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Weak Monotonicity Characterizes Deterministic Dominant Strategy Implementation

by

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Abstract

We characterize dominant strategy incentive compatibility with multi-dimensional types. A social choice mechanism is incentive compatible if and only if it is weakly monotone (W-Mon). W-Mon is the following requirement: if changing one agent's type (while keeping the types of other agents fixed) changes the outcome under the social choice function, then the resulting difference in utilities of the new and original outcomes evaluated at the new type of this agent must be no less than this difference in utilities evaluated at the original type of this agent.

JEL Classification Numbers: D44.

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This paper subsumes parts of “Towards a Characterization of Truthful Combinatorial Auctions,” by Lavi, Mu'alem, and Nisan, and of “Incentive Compatibility in Multi-unit Auctions,” by Bikhchandani, Chatterji, and Sen. We are grateful for helpful comments to Liad Blumrosen, Joe Ostroy, Moritz Meyer-ter-Vehn, Benny Moldovanu, Dov Monderer, Motty Perry, Phil Reny, Amir Ronen, and Rakesh Vohra. We are especially grateful to six referees and two editors whose comments led to substantial improvements in this paper. Bikhchandani was supported by National Science Foundation under grant no. SES-0422317, and Lavi, Mu'alem, and Nisan were supported by Israeli Science Foundation and USA-Israel Bi-National Science Foundation.

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1 Introduction

We characterize dominant strategy incentive compatibility of deterministic social choice functions in a model with multi-dimensional types. We show that incentive compatibility is characterized by a simple monotonicity property of the social choice function. This property, termed weak monotonicity (W-Mon), requires the following: if changing one agent's type (while keeping the types of other agents fixed) changes the outcome under the social choice function, then the resulting difference in utilities of the new and original outcomes evaluated at the new type of this agent must be no less than this difference in utilities evaluated at his original type. In effect W-Mon requires that the social choice function be sensitive to changes in differences in utilities.

We restrict attention to a private-values, quasilinear-preferences setting. The social choice function is deterministic in that randomization over the finite set of outcomes is not permitted. We take as primitive a preference order for each agent over the set of outcomes. These orders may be null, partial, or complete, and may differ across agents. This formulation incorporates multi-object auctions. The notion of incentive compatibility in this paper is dominant strategy, which is equivalent to requiring Bayesian incentive compatibility for all possible priors (see Ledyard (1978)). Thus, it is not necessary to assume that agents have priors over the types of all agents (let alone the mutual or common knowledge of such priors) for the mechanisms considered here. This weakening of common knowledge assumptions is in the spirit of the Wilson doctrine (see Wilson (1987)).

Myerson (1981) showed that in a single object auction, a random allocation function is Bayesian incentive compatible if and only if each buyers' probability of receiving the object is non-decreasing in his type. However this monotonicity condition applies only to single dimensional domains, in which a player's type is determined by a single real number. For multi-object auctions, as well as other general multi-dimensional domains, the necessary and sufficient condition for Bayesian incentive compatibility is that the random allocation rule be the subgradient of a convex function.¹ We show that when the incentive compatibility requirement is strengthened to dominant strategy and only deterministic mechanisms are considered, then incentive compatibility in a multi-dimensional types setting is characterized by W-Mon, which is much more intuitive than the subgradient condition. In particular, our condition is significantly simpler than the "cyclic monotonicity" condition of Rochet (1987). The resulting simplification of the constraint set for incentive compatibility should be

¹See, for example, Rochet (1987), McAfee and McMillan (1988), Williams (1999), Krishna and Perry (1997), Jehiel, Moldovanu, and Stacchetti (1996, 1999), Jehiel and Moldovanu (2001), Krishna and Maennar (2001), and Milgrom and Segal (2002).

helpful in applications such as in finding an expected revenue maximizing auction in the class of deterministic dominant strategy auctions.

Our characterization also bears upon a framework where the mechanism designer is interested in issues of efficiency rather than revenue. While the Vickery-Clarke-Groves (VCG) mechanism is ex post efficient, there are reasons to be interested in other (inefficient) incentive-compatible mechanisms. It is well known that because of its computational complexity the VCG auction is infeasible for selling more than a small number of objects. Several papers investigate computationally feasible (but inefficient) auctions in private-values settings (see Nisan and Ronen (2000), Lehman et al. (1999), and Holzman and Monderer (2004)). Characterizing the set of incentive-compatible auctions facilitates the selection of an auction that is preferable to the VCG auction on grounds of computational feasibility.

Roberts (1979) showed that in quasilinear environments with a complete domain, a condition called positive association of differences (PAD) is necessary and sufficient for dominant strategy incentive compatibility. Our paper considers a much more restrictive domain of preferences than Roberts assumes. In particular, the PAD condition is vacuous in our model as all social choice functions satisfy it.

The paper is organized as follows. The characterization of incentive compatibility for a single agent model is developed in Sections 2 and 3. In Section 4, we describe how this characterization extends easily to many agents. In Section 5, we discuss the connections of our paper to other characterizations of incentive compatibility with multi-dimensional types. We conclude in Section 6. Most proofs are given in an Appendix. A few related examples and results are in the Supplementary Materials to this paper, i.e., Bikhchandani et al. (2006).

2 A single agent model

Let $A = \{a_1, a_2, \dots, a_K\}$ be a finite set of possible outcomes. We assume that the agent has quasilinear preferences over outcomes and (divisible) money. The agent's type, which is his private information, determines his utility over outcomes. The utility of an agent of type V over outcome a and money m is:

$$\mathcal{U}(a, m, V) = U(a, V) + m, \quad a \in A.$$

The domain of V is $D \subseteq \mathfrak{R}_+^K$. It is convenient to assume that the agent's initial endowment of money is normalized to zero and he can supply any (negative) quantity required. We will sometimes write $V(a)$, $V'(a)$ instead of $U(a, V)$, $U(a, V')$ respectively.

A *social choice function* f is a function from the agent's report to an outcome in the set A . As we are interested in truth-telling social choice functions, by the revelation principle we restrict attention to direct mechanisms. Thus, $f : D \rightarrow A$. We assume, without loss of generality, that f is onto A . A *payment function* $p : D \rightarrow \mathfrak{R}$ is a function from the agent's reported type to a money payment by the agent. A *social choice mechanism* (f, p) consists of a social choice function f and a payment function p .

A social choice mechanism is *truth-telling* if truthfully reporting his type is optimal (i.e., is a dominant strategy) for the agent:

$$U(f(V), V) - p(V) \geq U(f(V'), V) - p(V'), \quad \forall V, V' \in D. \quad (1)$$

A social choice function f is *truthful* if there exists a payment function p such that (f, p) is truth-telling; p is said to *implement* f .

Consider the following restriction on the allocation mechanism. A social choice function f is *weakly monotone* (*W-Mon*) if for every V, V' ,

$$U(f(V'), V') - U(f(V), V') \geq U(f(V'), V) - U(f(V), V). \quad (2)$$

If f satisfies W-Mon, then the difference in the agent's utility between $f(V')$ and $f(V)$ at V' is greater than or equal to this difference at V .

W-Mon is a simple and intuitive condition on social choice functions. In effect, it is a requirement that the social choice function be sensitive to changes in differences in utilities. It is easy to see that W-Mon is a necessary condition for truth-telling:

Lemma 1 *If (f, p) is a truth-telling social choice mechanism then f is W-Mon.*

Proof: Let (f, p) be a truth-telling social choice mechanism. Consider two types V and V' of the agent. By the optimality of truth-telling at V and V' respectively, we have

$$\begin{aligned} U(f(V), V) - p(V) &\geq U(f(V'), V) - p(V') \\ \text{and} \quad U(f(V'), V') - p(V') &\geq U(f(V), V') - p(V) \end{aligned}$$

These two inequalities imply that

$$\begin{aligned} U(f(V'), V') - U(f(V), V') &\geq p(V') - p(V) \\ &\geq U(f(V'), V) - U(f(V), V). \end{aligned}$$

Hence f satisfies W-Mon. ■

Next, we obtain conditions on D , the domain of the agent's types, under which W-Mon is sufficient for truth-telling.

