA_k}(\bar{P}^S) = \frac{|A_k|}{n}$ for all $i \in I$ and $k \in \{1, \ldots, T\}$.

Let $S \subseteq \hat{I}$ with $|S| = l$. We will show $\varphi_{iA_k}(\bar{P}^S) = \frac{|A_k|}{n}$ for all $i \in I$ and all $k \in \{1, \ldots, T\}$. Given $j \in S$, by *sd-strategy-proofness* and induction hypothesis, $\sum_{t=1}^{k} \varphi_{jA_t}(\bar{P}^S) = \sum_{t=1}^{k} \varphi_{jA_t}(\bar{P}^{S \backslash \{j\}}) = \sum_{t=1}^{k} \frac{|A_t|}{n}$ for all $k \in \{1, \ldots, T\}$, which in turns implies $\varphi_{jA_k}(\bar{P}^S) = \frac{|A_k|}{n}$ for all $k \in \{1, \ldots, T\}$. Furthermore, *equal treatment of equals* implies $\varphi_{iA_k}(\bar{P}^S) = \frac{|A_k|}{n}$ for all $i \notin S$ and all $k \in \{1, \ldots, T\}$. Therefore, $\varphi_{iA_k}(\bar{P}^S) = \frac{|A_k|}{n}$ for all $i \in I$ and all $k \in \{1, \ldots, T\}$. This completes the verification of induction hypothesis. Therefore, $L_{iA_k} = \frac{|A_k|}{n}$ for all $i \in I$ and $k \in \{1, \ldots, T\}$. ■

Thus, random assignment L satisfies condition 1 of Fact 1.

Lemma 2 *Given $1 \leqslant k \leqslant T$, assume $A_k = \{a, b\}$ and let $I_k = \{i \in I | a \ \ P_i \ \ b\}$. The following statements hold.*

(i) *For all $i, j \in I_k$, $L_{ia} = L_{ja}$.*

(ii) *For all $i \in I_k$ and $j \in I \backslash I_k$, $L_{ia} \geqslant L_{ja}$ and $L_{ib} \leqslant L_{jb}$.*

Proof: Assume $|I_k| = l$. If $l = 0$, statements (i) and (ii) are satisfied vacuously. Henceforth, assume $1 \leqslant l \leqslant n$. We consider three cases.

Case 1: $l = 1$.

Statement (i) is satisfied vacuously. Assume $I_k = \{i\}$. By *sd-efficiency*, either $L_{ib} = 0$, or $L_{ja} = 0$ for all $j \neq i$. Suppose $L_{ib} > 0$. Then, $L_{ja} = 0$ for all $j \neq i$. Consequently, $L_{ia} = 1$ and $L_{ia} + L_{ib} > 1$. Contradiction! Therefore, $L_{ib} = 0$. Then, Lemma 1 implies $L_{ia} = \frac{2}{n}$. Moreover, since $L_{ja} + L_{jb} = \frac{2}{n}$ for all $j \in I \backslash I_k$, it is evident that $L_{ia} \geqslant L_{ja}$ and $L_{ib} \leqslant L_{jb}$ for all $i \in I_k$ and all $j \in I \backslash I_k$. This completes the verification of statement (ii) in Case 1.

Case 2: $l = n$.

Statement (ii) is satisfied vacuously. We focus on statement (i). Let $\bar{P} \in \mathbb{D}^n$ be such that $\bar{P}_i = \bar{P}_j$ for all $i, j \in I$ and $a \ \bar{P}_i \ b$ for all $i \in I$. Thus, according to P and $\bar{P}$, we can separate all agents into two groups: $\hat{I} = \{i \in I | P_i \neq \bar{P}_i\}$ and $I \backslash \hat{I} = \{i \in I | P_i = \bar{P}_i\}$. Given $S \subseteq \hat{I}$, let $\bar{P}^S \in \mathbb{D}^n$ be such that $\bar{P}_i^S = \bar{P}_i$ for all $i \in \hat{I} \backslash S$ and $\bar{P}_i^S = P_i$ for all $i \notin \hat{I} \backslash S$. Thus, $\bar{P}^{\emptyset} = \bar{P}$ and $\bar{P}^{\hat{I}} = P$. First, *equal treatment of equals* implies $\varphi_{ia}(\bar{P}) = \frac{1}{n}$ for all $i \in I = I_k$. Next, we provide an induction argument on S.

Induction Hypothesis: Given $0 < s \leqslant |\hat{I}|$, for all $S \subseteq \hat{I}$ with $0 \leqslant |S| < s$ and all $i \in I$, we have $\varphi_{ia}(\bar{P}^S) = \frac{1}{n}$.

Given $S \subseteq \hat{I}$ with $|S| = s$, we show $\varphi_{ia}(\bar{P}^S) = \frac{1}{n}$ for all $i \in I$. Given $i \in S$, *sd-strategy-proofness* and induction hypothesis imply $\varphi_{ia}(\bar{P}^S) = \varphi_{ia}(P_i, \bar{P}_{-i}^S) = \varphi_{ia}(\bar{P}_i, \bar{P}_{-i}^S) = \varphi_{ia}(\bar{P}^{S \backslash \{i\}}) = \frac{1}{n}$. Furthermore, in $\bar{P}^S$, for all $j \in I \backslash S$, *equal treatment of equals* implies

$\varphi_{ja}(\bar{P}^S) = \frac{1-\sum_{i\in S}\varphi_{ia}(\bar{P}^S)}{|I\backslash S|} = \frac{1-s\times\frac{1}{n}}{n-s} = \frac{1}{n}$. Therefore, $\varphi_{ia}(\bar{P}^S) = \frac{1}{n}$ for all $i \in I$. This completes the verification of induction hypothesis. Therefore, $L_{ia} = L_{ja}$ for all $i, j \in I_k = I$. This completes the verification of statement (i) in Case 2.

Case 3: $1 < l < n$.

First, *sd-efficiency* implies either $L_{ib} = 0$ for all $i \in I_k$, or $L_{ja} = 0$ for all $j \in I\backslash I_k$. If $L_{ib} = 0$ for all $i \in I_k$, then Lemma 1 implies $L_{ia} = \frac{2}{n}$ for all $i \in I_k$. Thus, $L_{ia} = L_{ja}$ for all $i, j \in I_k$. Moreover, since $L_{ja} + L_{jb} = \frac{2}{n}$ for all $j \in I\backslash I_k$, it is evident that $L_{ia} \geqslant L_{ja}$ and $L_{ib} \leqslant L_{jb}$ for all $i \in I_k$ and $j \in I\backslash I_k$.

Next, assume $L_{ja} = 0$ for all $j \in I\backslash I_k$, and $L_{ib} > 0$ for some $i \in I_k$. By Lemma 1, $L_{jb} = \frac{2}{n}$ for all $j \in I\backslash I_k$. Moreover, since $L_{ia} + L_{ib} = \frac{2}{n}$ for all $i \in I_k$, it is evident that $L_{ia} \geqslant L_{ja}$ and $L_{ib} \leqslant L_{jb}$ for all $i \in I_k$ and $j \in I\backslash I_k$. Hence, statement (ii) is verified.

Last, we verify statement (i). We first claim $l > \frac{n}{2}$. Suppose not, i.e., $l \leqslant \frac{n}{2}$ and hence $|I\backslash I_k| = n - l \geqslant \frac{n}{2}$. Since $L_{ja} = 0$ for all $j \in I\backslash I_k$, Lemma 1 implies $L_{jb} = \frac{2}{n}$ for all $j \in I\backslash I_k$. Consequently, $\sum_{i\in I} L_{ib} = \sum_{i\in I_k} L_{ib} + \sum_{j\in I\backslash I_k} L_{jb} = \sum_{i\in I_k} L_{ib} + (n-l)\frac{2}{n} > 1$. Contradiction! Therefore, $l > \frac{n}{2}$.

Let $\bar{P} \equiv (\bar{P}_{I_k}, P_{-I_k}) \in \mathbb{D}^n$ be such that $\bar{P}_i = \bar{P}_j$ for all $i, j \in I_k$, and $a \;\; \bar{P}_i \;\; b$ for all $i \in I_k$. We divide I_k into two groups: $\hat{I} = \{i \in I_k | P_i \neq \bar{P}_i\}$ and $I_k\backslash\hat{I} = \{i \in I_k | P_i = \bar{P}_i\}$. Given $S \subseteq \hat{I}$, let $\bar{P}^S \in \mathbb{D}^n$ be such that $\bar{P}^S_i = \bar{P}_i$ for all $i \in \hat{I}\backslash S$, and $\bar{P}^S_i = P_i$ for all $i \notin \hat{I}\backslash S$. Evidently, $\bar{P}^{\emptyset} = \bar{P}$ and $\bar{P}^{\hat{I}} = P$.

Since $l > \frac{n}{2}$, *sd-efficiency* implies $\varphi_{ja}(\bar{P}^{\emptyset}) = 0$ for all $j \in I\backslash I_k$, and hence $\sum_{i\in I_k}\varphi_{ia}(\bar{P}^{\emptyset}) = 1$. Moreover, since *equal treatment of equals* implies $\varphi_{ia}(\bar{P}^{\emptyset}) = \varphi_{ja}(\bar{P}^{\emptyset})$ for all $i, j \in I_k$, it is true that $\varphi_{ia}(\bar{P}^{\emptyset}) = \frac{1}{l}$ for all $i \in I_k$. Next, we provide an induction argument on S.

Induction Hypothesis: Given $0 < s \leqslant |\hat{I}|$, for all $S \subseteq \hat{I}$ with $0 \leqslant |S| < s$, $\varphi_{ia}(\bar{P}^S) = \frac{1}{l}$ for all $i \in I_k$.

Let $S \subseteq I_k$ with $|S| = s$. We show $\varphi_{ia}(\bar{P}^S) = \frac{1}{l}$ for all $i \in I_k$. Since $l > \frac{n}{2}$, *sd-efficiency* implies $\varphi_{ja}(\bar{P}^S) = 0$ for all $j \in I\backslash I_k$. Thus, $\sum_{i\in I_k}\varphi_{ia}(\bar{P}^S) = 1$. Given $i \in S$, *sd-strategy-proofness* and induction hypothesis imply $\varphi_{ia}(\bar{P}^S) = \varphi_{ia}(P_i, \bar{P}^S_{-i}) = \varphi_{ia}(\bar{P}_i, \bar{P}^S_{-i}) = \varphi_{ia}(\bar{P}^{S\backslash\{i\}}) = \frac{1}{l}$. Furthermore, in $\bar{P}^S$, for all $j \in I_k\backslash S$, *equal treatment of equals* implies $\varphi_{ja}(\bar{P}^S) = \frac{1-\sum_{i\in S}\varphi_{ia}(\bar{P}^S)}{l-s} = \frac{1-s\times\frac{1}{l}}{l-s} = \frac{1}{l}$. Therefore, $\varphi_{ia}(\bar{P}^S) = \frac{1}{l}$ for all $i \in I_k$. This completes the verification of induction hypothesis. Therefore, $L_{ia} = \frac{1}{l}$ for all $i \in I_k$, and hence, $L_{ia} = L_{ja}$ for all $i, j \in I_k$. This completes the verification of statement (i) in Case 3, and hence the lemma. ■

Lemma 3 *Random assignment L satisfies sd-envy-freeness.*

Proof: Given $a \in A$, assume $a \in A_k$. Given $i \in I$, assume $a = r_l(P_i)$. If $A_k = \{a\}$, or $|A_k| = 2$ and $a = \min(P_i, A_k)$, then Lemma 1 implies $\sum_{t=1}^{l} L_{ir_t(P_i)} = \sum_{t=1}^{k}\frac{|A_k|}{n} = \sum_{t=1}^{l} L_{jr_t(P_i)}$ for all $j \neq i$. If $|A_k| = 2$ and $a = \max(P_i, A_k)$, then Lemmas 1 and 2 imply $\sum_{t=1}^{l} L_{ir_t(P_i)} = \sum_{t=1}^{k-1}\frac{|A_k|}{n} + L_{ia} \geqslant \sum_{t=1}^{k-1}\frac{|A_k|}{n} + L_{ja} = \sum_{t=1}^{l} L_{jr_t(P_i)}$ for all $j \neq i$. Therefore, $L_i \;\; P^{sd}_i \;\; L_j$ for all $j \neq i$. Thus, φ satisfies *sd-envy-freeness*. ■

Now, given $1 \leqslant k \leqslant T$, assume $A_k = \{a, b\}$, and let $I_k \equiv \{i \in I | a \;\; P_i \;\; b\}$ and $l \equiv |I_k|$. By *sd-envy-freeness*, we first know that for each pair $i, j \in I_k$, or each pair $i, j \in I \backslash I_k$, $L_{ia} = L_{ja}$ and $L_{ib} = L_{jb}$. Next, by *sd-efficiency* and feasibility, we know that

(i) If $\frac{n}{2} \leqslant l \leqslant n$, then

- $L_{ia} = \frac{1}{l}$ and $L_{ib} = \frac{2}{n} - \frac{1}{l}$ for all $i \in I_k$;
- $L_{ja} = 0$ and $L_{jb} = \frac{2}{n}$ for all $j \in I \backslash I_k$.

(ii) If $0 \leqslant l < \frac{n}{2}$, then

- $L_{ia} = \frac{2}{n}$ and $L_{ib} = 0$ for all $i \in I_k$;
- $L_{ja} = \frac{2}{n} - \frac{1}{n-l}$ and $L_{jb} = \frac{1}{n-l}$ for all $j \in I \backslash I_k$.

Thus, random assignment L satisfies condition 2 of Fact 1. Therefore, L is induced by the PS rule. This completes the verification of the necessity part of Theorem 1. ■

According to the verification of Theorem 1, on a restricted tier domain, we also characterize the PS rule under *sd-efficiency* and *sd-envy-freeness*.

