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Dynamic regressions with variables observed at different frequencies

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Abstract: We consider the problem of formulating and estimating dynamic regression models with variables observed at different frequencies. The strategy adopted is to define the dynamics of the model in terms of the highest available frequency, and to apply certain lag polynomials to transform the dynamics so that the model is expressed solely in terms of observed variables. A general solution is provided for models with monthly and quarterly observations. We also show how the methods can be extended to models with quarterly and annual observations, and models combining monthly and annual observations.

Key Words: Variables of different frequencies, dynamic regressions, temporal aggregation, systematic sampling, lag polynomials, serial correlation.

JEL Classification: C22

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1. Introduction

Economic data are available in a variety of frequencies. Econometric models, on the other hand, are typically constructed for use with data observed at the same frequencies. Datasets for use in any one econometric application are thus assembled at the frequency of the lowest frequency variable, with the data series available at higher frequencies converted to the lower frequency through temporal aggregation or systematic sampling, depending on whether the corresponding variables are flow or stock variables respectively. A researcher may, for instance, be interested in modeling the relationship between output and employment: if output is observed quarterly and employment monthly, a model incorporating these two variables would have to be specified at a quarterly frequency, with quarterly employment figures systematically sampled from the monthly figures.

This paper develops a modeling strategy that avoids the need for all data series within an econometric application to be sampled at the same time intervals. Dynamic regression models are formulated which include variables observed at different frequencies. There are clear advantages to such a modeling approach. Consider the case where the dependent variable is available quarterly while the independent variable is observed monthly. By allowing the independent variable to be included in the model at the higher frequency, monthly multipliers would be available that would otherwise be lost had the monthly data been converted into quarterly observations. The model would permit updating of quarterly forecasts as monthly data becomes available. Including monthly dynamics may also improve one-quarter ahead forecasts.

A long history of papers has discussed the effects of systematic sampling and temporal aggregation on model structure, parameter estimates, forecasting and causal relationships (Zellner 1966, Brewer 1973, Wei 1981, Weiss 1984, among others), but these works focus on situations where all the variables in the model are available at one frequency whereas the theoretical model of interest is defined at a higher frequency. Our aim is to develop a way of including variables at their highest frequencies available, even if these frequencies are not the same across all variables. The strategy adopted in this paper is that of Abeysinghe (1998, 1999), which is to define an autoregressive distributed lag model with the dynamics of the model defined in terms of the highest frequency available among the variables. The problem then is one of missing observations, and our solution is to apply certain lag polynomials to transform the dynamics so that the model is expressed solely in terms of the observed variables. Abeysinghe (1998, 1999) considered a simple model with an AR(1) structure, with the dependent variable sampled less frequently than the independent variable. Our contribution in this paper is to provide a solution for the general AR(p) case for models combining monthly and quarterly, quarterly and annual, and monthly and annual observations. We also indicate how these results can be extended to other combinations of frequencies.

We begin by introducing the dynamic models that we consider in this paper. Focusing on the case where the model contains monthly and quarterly data, we show how a straightforward application of lag polynomials can transform the dynamic model so that only observed frequencies appear. The coefficients of these lag polynomials are simple functions of the autoregressive parameters in the original model. Estimation and testing issues are discussed. Section 3 extends the method to quarterly-annual and monthly-annual combinations, and we conclude in section 4.

2. The Basic Model

The basic autoregressive distributed lag model that we consider is

$$\phi(L)y_t = \alpha + \beta(L)x_t + \varepsilon_t, \quad \varepsilon_t \sim iid(0,\sigma^2) \tag{1}$$

where $\phi(L) = 1 + \phi_1 L + \phi_2 L^2 + ... + \phi_p L^p$, $\beta(L) = \beta_0 + \beta_1 L + \beta_2 L^2 + ... + \beta_r L^r$. We refer to this as ARX(p,r) model. The variables x_t and y_t are assumed to be available at different frequencies, and the time subscript t is defined in terms of the highest frequency. For example, if x_t is monthly and y_t is quarterly then $t=1,2,\ldots,T$ would represent months. The model can include more than one regressor though for expositional purposes we will stay with just one regressor. Our approach can also be extended to the ARMAX class of models, but we leave out the MA structures to keep the exposition clear. In all our examples we will assume that it is the dependent variable that is observed with the lower frequency, though our results can easily be adapted for the reverse case.