3 Sufficiency of W-Mon

If the domain of the agent's types, D , is not large enough then W-Mon is not sufficient for truth-telling. This is clear from the following example.

Example 1: There are three outcomes a_1 , a_2 , and a_3 . The agent's type is a vector representing his utilities for these outcomes. The agent has three possible types: $V^1 = (0, 55, 70)$, $V^2 = (0, 60, 85)$, $V^3 = (0, 40, 75)$.² That is, $D = \{V^1, V^2, V^3\}$.

The social choice function $f(V^1) = a_1$, $f(V^2) = a_2$, and $f(V^3) = a_3$ is W-Mon on the set D because:

$$\begin{aligned} V^2(a_2) - V^2(a_1) &= 60 - 0 \geq 55 - 0 = V^1(a_2) - V^1(a_1) \\ V^3(a_3) - V^3(a_2) &= 75 - 40 \geq 85 - 60 = V^2(a_3) - V^2(a_2) \\ V^1(a_1) - V^1(a_3) &= 0 - 70 \geq 0 - 75 = V^3(a_1) - V^3(a_3) \end{aligned}$$

However, there is no payment function that implements f . Suppose that the agent pays p^1 at report V^1 , p^2 at report V^2 , and p^3 at report V^3 . Without loss of generality, let $p^1 = 0$. For truth-telling we must have $p^2 \geq 55$, else type V^1 would report V^2 . Similarly, $p^3 - p^2 \geq 25$ else type V^2 would report V^3 . Therefore, we must have $p^3 \geq 80$. But then type V^3 would report V^1 .

Even if the domain of types is connected, W-Mon is not sufficient for truthfulness. Let \hat{D} be the sides of the triangle with corners V^1 , V^2 , and V^3 defined above. The allocation rule \hat{f} is as follows: $\hat{f}(V) = a_1$, $\forall V \in [V^1, V^3)$, $\hat{f}(V) = a_2$, $\forall V \in [V^2, V^1)$, and $\hat{f}(V) = a_3$, $\forall V \in [V^3, V^2)$.³ It may be verified that \hat{f} satisfies W-Mon but there are no payments that induce truth-telling under \hat{f} . \triangle

Requiring W-Mon on a larger domain (than in the example) strengthens this condition. To this end, we define order-based preferences over the possible outcomes.

Order-based domains: We restrict attention to domains $D \subseteq R_+^K$. In certain contexts, regardless of his type, the agent has an order of preference over some of the outcomes in the set A . In a multi-object auction, for instance, where an outcome is the bundle of objects allocated to the agent, if $a_\ell \subset a_k$, then under free disposal it is natural that the agent prefers a_k to a_ℓ and $V(a_\ell) \leq V(a_k)$ for all $V \in D$. Therefore, we take as a primitive the finite set of outcomes A and a (weak) order \succeq on it. This order may be null, partial, or complete.

A type V is *consistent* with respect to (A, \succeq) if $a_k \succeq a_\ell$ implies $V(a_k) \geq V(a_\ell)$. A domain of types D is *consistent* with respect to (A, \succeq) if every type in D is consistent

²Thus, $V^1(a_1) = 0$, $V^1(a_2) = 55$, and $V^1(a_3) = 70$ and so on.

³Here $[V^1, V^3)$ denotes the half-open line segment joining V^1 to V^3 etc.

with respect to (A, \succeq) . We will sometimes write domain D on (A, \succeq) to mean D is consistent with respect to (A, \succeq) .

If \succeq is null then D is an *unrestricted* domain in the sense that for any $a_k, a_\ell \in A$, there may exist $V, V' \in D$ such that $V(a_k) > V(a_\ell)$ and $V'(a_k) < V'(a_\ell)$. If, instead, \succeq is a partial order then D is a *partially ordered* domain: for any $a_k, a_\ell \in A$ if $a_k \succeq a_\ell$ then $V(a_k) \geq V(a_\ell)$ for all $V \in D$. If \succeq is a complete order then D is a *completely ordered* domain: for any $a_k, a_\ell \in A$ either $V(a_k) \geq V(a_\ell)$ for all $V \in D$ or $V(a_k) \leq V(a_\ell)$ for all $V \in D$ depending on whether $a_k \succeq a_\ell$ or $a_k \preceq a_\ell$.

Examples:

(i) As already mentioned, in a multi-object auction the set of outcomes A is a list of possible subsets of objects that the agent might be allocated. The order \succeq is the partial order induced by set inclusion.

(ii) A multi-unit auction is a special case of a multi-object auction in which all objects are identical. Let the outcomes be the number of objects allocated to the agent. Thus, for any $a_k, a_\ell \in A$ either $a_k \leq a_\ell$ or $a_\ell \leq a_k$; accordingly either $a_k \preceq a_\ell$ or $a_k \succeq a_\ell$ and \succeq is a complete order.

(iii) Another special case is when the agent has assignment model preferences over $K - 1$ heterogeneous objects. Let the outcome a_1 denote no object assigned to the agent, and let a_{k+1} , $k = 1, 2, \dots, K - 1$, denote the assignment of the k th object to the agent. The allocation of more than one object to the agent is not permitted. The underlying order is $a_k \succeq a_1$, for all $k \geq 2$, and $a_k \not\succeq a_\ell$, for all $k, \ell \geq 2$, $k \neq \ell$. \triangle

In an auction, there is an outcome at which the agent does not get any object; the utility of this outcome is 0 for all types of the agent. The proofs in Section 3.2 (but not in Section 3.1) require the existence of such an outcome.

The inverse of a social choice function f is

$$Y(k) \equiv \{V \in D \mid f(V) = a_k\}.$$

For any $k, \ell \in \{1, 2, \dots, K\}$, define⁴

$$\delta_{k\ell} \equiv \inf\{V(a_k) - V(a_\ell) \mid V \in Y(k)\}. \quad (3)$$

Note that $\delta_{kk} = 0$. The following lemma will be useful in the sequel.

Lemma 2 *For any social choice function f and $a_k, a_\ell, a_r \in A$ we have:*

- (i) *If $a_k \succeq a_\ell$ then $\delta_{rk} \leq \delta_{r\ell}$.*
- (ii) *W-Mon implies that $\delta_{k\ell} \geq -\delta_{\ell k}$.*

⁴The dependence of Y and of $\delta_{k\ell}$ on the social choice function f is suppressed for notational convenience.

Proof: (i) As $V(a_k) \geq V(a_\ell)$ for all V , including $V \in Y(r)$, we have $V(a_r) - V(a_k) \leq V(a_r) - V(a_\ell)$, $\forall V \in Y(r)$. Therefore, $\delta_{rk} \leq \delta_{r\ell}$.

(ii) By W-Mon, $V(a_k) - V(a_\ell) \geq V'(a_k) - V'(a_\ell)$, $\forall V \in Y(k), V' \in Y(\ell)$. Thus,

$$\begin{aligned} \delta_{k\ell} = \inf\{V(a_k) - V(a_\ell) \mid V \in Y(k)\} &\geq \sup\{V(a_k) - V(a_\ell) \mid V \in Y(\ell)\} \\ &= -\inf\{V(a_\ell) - V(a_k) \mid V \in Y(\ell)\} = -\delta_{\ell k} \end{aligned}$$

■

Next, we prove sufficiency of W-Mon for partially ordered domains.

3.1 Partially ordered domains

Recall that the set of outcomes is $A = (a_1, a_2, \dots, a_K)$. Throughout Section 3.1 we make the following assumption on the domain of types.

Rich domain assumption: Let D be a domain of types on (A, \succeq) . Then D is *rich* if every $V \in R_+^K$ that is consistent with (A, \succeq) belongs to D .