Corollary 1 *On a restricted tier domain, a rule is sd-efficient and sd-envy-free if and only if it is the PS rule.*

Proof: The sufficiency part holds evidently. We focus on the necessity part. Let $\varphi : \mathbb{D}^n \to \mathcal{L}$ be an *sd-efficient* and *sd-envy-free* rule. Fixing $P \equiv (P_1, \ldots, P_n) \in \mathbb{D}^n$, let $L \equiv \varphi(P)$. Fix $1 \leqslant k \leqslant T$. First, *sd-envy-freeness implies that* $L_{iA_k} = L_{jA_k}$ for all $i, j \in I$. Hence, feasibility implies that $L_{iA_k} = \frac{|A_k|}{n}$ for all $i \in I$. Thus, L satisfies condition 1 of Fact 1. Furthermore, in the proof of the necessity part of Theorem 1, note that the verification of Lemma 3 only relies on the application of *sd-efficiency* and *sd-envy-freeness*. Hence, L must also satisfy condition 2 of Fact 1. Therefore, φ is the PS rule. ■

Remark 4 Since every preference profile on a restricted tier domain has rich support on a partition (Heo (2014a)) and is recursively decomposable in the sense of Cho (2016b), by invoking either Theorem 1 in Heo (2014a) or Theorem 3 in Cho (2016b), we can also establish Corollary 1. We use the following examples to illustrate. Let $A \equiv \{a, b, c, d\}$, $\mathcal{P} \equiv (A_1, A_2)$ where $A_1 \equiv \{a, b\}$ and $A_2 \equiv \{c, d\}$), and $I \equiv \{1, 2, 3, 4\}$. Consider two preference profiles on the restricted tier domain $\mathbb{D}(\mathcal{P})$ specified below.

$$P = \begin{pmatrix} P_1 : a \succ b \succ c \succ d \\ P_2 : a \succ b \succ c \succ d \\ P_3 : b \succ a \succ c \succ d \\ P_4 : b \succ a \succ c \succ d \end{pmatrix} \qquad \bar{P} = \begin{pmatrix} \bar{P}_1 : a \succ b \succ c \succ d \\ \bar{P}_2 : a \succ b \succ c \succ d \\ \bar{P}_3 : b \succ a \succ d \succ c \\ \bar{P}_4 : a \succ b \succ d \succ c \end{pmatrix}$$

According to profile P, in the tier A_2, since every agent prefers c to d, we refine the tier structure $\mathcal{P}$ to $\mathcal{P}' \equiv (A_1, A_2^1, A_2^2) \equiv (\{a, b\}, \{c\}, \{d\})$. Thus, profile P has rich support on partition $\mathcal{P}'$, and hence the PS rule is the unique one satisfying *sd-efficiency* and *sd-envy-freeness*.

According to profile $\bar{P}$, we first partition the objects into $A_1 \equiv \{a, b\}$ and $A_2 \equiv \{c, d\}$ and construct a type 1 decomposition in Cho (2016b). Then, *sd-efficiency* and *sd-envy-freeness* requires that each agent should consume $\frac{1}{2}$ of $\{a, b\}$ and $\frac{1}{2}$ of $\{c, d\}$. Next, we partition $A_2 \equiv \{c, d\}$ into $\{c\}$ and $\{d\}$ and agents into $\{1, 2\}$ and $\{3, 4\}$. Thus, we construct a type 2 decomposition in Cho (2016b). Then, by *sd-efficiency* and *sd-envy-freeness*, agents 1 and 2 both consume $\frac{1}{2}$ of c; and agents 3 and 4 both consume $\frac{1}{2}$ of d. Last, we "partition" $\{a, b\}$ into $\{a, \frac{1}{2}$ of $b\}$, and $\{\frac{1}{2}$ of $b\}$, and partition agents into $\{1, 2, 4\}$ and $\{3\}$. In this way, we construct a type 3 decomposition in Cho (2016b). Then, *sd-efficiency* and *sd-envy-freeness* implies that agents $1, 2$ and 4 share a equally and each obtains $\frac{1}{6}$ of b, while agent 3 receives $\frac{1}{2}$ of b.

3.2 Necessity: A characterization of restricted tier domains

We have proposed a class of restricted domains, *restricted tier domains*, which is sufficient for the admission of an *sd-strategy-proof*, *sd-efficient* and *sd-envy-free* (or *equal-treatment-of-equals*) rule, specifically the PS rule. Despite of the significant restriction and small cardinality of *restricted tier domains* (recall Remark 3), we show in this section that a restricted tier structure is necessary for the existence of an *sd-strategy-proof*, *sd-efficient* and *sd-envy-free* (or *equal-treatment-of-equals*) rule, provided a mild richness condition.

We first introduce the richness condition: *connectedness* (Monjardet (2009)). Two preferences $P_i, P_i' \in \mathbb{D}$ are **adjacent**, denoted $P_i \sim^A P_i'$, if there exist $x, y \in A$ such that

(i) $x = r_k(P_i) = r_{k+1}(P_i')$ and $y = r_{k+1}(P_i) = r_k(P_i')$ for some $1 \leqslant k \leqslant n-1$; and

(ii) $r_l(P_i) = r_l(P_i')$ for all $l \neq k$.

Accordingly, a domain $\mathbb{D}$ is **connected** if for every pair of distinct preferences $P_i, P_i' \in \mathbb{D}$, there exists a sequence of consecutively adjacent preferences (in other words, a *path*) $\{P_i^k\}_{k=1}^t \subseteq \mathbb{D}$ connecting P_i and P_i', i.e., $P_i^1 = P_i$, $P_i^t = P_i'$ and $P_i^k \sim^A P_i^{k+1}$, $k = 1, \ldots, t-1$. Intuitively, connectedness implies that the difference of two preferences in the domain can be reconciled via a sequence of local switchings. When a domain is interpreted as a collection of opinions in a society (Puppe (2016)), connectedness implies that the society's opinions are sufficiently dispersed.

Remark 5 The notion of connectedness is introduced in Monjardet (2009) for the study of maximal Condorcet domains. Recently, it has been identified by Sato (2013) as a necessary condition for the equivalence of local and global strategy-proofness in deterministic voting. Note that many well studied domains are connected, including the universal domain (Gibbard (1973) and Satterthwaite (1975)), the single-peaked domain (Moulin (1980) and Demange (1982)), the single-dipped domain (Barberà et al. (2012)), and maximal single-crossing domains (Saporiti (2009) and Carroll (2012)).

We now present the domain characterization result.

Theorem 2 *If a connected domain admits an sd-strategy-proof, sd-efficient and sd-envy-free rule, it is a restricted tier domain.*

Proof: First, note that if $\mathbb{D}$ contains exactly two preferences, by connectedness, it is evident that it is a *restricted tier domain*. Henceforth, we assume that $\mathbb{D}$ contains at least three preferences. Let $\varphi : \mathbb{D}^n \to \mathcal{L}$ be an *sd-strategy-proof*, *sd-efficient* and *sd-envy-free* rule. To prove Theorem 2, we first introduce an important terminology, the *elevating property*.

Definition 2 *A domain satisfies the* ***elevating property*** *if there exist three preferences $\bar{P}_i, P_i, \hat{P}_i$, three objects a, b, c and a ranking position $1 \leqslant k \leqslant n-2$ such that the following three conditions are satisfied.*

1. $a = r_k(\bar{P}_i) = r_k(P_i) = r_{k+1}(\hat{P}_i)$.
2. $b = r_{k+2}(\bar{P}_i) = r_{k+1}(P_i) = r_k(\hat{P}_i)$.
3. $c = r_{k+1}(\bar{P}_i) = r_{k+2}(P_i) = r_{k+2}(\hat{P}_i)$.
4. $B(\bar{P}_i, a) = B(P_i, a) = B(\hat{P}_i, b)$.[10]

We use Table 1 below to illustrate the *elevating property*:

$$
\begin{array}{rccccccccc}
\text{Ranking:} & & & k & & k+1 & & k+2 & & \\
\bar{P}_i: & \underbrace{\cdots\cdots\cdots}_{B(\bar{P}_i, a)} & \succ & a & \succ & c & \succ & b & \succ & \cdots\cdots\cdots \\
 & \shortparallel & & & & & & & & \\
P_i: & \overbrace{\underbrace{\cdots\cdots\cdots}_{B(P_i, a)}}^{B(P_i, a)} & \succ & a & \succ & b & \succ & c & \succ & \cdots\cdots\cdots \\
 & \shortparallel & & & & & & & & \\
\hat{P}_i: & \overbrace{\cdots\cdots\cdots}^{B(\hat{P}_i, b)} & \succ & b & \succ & a & \succ & c & \succ & \cdots\cdots\cdots
\end{array}
$$

Table 1: The elevating property

Recall preferences P_1, P_2 and P_3 in Example 1. Note that objects a, b and c cluster in 3 ranking positions of these preferences; three corresponding upper contour sets are empty (and hence identical); and moreover, object b is elevated from the third ranking position in P_1 to the second in P_2, and then is successively elevated to the top of P_3. (This is a problem. Either remove this part or change the preference in Example 1.) Many well known voting domains satisfy the *elevating property*.[11] In a contrary, since each object takes at most two positions in *all* preferences of a *restricted tier domain*, it is evident that *restricted tier domains* always violate the elevating property. Lemma 4 below shows that domain $\mathbb{D}$ must violate the elevating property since it is the key for the incompatibility of *sd-strategy-proofness*, *sd-efficiency* and *sd-envy-freeness*.

[10] Note that within the upper contour set, the relative rankings of objects in three preferences are arbitrary.

[11] See for instance, the universal domain (Gibbard (1973)), the single-peaked domain (Moulin (1980) and Demange (1982)), the single-dipped domain (Barberà et al. (2012)), all the maximal single-crossing domains (Saporiti (2009)), the multi-dimensional single-peaked domain (Barberà et al. (1993)) and the separable domain (Le Breton and Sen (1999)), some linked domains (Aswal et al. (2003)) and some circular domains (Sato (2010)).

Lemma 4 *Domain $\mathbb{D}$ violates the elevating property.*

Proof: Suppose that $\mathbb{D}$ satisfy the *elevating property*. Specifically, assume that $\mathbb{D}$ contains three preferences in Table 1. Let $B \equiv B(\bar{P}_i, a) = B(P_i, a) = B(\hat{P}_i, b)$ for notational convenience. Thus, $|B| = k - 1$. In the detailed verification below, we consider four particular profiles:

(i) P, where every agent presents preference P_i in Table 1,

(ii) $(\bar{P}_1, P_{-1})$, where agent 1 deviates at P via $\bar{P}_i$ in Table 1,

(iii) $(\hat{P}_2, P_{-2})$, where agent 2 deviates at P via $\hat{P}_i$ in Table 1, and

(iv) $(\bar{P}_1, \hat{P}_2, P_{-\{1,2\}})$, where agent 2 deviates at $(\bar{P}_1, P_{-1})$ via $\hat{P}_i$ in Table 1.

First, at all four profiles, *sd-envy-freeness* and feasibility imply that the cumulative probability placed on subset B for each agent is fixed to $\frac{k-1}{n}$ which is identical to that given by the PS rule. Next, at all these four preference profiles, we only focus on the probabilities assigned to objects a and b. We first show that at profiles P, $(\bar{P}_1, P_{-1})$ and $(\hat{P}_2, P_{-2})$, these probabilities induced by φ are the same as those induced by the PS rule. Last, we show that, under $(\bar{P}_1, \hat{P}_2, P_{-\{1,2\}})$, *sd-strategy-proofness* implies that agent 2's probability of receiving a is the same as that given by the PS rule, while *sd-efficiency* requires that the probability of agent 2 getting b is higher than that given by the PS rule. Consequently, every agent other than 1 and 2 envies agent 2.

By *sd-envy-freeness* and feasibility, it is evident that $\sum_{x \in B} \varphi_{ix}(P) = \sum_{x \in B} \varphi_{ix}(\bar{P}_1, P_{-1}) = \sum_{x \in B} \varphi_{ix}(\hat{P}_2, P_{-2}) = \sum_{x \in B} \varphi_{ix}(\bar{P}_1, \hat{P}_2, P_{-\{1,2\}}) = \frac{k-1}{n}$ for all $i \in I$.

Now, we start with profile P. By *sd-envy-freeness*, $\varphi_{ix}(P) = \frac{1}{n}$ for all $i \in I$ and $x \in A$. Next, we consider profile $(\bar{P}_1, P_{-1})$.

Claim 1: The following two statements hold:

(i) $\varphi_{ia}(\bar{P}_1, P_{-1}) = \frac{1}{n}$ for all $i \in I$ by *sd-envy-freeness*.

(ii) $\varphi_{1b}(\bar{P}_1, P_{-1}) = 0$ by *sd-efficiency*, and $\varphi_{ib}(\bar{P}_1, P_{-1}) = \frac{1}{n-1}$ for all $i \neq 1$ by *sd-envy-freeness*.

Next, we consider profile $(\hat{P}_2, P_{-2})$.

Claim 2: The follow two statements hold:

(i) $\varphi_{2a}(\hat{P}_2, P_{-2}) = 0$ by *sd-efficiency*, and $\varphi_{2b}(\hat{P}_2, P_{-2}) = \frac{2}{n}$ by *sd-strategy-proofness* according to $\varphi_2(P)$.

(ii) $\varphi_{ia}(\hat{P}_2, P_{-2}) = \frac{1}{n-1}$ for all $i \neq 2$ and $\varphi_{ib}(\hat{P}_2, P_{-2}) = \frac{2}{n} - \frac{1}{n-1}$ for all $i \neq 2$ by *sd-envy-freeness* and Claim 2(i).

Last, we consider profile $(\bar{P}_1, \hat{P}_2, P_{-\{1,2\}})$.

Claim 3: The following two statements hold:

(i) $\varphi_{1a}(\bar{P}_1, \hat{P}_2, P_{-\{1,2\}}) = \frac{1}{n-1}$ by *sd-strategy-proofness* according to $\varphi_1(\hat{P}_2, P_{-2})$ and Claim 2, and $\varphi_{1b}(\bar{P}_1, \hat{P}_2, P_{-\{1,2\}}) = 0$ by *sd-efficiency*.

(ii) $\varphi_{2a}(\bar{P}_1, \hat{P}_2, P_{-\{1,2\}}) = 0$ by *sd-efficiency*, and $\varphi_{2b}(\bar{P}_1, \hat{P}_2, P_{-\{1,2\}}) = \frac{1}{n} + \frac{1}{n-1}$ by *sd-strategy-proofness* according to $\varphi_2(\bar{P}_1, P_{-1})$ and Claim 1.

Now, by feasibility and Claim 3 (i) and (ii), we know that for all $i \notin \{1,2\}$,

$$\begin{aligned}
\varphi_{ia}(\bar{P}_1, \hat{P}_2, P_{-\{1,2\}}) &= \frac{1}{n-2}\Big[1 - \sum_{j\in\{1,2\}} \varphi_{ja}(\bar{P}_1, \hat{P}_2, P_{-\{1,2\}})\Big] = \frac{1}{n-1}, \\
\varphi_{ib}(\bar{P}_1, \hat{P}_2, P_{-\{1,2\}}) &= \frac{1}{n-2}\Big[1 - \sum_{j\in\{1,2\}} \varphi_{jb}(\bar{P}_1, \hat{P}_2, P_{-\{1,2\}})\Big] = \frac{1}{n-2}(1 - \frac{1}{n} - \frac{1}{n-1}).
\end{aligned}$$

Consequently, between agent 2 and any agent $i \notin \{1,2\}$, we have $\sum\limits_{x\in B\cup\{a,b\}} \varphi_{2x}(\bar{P}_1, \hat{P}_2, P_{-\{1,2\}}) = \frac{k-1}{n} + 0 + (\frac{1}{n} + \frac{1}{n-1}) > \frac{k-1}{n} + \frac{1}{n-1} + \frac{1}{n-2}(1 - \frac{1}{n} - \frac{1}{n-1}) = \sum\limits_{x\in B\cup\{a,b\}} \varphi_{ix}(\bar{P}_1, \hat{P}_2, P_{-\{1,2\}})$. This contradicts *sd-envy-freeness* and we hence completes the verification of Lemma 4. ■

Henceforth, we will use the information of violating the *elevating property* in Lemma 4 to characterize the restricted tier structure in domain $\mathbb{D}$.