If the lower frequency variable y_t represents a stock variable, and x_t is observed at m times the frequency of y_t, then only every mth observation of y_t is available, and the observed data set would comprise $\{x_1, x_2, ..., x_T\}$ and $\{y_m, y_{2m}, ..., y_T\}$ where we have assumed for notational simplicity that the first available observation of y_t is at $t = m$ and that T is a multiple of m. In the quarterly-monthly case, $m = 3$. If, on the other hand, y_t represents a flow variable, then what is observed of y at every mth period is an aggregation of m flows recorded at the higher frequency. The ARX(p,r) can be modified to handle the case of a low-frequency flow variable by temporally aggregating the variables to obtain

$$\phi(L)Y_t = \alpha + \beta(L)X_t + (1 + L + L^2 + ... + L^{m-1})\varepsilon_t, \quad \varepsilon_t \sim iid(0,\sigma^2) \tag{2}$$

where $Y_t = (1 + L + L^2 + ... + L^{m-1}) y_t$ and $X_t = (1 + L + L^2 + ... + L^{m-1}) x_t$, and the lag polynomials $\phi(L)$ and $\beta(L)$ are as previously defined. Again, under our assumptions, what is observed of Y_t are the values at m, $2m$,…, T whereas X_t is available at all lags.

As the methods we propose are similar for both the stock as well as the flow variable cases, we will focus on the case where the low frequency variable is a stock variable, and refer to the flow variable case only when differences arise. Note that in the usual way of dealing with mismatched frequencies, the higher frequency data is systematically sampled, or temporally aggregated depending on whether the variable is a stock or a flow. In our framework, whether or not the higher frequency (independent) variable is aggregated depends on whether the low frequency (dependent) variable is a flow or a stock. The nature of the higher frequency data is inconsequential.

2.1 Monthly-Quarterly Data

Consider first the simple case with an ARX(1,r) structure

$$(1 + \phi L) y_t = \alpha + \beta(L) x_t + \varepsilon_t \tag{3}$$

where x_t ($t=1,2,\ldots,T$) is observed at monthly intervals whereas y_t is observed only quarterly, so only every third observation of y_t is available, i.e., the observed values of y_t comprise $\{y_3, y_6, ..., y_T\}$. The strategy adopted in Abeysinghe (1998) is to transform the model so that only the observed frequencies appear. This involves multiplying both sides of (3) by a lag polynomial $\lambda(L) = (1 + \lambda_1 L + \lambda_2 L^2) = (1 - \phi L + \phi^2 L^2)$ which will convert the model to[1]

$$(1 + \phi^3 L^3) y_t = (1 - \phi + \phi^2)\alpha + \beta(L)(1 - \phi L + \phi^2 L^2) x_t + \nu_t \tag{4}$$

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[1] Note that Abeysinghe (1998) adopted a fractional time subscript which we do not follow here.

to be estimated over $t = 3\tau$, $\tau = 1,2,...,T/3$. We will refer to the lag polynomial $\lambda(L)$ as the transformation polynomial, and the lower frequency as the observed frequency. In this case, the transformed error term $v_t = \lambda(L)\varepsilon_t$ still maintains the iid property at the observed frequency, and (4) can be estimated by a non-linear LS technique. One of the advantages of this approach is that although y_t is quarterly, the monthly multipliers or impulse responses can easily be worked out from (4) using $\phi(L)^{-1}\beta(L)$ once the parameters have been estimated.