Thus, if \succeq is null then $D = \mathfrak{R}_+^K$. If, instead, \succeq is a partial order then D is the largest subset of \mathfrak{R}_+^K satisfying inequalities $V(a_k) \geq V(a_\ell)$ whenever $a_k \succeq a_\ell$ for all $a_k, a_\ell \in A$. It is easily verified that the formulations of the auction examples of the previous section admit rich domains.

Next, we define a payment function that implements a social choice function satisfying W-Mon on a rich domain. Relabelling the outcomes if necessary, let a_K be an outcome such that no other outcome is always weakly preferred to it; that is, for each $a_\ell \in A$ there exists a $V \in D$ such that $V(a_K) > V(a_\ell)$.⁵ If \succeq is null, any outcome has this property. If \succeq is a non-null partial order then any outcome which is maximal under \succeq may be selected as a_K .⁶ Consider the payment function

$$p_k \equiv -\delta_{Kk}, \quad \forall k = 1, 2, \dots, K. \quad (4)$$

That is, if $V \in Y(k)$ then an agent who reports V pays p_k (and the outcome a_k is selected by f). We use the next result to show that p implements f .

Lemma 3 *Let f be a social choice function that is W-Mon. For any $a_\ell \in A$ and $V \in D$,*

- (i) *If $V(a_\ell) - p_\ell < V(a_K) - p_K$ then $f(V) \neq a_\ell$.*
- (ii) *If $V(a_\ell) - p_\ell > V(a_K) - p_K$ then $f(V) \neq a_K$.*

⁵Note that if $a_\ell \not\succeq a_K$ then, as D is rich, there exists such a V .

⁶In a multi-object auction, a_K is any maximal subset (with respect to set inclusion) in the range of the mechanism. Thus, if the outcome at which all objects are allocated to the agent is in the range of the mechanism then this outcome is a_K .

This leads to the main result for partially ordered domains.

Theorem 1 *A social choice function on a rich domain is truthful if and only if it is weakly monotone.*

As already observed, the smaller the domain of types on which the social choice mechanism satisfies W-Mon, the weaker the restriction imposed by W-Mon. Therefore, next we investigate whether W-Mon is sufficient for truth-telling when the domain is not rich, in particular the domain is bounded. To obtain a sufficiency result with smaller domain assumptions, we make the stronger assumption that the underlying order is complete.

3.2 Completely ordered domains

The order \succeq on the set of outcomes is complete. That is, for any $a_k, a_\ell \in A$, either $a_k \succeq a_\ell$ or $a_\ell \succeq a_k$ but not both.⁷ Thus, for any domain D consistent with (A, \succeq) either $V(a_k) \geq V(a_\ell)$ for all $V \in D$ or $V(a_\ell) \geq V(a_k)$ for all $V \in D$. We label the outcomes such that $a_k \succeq a_{k-1}$, $k = 1, 2, \dots, K$. Define for each type V the marginal (or incremental) utility of the k th outcome over the $(k - 1)$ th outcome:

$$v_k \equiv V(a_k) - V(a_{k-1}) \geq 0, \quad k = 1, 2, \dots, K.$$

For notational simplicity, we have $K + 1$ outcomes rather than K . Further, we assume that the utility of outcome a_0 is the same for each type in D , and we normalize $V(a_0) \equiv 0, \forall V \in D$.

A multi-unit auction has a completely ordered domain, with the number of units allocated to the buyer being the outcomes. Therefore, we denote the set of outcomes as $A = \{0, 1, 2, \dots, K\}$ (rather than $\{a_0, a_1, \dots, a_K\}$). It is convenient to define the agent's type in terms of marginal utilities $v = (v_1, v_2, \dots, v_K)$ for each successive unit (rather than total utilities $V = (V(1), V(2), \dots, V(K))$). The social choice and payment functions map marginal utilities to an outcome $k = 0, 1, \dots, K$ and to payments respectively. The inverse social choice function $Y(\cdot)$ maps integers $k = 0, 1, \dots, K$ to subsets of types (in marginal utility space).

Using (2), we see that in this context W-Mon may be restated as follows. A social choice rule f is W-Mon if for every v and v' ,

$$\text{If } f(v') > f(v) \text{ then } \sum_{\ell=f(v)+1}^{f(v')} v'_\ell \geq \sum_{\ell=f(v)+1}^{f(v')} v_\ell. \quad (5)$$

⁷If the agent is indifferent between any two outcomes, we can recombine them into one outcome.

Suppose that f is the allocation rule of a multi-unit auction and that the agent is allocated more units by the mechanism when his (reported) type is v' than when it is v . If f is W-Mon then the agent's valuation at v' for the additional units allocated at v' is at least as large as his valuation at v .

The domain in Example 1 is completely ordered but W-Mon is not sufficient for truthfulness; therefore, we need a larger domain. The following assumption encompasses both bounded utilities and diminishing marginal utilities.⁸

Bounded domain assumption: *There exist constants $\bar{a}_k \in (0, \infty)$, $k = 1, 2, \dots, K$, such that the domain of agent types, \mathcal{D} , satisfies either (A) or (B) below:*

A. $\mathcal{D} = \prod_{k=1}^K [0, \bar{a}_k]$

B. \mathcal{D} is the convex hull of points $(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_{k-1}, \bar{a}_k, 0, \dots, 0)$, $k = 0, 1, \dots, K$.

The assumption that $\bar{a}_k < \infty$ for all k is not essential, but does simplify the proofs. Domain assumption A does not restrict the marginal utilities to be decreasing (or increasing). We do not specifically assume that $\bar{a}_k \geq \bar{a}_{k+1}$, but when this inequality holds for all k and domain assumption B is satisfied, then we have diminishing marginal utilities; that is, $v_k \geq v_{k+1}$ for all $v \in \mathcal{D}$.⁹ Under domain assumption B, $v = (v_1, v_2, \dots, v_K) \in \mathcal{D}$ if and only if $0 \leq v_\ell \leq \bar{a}_\ell, \forall \ell$ and

$$\frac{v_\ell}{\bar{a}_\ell} \geq \frac{v_{\ell+1}}{\bar{a}_{\ell+1}} \quad \ell = 1, 2, \dots, K - 1. \quad (6)$$

Recalling the definition in (3), note that

$$\begin{aligned} \delta_{k \ k-1} &= \inf\{v_k \mid v \in Y(k)\} \\ \delta_{k-1 \ k} &= -\sup\{v_k \mid v \in Y(k-1)\}. \end{aligned} \quad (7)$$

Next, a ‘‘tie-breaking at boundaries’’ assumption, TBB, is invoked to deal with difficulties at the boundary of the domain.

Tie-breaking at boundaries (TBB): *A social choice function f satisfies TBB if:*

- (i) $v_k > 0$ for all $v \in Y(k)$, and
- (ii) $v_k < \bar{a}_k$ for all $v \in Y(k-1)$.

⁸The domain of types is referred to by \mathcal{D} rather than D as types now specify marginal utilities rather than total utilities. However, we continue to use the symbol $Y(\cdot)$ for the inverse social choice function, which now maps outcomes into subsets of \mathcal{D} .

⁹A straightforward modification extends the proofs to the case of increasing marginal utilities, i.e., when \mathcal{D} is the convex hull of points $(0, 0, \dots, 0, \bar{a}_k, \bar{a}_{k+1}, \dots, \bar{a}_K)$, $k = 1, \dots, K$ and $(0, 0, \dots, 0)$. The assumption of increasing marginal utilities obtains when the objects are complements, such as airwave spectrum rights.

Consider TBB(i). If $\delta_{kk-1} > 0$ then TBB(i) imposes no restriction. If, instead, $\delta_{kk-1} = 0$ then there exists a sequence $v^n \in Y(k)$ such that $\lim_{n \rightarrow \infty} v_k^n = 0$; the existence of a point $v \in Y(k)$ at which $v_k = \delta_{kk-1} = 0$ is precluded by TBB(i). Similarly, TBB(ii) imposes no restriction if $-\delta_{k-1k} < \bar{a}_k$, and if, instead, $-\delta_{k-1k} = \bar{a}_k$ it requires that for any $v \in Y(k-1)$, we have $v_k < \bar{a}_k$.