Lemma 5 *For every path $\{P_i^k\}_{k=1}^t$, if there exists $1 \leqslant l \leqslant n-1$ such that $r_l(P_i^1) = a$, $r_{l+1}(P_i^1) = b$ and $r_l(P_i^k) = b$ for all $k = 2, \ldots, t$, then $r_{l+1}(P_i^k) = a$ for all $k = 2, \ldots, t$.*

Proof: Given a path $\{P_i^k\}_{k=1}^t$, consider $l = n-1$. Thus, $r_{n-1}(P_i^1) = a$, $r_n(P_i^1) = b$, and $r_{n-1}(P_i^k) = b$ for all $k = 2, \ldots, t$. Suppose $r_n(P_i^2) \equiv c \neq a$. Thus, $cP_i^1 b$ (recall that b is the bottom ranked object) and $bP_i^2 c$. Therefore, the local switching pair in P_i^1 and P_i^2 is b and c. Consequently, it must be the case that $r_{n-1}(P_i^1) = c \neq a$. Contradiction! Therefore, $r_n(P_i^2) = a$. Next, consider P_i^3, and suppose $r_n(P_i^3) \equiv c \neq a$. Thus, $cP_i^2 a$ (recall that a is the bottom ranked object) and $aP_i^3 c$. Therefore, the local switching pair in P_i^2 and P_i^3 is c and a. Consequently, it must be the case that $r_{n-1}(P_i^2) = c \neq b$. Contradiction! Therefore, $r_n(P_i^3) = a$. Applying the same argument along the path, we can show that $r_n(P_i^k) = a$ for all $k = 4, \ldots, t$. We next adopt an induction argument.

Induction Hypothesis: Given $1 \leqslant l \leqslant n-1$, for every path $\{P_i^k\}_{k=1}^t$, if there exists $l < l' \leqslant n-1$ such that $r_{l'}(P_i^1) = a$, $r_{l'+1}(P_i^1) = b$, and $r_{l'}(P_i^k) = b$ for all $k = 2, \ldots, t$, then $r_{l'+1}(P_i^k) = a$ for all $k = 2, \ldots, t$.

Now, given a path $\{P_i^k\}_{k=1}^t$, assume that $r_l(P_i^1) = a$, $r_{l+1}(P_i^1) = b$; and $r_l(P_i^k) = b$ for all $k = 2, \ldots, t$. We will show that $r_{l+1}(P_i^k) = a$ for all $k = 2, \ldots, t$.

Since $P_i^1 \sim^A P_i^2$, it is evident that $r_{l+1}(P_i^2) = a$. Suppose that there exists $2 \leqslant \bar{k} \leqslant t$ such that $r_{l+1}(P_i^{\bar{k}}) \neq a$. Assume $r_{l+1}(P_i^{\bar{k}}) = c$. Evidently, $\bar{k} > 2$ and $c \notin \{a,b\}$. Moreover, we can assume $r_{l+1}(P_i^k) = a$ for all $2 \leqslant k \leqslant \bar{k}-1$. Since $P_i^{\bar{k}-1} \sim^A P_i^{\bar{k}}$, $r_l(P_i^{\bar{k}-1}) = r_l(P_i^{\bar{k}}) = b$, and $r_{l+1}(P_i^{\bar{k}-1}) = a \neq c = r_{l+1}(P_i^{\bar{k}})$, it must be the case that $r_{l+2}(P_i^{\bar{k}-1}) = c$ and $r_{l+2}(P_i^{\bar{k}}) = a$. Now, consider the path $\{P_i^{\bar{k}}, P_i^{\bar{k}-1}, \ldots, P_i^2\}$. Since $r_{l+1}(P_i^{\bar{k}}) = c$, $r_{l+2}(P_i^{\bar{k}}) = a$ and $r_{l+1}(P_i^k) = a$ for all $k = \bar{k}-1, \ldots, 2$, induction hypothesis implies $r_{l+2}(P_i^k) = c$ for all $k = \bar{k}-1, \ldots, 2$. Furthermore, since $P_i^1 \sim^A P_i^2$, $r_{l+2}(P_i^1) = r_{l+2}(P_i^2) = c$. Along the sub-path $\{P_i^k\}_{k=1}^{\bar{k}}$, since a, b and c take positions l, $l+1$ and $l+2$ in every preference, it is easy to verify that the sets of top $l-1$ ranked objects are identical for all preferences. Thus, $B(b, P_i^{\bar{k}}) = B(b, P_i^{\bar{k}-1}) = B(a, P_i^1)$. Consequently, preferences $P_i^{\bar{k}}, P_i^{\bar{k}-1}$ and P_i^1 indicates that domain $\mathbb{D}$ satisfies the *elevating property* (see the table below). Contradiction to Lemma 4!

$$\begin{array}{rcccccccccc}
\text{Ranking:} & & & l & & l+1 & & l+2 & & \\
P_i^{\bar{k}}: & \underbrace{\cdots\cdots\cdots}_{B(P_i^{\bar{k}},b)} & \succ & b & \succ & c & \succ & a & \succ & \cdots\cdots\cdots \\
 & \parallel & & & & & & & & \\
P_i^{\bar{k}-1}: & \overbrace{\underbrace{\cdots\cdots\cdots}_{B(P_i^{\bar{k}-1},b)}}^{B(P_i^{\bar{k}-1},b)} & \succ & b & \succ & a & \succ & c & \succ & \cdots\cdots\cdots \\
 & \parallel & & & & & & & & \\
P_i^{1}: & \overbrace{\cdots\cdots\cdots}^{B(P_i^{1},a)} & \succ & a & \succ & b & \succ & c & \succ & \cdots\cdots\cdots
\end{array}$$

Therefore, $r_{l+1}(P_i^k) = a$ for all $k = 2, \ldots, t$. This completes the verification of induction hypothesis and hence the lemma. ■

Lemma 6 *For every path $\{P_i^k\}_{k=1}^t$, if there exists $1 \leqslant l \leqslant n-1$ such that $r_l(P_i^1) = a$, $r_{l+1}(P_i^1) = b$, and $r_{l+1}(P_i^k) = a$ for all $k = 2, \ldots, t$, then $r_l(P_i^k) = b$ for all $k = 2, \ldots, t$.*

Proof: The verification of this lemma is symmetric to Lemma 5. The induction argument in the proof of Lemma 5 starts from the bottom (i.e., $l = n-1$) and proceeds successively up to the top (i.e., $l = 1$). To verify this lemma, an analogous induction argument can be adopted from the top (i.e., $l = 1$) down to the bottom (i.e., $l = n-1$). ■

Lemma 7 *Given $P_i, P_i' \in \mathbb{D}$, assume $P_i \sim^A P_i'$, $a = r_l(P_i) = r_{l+1}(P_i')$ and $b = r_{l+1}(P_i) = r_l(P_i')$. In every preference, objects a and b occupy positions l and $l+1$, i.e., $\{r_l(P_j), r_{l+1}(P_j)\} = \{a, b\}$ for all $P_j \in \mathbb{D}$.*

Proof: It is evident that $\{r_l(P_i), r_{l+1}(P_i)\} = \{a, b\}$ and $\{r_l(P_i'), r_{l+1}(P_i')\} = \{a, b\}$. Next, fix an arbitrary $P_j \in \mathbb{D}\backslash\{P_i, P_i'\}$, and we show $\{r_l(P_j), r_{l+1}(P_j)\} = \{a, b\}$. Since $\mathbb{D}$ is connected and $P_i \sim^A P_i'$, it is true that there exists a path $\{P_i^k\}_{k=1}^t \subseteq \mathbb{D}$ such that $\{P_i^1, P_i^2\} = \{P_i, P_i'\}$ and $P_i^t = P_j$. We assume $P_i^1 = P_i$ and $P_i^2 = P_i'$. The verification of the situation $P_i^1 = P_i'$ and $P_i^2 = P_i$ is symmetric and we hence omit it.

If $r_l(P_i^k) = b$ for all $k = 3, \ldots, t$, then Lemma 5 implies $\{r_l(P_j), r_{l+1}(P_j)\} = \{a, b\}$. Next, we assume that there exists $3 \leqslant k \leqslant t$ such that $r_l(P_i^k) \neq b$. We highlight the subset $\{k_j\}_{j=1}^{\nu} \subseteq \{3, \ldots, t\}$ such that $r_l(P_i^{k_j}) \neq r_l(P_i^{k_j-1})$, $j = 1, \ldots, \nu$. Since there exists $3 \leqslant k \leqslant t$ such that $r_l(P_i^k) \neq b$, the set of preferences $\{P_i^{k_j}\}_{j=1}^{\nu}$ is not empty. Moreover, we can separate the path $\{P_i^k\}_{k=1}^t$ into $\nu+1$ parts according to the l-th ranked object in each preference, i.e., $r_l(P_i^2) = \cdots = r_l(P_i^{k_1-1})$, $r_l(P_i^{k_1}) = \cdots = r_l(P_i^{k_2-1})$, ... , $r_l(P_i^{k_{\nu}-1}) = \cdots = r_l(P_i^{k_{\nu}-1})$, and $r_l(P_i^{k_\nu}) = \cdots = r_l(P_i^t)$ (see the table below).

$$\underbrace{P_i^2,\dots,P_i^{k_1-1}}_{\substack{\text{the same } l\text{th}\\ \text{ranked alternative}}} \;;\; \underbrace{P_i^{k_1},\dots,P_i^{k_2-1}}_{\substack{\text{the same } l\text{th}\\ \text{ranked alternative}}} \;;\; P_i^{k_2},\dots\dots,P_i^{k_{\nu-1}-1};\; \underbrace{P_i^{k_{\nu-1}},\dots,P_i^{k_\nu-1}}_{\substack{\text{the same } l\text{th}\\ \text{ranked alternative}}} \;;\; \underbrace{P_i^{k_\nu},\dots,P_i^{t}}_{\substack{\text{the same } l\text{th}\\ \text{ranked alternative}}}$$

Evidently, $r_l(P_i^2) = \cdots = r_l(P_i^{k_1-1}) = b$. Then, we apply Lemma 5 on the sub-path $\{P_i^1, P_i^2, \dots P_i^{k_1-1}\}$ and obtain $r_{l+1}(P_i^k) = a$ for all $k = 2, \dots, k_1 - 1$.

Claim 1: $r_l(P_i^{k_1}) = a$.

Evidently, $r_l(P_i^{k_1}) \neq r_l(P_i^{k_1-1}) = b$. Suppose $r_l(P_i^{\bar{k}}) = c \neq a$. Thus, $c \notin \{a, b\}$. Since $P_i^{k_1-1} \sim^A P_i^{k_1}$ and $r_l(P_i^{k_1-1}) = b \neq c = r_l(P_i^{k_1})$, it must be the case that $r_{l+1}(P_i^{k_1}) = a$. Furthermore, since $r_{l+1}(P_i^{k_1-1}) = r_{l+1}(P_i^{k_1}) = a$, it is true that $r_{l-1}(P_i^{k_1}) = b$ and $r_{l-1}(P_i^{k_1-1}) = c$. Now, we can apply Lemma 6 on the sub-path $\{P_i^{k_1}, P_i^{k_1-1}, \dots, P_i^2\}$ and obtain $r_{l-1}(P_i^k) = c$ for all $k = k_1 - 1, \dots, 2$. Moreover, since $P_i^1 \sim^A P_i^2$, $r_{l-1}(P_i^1) = r_{l-1}(P_i^2) = c$. Furthermore, it is easy to verify that the set of top $l - 2$ ranked objects in each preference of the sub-path $\{P_i^k\}_{k=1}^{k_1}$ is identical. Thus, $B(c, P_i^1) = B(c, P_i^{k_1-1}) = B(b, P_i^{k_1})$. Consequently, preferences $P_i^1, P_i^{k_1-1}$ and $P_i^{k_1}$ indicates that domain $\mathbb{D}$ satisfies the *elevating property* (see the table below). Contradiction to Lemma 4!

$$\begin{array}{lcccccccl}
\text{Ranking:} & & & l-1 & & l & & l+1 & \\
P_i^1: & \underbrace{\cdots\cdots\cdots}_{B(P_i^1, c)} & \succ & c & \succ & a & \succ & b & \succ \cdots\cdots\cdots \\
& || & & & & & & & \\
P_i^{k_1-1}: & \overbrace{\underbrace{\cdots\cdots\cdots}_{B(P_i^{k_1-1}, c)}}^{B(P_i^{k_1-1}, c)} & \succ & c & \succ & b & \succ & a & \succ \cdots\cdots\cdots \\
& || & & & & & & & \\
P_i^{k_1}: & \overbrace{\cdots\cdots\cdots}^{B(P_i^{k_1}, b)} & \succ & b & \succ & c & \succ & a & \succ \cdots\cdots\cdots
\end{array}$$

This completes the verification of the claim.

Now, we know $r_l(P_i^{k_1}) = \cdots = r_l(P_i^{k_2-1}) = a$. Applying Lemma 5 on $\{P_i^{k_1-1}, P_i^{k_1}, \dots, P_i^{k_2-1}\}$, we have $r_{l+1}(P_i^{k_1}) = \cdots = r_{l+1}(P_i^{k_2-1}) = b$. Along the path $\{P_i^k\}_{k=1}^t$, repeatedly applying the symmetric argument above, we finally have $\{r_l(P_j), r_{l+1}(P_j)\} = \{a, b\}$. ■

Now, we are ready to reveal the *restricted tier structure* in domain $\mathbb{D}$. If there exists $a \in A$ such that $r_1(P_i) = a$ for all $P_i \in \mathbb{D}$, let $A_1 = \{a\}$. If there exist $P_i, P_i' \in \mathbb{D}$ such that $r_1(P_i) \equiv a \neq b \equiv r_1(P_i')$, connectedness implies that there must exist $\bar{P}_i, \bar{P}_i' \in \mathbb{D}$ such that $\bar{P}_i \sim^A \bar{P}_i'$, $r_1(\bar{P}_i) = a$ and $r_1(\bar{P}_i') = b$. Thus, $r_2(\bar{P}_i) = b$ and $r_2(\bar{P}_i') = a$. Then, Lemma 7 implies $\{r_1(P_i), r_2(P_i)\} = \{a, b\}$ for all $P_i \in \mathbb{D}$. Then, let $A_1 = \{a, b\}$.