In the general ARX(p,r) case the necessary transformation polynomial will be a lag polynomial of order 2p, $\lambda(L) = (1 + \lambda_1 L + \lambda_2 L^2 + ... + \lambda_{2p} L^{2p})$. Applying this transformation to (1) gives

$$\lambda(L)\phi(L)y_t = \lambda(1)\alpha + \lambda(L)\beta(L)x_t + v_t \tag{5}$$

where $v_t = \lambda(L)\varepsilon_t$. Note that the polynomial $\pi(L) = \lambda(L)\phi(L)$ is of order 3p. Setting the coefficients of the unobserved lags of this polynomial to zero, i.e., $\pi_{3j-1} = \pi_{3j-2} = 0$, j = 1, 2,…, p, will provide 2p relationships from which we can solve for the 2p coefficients of $\lambda(L)$ in terms of the ϕ's.

For illustration, consider the ARX(2,r) case where $\phi(L) = (1 + \phi_1 L + \phi_2 L^2)$. Multiplying this polynomial with the transformation polynomial $\lambda(L)$ of order 4 will give us the following lag polynomial of order 6:

$$\begin{aligned} &1 + (\lambda_1 + \phi_1)L + (\lambda_2 + \phi_1\lambda_1 + \phi_2)L^2 + (\lambda_3 + \phi_1\lambda_2 + \phi_2\lambda_1)L^3 \\ &+ (\lambda_4 + \phi_1\lambda_3 + \phi_2\lambda_2)L^4 + (\phi_1\lambda_4 + \phi_2\lambda_3)L^5 + \phi_2\lambda_4 L^6 \end{aligned}$$

Setting the coefficients of lags 1, 2, 4 and 5 to zero and solving for the λ's will yield the following solution

$$\lambda_1 = -\phi_1,$$
$$\lambda_2 = \phi_1^2 - \phi_2,$$
$$\lambda_3 = -\phi_1\phi_2,$$
$$\lambda_4 = \phi_2^2.$$

Thus the ARX(2,*r*) model $(1+\phi_1 L + \phi_2 L^2)y_t = \alpha + \beta(L)x_t + \varepsilon_t$ can be expressed in observed frequencies as

$$(1+(\phi_1^3 - 3\phi_1\phi_2)L^3 + \phi_2^3 L^6)y_t = \lambda(1)\alpha + \lambda(L)\beta(L)x_t + \lambda(L)\varepsilon_t \qquad (6)$$

where $\lambda(L) = 1 + \lambda_1 L + \lambda_2 L^2 + \lambda_3 L^3 + \lambda_4 L^4$ with the λ's as defined above.

The following theorem provides the general solution to the problem of finding the coefficients of the lag transformation polynomial $\lambda(\mathrm{L}) = (1+\lambda_1 L + \lambda_2 L^2 + \ldots + \lambda_{2p} L^{2p})$ for the ARX(*p*,*r*) case.

Theorem 1: *Let* $\phi_0 = 1$ and $\phi_{p+1} = \phi_{p+2} = \ldots = \phi_{2p} = 0$. *If*

$$\lambda_i = -\frac{1}{2}\sum_{j=0}^{i} c_{i,j}\phi_j\phi_{i-j}\ , \ \ i = 0, 1, 2, \ldots, 2p,$$

where $c_{i,j} = \begin{cases} -2 & \text{if } rem\left(\dfrac{i-2j}{3}\right) = 0 \\ 1 & \text{otherwise} \end{cases}$,

then $(1+\phi_1 L + \phi_2 L^2 + \ldots + \phi_p L^p)(1+\lambda_1 L + \lambda_2 L^2 + \ldots + \lambda_{2p} L^{2p}) = (1+\pi_1 L + \pi_2 L^2 + \ldots + \pi_{3p} L^{3p})$

where $\pi_k = 0$ *if k = 3j – 1 or 3j – 2 for some j = 1, 2, …, p.*

Proof: See Appendix A1.

The term $rem\left(\frac{i-2j}{3}\right)$ refers to the remainder of quotient $\frac{i-2j}{3}$, i.e., we have $c_{i,j} = -2$ if the difference between the subscripts of ϕ_j and ϕ_{i-j} is divisible by 3, and 1 otherwise. For convenience, the coefficients of λ*(L)* for the AR(1) through to the AR(5) case are tabulated in Appendix A2.