First, we prove sufficiency of W-Mon and TBB (Lemmas 4 and 5) for truth-telling. We then show (Lemma 6) that (i) for any W-Mon social choice function f there exists a social choice function f' that satisfies W-Mon and TBB and agrees with f almost everywhere, and (ii) the money payments which truthfully implement f' also truthfully implement f .

By Lemma 2(ii), W-Mon implies $\delta_{kk-1} \geq -\delta_{k-1k}$. If in addition TBB is satisfied then the weak inequality is replaced by an equality:

Lemma 4 *Let f be a social choice function on a completely ordered, bounded domain. If f satisfies W-Mon and TBB then $\bar{a}_k \geq \delta_{kk-1} = -\delta_{k-1k} \geq 0$ for all k .*

It is clear from (7) that $v_\ell \geq \delta_{\ell\ell-1}$ for any $v \in Y(\ell)$ and $v'_\ell \leq -\delta_{\ell-1\ell}$ for any $v' \in Y(\ell-1)$. This, together with Lemma 4, implies that $v'_\ell \leq -\delta_{\ell-1\ell} = \delta_{\ell\ell-1} \leq v_\ell$. In fact, there exist $v \in Y(\ell)$ and $v' \in Y(\ell-1)$ such that v_ℓ and v'_ℓ are arbitrarily close to $\delta_{\ell\ell-1}$. In other words, the hyperplane $v_\ell = \delta_{\ell\ell-1}$ weakly separates $Y(\ell)$ and $Y(\ell-1)$.

Construct a payment function using the $\delta_{\ell\ell-1}$'s as follows:

$$p_k \equiv \begin{cases} \sum_{\ell=1}^k \delta_{\ell\ell-1}, & \text{if } \ell = 1, 2, \dots, K \\ 0, & \text{if } \ell = 0. \end{cases} \quad (8)$$

This payment function is shown to truthfully implement a social choice function satisfying W-Mon and TBB.

Lemma 5 *A social choice function on a completely ordered, bounded domain is truthful if it satisfies W-Mon and TBB.*

The next lemma allows us to dispense with TBB in the sufficient condition for truth-telling. The proof is in the Supplementary Material.

Lemma 6 *If a social choice mechanism f satisfies W-Mon then there exists an allocation mechanism f' which satisfies W-Mon and TBB such that $f(v) = f'(v)$, for almost all $v \in \mathcal{D}$. Moreover, the payment function p'_k defined as in (8) with respect to f' truthfully implements f .*

Lemma 6 assures us that given any social choice function f that satisfies W-Mon we can construct another social choice function f' which is W-Mon and TBB. By

Lemma 5, f' is truthful and by Lemma 6 the payment function which implements f' also implements f . Thus, W-Mon is sufficient for truth-telling. This leads to the main result for completely ordered domains.

Theorem 2 *A social choice function on a completely ordered bounded domain is truthful if and only if it is weakly monotone.*

An alternative characterization for the single agent, completely ordered domain model is through the payment function rather than the social choice function. Consider a multi-unit auction with one buyer. The allocation rule “induced” by any increasing payment function ($p_k \geq p_{k-1} \geq 0$) is implementable. We note that this characterization becomes considerably more complex when one considers two or more buyers. This is because each buyer’s payment function will, in general, depend on others’ reported types and for each vector of types, it must be verified that the induced allocation rule does not distribute more units than are available. Our characterization based on W-Mon is easily generalized to multi-agent settings, both for completely ordered and partially ordered domains.

4 Extension to multiple agents

We extend the results of the single agent model to multiple agents, with each agent having private values over the possible outcomes. For concreteness, we use the set-up of Section 3.1; an identical argument extends the results of Section 3.2.

There are $i = 1, 2, \dots, n$ agents and the finite set of outcomes is $A = \{a_1, a_2, \dots, a_L\}$. Agent i ’s type is denoted by $V_i = (V_{i1}, V_{i2}, \dots, V_{i\ell}, \dots, V_{iL})$, where each $V_i \in D_i \subseteq \mathbb{R}_+^L$. The characteristics of all the agents are denoted by $V = (V_1, V_2, \dots, V_i, \dots, V_n)$.¹⁰ The private-values assumption is that each agent’s utility function depends only on his type. Thus, when the types are $V = (V_i, V_{-i})$, agent i ’s utility over the outcome a and m units of money is $\mathcal{U}_i(a, m, (V_i, V_{-i})) = U_i(a, V_i) + m$, $a \in A$.

The outcome set A is endowed with (partial) orders, \succeq_i , $i = 1, 2, \dots, n$, one for each agent. The domain of agents’ types, $D = D_1 \times D_2 \times \dots \times D_n$, is consistent with $(A, \succeq_1, \succeq_2, \dots, \succeq_n)$ if each D_i , the domain of agent i ’s types, is consistent with (A, \succeq_i) . Further, D is rich if each D_i is rich (as defined in Section 3.1).

In an auction, A represents the set of possible assignments of objects to agents (buyers). If buyer i cares only about the objects allocated to him, then the partial order \succeq_i is determined by set inclusion on the respective allocations to buyer i at

¹⁰In a departure from the notation of Section 3, V now refers to a profile of utilities for n agents rather than for a single agent.

$a, a' \in A$. Thus, $a \sim_i a'$, (i.e., $U_i(a, V_i) = U_i(a', V_i)$, $\forall V_i \in D_i$) whenever a and a' allocate the same bundle of objects to buyer i .

A social choice function f is a mapping from the domain of all agents' (reported) types onto A , $f : D \rightarrow A$. For each agent i there is a payment function $p_i : D \rightarrow \mathfrak{R}$. Let $p = (p_1, p_2, \dots, p_n)$. The pair (f, p) is a social choice mechanism. A social choice mechanism is *dominant strategy incentive compatible* if truthfully reporting one's type is a dominant strategy for each agent. That is, for every i , V_i, V'_i, V_{-i} ,

$$U_i(f(V_i, V_{-i}), V_i) - p_i((V_i, V_{-i})) \geq U_i(f(V'_i, V_{-i}), V_i) - p_i(V'_i, V_{-i}). \quad (9)$$

A social choice function f is *dominant strategy implementable* if there exist payment functions p such that (f, p) is dominant strategy incentive compatible.

The following definition generalizes weak monotonicity to a multiple agent setting. A social choice function f is *weakly monotone (W-Mon)* if for every i , V_i, V'_i, V_{-i} ,

$$U_i(f(V'_i, V_{-i}), V'_i) - U_i(f(V_i, V_{-i}), V'_i) \geq U_i(f(V'_i, V_{-i}), V_i) - U_i(f(V'_i, V_{-i}), V_i). \quad (10)$$

Observe that the requirement of dominant strategy, (9), is the same as requiring truth-telling (i.e. (1)) for each agent i , for each value of V_{-i} . Further, (10) is equivalent to requiring (2) for each agent i , for each value of V_{-i} . Thus, Theorem 1 (and similarly also Theorem 2) generalize:

Theorem 3 (i) *A social choice function on a rich domain is dominant strategy implementable if and only if it is weakly monotone.*

(ii) *A social choice function on a completely ordered, bounded domain is dominant strategy implementable if and only if it is weakly monotone.*

5 Relationship to earlier work

In his seminal paper, Myerson (1981) showed that a necessary and sufficient condition for incentive compatibility of a single object auction is that each buyer's probability of receiving the object is non-decreasing in his reported valuation.¹¹ Several authors, including Rochet (1987), McAfee and McMillan (1988), Williams (1999), Krishna and Perry (1997), Jehiel, Moldovanu, and Stacchetti (1996, 1999), Jehiel and Moldovanu (2001), Krishna and Maennar (2001), and Milgrom and Segal (2002), have extended Myerson's analysis to obtain necessary and sufficient conditions for Bayesian incentive-compatible mechanisms in the presence of multi-dimensional types. These results are easily adapted to dominant strategy mechanisms.