Assume $|A_1| = l$ (either $l = 1$ or $l = 2$). If there exists $x \in A$ such that $r_{l+1}(P_i) = x$ for all $P_i \in \mathbb{D}$, let $A_2 = \{x\}$. Next, assume that there exist $P_i, P_i' \in \mathbb{D}$ such that $r_{l+1}(P_i) \equiv x \neq y \equiv r_{l+1}(P_i')$. Since the set of top l ranked objects in every preference is identical, connectedness implies that there must exist $\bar{P}_i, \bar{P}_i' \in \mathbb{D}$ such that $\bar{P}_i \sim^A \bar{P}_i'$, $r_{l+1}(\bar{P}_i) = x$ and $r_{l+1}(\bar{P}_i') = y$. Thus, $r_{l+2}(\bar{P}_i) = y$ and $r_{l+2}(\bar{P}_i') = x$. Then, Lemma 7 implies $\{r_{l+1}(P_i), r_{l+2}(P_i)\} = \{x, y\}$ for all $P_i \in \mathbb{D}$. Then, let $A_2 = \{x, y\}$.

Applying the symmetric argument repeatedly, since A is finite, we can generate tiers $A_1, A_2, \ldots, A_T$ such that (i) $A_k \cap A_{k'} = \emptyset$ for all $1 \leqslant k < k' \leqslant T$ and $\cup_{k=1}^{T} A_k = A$, (ii) $1 \leqslant |A_k| \leqslant 2$ for all $1 \leqslant k \leqslant T$, and (iii) for all $1 \leqslant k < k' \leqslant T$, $[a \in A_k$ and $b \in A_{k'}] \Rightarrow [aP_ib$ for all $P_i \in \mathbb{D}]$. In conclusion, domain $\mathbb{D}$ is a *restricted tier domain*. ■

We now weaken *sd-envy-freeness* to *equal treatment of equals*, and investigate the connected domains which admit an *sd-strategy-proof*, *sd-efficient*, and *equal-treatment-of-equals* rule. Surprisingly, such weakening does not expand the characterized domains, i.e., they are still *restricted tier domains*.

Theorem 3 *If a connected domain admits an sd-strategy-proof, sd-efficient and equal-treatment-of-equals rule, it is a restricted tier domain.*

Lemmas 5 - 7 remain valid for the proof of Theorem 3. However, the proof of Lemma 4 becomes significantly complicated as we weaken *sd-envy-freeness* to *equal treatment of equals*. Therefore, we relegate the proof of Theorem 3 to the Appendix.

As restricted tier domains are characterized in Theorems 2 and 3 under different fairness axioms, it suggests that the source of restriction power that pins down the *restricted tier domains* arises mainly from the resolution of the conflict between *sd-strategy-proofness* and *sd-efficiency* under the *elevating property*.

Since the weakening of fairness axiom in Theorem 3 does not expand the characterized domain in Theorem 2 and more importantly the proof of Theorem 2 is significantly simpler and conveys the central logic of the proof of Theorem 3, we believe that Theorem 2 is of special interest and present it in the first place.

Our proof of Theorem 3 uses the proof strategy introduced by Chang and Chun (2016) in their impossibility which says that there is no *sd-strategy-proof*, *sd-efficient*, and *equal-treatment-of-equals* rule on a domain that includes three particular preferences such that one object takes the last three ranking positions respectively and all the other objects are identically ranked in these three preferences. Their preference structure is actually a special case of our *elevating property*! In the Appendix, we establish the impossibility of the existence of an *sd-strategy-proof*, *sd-efficient* and *equal-treatment-of-equals* rule on a domain satisfying the *elevating property* which hence generalizes the impossibility theorem in Chang and Chun (2016). We believe that such a generalization is significant since it allows first the elevated object to take arbitrary three consecutive positions; second the other objects to be arbitrarily ranked as long as the truncation sets up to the elevating positions in three preferences are the same; and more importantly, our impossibility result under the *elevating property* appears to be informative and is hence repeatedly referred to for establishing Theorem 3. Last, our proof slightly improves theirs in logical conciseness and fluency (see for instance, footnote **13**).

We conclude this section by emphasizing insightful light shed by our domain characterization results on the direction of identifying a unified necessary and sufficient condition for the existence of an *sd-strategy-proof*, *sd-efficient* and *sd-envy-free* or *equal-treatment-of-equals* rule. When we encounter with a preference domain which fails connectedness but admits an *sd-strategy-proof*, *sd-efficient* and *sd-envy-free* or *equal-treatment-of-equals* rule, we first partition the domain into several connected subdomains. Thus, Theorem 2 or 3 implies that each subdomain must be a *restricted tier domain*. Therefore, to completely reveal the domain structure, one needs to resolve this problem: *what are the relations among the restricted tier structures of these subdomains*? For instance, more specifically, if two *restricted tier subdomains* share an identical set of tiers, how is this set of tiers systematically organized in two distinct restricted tier structures? Recently, an independent research of Liu (2016) provides a clear answer to this question.

4 A generalized model with outside options

In this section, we extend our model to situations in which the number of agents may differ from the number of objects, and each agent has an outside option. In the generalized model, the characterizations of the PS rule in Theorem 1 still hold. This extension can be viewed as a strengthening of Bogomolnaia and Moulin (2002) since their domain is strictly nested in the class of domains investigated in this section.

Let $m \equiv |A|$ and $n \equiv |I|$. Moreover, there is an object $\varnothing$ with at least n copies. Object $\varnothing$ can be interpreted as an individual outside option for each agent. Each agent i has a strict preference order P_i over $A \cup \{\varnothing\}$. An object $a \in A$ is *acceptable* if $a \ P_i \ \varnothing$. Let $\mathcal{A}(P_i)$ denote the set of acceptable objects in P_i.

Since the number of agents may differ from the number of objects, it may be that an object is not fully shared by all agents. Accordingly, the definition of an assignment $[L_{ia}]_{i \in I, a \in A \cup \{\varnothing\}}$ is modified in such a way that (i) $L_{ia} \geqslant 0$ for all $i \in I$ and $a \in A \cup \{\varnothing\}$, (ii) $\sum_{a \in A \cup \{\varnothing\}} L_{ia} = 1$ for all $i \in I$, and (iii) $0 \leqslant \sum_{i \in I} L_{ia} \leqslant 1$ for all $a \in A$.

All original axioms of *sd-strategy-proofness*, *sd-efficiency*, *sd-envy-freeness*, and *equal treatment of equals* apply without any modification. Also, the definition of the PS rule remains unchanged. Evidently, the PS rule remains *sd-efficient* and *sd-envy-free*.

However, we need to modify the definition of a *restricted tier domain* of preferences. Notably, we require restricted tier structure only on the acceptable objects.

Definition 3 *A domain $\mathbb{D}$ is an **augmented restricted tier domain** if there exists a restricted tier structure $\mathcal{P} \equiv (A_k)_{k=1}^T$ (over A, not $A \cup \{\varnothing\}$) such that*

1. *For all $1 \leqslant k \leqslant T$, $|A_k| \leqslant 2$;*

2. *Given $P_i \in \mathbb{D}$, $\mathcal{A}(P_i) = \cup_{k=1}^t A_k$ for some $0 \leqslant t \leqslant T$;*

3. *Given $P_i \in \mathbb{D}$ and $a, b \in A$, $[a \in A_k, b \in A_{k'}, a, b \in \mathcal{A}(P_i)$ and $k < k'] \Rightarrow [a \ \ P_i \ \ b]$.*

Example 2 Let $|A| = 5$ and $\mathcal{P} \equiv (A_1, A_2, A_3)$ where $A_1 = \{a_1, a_2\}$, $A_2 = \{a_3\}$, and $A_3 = \{a_4, a_5\}$. Then $\mathbb{D} = \{P_1, P_2, P_3, P_4\}$ is an *augmented restricted tier domain* associated to $\mathcal{P}$.

$$
\begin{array}{lccccccccccc}
P_1: & \varnothing & \succ & \cdots & & & & & & & & \\
P_2: & a_1 & \succ & a_2 & \succ & \varnothing & \succ & \cdots & & & & \\
P_3: & a_2 & \succ & a_1 & \succ & a_3 & \succ & \varnothing & \succ & \cdots & & \\
P_4: & a_1 & \succ & a_2 & \succ & a_3 & \succ & a_5 & \succ & a_4 & \succ & \varnothing
\end{array}
$$

In words, an agent with P_1 perceives every object as unacceptable, and all unacceptable objects are ranked arbitrarily. An agent with P_2 perceives only A_1 as acceptable and she prefers a_1 to a_2. An agent with P_3 perceives A_1 and A_2 as acceptable and A_3 as unacceptable. In addition, she prefers all objects in A_1 to all objects in A_2 according to the tier structure. Last, in P_4, all objects are acceptable, and ranked according to the tier structure restriction. $\square$

Analogous to Theorems 1, Theorem 4 below characterizes the PS rule on *augmented restricted tier domains*.

Theorem 4 *On an augmented restricted tier domain, a rule is sd-strategy-proof, sd-efficient and equal-treatment-of-equals if and only if it is the PS rule.*

Proof: Given $\mathcal{P} \equiv (A_k)_{k=1}^T$, let $\mathbb{D} \subseteq \mathbb{D}(\mathcal{P})$ be a *augmented restricted tier domain*.

Given $P \in \mathbb{D}^n$ and $1 \leqslant k \leqslant T$, let $N_k \equiv \{i \in I | A_k \subseteq \mathcal{A}(P_i)\}$ denote the set of agents whose acceptable set includes tier A_k, and $n_k \equiv |N_k|$. Given $1 \leqslant k \leqslant T$, if $n_k > 0$ (equivalently, tier A_k is acceptable for some agent), it is true that $n_{k'} > 0$ for all $1 \leqslant k' \leqslant k-1$ (equivalently, each tier $A_{k'}$, $1 \leqslant k' < k-1$, is also acceptable for some agent). Therefore, given $1 \leqslant k \leqslant T$ with $n_k > 0$, we can define $r_{k'} = \sum_{t=1}^{k'-1} \frac{|A_t|}{n_t}$ for all $1 \leqslant k' \leqslant k+1$. Note that it is either $r_k \leqslant 1$ or $r_k \geqslant 1$.[12]

Due to the augmented restricted tier structure embedded in $\mathbb{D}$, at each preference profile, we can clearly specify the random assignment induced by the PS rule as shown in Fact 2 below.

Fact 2 *Given a profile $P \in \mathbb{D}^n$, let L be the random assignment induced by the PS rule. Then, the following five conditions hold.*

1. *Given $i \in I$, assume $\mathcal{A}(P_i) = \cup_{t=1}^k A_t$. Then, $L_{i\varnothing} = \max(0, 1 - r_{k+1})$ and $L_{ia} =$ for all $a \notin \mathcal{A}(P_i)$.*

Given $1 \leqslant k \leqslant T$, assume $A_k = \{a\}$ or $A_k = \{a, b\}$, and $n_k > 0$. If $A_k = \{a, b\}$, let $I_k = \{i \in I | a \; P_i \; b\}$ and $|I_k \cap N_k| = l$.

2. *If $r_k \geqslant 1$, then $L_{iA_k} = 0$ (equivalently, $L_{ia} = 0$ for all $a \in A_k$) for all $i \in N_k$.*

3. *If $r_k < 1$ and $A_k = \{a\}$, then $L_{ia} = \min(\frac{1}{n_k}, 1 - r_k)$ for all $i \in N_k$.*

4. *If $r_k < r_{k+1} < 1$ and $A_k = \{a, b\}$, we have*

   - $[\frac{n_k}{2} < l \leqslant n_k] \Rightarrow \begin{cases} L_{ia} = \frac{1}{l} \text{ and } L_{ib} = \frac{2}{n_k} - \frac{1}{l} & \text{for all } i \in N_k \cap I_k; \\ L_{ia} = 0 \text{ and } L_{ib} = \frac{2}{n_k} & \text{for all } i \in N_k \backslash I_k. \end{cases}$

   - $[0 \leqslant l \leqslant \frac{n_k}{2}] \Rightarrow \begin{cases} L_{ia} = \frac{2}{n_k} \text{ and } L_{ib} = 0 & \text{for all } i \in N_k \cap I_k; \\ L_{ia} = \frac{2}{n_k} - \frac{1}{n_k - l} \text{ and } L_{ib} = \frac{1}{n_k - l} & \text{for all } i \in N_k \backslash I_k. \end{cases}$

[12] For instance, in profile P, if all tiers are acceptable for all agents, then $r_k \leqslant 1$ for all $1 \leqslant k \leqslant T+1$. If one agent accepts all tiers and all others do not accept any tier, then $r_k \geqslant 1$ for all $1 \leqslant k \leqslant T+1$. In particular, recall the consumption procedure at profile P in the PS rule, and note that if $0 \leqslant r_k < 1$, then r_k is identical to the time at which all tiers $A_1, \ldots, A_{k-1}$ are exhausted, and all agents in N_k are about to consume A_k.

5. *If $r_k < 1 \leqslant r_{k+1}$ and $A_k = \{a, b\}$, we have*

- $[\frac{n_k}{2} < l \leqslant n_k] \Rightarrow \begin{cases} L_{ia} = \min(\frac{1}{l}, 1 - r_k) \text{ and } L_{ib} = \max(1 - r_k - \frac{1}{l}, 0) & \text{for all } i \in N_k \cap I_k; \\ L_{ia} = 0 \text{ and } L_{ib} = 1 - r_k & \text{for all } i \in N_k \backslash I_k. \end{cases}$

- $[0 \leqslant l \leqslant \frac{n_k}{2}] \Rightarrow \begin{cases} L_{ia} = 1 - r_k \text{ and } L_{ib} = 0 & \text{for all } i \in N_k \cap I_k; \\ L_{ia} = \max(1 - r_k - \frac{1}{n_k - l}, 0) \text{ and } L_{ib} = \min(\frac{1}{n_k - l}, 1 - r_k) & \text{for all } i \in N_k \backslash I_k. \end{cases}$

The verification of Fact 2 is routine, and we hence omit it.

It is evident that the PS rule satisfies *sd-efficiency* and *equal treatment of equals*. We first show that the PS rule is *sd-strategy-proof* on $\mathbb{D}$.

Fix $i \in I$, $P \in \mathbb{D}^n$ and $P_i' \in \mathbb{D}$. Assume $\mathcal{A}(P_i) = \cup_{t=1}^{k} A_t$ and $\mathcal{A}(P_i') = \cup_{t=1}^{k'} A_t$. Let L and L' be two random assignments induced by the PS rule at profiles P and (P_i', P_{-i}) respectively. We show $L_i \;\, P_i^{sd} \;\, L_i'$.

According Fact 2, we know $L_{iA_t} = L'_{iA_t}$ for all $1 \leqslant t \leqslant \min(k, k')$. Moreover, given $1 \leqslant t \leqslant \min(k, k')$, assume $A_t = \{a, b\}$ and $a \;\, P_i \;\, b$. By a similar argument in verifying *sd-strategy-proofness* of the PS rule in Theorem 1, we have $L_{ia} \geqslant L'_{ia}$.