The case where y_t contains a unit root (at the higher frequency) can easily be handled. A process with a unit root at the higher frequency will display a unit root at the lower frequency after application of the transformation polynomials. In the quarterly-monthly ARX(2,*r*) case, this can be verified by simply substituting $\phi_2 = -1 - \phi_1$ into the AR polynomial in (6) and setting $L = 1$. The unit root ARX(*p*,*r*) case can be handled by factoring $1 - L$ out of the *p*-order AR polynomial in (1), and applying the transformation for the ARX(p–1) case followed by the transformation for ARX(1) with $\phi_1 = -1$. We illustrate this procedure in the quarterly-monthly ARX(3,*r*) case with a unit root. Let $\phi(L) = (1 + \phi_1 L + \phi_2 L^2)(1 - L)$. Multiplying this polynomial with the transformation polynomial $\lambda'(L)$ of order 4 as in the ARX(2,*r*) case will give us the following lag polynomial:

$$(1 + (\phi_1^3 - 3\phi_1\phi_2)L^3 + \phi_2^3 L^6)(1 - L)y_t = \lambda'(1)\alpha + \lambda'(L)\beta(L)x_t + \lambda'(L)\varepsilon_t \tag{7}$$

Multiplying (7) by $\lambda''(L) = (1 + L + L^2)$ gives

$$(1 + (\phi_1^3 - 3\phi_1\phi_2)L^3 + \phi_2^3 L^6)(1 - L^3)y_t = \lambda'(1)\lambda''(1)\alpha + \lambda'(L)\beta(L)\lambda''(L)x_t + \lambda'(L)\lambda''(L)\varepsilon_t \tag{8}$$

where $\lambda''(L)x_t = X_t$ is a moving sum of x_t.

The formulation in (8) is suitable for the situation where x_t is a stationary variable. For example, $(1 - L^3)y_t$ may be the quarterly inflation rate and x_t the monthly unemployment rate. If

x_t is also a unit root process but not cointegrated with y_t, the (1 – L) operator must be applied throughout equation (7) and as a result $\lambda''(L)(1-L)x_t$ reduces to $(1-L^3)x_t$, and $(1-L)\varepsilon_t$ becomes the white noise process. In this case modeling is done using the quarterly differences of both y_t and x_t. If y_t and x_t are I(1) processes and cointegrated, then the model reverts back to the original form (6) and can be estimated in level form without imposing the cointegrating restriction. Being a dynamic model, standard *t* tests apply (Sims et al., 1990).

2.2 Estimation and the Autocorrelation Problem

We have noted in the quarterly-monthly ARX(1,*r*) stock variable case that the transformed error process $v_t = \lambda(L)\varepsilon_t$ is not serially correlated at the observed lags. Estimation of the model parameters can therefore be carried out using a non-linear least squares method. However, the transformed errors will be autocorrelated in the general quarterly-monthly ARX(*p*,*r*) flow variable case for $p \geq 1$ as well as the quarterly-monthly ARX(*p*,*r*) stock variable case for $p \geq 2$. In the stock variable case, $\lambda(L)$ is of order 2*p* and therefore $v_t = \lambda(L)\varepsilon_t$ systematically sampled at every 3rd observation will be an MA(*q*) process where $q \leq \text{int}[2p/3]$ where int[.] is the integer operator (Brewer, 1973). For the flow variable case, $v_t = \lambda(L)(1+L+L^2)\varepsilon_t$ and so will follow an MA(*q*) process with $q \leq \text{int}[2(p+1)/3]$.

To get a feel for the size of the autocorrelations involved we explore some simple cases below. For a general MA($\tilde{q}$) process $v_t = (1+\theta_1 L + ... + \theta_{\tilde{q}} L^{\tilde{q}})\varepsilon_t$ systematically sampled at every *m* periods, the *j*th autocorrelation at the observed frequency, ρ_{mj}, can be computed as $\rho_{mj} = \gamma_{mj} / \gamma_0$ where $\gamma_{mj} = E(\varepsilon_t \varepsilon_{t-mj}) = \sigma^2 \sum_{i=0}^{\tilde{q}-mj} \theta_i \theta_{i+mj}$, $\theta_0 = 1$, j = 0, 1, 2, …, int[$\tilde{q}$ /*m*]. In the quarterly-monthly ARX(1,r) flow variable case, the transformed errors $v_t = \lambda(L)(1+L+L^2)\varepsilon_t$ will follow an MA(1) process at the observed frequency. After substituting for the original AR parameters, the observed frequency-first order autocorrelation of v_t is

$$\rho_{m1} = \frac{-\phi_1(1-\phi_1)^2}{3-4\phi_1+5\phi_1^2-4\phi_1^3+3\phi_1^4}$$