¹¹Myerson characterized Bayesian incentive compatibility when agents' types are one dimensional; simple modifications to his proofs yield a similar characterization for dominant strategy incentive compatibility. Myerson's characterization coincides with W-Mon applied to one dimensional types.

To place our results in the context of this earlier work, let G be a (random) social choice function that maps the domain of agents' types D to a probability distribution over the set of outcomes $A = \{a_1, a_2, \dots, a_L\}$. Thus, for each $V \in D$, $G(V) = (g_1(V), g_2(V), \dots, g_\ell(V), \dots, g_L(V))$ is a probability distribution. Recall that the payment functions are $p = (p_1, p_2, \dots, p_n)$. A social choice mechanism (G, p) induces the following payoff function for agent i :

$$\Pi_i(V_i, V_{-i}) \equiv G(V_i, V_{-i}) \cdot V_i - p_i(V_i, V_{-i}),$$

where $x \cdot y$ denotes the dot product of two vectors x and y . Dominant strategy incentive compatibility implies that for all i, V_i, V'_i, V_{-i} ,

$$\begin{aligned} \Pi_i(V_i, V_{-i}) &\geq G(V'_i, V_{-i}) \cdot V_i - p_i(V'_i, V_{-i}) \\ &= \Pi_i(V'_i, V_{-i}) + G(V'_i, V_{-i}) \cdot (V_i - V'_i), \quad (11) \\ \implies \Pi_i(V_i, V_{-i}) &= \max_{V'_i} \{G(V'_i, V_{-i}) \cdot V_i - p_i(V'_i, V_{-i})\} \end{aligned}$$

As $\Pi_i(\cdot, V_{-i})$ is the maximum of a family of linear functions, it is a convex function of V_i . Further, for each i and V_{-i} , $G(\cdot, V_{-i})$ is a subgradient of $\Pi_i(\cdot, V_{-i})$. This leads to the following characterization: A social choice function G is dominant strategy implementable if and only if for each V_{-i} , $G(\cdot, V_{-i})$ is a subgradient of a convex function from D_i to \mathfrak{R} .

A function $G(\cdot, V_{-i}) : D_i \rightarrow \mathfrak{R}^L$, $D_i \subseteq \mathfrak{R}^L$, is *cyclically monotone* if for every finite selection $V_i^j \in D_i$, $j = 1, 2, \dots, m$, with $V_i^{m+1} = V_i^1$,

$$\sum_{j=1}^m V_i^j \cdot [G(V_i^j, V_{-i}) - G(V_i^{j+1}, V_{-i})] \geq 0. \quad (12)$$

A function is a subgradient of a convex function if and only if it is cyclically monotone (Rockafellar (1970, p. 238)). Thus, cyclic monotonicity of the social choice function also characterizes dominant strategy implementability. The rationalizability condition of Rochet (1987) generalizes the cyclic monotonicity characterization of incentive compatibility to settings where the utility function is possibly non-linear.

W-Mon is a weaker condition than cyclic monotonicity in that W-Mon requires (12) only for $m = 2$.¹² Thus, our contribution is to show that when one restricts attention to deterministic social choice functions, dominant strategy incentive compatibility is characterized by the simpler condition of W-Mon. Rochet's cyclic monotonicity condition requires that inequality (12) be checked for all finite selections of types, whereas W-Mon requires the inequality to be verified only for every pair of types.

¹²Note that if $m = 2$ then (12) may be restated as $[G(V'_i, V_{-i}) - G(V_i, V_{-i})] \cdot (V'_i - V_i) \geq 0$ for all V_i, V'_i . This is the same as (10), with $U_i(G(V_i, V_{-i}), V_i) = G(V_i, V_{-i}) \cdot V_i$, etc.

It is well known that with multi-dimensional types, characterizations of incentive compatibility are complex. For one dimensional types, cyclic monotonicity is equivalent to W-Mon which is equivalent to a non-decreasing subgradient function (Rockafellar (1970, p. 240)). Hence, Myerson’s characterization of incentive compatibility as a non-decreasing allocation function. W-Mon, which generalizes the concept of a non-decreasing function, does not characterize incentive compatibility in a multi-dimensional setting with random mechanisms; the more complex condition of cyclic monotonicity is needed. However, when attention is restricted to deterministic mechanisms then we show that the simpler condition of W-Mon is enough. Thus our paper helps delineate the boundaries of multi-dimensional models which permit a simple characterization of incentive compatibility.

Although our characterization is significantly simpler, the restriction to deterministic mechanisms may be a limitation. Manelli and Vincent (2003) and Thanassoulis (2004) show that a multi-product monopolist can strictly increase profits by using a random, rather than deterministic, mechanism. Example S1 in the Supplementary Materials establishes that for random social choice functions W-Mon is not sufficient for dominant strategy implementability.¹³ Whether there is an intuitive condition, which in conjunction with W-Mon, is sufficient for incentive compatibility of random social choice functions is an open question.

Roberts (1979) characterizes deterministic dominant strategy mechanisms in quasi-linear environments with a “complete” domain. Roberts identifies a condition called *positive association of differences* (PAD) which is satisfied by a social choice function f if for all $V = (V_1, V_2, \dots, V_n)$ and $V' = (V'_1, V'_2, \dots, V'_n)$

$$\begin{aligned} &\text{if } U_i(f(V), V'_i) - U_i(a, V'_i) > U_i(f(V), V_i) - U_i(a, V_i), \quad \forall a \neq f(V), \forall i, \\ &\text{then } f(V') = f(V). \end{aligned} \tag{13}$$

An allocation rule f is an *affine maximizer* if there exist constants $\gamma_i \geq 0$, with at least one $\gamma_i > 0$, and a function $U_0 : A \rightarrow \Re$ such that

$$f(V) \in \arg \max_{a \in A} \left(U_0(a) + \sum_{i=1}^n \gamma_i U_i(a, V_i) \right).$$

Roberts (1979) shows that f is a (deterministic) dominant strategy mechanism if and only if f satisfies PAD if and only if f is an affine maximizer.

What is the relationship between Roberts’ work and ours? The fundamental difference is that Roberts assumes an unrestricted domain of preferences while we operate in a restricted domain. In particular, Roberts requires that for all $a \in A$, any real number α , and any agent i , there exists a type V_i of agent i such that $U_i(a, V_i) = \alpha$.

¹³We are grateful to an anonymous referee for this example.

Thus, taking $(A, \succeq_1, \succeq_2, \dots, \succeq_n)$ and the domain of types as primitives of the two models, in Roberts' model \succeq_i is a null order and $D_i = \mathfrak{R}^L$ for each agent i ,¹⁴ whereas we allow each \succeq_i to be a non-null (even complete) order and the corresponding D_i to be a strict subset of \mathfrak{R}_+^L . Thus, an auction or any mechanism that allocates (private) goods does not satisfy Roberts' domain assumptions as they preclude free disposal and no externalities in consumption. Indeed, in an auction with two or more buyers PAD is vacuous in that all mechanisms satisfy PAD.¹⁵ W-Mon, however, is not vacuous in this setting and is the appropriate condition for incentive compatibility.¹⁶ Because a smaller domain (than Roberts') is sufficient for our characterization, one may suspect that W-Mon is stronger than PAD. This is proved in Lemma S1 in the Supplementary Materials. As already noted, in multi-agent models PAD does not imply W-Mon. Example S2 in the Supplementary Materials presents a single agent model in which a social choice mechanism satisfies PAD but not W-Mon; this mechanism is, of course, not incentive compatible. An important difference between these two conditions is that PAD is defined for changes in types of any combination of players, while W-Mon is defined for changes in exactly one player's type.