Let $\underline{l} \equiv \sum_{t=1}^{\min(k,k')} |A_t|$ and $\bar{l} \equiv \sum_{t=1}^{\max(k,k')} |A_t|$. Given $x \in A \cup \{\varnothing\}$, assume $x = r_l(P_i)$. If $1 \leqslant l \leqslant \underline{l}$, then $\sum_{t=1}^{l} L_{ir_t(P_i)} \geqslant \sum_{t=1}^{l} L'_{ir_t(P_i)}$. If $l > \underline{l}$ and $k \leqslant k'$, it is evident that $\sum_{t=1}^{l} L_{ir_t(P_i)} = 1 \geqslant \sum_{t=1}^{l} L'_{ir_t(P_i)}$ by condition 1 of Fact 2.

Last, assume $l > \underline{l}$ and $k > k'$. Observe that $L'_{iz} = 0$ for all $z \in \cup_{t=k'+1}^{k} A_t$ by condition 1 of Fact 2. Therefore, if $\underline{l} < l \leqslant \bar{l}$, we have $\sum_{t=1}^{l} L_{ir_t(P_i)} = \sum_{t=1}^{\underline{l}} L_{ir_t(P_i)} + \sum_{t=\underline{l}+1}^{l} L_{ir_t(P_i)} \geqslant \sum_{t=1}^{\underline{l}} L'_{ir_t(P_i)} + \sum_{t=\underline{l}+1}^{l} L'_{ir_t(P_i)} = \sum_{t=1}^{l} L'_{ir_t(P_i)}$. Furthermore, if $\bar{l} < l \leqslant |A| + 1$, it is evident that $\sum_{t=1}^{l} L_{ir_t(P_i)} = 1 \geqslant \sum_{t=1}^{l} L'_{ir_t(P_i)}$ by condition 1 of Fact 2.

Therefore, $\sum_{t=1}^{l} L_{ir_t(P_i)} \geqslant \sum_{t=1}^{l} L'_{ir_t(P_i)}$ for all $1 \leqslant l \leqslant |A| + 1$. Hence, $L_i \;\, P_i^{sd} \;\, L_i'$ as required. In conclusion, the PS rule is *sd-strategy-proof* on domain $\mathbb{D}$.

Henceforth, we prove that on domain $\mathbb{D}$, the PS rule is the unique one satisfying *sd-strategy-proofness*, *sd-efficiency* and *equal treatment of equals*. Let $\varphi : \mathbb{D}^n \to \mathcal{L}$ a rule which satisfies all three axioms.

Fixing $P \equiv (P_1, \ldots, P_n) \in \mathbb{D}^n$, let $L \equiv \varphi(P)$. We first show that L satisfies *sd-envy-freeness*, and then show that L satisfies all five conditions of Fact 2. Given $1 \leqslant k \leqslant T$, recall $N_k \equiv \{i \in I | A_k \subseteq \mathcal{A}(P_i)\}$ and $n_k \equiv |N_k|$. Moreover, let $k^* \equiv \max\big\{k \in \{1, \ldots, T\} | L_{iA_k} > 0 \text{ for some } i \in I\big\}$ be the *maximum* index in $\{1, \ldots, T\}$ such that some agent consumes strictly positive proportion of A_{k^*}. Consequently, $n_{k^*} > 0$. Hence, $n_k > 0$ for all $1 \leqslant k \leqslant k^*$, and $r_k = \sum_{t=1}^{k-1} \frac{|A_t|}{n_k}$, $1 \leqslant k \leqslant k^* + 1$, is well-defined.

First, taking each tier A_k as one combined object and applying Theorem 5.1 in Bogomolnaia and Moulin (2002), we have the following three statements.

(i) Given $i \in I$, assume $\mathcal{A}(P_i) = \cup_{t=1}^{k} A_t$. Then, we have

- $L_{i\varnothing} = \max(0, 1 - r_{k+1})$ and $L_{ia} = 0$ for all $a \notin \mathcal{A}(P_i)$.
- $\sum_{t=1}^{k'} L_{iA_t} \geqslant \sum_{t=1}^{k'} L_{jA_t}$ for all $0 \leqslant k' \leqslant \min(k, k^*)$ and $j \neq i$.

(ii) Given $1 \leqslant k < k^*$, $L_{iA_k} = \frac{|A_k|}{n_k}$ for all $i \in N_k$.

(iii) $L_{iA_{k^*}} = 1 - r_{k^*} \leqslant \frac{|A_{k^*}|}{n_{k^*}}$ for all $i \in N_{k^*}$.

According to the first part of statement (i) above, condition 1 of Fact 2 is satisfied in L.

Lemma 8 *Given $1 \leqslant k \leqslant k^*$, assume $A_k = \{a, b\}$ and let $I_k = \{i \in I | a \ P_i \ b\}$. The following two statements hold.*

(1) *For all $i, j \in N_k \cap I_k$, $L_{ia} = L_{ja}$.*

(2) *For all $i \in N_k \cap I_k$ and $j \in N_k \backslash I_k$, $L_{ia} \geqslant L_{ja}$ and $L_{ib} \leqslant L_{jb}$.*

Proof: The verification of this lemma follows from a modification of the proof of Lemma 2. Specifically, fix all preferences in profile P whose acceptable set do not include A_k, and apply all proofs of Lemma 2 with respect to the remaining preferences in P with the following modifications:

- Change I in the proof of Lemma 2 to N_k.
- Change I_k in the proof of Lemma 2 to $N_k \cap I_k$.
- Change $I \backslash I_k$ in the proof of Lemma 2 to $N_k \backslash I_k$.
- Change n in the proof of Lemma 2 to n_k.
- If $k < k^*$, change $\frac{2}{n}$, $\frac{n}{2}$ and $\frac{1}{n}$ in the proof of Lemma 2 to $\frac{2}{n_k}$, $\frac{n_k}{2}$ and $\frac{1}{n_k}$ respectively. Moreover, whenever Lemma 1 is referred to in the proof of Lemma 2, change it to statement (ii) above.
- If $k = k^*$, change $\frac{2}{n}$, $\frac{n}{2}$ and $\frac{1}{n}$ in the proof of Lemma 2 to $1 - r_k$, $\frac{1}{1-r_k}$ and $\min(\frac{1}{n_k}, 1 - r_k)$ respectively. Moreover, whenever Lemma 1 is referred to in the proof of Lemma 2, change it to statement (iii) above.

■

Lemma 9 *Random assignment L satisfies sd-envy-freeness.*

Proof: Fix $i \in I$ and assume $\mathcal{A}(P_i) = \cup_{t=1}^{k} A_t$. Thus, $\varnothing = r_{l^*}(P_i)$ where $l \equiv \sum_{t=1}^{k} |A_t| + 1$. Evidently, *sd-efficiency* implies $\sum_{t=1}^{l^*} L_{ir_t(P_i)} = 1 \geqslant \sum_{t=1}^{l^*} L_{jr_t(P_i)}$ for all $j \neq i$.

Next, given $a \in A$, assume $a \in A_s$ and $a = r_l(P_i)$. We consider three cases.

Case 1: $s > \min(k, k^*)$.

Statements (i) - (iii) above imply $\sum_{t=1}^{l} L_{ir_t(P_i)} = 1 \geqslant \sum_{t=1}^{l} L_{jr_t(P_i)}$ for all $j \neq i$.

Case 2: $s \leqslant \min(k, k^*)$ and moreover, either $A_s = \{a\}$, or $|A_s| = 2$ and $a = \min(P_i, A_s)$.

The second part of statement (i) above implies $\sum_{t=1}^{l} L_{ir_t(P_i)} = \sum_{t=1}^{s} L_{iA_s} \geqslant \sum_{t=1}^{s} L_{jA_s} = \sum_{t=1}^{l} L_{jr_t(P_i)}$ for all $j \neq i$.

Case 3: $s \leqslant \min(k, k^*)$ and moreover, $|A_s| = 2$ and $a = \max(P_i, A_s)$.

Let $j \neq i$. If A_s is not included in $\mathcal{A}(P_j)$, the first part of statement (i) above implies $L_{ja} = 0$. If A_s is included in $\mathcal{A}(P_j)$, Lemma 8 implies $L_{ia} \geqslant L_{ja}$. Therefore, $L_{ia} \geqslant L_{ja}$. Furthermore, since $\sum_{t=1}^{s-1} L_{iA_t} \geqslant \sum_{t=1}^{s-1} L_{jA_t}$ by the first part of statement (i) above, we have $\sum_{t=1}^{l} L_{ir_t(P_i)} = \sum_{t=1}^{s-1} L_{iA_t} + L_{ia} \geqslant \sum_{t=1}^{s-1} L_{jA_t} + L_{ja} = \sum_{t=1}^{l} L_{jr_t(P_i)}$.

In conclusion, $\sum_{t=1}^{l} L_{ir_t(P_i)} \geqslant \sum_{t=1}^{l} L_{jr_t(P_i)}$ for all $1 \leqslant l \leqslant |A| + 1$ and $j \neq i$. Therefore, L satisfies *sd-envy-freeness*. ■

Last, we use the following 5 claims to show that conditions 2 - 5 of Fact 2 are satisfied in L.

Claim 1: $r_{k^*} < 1 \leqslant r_{k^*+1}$.

According to the definition of k^*, there exist $i \in N_{k^*}$ such that $L_{iA_{k^*}} > 0$. Fix such an agent i. By statement (ii) above, we know $r_{k^*} = \sum_{t=1}^{k^*-1} \frac{|A_t|}{n_t} = \sum_{t=1}^{k^*-1} L_{iA_t} < 1$. Moreover, by statement (iii), we have $r_{k^*+1} = \sum_{t=1}^{k^*} \frac{|A_t|}{n_t} = r_{k^*} + \frac{|A_{k^*}|}{n_{k^*}} \geqslant 1$. This completes the verification of the claim.

Given $1 \leqslant k \leqslant T$, assume $A_k = \{a\}$ or $A_k = \{a, b\}$, and $n_k > 0$. If $A_k = \{a, b\}$, let $I_k = \{i \in I | a \ P_i \ b\}$ and $|I_k \cap N_k| = l$.

Claim 2: Condition 2 of Fact 2 is satisfied in L.

According to the hypothesis of condition 2, since $r_k \geqslant 1$, Claim 1 implies $k \geqslant k^* + 1$. Then, the definition of k^* implies $L_{iA_k} = 0$ for all $i \in I$ (hence for all $i \in N_k$). This completes the verification of the claim.

Claim 3: Condition 3 of Fact 2 is satisfied in L.

According to the hypothesis of condition 3, since $r_k < 1$, Claim 1 implies $k \leqslant k^*$. Fix $i \in N_k$. If $k < k^*$, then statement (ii) above implies $L_{ia} = \frac{|A_k|}{n_k} = \frac{1}{n_k}$. Since $k + 1 \leqslant k^*$ and $r_{k^*} < 1$, it must be the case that $r_k + \frac{|A_k|}{n_k} = r_k + \frac{1}{n_k} = r_{k+1} \leqslant r_{k^*} < 1$. Thus, $\frac{1}{n_k} < 1 - r_k$, and hence $L_{ia} = \min(\frac{1}{n_k}, 1 - r_k)$. If $k = k^*$, statement (iii) above implies $L_{ia} = 1 - r_{k^*} = \min(\frac{1}{n_{k^*}}, 1 - r_{k^*})$. In conclusion, $L_{ia} = \min(\frac{1}{n_k}, 1 - r_k)$ for all $i \in N_k$. This completes the verification of the claim.

Claim 4: Condition 4 of Fact 2 is satisfied in L.

According to the hypothesis of condition 4, since $r_{k+1} < 1$, Claim 1 implies $k + 1 \leqslant k^*$, and hence $k < k^*$. Therefore, $L_{ia} + L_{ib} = L_{iA_k} = \frac{|A_k|}{n_k} = \frac{2}{n_k}$ for all $i \in N_k$ by statement (ii) above. Recall $I_k = \{i \in I | a \ \ P_i \ \ b\}$ and $|N_k \cap I_k| = l$. Then, *sd-efficiency* and *sd-envy-freeness* (recall Lemma 9) imply

- $[\frac{n_k}{2} < l \leqslant n_k] \Rightarrow \begin{cases} L_{ia} = \frac{1}{l} \text{ and } L_{ib} = \frac{2}{n_k} - \frac{1}{l} & \text{for all } i \in N_k \cap I_k; \\ L_{ia} = 0 \text{ and } L_{ib} = \frac{2}{n_k} & \text{for all } i \in N_k \backslash I_k. \end{cases}$

- $[0 \leqslant l \leqslant \frac{n_k}{2}] \Rightarrow \begin{cases} L_{ia} = \frac{2}{n_k} \text{ and } L_{ib} = 0 & \text{for all } i \in N_k \cap I_k; \\ L_{ia} = \frac{2}{n_k} - \frac{1}{n_k - l} \text{ and } L_{ib} = \frac{1}{n_k - l} & \text{for all } i \in N_k \backslash I_k. \end{cases}$

Claim 5: Condition 5 of Fact 2 is satisfied in L.

According to the hypothesis of condition 5, since $r_k < 1 \leqslant r_{k+1}$, Claim 1 implies $k = k^*$. Thus, by statement (iii), $L_{ia} + L_{ib} = L_{iA_{k^*}} = 1 - r_{k^*} \leqslant \frac{|A_{k^*}|}{n_{k^*}} = \frac{2}{n_{k^*}}$ for all $i \in N_{k^*}$. Recall $I_{k^*} \equiv \{i \in I | a \ \ P_i \ \ b\}$ and $l \equiv |N_{k^*} \cap I_{k^*}|$. We know either $\frac{n_{k^*}}{2} < l \leqslant n_{k^*}$ or $0 \leqslant l \leqslant \frac{n_{k^*}}{2}$.

First, assume $\frac{n_{k^*}}{2} < l \leqslant n_{k^*}$. Subsequently, two cases are separately considered.

Case 1: $\frac{n_{k^*}}{2} < l \leqslant n_{k^*}$ and $\frac{1}{l} \leqslant 1 - r_{k^*}$.

If $l = n_{k^*}$, then *sd-envy-freeness* implies $L_{ia} = \frac{1}{l}$, and hence $L_{ib} = 1 - r_{k^*} - \frac{1}{l}$ for all $i \in N_k \cap I_k$. If $\frac{n_{k^*}}{2} < l < n_{k^*}$, *sd-efficiency* first implies $L_{ia} = 0$, and hence $L_{ib} = 1 - r_{k^*}$ for all $i \in N_k \backslash I_k$. Consequently, *sd-envy-freeness* implies $L_{ia} = \frac{1}{l}$ for all $i \in N_k \cap I_k$. Hence, $L_{ib} = 1 - r_{k^*} - \frac{1}{l}$ for all $i \in N_k \cap I_k$.