Figure 1 plots this autocorrelation for stationary values of ϕ_1. The autocorrelation problem appears to be small; for values of $\phi_1 \in (-1,0)$, which is the more likely region for economic data (recall that our AR coefficients have signs that are the reverse of the conventional specification), ρ_{m1} is less than 0.21. Unfortunately, there is no reason to expect the autocorrelation problem to be small for the other cases. Figure 2 plots the first autocorrelation of $v_t = \lambda(L)\varepsilon_t$ for the quarterly-monthly ARX(2,r) stock variable case, which also follows an MA(1) process when systematically sampled at the observed frequency. The autocorrelation is seen to lie between -0.5 and 0.5 for values of ϕ_1 and ϕ_2 in the stationary range. A plot of ρ_{m1} in the ARX(2,r) flow variable case shows this autocorrelation to range from about -0.6 to 0.6.

The major obstacle posed by the autocorrelation problem is the inconsistency of the non-linear LS estimator of the transformed model. Since the autocorrelations, and therefore the MA parameters, depend on the AR parameters a simple alternative to least squares is to use a non-linear IV estimator. After computing the autocorrelations from the estimated ϕ's, the MA parameters can be derived by solving the set of non-linear equations given in Box et al. (1994, p. 202, eq. 6.3.1). The same procedure can be used to estimate $\sigma^2 = \text{var}(\varepsilon_t)$ and the standard errors of the IV estimator can be recomputed by replacing σ_v^2 by σ^2 (note that $\sigma^2 \le \sigma_v^2$). One has to go through the trouble of deriving the MA parameters only if the model is designed for forecasting. If the objective is to derive the impulse responses, then the MA parameters do not enter the calculations and can be ignored.

The success of the IV estimator depends on the quality of the instrument used. One possibility is to use lagged dependent variables $y_{t-(p+j)}$, j=1,2,.. as instruments, although this may not work well if p is large. Monte Carlo studies carried out in relation to a flow ARX(1,1) model shows that in small samples the LS and IV bias could be similar and may be negligible if the autocorrelation is small (Abeysinghe, 1999).

Another practical problem is the choice of the lag orders p and r. As observed in Abeysinghe (1998) if p is known the choice of r is not difficult. Starting with a large value for r one can test downward to choose an appropriate value for r. Complications arise in the choice of p because the form of the transformation polynomial $\lambda(L)$ depends on p. One possibility is to treat (5) as a reduced form and estimate it as a linear model. The number of significant lags would indicate the appropriate order of the lag polynomial $\phi(L)$. If, for instance, the coefficient on y_{t-6} is significantly different from zero while those of y_{t-9}, y_{t-12}, …are not, this would imply p = 2. If r* lags of x_t are significant, this would suggest r = r* – 2p (inclusion of y_{t-6} would necessarily imply the inclusion of at least four lags of x_t). The disadvantage of this approach is that some reduced form parameters might be very small, even if the original structural parameters are not, and in small samples these parameter estimates may turn out to be statistically insignificant.

In summary, the practical implementation of our modeling approach might take the following form: if unit root variables are involved, test for cointegration by converting all high frequency variables to the low frequency available[2]. If cointegration cannot be rejected, use the level variables for modeling, otherwise use differenced data. Estimating (5) as a reduced form, as described in the previous paragraphs, would suggest suitable values of *p* and *r,* after which (5) can be estimated using a non-linear IV technique. We suggest overfitting to see if the chosen *p* and *r* are sufficient. Note that the standard *t* test is applicable here. If the residuals appear to be empirically white noise, ignoring the MA structure of the transformed model would probably be inconsequential, and the estimated model may be put to use. In this case a non-linear LS estimation of the model might be better as the LS estimator is more efficient than the IV estimator; if the residuals remain white noise under the LS method, the LS estimates would be preferable for inference. If residual autocorrelation is present, the MA parameters can be derived as described earlier in this section. An alternative is to identify an ARMA model for the error term and estimate them together with the model parameters as in the Box-Jenkins transfer function noise model approach, i.e., generalize the ARX model to an ARMAX structure.