Thus, W-Mon and PAD are not equivalent. Further, because of the domain restrictions inherent in our model, our result is not a consequence of the characterization result of Roberts. It may be useful conceptually to draw an approximate parallel with the results on dominant strategy incentive compatibility in various domains. According to the Gibbard-Satterthwaite Theorem, dominant strategy is equivalent to dictatorship in an unrestricted domain (subject to a range assumption). In the quasilinear model (with otherwise unrestricted domain), Roberts showed that dominant strategy, PAD, and the existence of affine maximizers are equivalent. In the more restricted economic environments of auctions, where agents care only about their private consumption, the equivalence of these three concepts breaks down; in particular, PAD is necessary (and in the multi-buyer case vacuously so) but not sufficient for dominant strategy. The domain restrictions inherent in auctions imply that a wider class of allocation rules is incentive compatible. But if PAD is strengthened to W-Mon, then we recover equivalence between dominant strategy and W-Mon.¹⁷ Although it is stronger than PAD, W-Mon is much weaker than cyclic monotonicity which has been used to characterize incentive compatibility in multi-dimensional settings (Rochet (1987)).

¹⁴ $D_i = \mathfrak{R}^L$, for all i , is essential for Roberts' proofs.

¹⁵Let a differ from $f(V)$ in the allocation to exactly one buyer. Then the hypothesis in (13) is false as the inequality holds for at most one and not for all buyers.

¹⁶In our search for conditions that might be necessary and sufficient on even smaller domains than considered here, we examined two conditions that strengthen W-Mon in a natural way. However, neither of these two conditions is necessary. See Example S3 in Supplementary Materials.

¹⁷W-Mon by itself does not imply affine maximization. Lavi, Mu'alem, and Nisan (2003) identify an additional property which together with W-Mon implies affine maximization.

Chung and Ely (2002) obtain a characterization of incentive compatibility which they call *pseudo-efficiency*. They show that f is dominant strategy implementable if and only if there exist real-valued functions $w_i(a, V_i)$ such that for each V ,

$$f(V) \in \arg \max_{a \in A} \left(U_i(a, V_i) + w_i(a, V_i) \right), \quad \forall i.$$

W-Mon must therefore be equivalent to pseudo-efficiency. However, we believe that W-Mon is, in some ways, a more insightful condition than pseudo-efficiency. For instance, the definition of the latter involves an existential quantifier which makes it hard to verify.

6 Concluding remarks

Characterizations of incentive compatibility with multi-dimensional types are far from simple. By restricting attention to deterministic, dominant strategy mechanisms we obtain a substantial simplification. In particular, the W-Mon condition clarifies the structure of incentive-compatible auctions. The resulting simplification of the constraint set for incentive compatibility will be of use in identifying revenue-maximizing auctions within the class of deterministic dominant strategy mechanisms. Another possible application of our characterization is to practical auctions. Starting with the auction of spectrum rights in the U.S. in the mid-nineties, new market institutions have been proposed for selling multiple heterogeneous objects. Simple characterizations such as ours can be used to check the incentive properties of these institutions.¹⁸

Our strategy has been to start with W-Mon, a monotonicity condition necessary for dominant strategy incentive compatibility, and show that when applied to a large enough domain of agent types, W-Mon is also sufficient. In principle, this approach can also be applied to Bayesian incentive compatibility. However, first the problem of extending this approach to random mechanisms must be solved (see Section 5).¹⁹

¹⁸To our knowledge, all the recently proposed market mechanisms are deterministic except, of course, when there are ties. Thus a restriction to deterministic mechanisms does not seem to be a limitation in these applications.

¹⁹The distinction between random and deterministic social choice functions is less useful for Bayesian mechanisms. A Bayesian agent views a social choice function as a function of his type alone, which is a probability distribution over outcomes.

7 Appendix: Proofs of Sections 3.1 & 3.2

Proof of Lemma 3: (i) By definition, $p_K = 0$ and $p_\ell = -\delta_{K\ell}$. Therefore, $V(a_\ell) - V(a_K) < -\delta_{K\ell} \leq \delta_{\ell K}$, where the second inequality follows from Lemma 2(ii). The definition of $\delta_{\ell K}$ implies that $f(V) \neq a_\ell$.

(ii) In the other direction we have $V(a_K) - V(a_\ell) < p_K - p_\ell = \delta_{K\ell}$ implies $f(V) \neq a_K$. ■

Proof of Theorem 1: In view of Lemma 1, we need only show sufficiency of W-Mon. In particular, we show that the payment function defined in (4) truthfully implements any social choice function f which is W-Mon. Suppose to the contrary that there exists k^*, k and $V \in Y(k^*)$ such that $V(a_{k^*}) - p_{k^*} < V(a_k) - p_k$. Lemma 3(i) and (ii) imply that $k \neq K$ and $k^* \neq K$ respectively (else it would contradict $V \in Y(k^*)$). Further, Lemma 3(i) implies that $V(a_{k^*}) - p_{k^*} \geq V(a_K) - p_K (= V(a_K))$. Choose a $\gamma > 0$ and a small enough $\epsilon > 0$ such that

$$V(a_{k^*}) + \epsilon - p_{k^*} < V(a_K) + \gamma - p_K < V(a_k) - p_k.$$

Note that $\gamma > \epsilon$. Define $T = \{a_{k^*}\} \cup \{a_\ell \in A \mid a_\ell \succeq a_{k^*} \text{ and } V(a_\ell) = V(a_{k^*})\}$. Let V' be the following type:

$$V'(a_r) \equiv \begin{cases} V(a_r) + \epsilon, & \text{if } a_r \in T \setminus \{a_K\} \\ V(a_r) + \gamma, & \text{if } a_r = a_K \\ V(a_r), & \text{otherwise.} \end{cases}$$

As V' is consistent with the underlying order \succeq and D is rich, $V' \in D$.²⁰

By Lemma 2(i), $p_{k^*} \leq p_\ell$ for any $a_\ell \in T$. Therefore, as $V'(a_\ell) = V'(a_{k^*})$, for all $a_\ell \in T \setminus \{a_K\}$, we have

$$V'(a_\ell) - p_\ell \leq V'(a_{k^*}) - p_{k^*} < V'(a_K) - p_K, \quad \forall a_\ell \in T \setminus \{a_K\}.$$

Thus, $a_k \notin T$ and Lemma 3(i) implies that $f(V') \neq a_\ell$ for any $\ell \in T \setminus \{a_K\}$. As $V'(a_K) - p_K < V'(a_k) - p_k$, $f(V') \neq a_K$ by Lemma 3(ii). Thus, $f(V') = a_{k'} \notin T \cup \{a_K\}$. But then,

$$0 = V'(a_{k'}) - V(a_{k'}) < V'(a_{k^*}) - V(a_{k^*}) = \epsilon$$

which violates W-Mon. ■

Proof of Lemma 4: By Lemma 2(ii) and the fact that $\bar{a}_k \geq v_k \geq 0$ for all $v \in \mathcal{D}$, we have $\bar{a}_k \geq \delta_{kk-1} \geq -\delta_{k-1k} \geq 0$.

²⁰To verify the consistency of V' note the following. If $a_{\ell'} \succeq a_\ell$ and $a_\ell \in T$, $a_{\ell'} \notin T$, $a_{\ell'} \neq a_K$ then select $\epsilon > 0$ small enough so that $V'(a_{\ell'}) = V(a_{\ell'}) \geq V(a_\ell) + \epsilon = V'(a_\ell)$. If $a_K \succeq a_\ell$, $a_\ell \in T$, then as $\gamma > \epsilon$, we have $V'(a_K) \geq V'(a_\ell)$ if $V(a_K) \geq V(a_\ell)$. Further, a_K was chosen so that $a_\ell \not\succeq a_K$ for any $\ell \neq K$.