Case 2: $\frac{n_{k^*}}{2} < l \leqslant n_{k^*}$ and $\frac{1}{l} > 1 - r_{k^*}$.

If $l = n_{k^*}$, *sd-envy-freeness* implies $L_{ia} = L_{ja}$ for all $i, j \in N_k \cap I_k$. Moreover, since $\frac{1}{l} > 1 - r_{k^*}$, it is true that $L_{ia} = 1 - r_{k^*}$, and hence $L_{ib} = 0$ for all $i \in N_k \cap I_k$. If $\frac{n_{k^*}}{2} < l < n_{k^*}$, *sd-efficiency* first implies $L_{ia} = 0$, and hence $L_{ib} = 1 - r_{k^*}$ for all $i \in N_k \backslash I_k$. Next, *sd-envy-freeness* implies $L_{ia} = L_{ja}$ for all $i, j \in N_k \cap I_k$. Since $\frac{1}{l} > 1 - r_{k^*}$, it is true that $L_{ia} = 1 - r_{k^*}$ for all $i \in N_k \cap I_k$. Hence, $L_{ib} = 0$ for all $i \in N_k \cap I_k$.

In conclusion, if $\frac{n_{k^*}}{2} < l \leqslant n_{k^*}$,

$$L_{ia} = \min(\frac{1}{l}, 1 - r_{k^*}) \text{ and } L_{ib} = \max(1 - r_{k^*} - \frac{1}{l}, 0) \qquad \text{for all } i \in N_k \cap I_k;$$

$$L_{ia} = 0 \text{ and } L_{ib} = 1 - r_{k^*} \qquad \text{for all } i \in N_k \backslash I_k.$$

By a symmetric argument, if $0 \leqslant l \leqslant \frac{n_{k^*}}{2}$,

$$L_{ia} = 1 - r_{k^*} \text{ and } L_{ib} = 0 \qquad \text{for all } i \in N_k \cap I_k;$$

$$L_{ia} = \max(1 - r_{k^*} - \frac{1}{n_{k^*} - l}, 0) \text{ and } L_{ib} = \min(\frac{1}{n_{k^*} - l}, 1 - r_{k^*}) \qquad \text{for all } i \in N_k \backslash I_k.$$

Thus, all five conditions of Fact 2 are verified. Therefore, φ is the PS rule. This completes the verification of Theorem 4. ■

Analogous to Corollary 1, the verification of Theorem 4 implies that the PS rule is the unique one satisfying *sd-efficiency* and *sd-envy-freeness* on an augmented restricted tier domain.

Corollary 2 *Let $\mathbb{D}$ be an augmented restricted tier domain. A rule is sd-efficient and sd-envy-free if and only if it is the PS rule.*

Proof: The sufficiency part hold evidently. We focus on the necessity part. Let $\varphi : \mathbb{D}^n \to \mathcal{L}$ be an *sd-efficient* and *sd-envy-free* rule. Fixing $P \equiv (P_1, \ldots, P_n) \in \mathbb{D}^n$, let $L \equiv \varphi(P)$. First, according to Theorem 4.1 in Bogomolnaia and Moulin (2002), we have statements (i) - (iii) in the proof of the necessity part of Theorem 4. Furthermore, in the proof of the necessity part of Theorem 4, note that the verification of Claims 1 - 5 only relies on the application of statements (i) - (iii) and the axioms of *sd-efficiency* and *sd-envy-freeness*. Therefore, we assert that φ is the PS rule. ■

5 Conclusion

In this paper, we have shown that if a connected domain admits an *sd-strategy-proof*, *sd-efficient* and *equal-treatment-of-equals* (or *sd-envy-free*) rule, the domain is a *restricted tier domain*, and this rule must be the PS rule.

Our results may be interpreted as both negative and positive. On the one hand, a *restricted tier domain* is restrictive, and does not give much freedom for agents to spell their preferences. On the other hand, in some realistic situations, for example the house allocation in a skyscraper or along a road, the restricted tier structure seems to be an appropriate assumption. Then our characterization of the PS rule supports its application in these situations.

More importantly, we identify the *restricted tier domain* as a boundary for the compatibility of these canonical axioms. Since connectedness is a mild and economically reasonable domain richness assumption and the axioms we impose are both canonical and normatively desirable, our characterizations suggest that a restricted tier structure must be embedded and the PS rule should be used to determine random assignments.

For further research, it would be interesting to investigate the analogous characterization problem for more general class of domains beyond connectedness. Another interesting problem is related to the domain characterization under different preference extension approaches (e.g., Cho (2012) and Aziz et al. (2014)) other than the stochastic-dominance extension.

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A Appendix: Proof of Theorem 3

To prove Theorem 3, it suffices to show that if domain $\mathbb{D}$ satisfies the *elevating property*, there exists no *sd-strategy-proof*, *sd-efficient* and equal-treatment-of-equals rule.

Suppose that $\mathbb{D}$ satisfy the *elevating property*, e.g., domain $\mathbb{D}$ contains preferences $\bar{P}_i, P_i, \hat{P}_i$ in Table 1. Let $\varphi : \mathbb{D}^n \to \mathcal{L}$ be a rule satisfying *sd-strategy-proofness*, *sd-efficiency*, and *equal treatment of equals*. Let $\bar{n} \equiv \frac{n}{2}$ if n is even, and $\bar{n} \equiv \frac{n-1}{2}$ if n is odd. We search for a contradiction. We first provide the sketch of proofs.

We consider the following four groups of preference profiles: **Profile Groups I - IV**. In particular, for the case of odd number of agents, we consider two additional groups of preference profiles: **Profile Groups V** and **VI**. See Table 2 below. Note that every preference profile in these groups consists of only preferences of $\bar{P}_i$, P_i and $\hat{P}_i$.

Profile Group I: n **is either even or odd**	**Profile Group II:** n **is either even or odd**
$P^{1,0} = (P_1, P_2, \ldots, P_n)$ $P^{1,1} = (\hat{P}_1, P_2, \ldots, P_n)$ $\vdots$ $\vdots$ $P^{1,m} = (\hat{P}_1, \ldots, \hat{P}_m, P_{m+1}, \ldots, P_n)$ $\vdots$ $P^{1,\bar{n}} = (\hat{P}_1, \ldots, \hat{P}_{\bar{n}}, P_{\bar{n}+1}, \ldots, P_n)$	$P^{2,1} = (P_1, P_2, \ldots, P_{n-1}, \bar{P}_n)$ $P^{2,2} = (\hat{P}_1, P_2, \ldots, P_{n-1}, \bar{P}_n)$ $\vdots$ $P^{2,m} = (\hat{P}_1, \ldots, \hat{P}_{m-1}, P_m, \ldots, P_{n-1}, \bar{P}_n)$ $\vdots$ $P^{2,\bar{n}} = (\hat{P}_1, \ldots, \hat{P}_{\bar{n}-1}, \mathbf{P}_{\bar{\mathbf{n}}}, P_{\bar{n}+1}, \ldots, P_{n-1}, \bar{P}_n)$ $P^{2,\bar{n}+1} = (\hat{P}_1, \ldots, \hat{P}_{\bar{n}-1}, \hat{P}_{\bar{n}}, P_{\bar{n}+1}, \ldots, P_{n-1}, \bar{P}_n)$
Profile Group III: n **is either even or odd**	**Profile Group IV:** n **is either even or odd**
$P^{3,0} = (\hat{P}_1, \ldots, \hat{P}_{n-1}, \hat{P}_n)$ $P^{3,1} = (\hat{P}_1, \ldots, \hat{P}_{n-1}, P_n)$ $\vdots$ $\vdots$ $P^{3,m} = (\hat{P}_1, \ldots, \hat{P}_{n-m}, P_{n-m+1}, \ldots, P_n)$ $\vdots$ $P^{3,\bar{n}} = (\hat{P}_1, \ldots, \hat{P}_{n-\bar{n}}, P_{n-\bar{n}+1}, \ldots, P_n)$	$P^{4,1} = (\hat{P}_1, \ldots, \hat{P}_{n-2}, \hat{P}_{n-1}, \bar{P}_n)$ $P^{4,2} = (\hat{P}_1, \ldots, \hat{P}_{n-2}, P_{n-1}, \bar{P}_n)$ $\vdots$ $P^{4,m} = (\hat{P}_1, \ldots, \hat{P}_{n-m}, P_{n-m+1}, \ldots, P_{n-1}, \bar{P}_n)$ $\vdots$ $P^{4,\bar{n}} = (\hat{P}_1, \ldots, \hat{\mathbf{P}}_{\mathbf{n}-\bar{\mathbf{n}}}, P_{n-\bar{n}+1}, \ldots, P_{n-1}, \bar{P}_n)$
Profile Group V: n **is odd**	**Profile Group VI:** n **is odd**
$P^{5,1} = (P_1, P_2, \ldots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n)$ $P^{5,2} = (\hat{P}_1, P_2, \ldots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n)$ $\vdots$ $\vdots$ $P^{5,m} = (\hat{P}_1, \ldots, \hat{P}_{m-1}, P_m, \ldots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n)$ $\vdots$ $P^{5,\bar{n}} = (\hat{P}_1, \ldots, \hat{P}_{\bar{n}-1}, P_{\bar{n}}, P_{\bar{n}+1}, \ldots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n)$ $P^{5,\bar{n}+1} = (\hat{P}_1, \ldots, \hat{P}_{\bar{n}-1}, \hat{P}_{\bar{n}}, \mathbf{P}_{\bar{\mathbf{n}}+\mathbf{1}}, \ldots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n)$	$P^{6,1} = (\hat{P}_1, \ldots, \hat{P}_{n-3}, \hat{P}_{n-2}, P_{n-1}, \bar{P}_n)$ $P^{6,2} = (\hat{P}_1, \ldots, \hat{P}_{n-3}, \hat{P}_{n-2}, \bar{P}_{n-1}, \bar{P}_n)$ $P^{6,3} = (\hat{P}_1, \ldots, \hat{P}_{n-3}, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n)$ $\vdots$ $P^{6,m} = (\hat{P}_1, \ldots, \hat{P}_{n-m}, P_{n-m+1}, \ldots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n)$ $\vdots$ $P^{6,\bar{n}} = (\hat{P}_1, \ldots, \hat{\mathbf{P}}_{\mathbf{n}-\bar{\mathbf{n}}}, P_{n-\bar{n}+1}, \ldots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n)$

Table 2: Preference Profile Groups

We first show that for every preference profile of each profile group and every agent, the sum of probabilities over objects a, b and c equals to $\frac{3}{n}$ (see Lemma 10). Then, in the rest of verification, we only focus on the random assignments of objects a, b and c.

At every preference profile in profile groups I - IV, we fully characterize the random assignment of objects a, b and c(see Claims 1 - 5). Then, we realize that when n is an even number, the probability of assigning object c to agent $\bar{n}$ under profile $P^{2,\bar{n}}$ is distinct from that under profile $P^{4,\bar{n}}$. This formulates a contradiction against *sd-strategy-proofness* since from $P^{2,\bar{n}}$ to $P^{4,\bar{n}}$, agent $\bar{n}$ unilaterally deviates from P_i to $\hat{P}_i$, and object c shares the same upper contour set in both P_i and $\hat{P}_i$.

When n is an odd number, in addition to profile groups I - IV, we consider profile groups V and VI. At every preference profile in both profile groups V and VI, we focus on characterizing probabilities of assigning object c to every agent (see Claims 6 - 8). Eventually, we observe that the probability of assigning object c to agent $\bar{n}+1$ under profile $P^{5,\bar{n}+1}$ is distinct from that under profile $P^{6,\bar{n}}$. This formulates a similar contradiction against *sd-strategy-proofness*.

Lemma 10 *For every profile P in profile groups I - VI, $\varphi_{ia}(P) + \varphi_{ib}(P) + \varphi_{ic}(P) = \frac{3}{n}$.*

Proof: The verification of this lemma is routine. In each profile group, repeatedly applying *sd-strategy-proofness* and *equal treatment of equals*, we have the result. Due to the tediousness, we omit the detailed proof. ■

Now, we consider profile groups I - IV. According to Lemma 10, we only focus on the random assignments over a, b and c in each preference profile.

Claim 1 *In profile group I, for each $m = 0, 1, \ldots, \bar{n}$, the random assignment $\varphi(P^{1,m})$ over a, b and c is specified below*

	a	b	c
1	0	$\frac{2}{n}$	$\frac{1}{n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
m	0	$\frac{2}{n}$	$\frac{1}{n}$
$m+1$	$\frac{1}{n-m}$	$\frac{n-2m}{n(n-m)}$	$\frac{1}{n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$\frac{1}{n-m}$	$\frac{n-2m}{n(n-m)}$	$\frac{1}{n}$

Proof: For $m = 0$, *equal treatment of equals* implies $\varphi_{ix}(P^{1,0}) = \frac{1}{n}$ for all $i \in I$ and $x \in \{a, b, c\}$.

Next, we show $\varphi_{ic}(P^{1,m}) = \frac{1}{n}$ for all $i \in I$ and $m = 1, \ldots, \bar{n}$. We specify an induction hypothesis: given $1 \leqslant m \leqslant \bar{n}$, for all $0 \leqslant l < m$, $\varphi_{ic}(P^{1,l}) = \frac{1}{n}$ for all $i \in I$. We will show $\varphi_{ic}(P^{1,m}) = \frac{1}{n}$ for all $i \in I$. Notice that profiles $P^{1,m-1}$ and $P^{1,m}$ are different only in agent m's preference, i.e., $P^{m-1}_m = P_i$ and $P^m_m = \hat{P}_i$ in Table 1. Then *sd-strategy-proofness* and induction hypothesis imply $\varphi_{mc}(P^{1,m}) = \varphi_{mc}(P^{1,m-1}) = \frac{1}{n}$. Moreover, *equal treatment of equals* and feasibility imply $\varphi_{ic}(P^{1,m}) = \frac{1}{n}$ for all $i \in I$.

Last, for $m = 1, \ldots, \bar{n}$, note that $|\{1, \ldots, m\}| \leqslant \frac{n}{2}$ and all agents in $\{m+1, \ldots, n\}$ prefer a to b in profile $P^{1,m}$. Consequently, by *sd-efficiency*, feasibility and Lemma 10, we have $\varphi_{ia}(P^{1,m}) = 0$ for all $i = 1, \ldots, m$.