Let $v^k \equiv (\bar{a}_1, \bar{a}_2, \dots, \bar{a}_k, 0, \dots, 0)$ for any $k = 0, 1, 2, \dots, K$ (with $v^0 \equiv (0, 0, \dots, 0)$). Observe that under either bounded domain assumption A or B, $v^k \in \mathcal{D}$. Thus, $v^k \in Y(q)$ for some $q = 0, 1, \dots, K$. TBB(i) implies that $v^k \notin Y(q)$ for any $q > k$, and TBB(ii) implies that $v^k \notin Y(q)$ for any $q < k$. Therefore, $v^k \in Y(k)$. Next, let $v(t) = (1-t)v^{k-1} + tv^k$, $t \in [0, 1]$, be a point on the straight line joining v^k and v^{k-1} , $k \geq 1$. Observe that $v(t) = (\bar{a}_1, \dots, \bar{a}_{k-1}, t\bar{a}_k, 0, \dots, 0) \in \mathcal{D}$, $\forall t \in [0, 1]$. Thus, $v(t) \in Y(q)$ for some q . TBB implies that $v(t) \in Y(k-1) \cup Y(k)$. Because $v_k(t) = t\bar{a}_k$ increases in t , there exists a $t^* \in [0, 1]$ such that $v(t) \in Y(k-1)$ for all $t < t^*$ and $v(t) \in Y(k)$ for all $t > t^*$. Thus,

$$t^*\bar{a}_k = \lim_{t \uparrow t^*} v_k(t) \leq -\delta_{k-1k} \leq \delta_{kk-1} \leq \lim_{t \downarrow t^*} v_k(t) = t^*\bar{a}_k$$

Hence, $\delta_{kk-1} = -\delta_{k-1k}$. ■

Proof of Lemma 5: W-Mon implies that

$$\text{If } \sum_{\ell=f(v)+1}^q v'_\ell < \sum_{\ell=f(v)+1}^q v_\ell, \quad \forall q > f(v) \text{ then } f(v') \leq f(v). \quad (14)$$

$$\text{If } \sum_{\ell=q+1}^{f(v)} v'_\ell > \sum_{\ell=q+1}^{f(v)} v_\ell, \quad \forall q < f(v) \text{ then } f(v') \geq f(v). \quad (15)$$

Observe that if v' , v , satisfy the hypotheses in (14) and (15) then $f(v') = f(v)$.

First, we prove that for any $k = 0, 1, 2, \dots, K$,

$$\left\{ v \in \mathcal{D} \mid \sum_{\ell=q}^k v_\ell \geq \sum_{\ell=q}^k \delta_{\ell\ell-1}, \forall q \leq k, \sum_{\ell=k+1}^q v_\ell \leq \sum_{\ell=k+1}^q \delta_{\ell\ell-1}, \forall q > k \right\} \subseteq \text{cl}[Y(k)]. \quad (16)$$

There are two cases to consider.

Case A: $(\delta_{10}, \delta_{21}, \dots, \delta_{KK-1}) \in \mathcal{D}$.²¹

Consider the point $\hat{v}^k(\epsilon) = (\delta_{10} + \epsilon_1, \dots, \delta_{kk-1} + \epsilon_k, \delta_{k+1k} - \epsilon_{k+1}, \dots, \delta_K - \epsilon_K)$ where $\epsilon_1, \epsilon_2, \dots, \epsilon_K$ satisfy the following conditions:

- (i) If $[q \leq k \text{ and } \delta_{qq-1} = \bar{a}_q]$ or $[q > k \text{ and } \delta_{qq-1} = 0]$ then $\epsilon_q = 0$.
- (ii) If $[q \leq k \text{ and } \delta_{qq-1} < \bar{a}_q]$ or $[q > k \text{ and } \delta_{qq-1} > 0]$ then $\epsilon_q > 0$.

As $(\delta_{10}, \delta_{21}, \dots, \delta_{KK-1}) \in \mathcal{D}$, there exist $\epsilon_1, \epsilon_2, \dots, \epsilon_K$ satisfying (i) and (ii) above such that $\hat{v}^k(\epsilon) \in \mathcal{D}$.²² Consider any $q < k$. If $\delta_{q+1q} < \bar{a}_{q+1}$ then as $\hat{v}_{q+1}^k(\epsilon) > \delta_{q+1q}$, we know that $\hat{v}^k(\epsilon) \notin Y(q)$. If, instead, $\delta_{q+1q} = \bar{a}_{q+1}$ then (as $\epsilon_{q+1} = 0$) we have $\hat{v}_{q+1}^k(\epsilon) = \bar{a}_{q+1}$. Thus, TBB(ii) implies that $\hat{v}^k(\epsilon) \notin Y(q)$. Similarly, TBB(i) implies

²¹Lemma 4 implies that if domain assumption A is satisfied then we are in Case A.

²²If domain assumption A is satisfied, this is easy to verify. If, instead, domain assumption B is satisfied then $\epsilon_1, \epsilon_2, \dots, \epsilon_K$ must be chosen to ensure that $\hat{v}^k(\epsilon)$ satisfies (6).

that $\hat{v}^k(\epsilon) \notin Y(q)$ for $q > k$. Hence $\hat{v}^k(\epsilon) \in Y(k)$. Therefore, (14) and (15) imply that²³

$$\left\{ v \in \mathcal{D} \mid \sum_{\ell=q}^k v_\ell > \sum_{\ell=q}^k (\delta_{\ell\ell-1} + \epsilon_\ell), \forall q \leq k, \sum_{\ell=k+1}^q v_\ell < \sum_{\ell=k+1}^q (\delta_{\ell\ell-1} - \epsilon_\ell), \forall q > k \right\} \subset Y(k).$$

One can construct a sequence $(\epsilon_1^n, \epsilon_2^n, \dots, \epsilon_K^n) \rightarrow 0$ such that $\hat{v}^k(\epsilon^n) \in \mathcal{D}$. Taking limits as $\epsilon^n \rightarrow 0$, we get

$$\left\{ v \in \mathcal{D} \mid \sum_{\ell=q}^k v_\ell > \sum_{\ell=q}^k \delta_{\ell\ell-1}, \forall q \leq k, \sum_{\ell=k+1}^q v_\ell < \sum_{\ell=k+1}^q \delta_{\ell\ell-1}, \forall q > k \right\} \subset Y(k),$$

which in turn implies (16).

Case B: $(\delta_{10}, \delta_{21}, \dots, \delta_{K K-1}) \notin \mathcal{D}$.²⁴

For each $k = 0, 1, 2, \dots, K$ define

$$\mathbf{v}^k(\epsilon) = \left\{ v \mid v_q = \max\left[v_{q+1} \frac{\bar{a}_q}{\bar{a}_{q+1}}, \delta_{qq-1} + \epsilon_q\right], \forall q < k, \delta_{kk-1} + \epsilon_k \leq v_k \leq \bar{a}_k, \right. \\ \left. v_q = \min\left[v_{q-1} \frac{\bar{a}_q}{\bar{a}_{q-1}}, \delta_{qq-1} - \epsilon_q\right], \forall q > k \right\}.$$

Any $v \in \mathbf{v}^k(\epsilon)$ satisfies (6). Thus, provided $\epsilon_1, \epsilon_2, \dots, \epsilon_K$ satisfy (i) and (ii) defined in Case A, and are small enough, $\mathbf{v}^k(\epsilon) \subset \mathcal{D} [= \cup_{q=0}^K Y(q)]$. For any $v \in \mathbf{v}^k(\epsilon)$, we have $v_q \geq \delta_{qq-1} + \epsilon_q$ for any $q \leq k$; thus $\mathbf{v}^k(\epsilon) \cap Y(q-1) = \emptyset$. Similarly, for any $q > k$, $\mathbf{v}^k(\epsilon) \cap Y(q) = \emptyset$. Thus, $\mathbf{v}^k(\epsilon) \subset Y(k)$ for small enough ϵ_ℓ 's. From (14) and (15) applied to each $v \in \mathbf{v}^k(\epsilon)$, we know that (with the qualification in footnote 23)

$$\left\{ v \in \mathcal{D} \mid v_k > \delta_{kk-1} + \epsilon_k, \sum_{\ell=q}^k v_\ell > \delta_{kk-1} + \epsilon_k + \sum_{\ell=q}^{k-1} \max\left[v_{\ell+1} \frac{\bar{a}_\ell}{\bar{a}_{\ell+1}}, \delta_{\ell\ell-1} + \epsilon_\ell\right], \forall q < k, \right. \\ \left. \sum_{\ell=k+1}^q v_\ell < \sum_{\ell=k+1}^q \min\left[v_{\ell-1} \frac{\bar{a}_\ell}{\bar{a}_{\ell-1}}, \delta_{\ell\ell-1} - \epsilon_\ell\right], \forall q > k \right\} \subset Y(k).$$

Taking limits as $(\epsilon_1, \epsilon_2, \dots, \epsilon_K) \rightarrow 0$, we see that

$$\left\{ v \in \mathcal{D} \mid v_k > \delta_{kk-1}, \sum_{\ell=q}^k v_\ell > \delta_{kk-1} + \sum_{\ell=q}^{k-1} \max\left[v_{\ell+1} \frac{\bar{a}_\ell}{\bar{a}_{\ell+1}}, \delta_{\ell\ell-1}\right], \forall q < k, \right. \\ \left. \sum_{\ell=k+1}^q v_\ell < \sum_{\ell=k+1}^q \min\left[v_{\ell-1} \frac{\bar{a}_\ell}{\bar{a}_{\ell-1}}, \delta_{\ell\ell-1}\right], \forall q > k \right\} \subseteq Y(k)$$

²³If for some $q \leq k$, $\delta_{\ell\ell-1} = \bar{a}_\ell$ for all $\ell = q, q+1, \dots, k$ then the corresponding strict inequality in the set on the left hand side is replaced by a weak inequality. A similar change is made if for some $q > k$, $\delta_{\ell\ell-1} = 0$ for all $\ell = q, q+1, \dots, k$. In either case, (i) implies that $\epsilon_\ell = 0$ in the relevant range. This ensures that the set on the left hand side is non-empty; the inclusion of this set in $Y(k)$ is implied by TBB together with (14) and (15).

²⁴Domain assumption B must hold and (6) is violated by $(\delta_{10}, \delta_{21}, \dots, \delta_{K K-1})$.

and therefore

$$\left\{ v \in \mathcal{D} \mid v_k \geq \delta_{kk-1}, \sum_{\ell=q}^k v_\ell \geq \delta_{kk-1} + \sum_{\ell=q}^{k-1} \max[v_{\ell+1} \frac{\bar{a}_\ell}{\bar{a}_{\ell+1}}, \delta_{\ell\ell-1}], \forall q < k, \right. \\ \left. \sum_{\ell=k+1}^q v_\ell \leq \sum_{\ell=k+1}^q \min[v_{\ell-1} \frac{\bar{a}_\ell}{\bar{a}_{\ell-1}}, \delta_{\ell\ell-1}], \forall q > k \right\} \subseteq \text{cl}[Y(k)].$$

That this last set inclusion is equivalent to (16) follows from the observation that (6) implies that if $\delta_{kk-1} + \sum_{\ell=q}^{k-1} \max[v_{\ell+1} \frac{\bar{a}_\ell}{\bar{a}_{\ell+1}}, \delta_{\ell\ell-1}] > \sum_{\ell=q}^k v_\ell \geq \sum_{\ell=q}^k \delta_{\ell\ell-1}$ for some $q < k$ or if $\sum_{\ell=q+1}^k \min[v_{\ell-1} \frac{\bar{a}_\ell}{\bar{a}_{\ell-1}}, \delta_{\ell\ell-1}] < \sum_{\ell=q+1}^k v_\ell \leq \sum_{\ell=q+1}^k \delta_{\ell\ell-1}$ for some $q > k$, then $v \notin \mathcal{D}$. This establishes (16) for Case B.

Next, suppose that the set inclusion in (16) is strict. In particular, there exists $k, v \in \text{cl}[Y(k)]$ such that $\sum_{\ell=q'}^k v_\ell < \sum_{\ell=q'}^k \delta_{\ell\ell-1}$, for some $q' < k$.²⁵ We may assume WLOG that $v \in Y(k)$ ²⁶ and that $\sum_{\ell=q}^k v_\ell \geq \sum_{\ell=q}^k \delta_{\ell\ell-1}, \forall q = q' + 1, \dots, k$. Therefore, $v_{q'} < \delta_{q'q'-1} \leq \bar{a}_{q'}$ and $\sum_{\ell=q'}^q v_\ell < \sum_{\ell=q'}^q \delta_{\ell\ell-1}, \forall q = q', q' + 1, \dots, k$. Consider the point $\hat{v} \equiv (\bar{a}_1, \bar{a}_2, \dots, \bar{a}_{q'-1}, v_{q'} + \hat{\epsilon}, v_{q'+1}, \dots, v_k, 0, \dots, 0)$ where $\hat{\epsilon} > 0$ is small enough that $\hat{v} \in \mathcal{D}$ and $\sum_{\ell=q'}^q \hat{v}_\ell < \sum_{\ell=q'}^q \delta_{\ell\ell-1}, \forall q = q', q' + 1, \dots, k$. Thus, (16) implies that $\hat{v} \in \text{cl}[Y(q' - 1)]$. Suppose that $\hat{v} \in Y(q' - 1)$. But this violates (5) as $\sum_{\ell=q'}^k \hat{v}_\ell > \sum_{\ell=q'}^k v_\ell$ and $v \in Y(k)$. If, instead, $\hat{v} \in \text{cl}[Y(q' - 1)] \setminus Y(q' - 1)$ then there exists $v^* \in Y(q' - 1)$ which is arbitrarily close to \hat{v} and (5) is violated. Thus, for any $v \in \text{cl}[Y(k)]$ we have $\sum_{\ell=q}^k v_\ell \geq \sum_{\ell=q}^k \delta_{\ell\ell-1}, \forall q \leq k$. A similar proof establishes that if $v \in \text{cl}[Y(k)]$ then $\forall q > k, \sum_{\ell=k+1}^q v_\ell \leq \sum_{\ell=k+1}^q \delta_{\ell\ell-1}$. Therefore, the set inclusion in (16) can be replaced by an equality, i.e.,

$$\text{cl}[Y(k)] = \left\{ v \in \mathcal{D} \mid \sum_{\ell=q}^k v_\ell \geq \sum_{\ell=q}^k \delta_{\ell\ell-1}, \forall q \leq k, \& \sum_{\ell=k+1}^q v_\ell \leq \sum_{\ell=k+1}^q \delta_{\ell\ell-1}, \forall q > k \right\}. \quad (17)$$

For any $v \in Y(k)$ and any $q < k$,

$$\sum_{\ell=1}^k v_\ell - \sum_{\ell=1}^k \delta_{\ell\ell-1} \geq \sum_{\ell=1}^q v_\ell - \sum_{\ell=1}^q \delta_{\ell\ell-1} \quad (18) \\ \iff \sum_{\ell=q+1}^k v_\ell \geq \sum_{\ell=q+1}^k \delta_{\ell\ell-1}.$$

The last inequality follows from (17). Thus, (18) is true; when $v \in Y(k)$ the agent cannot increase his payoffs by reporting a type $v' \in Y(q), q < k$. Similarly, (18) is true for $q > k$. Thus, the payment function p_k defined in (8) implements f . \blacksquare

²⁵From the definition of δ_{kk-1} we know that $q' \neq k$.

²⁶If $v \in \text{cl}[Y(k)] \setminus Y(k)$, then there exists $v' \in Y(k), v'$ close to v , such that $\sum_{\ell=q'}^k v'_\ell < \sum_{\ell=q'}^k \delta_{\ell\ell-1}$.

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