Finally, by Lemma 10, feasibility and *equal treatment of equals*, we have the claim. ■

Claim 2 *In profile group II, the random assignment $\varphi(P^{2,1})$ over a, b and c is specified below*

$$\begin{array}{cccc} & a & b & c \\ 1 & \frac{1}{n} & \frac{1}{n-1} & \frac{n-2}{n(n-1)} \\ \vdots & \vdots & \vdots & \vdots \\ n-1 & \frac{1}{n} & \frac{1}{n-1} & \frac{n-2}{n(n-1)} \\ n & \frac{1}{n} & 0 & \frac{2}{n} \end{array}$$

Proof: The verification is routine and we hence omit it. ■

Claim 3 *In profile group II, for each $m = 2, \ldots, \bar{n}$ (if n is even), and for each $m = 2, \ldots, \bar{n}, \bar{n}+1$ (if n is odd), the random assignment $\varphi(P^{2,m})$ over a, b and c is specified below*

$$\begin{array}{cccc} & a & b & c \\ 1 & 0 & \frac{3}{n} - \alpha(m) & \alpha(m) \\ \vdots & \vdots & \vdots & \vdots \\ m-1 & 0 & \frac{3}{n} - \alpha(m) & \alpha(m) \\ m & \frac{1}{n-(m-1)} & \frac{1-(m-1)[\frac{3}{n}-\alpha(m)]}{n-m} & \frac{1-[\frac{3}{n}-\frac{1}{n-(m-1)}]-(m-1)\alpha(m)}{n-m} \\ \vdots & \vdots & \vdots & \vdots \\ n-1 & \frac{1}{n-(m-1)} & \frac{1-(m-1)[\frac{3}{n}-\alpha(m)]}{n-m} & \frac{1-[\frac{3}{n}-\frac{1}{n-(m-1)}]-(m-1)\alpha(m)}{n-m} \\ n & \frac{1}{n-(m-1)} & 0 & \frac{3}{n} - \frac{1}{n-(m-1)} \end{array}$$

where $\alpha(m) = \frac{n^2-(m+1)n+(3m-4)}{n(n-1)[n-(m-1)]}$

Proof: The verification of this claim consists of 4 steps. The first 3 steps are valid for $m = 2, \ldots, \bar{n}, \bar{n}+1$ no matter n is even or odd. The last steps is verified under the cases of even and odd number of agents separately.

Step 1, we show $\varphi_{na}(P^{2,m}) = \frac{1}{n-(m-1)}$ for all $m = 2, \ldots, \bar{n}, \bar{n}+1$. Notice that $P^{2,m}$ and $P^{1,m-1}$ are different merely in agent n's preferences, i.e., $P^{2,m}_n = \bar{P}_i$ and $P^{1,m-1}_n = P_i$ in Table 1. Then, *sd-strategy-proofness* implies $P^{2,m}_{na} = P^{1,m-1}_{na} = \frac{1}{n-(m-1)}$. This completes the verification of step 1.

Step 2, we show $\varphi_{nb}(P^{2,m}) = 0$ and $\varphi_{nc}(P^{2,m}) = \frac{3}{n} - \frac{1}{n-(m-1)}$ for all $m = 2, \ldots, \bar{n}, \bar{n}+1$. Given $m \in \{1, \ldots, \bar{n}, \bar{n}+1\}$, since all agents other than n prefer b to c, *sd-efficiency* and feasibility imply $\varphi_{nb}(P^{2,m}) = 0$. Then, by Lemma 10 and Step 1, we have $\varphi_{nc}(P^{2,m}) = \frac{3}{n} - \frac{1}{n-(m-1)}$. This completes the verification of step 2.

Step 3, we show $\varphi_{ic}(P^{2,m}) = \alpha(m)$ for all $i = 1, \ldots, m-1$ and $m = 2, \ldots, \bar{n}, \bar{n}+1$. By *equal treatment of equals*, it suffices to show $\varphi_{m-1,c}(P^{2,m}) = \alpha(m)$ for all $m = 2, \ldots, \bar{n}, \bar{n}+1$.

Notice that for all $m = 2, \ldots, \bar{n}, \bar{n}+1$, profiles $P^{2,m}$ and $P^{2,m-1}$ are different merely in agent $m-1$'s preferences, i.e., $P^{2,m-1}_{m-1} = P_i$ and $P^{2,m}_{m-1} = \hat{P}_i$ in Table 1.

Now, we prove Step 3 by an induction argument on $m = 2, \ldots, \bar{n}, \bar{n}+1$.

Initial statement: for $m = 2$, by *sd-strategy-proofness* and Claim 2, we have

$$\varphi_{1,c}(P^{2,2}) = \varphi_{1,c}(P^{2,1}) = \frac{n-2}{n(n-1)} = \frac{n^2-(2+1)n+(3\times 2-4)}{n(n-1)[n-(2-1)]} = \alpha(2).$$

Induction Hypothesis: Given $2 \leqslant m \leqslant \bar{n}$, for all $2 \leqslant l < m+1$, we have $\varphi_{l-1,c}(P^{2,l}) = \alpha(l)$.

We show $\varphi_{m,c}(P^{2,m+1}) = \alpha(m+1)$ by the following elaboration.

$$\begin{aligned}
\varphi_{m,c}(P^{2,m+1}) &= \varphi_{m,c}(P^{2,m}) && \text{by } \textit{sd-strategy-proofness}\\
&= \tfrac{1-\varphi_{nc}(P^{2,m})-\sum_{i=1}^{m-1}\varphi_{ic}(P^{2,m})}{n-m} && \text{by } \textit{equal treatment of equals} \text{ and feasibility}\\
&= \tfrac{1-[\frac{3}{n}-\frac{1}{n-(m-1)}]-(m-1)\varphi_{m-1,c}(P^{2,m})}{n-m} && \text{by Step 2 and } \textit{equal treatment of equals}\\
&= \tfrac{1-[\frac{3}{n}-\frac{1}{n-(m-1)}]-(m-1)\alpha(m)}{n-m} && \text{by induction hypothesis}\\
&= \alpha(m+1) && \text{by simplifying the expression}
\end{aligned}$$

This completes the verification of induction hypothesis and hence step 3.

Step 4, we show $\varphi_{ia}(P^{2,m}) = 0$ for all $i = 1, \ldots, m-1$ and $m = 2, \ldots, \bar{n}$ (if n is even) or $m = 2, \ldots, \bar{n}, \bar{n}+1$ (if n is odd).[13] Given $m \in \{2, \ldots, \bar{n}\}$ (if n is even), or $m \in \{2, \ldots, \bar{n}, \bar{n}+1\}$ (if n is odd), suppose that $\varphi_{ia}(P^{2,m}) = \beta > 0$ for some $i = 1, \ldots, m-1$. Thus, *sd-efficiency* implies that $\varphi_{jb}(P^{2,m}) = 0$ for all $j = m, \ldots, n-1$. Consequently, since $\varphi_{nb}(P^{2,m}) = 0$ by Step 2, we know $\sum_{i=1}^{m-1} \varphi_{ib}(P^{2,m}) = 1$.

Evidently, *equal treatment of equals* implies $\varphi_{ia}(P^{2,m}) = \beta$ for all $i = 1, \ldots, m-1$. Thus, by Lemma 10 and Step 3, we have $\varphi_{ib}(P^{2,m}) = \frac{3}{n} - \alpha(m) - \beta$ for all $i = 1, \ldots, m-1$. Therefore, *equal treatment of equals* implies

$$\sum_{i=1}^{m-1} \varphi_{ib}(P^{2,m}) = (m-1)\times\left[\frac{3}{n} - \alpha(m) - \beta\right] < (m-1)\times\left[\frac{3}{n} - \frac{n^2-(m+1)n+(3m-4)}{n(n-1)[n-(m-1)]}\right].$$

To induce the contradiction $\sum_{i=1}^{m-1} \varphi_{ib}(P^{2,m}) < 1$, we show $(m-1)\times\left[\frac{3}{n} - \frac{n^2-(m+1)n+(3m-4)}{n(n-1)[n-(m-1)]}\right] \leqslant 1$. Equivalently, we show $-2m^2n + 3mn^2 + m - n^3 - 2n^2 + 2nm - 1 \leqslant 0$.

Consider the function $f(\theta) = -2\theta^2 n + 3\theta n^2 + \theta - n^3 - 2n^2 + 2n\theta - 1$, $\theta \in \mathbb{R}$. We know $f'(\theta) = -4n\theta + 3n^2 + 1 + 2n$ and $f''(\theta) = -4n < 0$ for all $\theta \in \mathbb{R}$. It is evident that $f'(\theta)$ is a strictly decreasing function on $\mathbb{R}$. Now, we consider the case n is even and the case n is odd *separately*.

[13] In Chang and Chun (2016), the verification related to this step is simply an application of *sd-efficiency*. However, due to the complexity of $\alpha(m)$, mere *sd-efficiency* is not enough for the verification.

Case 1: n is even. Thus, $\bar{n} = \frac{n}{2}$. Since $f'(\frac{n}{2}) = (n+1)^2 > 0$, it must be the case that $f'(\theta) > 0$ for all $2 \leqslant \theta \leqslant \frac{n}{2}$. Therefore, f is a strictly increasing function on $2 \leqslant \theta \leqslant \frac{n}{2}$. Next, since $f(\frac{n}{2}) = -(n-\frac{1}{4})^2 - \frac{15}{16} < 0$, we have $f(\theta) < 0$ for all $2 \leqslant \theta \leqslant \frac{n}{2}$.

Case 2: n is odd. Thus, $\bar{n} = \frac{n-1}{2}$. Since $f'(\frac{n+1}{2}) = n^2 + 1 > 0$, it must be the case that $f'(\theta) > 0$ for all $2 \leqslant \theta \leqslant \frac{n+1}{2}$. Therefore, f is a strictly increasing function on $2 \leqslant \theta \leqslant \frac{n+1}{2}$. Since $f(\frac{n+1}{2}) = -\frac{1}{2}(n-1)^2 < 0$, we have $f(\theta) < 0$ for all $2 \leqslant \theta \leqslant \frac{n+1}{2}$.

In conclusion, no matter n is even or odd, we have $-2m^2n + 3mn^2 + m - n^3 - 2n^2 + 2nm - 1 = f(m) < 0$, and hence, $\sum_{i=1}^{m-1} \varphi_{ib}(P^{2,m}) < 1$. Contradiction! This completes the verification of step 4.

Finally, Lemma 10, feasibility and *equal treatment of equals* give the rest of characterizations in the claim. ■

Claim 4 *In profile group III, for each $m = 0, 1, \ldots, \bar{n}$, the random assignment $\varphi(P^{3,m})$ over a, b and c is specified below*

$$\begin{array}{rccc}
 & a & b & c \\
1 & \frac{n-2m}{n(n-m)} & \frac{1}{n-m} & \frac{1}{n} \\
\vdots & \vdots & \vdots & \vdots \\
n-m & \frac{n-2m}{n(n-m)} & \frac{1}{n-m} & \frac{1}{n} \\
n-m+1 & \frac{2}{n} & 0 & \frac{1}{n} \\
\vdots & \vdots & \vdots & \vdots \\
n & \frac{2}{n} & 0 & \frac{1}{n}
\end{array}$$

Proof: The verification follows from a similar argument in the proof of Claim 1. ■

Claim 5 *In profile group IV, for each $m = 1, \ldots, \bar{n}$, the random assignment $\varphi(P^{4,m})$ over a, b and c is specified below*

$$\begin{array}{rccc}
 & a & b & c \\
1 & \frac{n-2m}{n(n-m)} & \frac{1}{n-m} & \frac{1}{n} \\
\vdots & \vdots & \vdots & \vdots \\
n-m & \frac{n-2m}{n(n-m)} & \frac{1}{n-m} & \frac{1}{n} \\
n-m+1 & \frac{2}{n} & 0 & \frac{1}{n} \\
\vdots & \vdots & \vdots & \vdots \\
n-1 & \frac{2}{n} & 0 & \frac{1}{n} \\
n & \frac{2}{n} & 0 & \frac{1}{n}
\end{array}$$

Proof: The verification of the claim consists of 4 steps.

Step 1, we show $\varphi_{n,a}(P^{4,m}) = \frac{2}{n}$ for all $m = 1, \ldots, \bar{n}$. Notice that, for all $m = 1, \ldots, \bar{n}$, $P^{4,m}$ and $P^{3,m}$ are different merely in agent n's preferences, i.e., $P^{4,m}_n = \bar{P}_i$ and $P^{3,m}_n = P_i$ in Table 1. By *sd-strategy-proofness*, we have $\varphi_{n,a}(P^{4,m}) = \varphi_{n,a}(P^{3,m}) = \frac{2}{n}$. This completes the verification of step 1.

Step 2, we show $\varphi_{n,b}(P^{4,m}) = 0$ and $\varphi_{n,c}(P^{4,m}) = \frac{1}{n}$ for all $m = 1, \ldots, \bar{n}$. The verification of this step follows from the same verification of Step 2 in the proof of Claim 3.

Step 3, we show $\varphi_{ic}(P^{4,m}) = \frac{1}{n}$ for all $i \in I$ and $m = 1, \ldots, \bar{n}$. First, since $\varphi_{n,c}(P^{4,1}) = \frac{1}{n}$ by Step 2, feasibility and *equal treatment of equals* imply $\varphi_{ic}(P^{4,1}) = \frac{1}{n}$ for all $i = 1, \ldots, n-1$. Therefore, $\varphi_{ic}(P^{4,1}) = \frac{1}{n}$ for all $i \in I$.

Next, we specify an induction hypothesis: given $2 \leqslant m \leqslant \bar{n}$, for all $1 \leqslant l < m$, $\varphi_{ic}(P^{4,l}) = \frac{1}{n}$ for all $i \in I$. We will show $\varphi_{ic}(P^{4,m}) = \frac{1}{n}$ for all $i \in I$. Notice that, $P^{4,m}$ and $P^{4,m-1}$ are different merely in agent $(n-m+1)$'s preferences, i.e., $P^{4,m}_{n-m+1} = P_i$ and $P^{4,m-1}_{n-m+1} = \hat{P}_i$ in Table 1. Then *sd-strategy-proofness* and induction hypothesis imply $\varphi_{n-m+1,c}(P^{4,m}) = \varphi_{n-m+1,c}(P^{4,m-1}) = \frac{1}{n}$. Then, by *equal treatment of equals*, we know $\varphi_{ic}(P^{4,m}) = \frac{1}{n}$ for all $i = n-m+1, \ldots, n-1$. Moreover, by Step 2 and feasibility, we have $\varphi_{jc}(P^{4,m}) = \frac{1}{n}$ for all $j = 1, \ldots, n-m$. Therefore, $\varphi_{ic}(P^{4,m}) = \frac{1}{n}$ for all $i \in I$. This completes the verification of induction hypothesis and hence step 3.

Step 4, we show $\varphi_{i,b}(P^{4,m}) = 0$ for all $i = n-m+1, \ldots, n-1$ and $m = 2, \ldots, \bar{n}$. Given $m \in \{2, \ldots, \bar{n}\}$, suppose $\varphi_{i,b}(P^{4,m}) = \alpha > 0$ for some $i \in \{n-m+1, \ldots, n-1\}$. Then *equal treatment of equals* and Step 4 imply $\varphi_{i,b}(P^{4,m}) = \alpha$ and $\varphi_{i,a}(P^{4,m}) = \frac{2}{n} - \alpha$ for all $i = n-m+1, \ldots, n-1$. Moreover, by *sd-efficiency*, it must be the case that $\varphi_{i,a}(P^{4,m}) = 0$ for all $i = 1, \ldots, n-m$. Then, Lemma 10 and Step 3 imply $\varphi_{i,b}(P^{4,m}) = \frac{2}{n}$ for all $i = 1, \ldots, n-m$ Thus, the feasibility of a implies $\alpha = \frac{1}{m-1}\frac{2}{n}(m - \frac{n}{2}) \leqslant 0$ since $m \leqslant \bar{n} \leqslant \frac{n}{2}$. Contradiction!

Finally, Lemma 10, feasibility and *equal treatment of equals* give the rest of characterizations in the claim. ■

Now we have the contradiction for the case of even number of agents. Let n be even. Notice that $P^{2,\bar{n}}$ and $P^{4,\bar{n}}$ are different merely in agent $\bar{n}$'s preference, i.e., $P^{2,\bar{n}}_{\bar{n}} = P_i$ and $P^{4,\bar{n}}_{\bar{n}} = \hat{P}_i$ where P_i and $\hat{P}_i$ are from Table 1. Then *sd-strategy-proofness* requires $\varphi_{\bar{n},c}(P^{2,\bar{n}}) = \varphi_{\bar{n},c}(P^{4,\bar{n}})$. Thus, we have

$$\frac{1 - [\frac{3}{n} - \frac{1}{n-(\frac{n}{2}-1)}] - (\frac{n}{2}-1)\frac{n^2-(\frac{n}{2}+1)n+(3\times\frac{n}{2}-4)}{n(n-1)[n-(\frac{n}{2}-1)]}}{n - \frac{n}{2}} = \frac{1}{n} \Leftrightarrow n^2 - n - 2 = n^2 - n. \text{ Contradiction!}$$

When n is odd, profiles $P^{2,\bar{n}+1}$ and $P^{4,\bar{n}}$ are different merely in agent $(\bar{n}+1)$'s preferences, i.e., $P^{2,\bar{n}+1}_{\bar{n}+1} = P_i$ and $P^{4,\bar{n}}_{\bar{n}+1} = \hat{P}_i$ in Table 1. However, we cannot induce a contradiction similar to that above since we can verify that $\varphi_{\bar{n}+1,c}(P^{2,\bar{n}+1}) = \frac{1}{n} = \varphi_{\bar{n}+1,c}(P^{4,\bar{n}})$. Henceforth, we assume that n is an odd number. Hence, $n \geqslant 5$. We proceed the verification on profile groups V and VI.

Claim 6 *In profile group V, the random assignment $\varphi(P^{5,1})$ over a, b and c is specified below*

$$\begin{array}{cccc} & a & b & c \\ 1 & \frac{1}{n} & \frac{1}{n-2} & \frac{2}{n}-\frac{1}{n-2} \\ \vdots & \vdots & \vdots & \vdots \\ n-2 & \frac{1}{n} & \frac{1}{n-2} & \frac{2}{n}-\frac{1}{n-2} \\ n-1 & \frac{1}{n} & 0 & \frac{2}{n} \\ n & \frac{1}{n} & 0 & \frac{2}{n} \end{array}$$

Proof: The verification is routine and we hence omit it. ■

Claim 7 *In profile group V, for each $m = 2, \ldots, \bar{n}, \bar{n}+1$, the random assignment $\varphi(P^{5,m})$ over a, b and c is specified below*[14]

$$\begin{array}{cccc} & a & b & c \\ 1 & - & - & \gamma(m) \\ \vdots & \vdots & \vdots & \vdots \\ m-1 & - & - & \gamma(m) \\ m & - & - & \frac{1-2\times(\frac{3}{n}-\frac{1}{n-(m-1)})-(m-1)\gamma(m)}{n-(m+1)} \\ \vdots & \vdots & \vdots & \vdots \\ n-2 & - & - & \frac{1-2\times(\frac{3}{n}-\frac{1}{n-(m-1)})-(m-1)\gamma(m)}{n-(m+1)} \\ n-1 & \frac{1}{n-(m-1)} & 0 & \frac{3}{n}-\frac{1}{n-(m-1)} \\ n & \frac{1}{n-(m-1)} & 0 & \frac{3}{n}-\frac{1}{n-(m-1)} \end{array}$$

where $\gamma(m) = \frac{n^4-2(m+2)n^3+(m^2+11m-5)n^2-(7m^2+m-8)n+(6m^2-6m-4)}{n(n-1)(n-2)(n-(m-1))(n-m)}$.

Proof: The verification of this claim consists of 3 steps.

Step 1, we show $\varphi_{ia}(P^{5,m}) = \frac{1}{n-(m-1)}$ for $i = n-1, n$ and all $m = 2, \ldots, \bar{n}, \bar{n}+1$. Notice that $P^{5,m}$ and $P^{2,m}$ are different merely in agent $(n-1)$'s preferences, i.e., $P^{5,m}_{n-1} = \bar{P}_i$ and $P^{2,m}_{n-1} = P_i$ in Table 1. Then *sd-strategy-proofness* implies $\varphi_{n-1,a}(P^{5,m}) = \varphi_{n-1,a}(P^{2,m}) = \frac{1}{n-(m-1)}$. This completes the verification of step 1.

Step 2, we show $\varphi_{ib}(P^{5,m}) = 0$ and $\varphi_{ic}(P^{5,m}) = \frac{3}{n} - \frac{1}{n-(m-1)}$ for $i = n-1, n$ and all $m = 2, \ldots, \bar{n}, \bar{n}+1$. The verification simply follows from an application of *sd-efficiency*, *equal treatment of equals*, feasibility and Lemma 10. Therefore, we omit the details.

Step 3, we show $\varphi_{ic}(P^{5,m}) = \gamma(m)$ for all $i = 1, \ldots, m-1$ and $m = 2, \ldots, \bar{n}, \bar{n}+1$. By *equal treatment of equals*, it suffices to show $\varphi_{m-1,c}(P^{5,m}) = \gamma(m)$ for all $m = 2, \ldots, \bar{n}, \bar{n}+1$.

[14] A dash "–" in the random assignment matrix represents that the probability of assigning one object to a corresponding agent is *not* specified.

First, notice that for all $m = 2, \dots, \bar{n}, \bar{n}+1$, profiles $P^{5,m}$ and $P^{5,m-1}$ are different merely in agent $m-1$'s preferences, i.e., $P^{5,m-1}_{m-1} = P_i$ and $P^{5,m}_{m-1} = \hat{P}_i$ in Table 1.

Now, we prove Step 3 by an induction argument on $m = 2, \dots, \bar{n}, \bar{n}+1$.

Initial statement: for $m = 2$, by *sd-strategy-proofness* and Claim 6, we have

$$\begin{aligned} \varphi_{1,c}(P^{5,2}) &= \varphi_{1,c}(P^{5,1}) \\ &= \frac{2}{n} - \frac{1}{n-2} \\ &= \frac{n^4 - 2\times(2+2)n^3 + (2^2 + 11\times 2 - 5)n^2 - (7\times 2^2 + 2 - 8)n + (6\times 2^2 - 6\times 2 - 4)}{n(n-1)(n-2)(n-(2-1))(n-(2+1))} \\ &= \gamma(2). \end{aligned}$$

Induction Hypothesis: Given $2 \leqslant m \leqslant \bar{n}$, for all $2 \leqslant l < m+1$, we have $\varphi_{l-1,c}(P^{5,l}) = \gamma(l)$.

We show $\varphi_{m,c}(P^{5,m+1}) = \gamma(m+1)$ by the following elaboration.

$$\begin{aligned} \varphi_{m,c}(P^{5,m+1}) &= \varphi_{m,c}(P^{5,m}) && \text{by } \textit{sd-strategy-proofness} \\ &= \tfrac{1-\varphi_{n-1,c}(P^{5,m})-\varphi_{nc}(P^{5,m})-\sum_{i=1}^{m-1}\varphi_{ic}(P^{5,m})}{n-(m+1)} && \text{by } \textit{equal treatment of equals} \text{ and feasibility} \\ &= \tfrac{1-2\times[\frac{3}{n}-\frac{1}{n-(m-1)}]-(m-1)\varphi_{m-1,c}(P^{5,m})}{n-(m+1)} && \text{by Step 2 and } \textit{equal treatment of equals} \\ &= \tfrac{1-2\times[\frac{3}{n}-\frac{1}{n-(m-1)}]-(m-1)\gamma(m)}{n-(m+1)} && \text{by induction hypothesis} \\ &= \gamma(m+1) && \text{by simplifying the expression} \end{aligned}$$

This completes the verification of induction hypothesis and hence step 3.

Finally, by feasibility and *equal treatment of equals*, we have $\varphi_{ic}(P^{5,m}) = \frac{1-2\times[\frac{3}{n}-\frac{1}{n-(m-1)}]-(m-1)\gamma(m)}{n-(m+1)}$ for all $i = m, \dots, n-2$ and $m = 2, \dots, \bar{n}, \bar{n}+1$. This completes the verification of the claim. ■

Claim 8 *In profile group VI, for each $m = 2, \dots, \bar{n}$, the random assignment $\varphi(P^{6,m})$ over a, b and c is specified below*

	a	b	c
1	$-$	$-$	$\frac{1}{n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-m$	$-$	$-$	$\frac{1}{n}$
$n-m+1$	$-$	$-$	$\frac{1}{n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-2$	$-$	$-$	$\frac{1}{n}$
$n-1$	$\frac{2}{n}$	0	$\frac{1}{n}$
n	$\frac{2}{n}$	0	$\frac{1}{n}$

Proof: The verification of this claim consists of 3 steps.

Step 1, we show $\varphi_{ia}(P^{6,m}) = \frac{2}{n}$ for $i = n-1, n$ and all $m = 2, \dots, \bar{n}$. For each $m = 2, \dots, \bar{n}$, notice that $P^{6,m}$ and $P^{4,m}$ are different merely in agent $(n-1)$'s preferences, i.e.,

$P^{6,m}_{n-1} = P_i$ and $P^{4,m}_{n-1} = \bar{P}_i$ in Table 1. Then, *sd-strategy-proofness* implies $\varphi_{n-1,a}(P^{6,m}) = \varphi_{n-1,a}(P^{4,m}) = \frac{2}{n}$. Then, *equal treatment of equals* implies $\varphi_{na}(P^{6,m}) = \frac{2}{n}$. This completes the verification of step 1.

Step 2, we show $\varphi_{ib}(P^{6,m}) = 0$ and $\varphi_{ic}(P^{6,m}) = \frac{1}{n}$ for $i = n-1, n$ and all $m = 2, \dots, \bar{n}$. The verification simply follows from an application of *sd-efficiency*, *equal treatment of equals*, feasibility and Lemma 10. Therefore, we omit the details.

Step 3, we show $\varphi_{ic}(P^{6,m}) = \frac{1}{n}$ for all $i = 1, \dots, n-2$ and $m = 3, \dots, \bar{n}$. First, in $P^{6,2}$, according to Step 2, by feasibility and *equal treatment of equals*, we have $\varphi_{ic}(P^{6,2}) = \frac{1}{n}$ for all $i = 1, \dots, n-2$.

Next, notice that $P^{6,3}$ and $P^{6,2}$ are different merely in agent $n-2$'s preferences, i.e., $P^{6,3}_{n-2} = P_i$ and $P^{6,2}_{n-2} = \hat{P}_i$ where P_i and $\hat{P}_i$ are from Table 1. Then, *sd-strategy-proofness* implies $\varphi_{n-2,c}(P^{6,3}) = \varphi_{n-2,c}(P^{6,2}) = \frac{1}{n}$. Last, by feasibility, *equal treatment of equals* and Step 2, we know $\varphi_{ic}(P^{6,3}) = \frac{1}{n}$ for all $i = 1, \dots, n-3$. Next, we provide an induction argument.

Induction Hypothesis: Given $4 \leqslant m \leqslant \bar{n}$, for all $3 \leqslant l < m$, $\varphi_{ic}(P^{6,l}) = \frac{1}{n}$ for all $i = 1, \dots, n-2$.

We will show $\varphi_{ic}(P^{6,m}) = \frac{1}{n}$ for all $i = 1, \dots, n-2$. Notice that $P^{6,m}$ and $P^{6,m-1}$ are different merely in agent $n-m+1$'s preference, i.e., $P^{6,m}_{n-m+1} = P_i$ and $P^{6,m-1}_{n-m+1} = \hat{P}_i$ in Table 1. Then, *sd-strategy-proofness* and induction hypothesis imply $\varphi_{n-m+1,c}(P^{6,m}) = \varphi_{n-m+1,c}(P^{6,m-1}) = \frac{1}{n}$. Furthermore, *equal treatment of equals* implies $\varphi_{i,c}(P^{6,m}) = \frac{1}{n}$ for all $i = n-m+1, \dots, n-2$. Last, by feasibility, *equal treatment of equals* and Step 2, we have $\varphi_{ic}(P^{6,m}) = \frac{1}{n}$ for all $i = 1, \dots, n-m$. This completes the verification of induction hypothesis, hence Step 3 and the claim. ■

Now we have the contradiction for the case of odd number of agents. Now, $\bar{n} = \frac{n-1}{2}$. Notice that $P^{5,\bar{n}+1}$ and $P^{6,\bar{n}}$ are different only in agent $(\bar{n}+1)$'s preference, i.e., $P^{5,\bar{n}+1}_{\bar{n}+1} = P_i$ and $P^{6,\bar{n}}_{\bar{n}+1} = \hat{P}_i$ where P_i and $\hat{P}_i$ are from Table 1. Then *sd-strategy-proofness* requires $\varphi_{\bar{n}+1,c}(P^{5,\bar{n}+1}) = \varphi_{\bar{n}+1,c}(P^{6,\bar{n}})$. Thus, we have

$$\frac{1 - 2 \times \left[\frac{3}{n} - \frac{1}{n-[(\bar{n}+1)-1]}\right] - [(\bar{n}+1)-1]\gamma(\bar{n}+1)}{n - [(\bar{n}+1)+1]} = \frac{1}{n} \Leftrightarrow \frac{n^3 - 6n^2 + 11n - 2}{n(n^3 - 6n^2 + 11n - 6)} = \frac{1}{n}. \text{ Contradiction!}$$

In conclusion, a domain satisfying the *elevating property* admits no *sd-strategy-proof*, *sd-efficient* and *equal-treatment-of-equals* rule. Therefore, the connected domain $\mathbb{D}$ in Theorem 3 must violate the *elevating property*. Last, applying Lemmas 5 - 7, we show that domain $\mathbb{D}$ is a *restricted tier domain*. This completes the verification of Theorem 3.