3. Extensions to Quarterly-Annual and Monthly-Annual Cases

Another empirically important case is where the dependent variable is observed annually and the independent variable is observed quarterly. The general strategy in this case will be to apply the transformation given in the following theorem twice. The first transformation will convert the quarterly lag structure into biannual terms, and the second transformation will convert the biannual structure into an annual structure.

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[2] Integration and cointegration are invariant to temporal aggregation and systematic sampling (Marcellino, 1999).

__Theorem 2:__ Let

$$\lambda_i = (-1)^i \phi_i \ , \ \ i = 1, 2, \ldots, p,$$

then $$(1+\phi_1 L+\phi_2 L^2+\ldots+\phi_p L^p)(1+\lambda_1 L+\lambda_2 L^2+\ldots+\lambda_p L^p) = (1+\pi_1 L+\pi_2 L^2+\ldots+\pi_{2p} L^{2p})$$

where $\pi_k = 0$ *if k = 2j – 1 for some j = 1, 2, …, p.*

Proof: See appendix A1.

For example, consider the AR(2) case $(1+\phi_1 L+\phi_2 L^2)y_t = \alpha+\beta(L)x_t+\varepsilon_t$. We have to convert this model to a form in which the lag structure on y_t only contains the lags in multiples of 4. Theorem 2 suggests applying the transformation $\lambda(L) = (1-\phi_1 L+\phi_2 L^2)$ once to obtain a lag structure in multiples of 2 for y_t to obtain:

$$(1+(2\phi_2-\phi_1{}^2)L^2+\phi_2{}^2 L^4)y_t = (1-\phi_1+\phi_2)\alpha+\beta(L)(1-\phi_1 L+\phi_2 L^2)x_t+(1-\phi_1 L+\phi_2 L^2)\varepsilon_t \,.$$

Applying a second transformation $(1-(2\phi_2-\phi_1{}^2)L^2+\phi_2{}^2 L^4)$ gives us

$$\begin{aligned}(1+(2\phi_1{}^2\phi_2-\phi_1{}^4-2\phi_2{}^2)L^4+\phi_2{}^2 L^8)y_t \\ = (1-\phi_1+\phi_2)(1-(2\phi_2-\phi_1{}^2)+\phi_2{}^2)\omega+\beta(1-\phi_1 L+\phi_2 L^2)(1-(2\phi_2-\phi_1{}^2)L^2+\phi_2{}^2 L^4)x_t \\ +(1-(2\phi_2-\phi_1{}^2)L^2+\phi_2{}^2 L^4)(1-\phi_1 L+\phi_2 L^2)\varepsilon_t\end{aligned}$$

A similar idea can be applied to the monthly-annual case: first transform the lag structure on y_t to the bimonthly form (using Theorem 2), followed by a transformation to the biannual form (using Theorem 1) and finally to the annual form (again using Theorem 2).

As in the monthly-quarterly case, these transformations create a problem of autocorrelation of the transformed error term; the transformed error term follows an MA process at the observed frequencies in all cases. For each p, the final transformation matrix will be of order 3p, and the transformed error will follow, at the observed lags, an MA(q) process where $q \le \text{int}[3p/4]$ for the stock variable case and $q \le \text{int}[3(p+1)/4]$ for the flow variable case.

Finally, we note that the above transformations can easily be adapted to the case where the independent variable is observed less frequently than the dependent variable. Now the transformation polynomial $\lambda(L)$ has to be worked out in relation to $\beta(L)$ in (1). To apply the previous results $\beta(L)$ can be written as $\beta(L) = \beta_0(1 + \beta_1^* L + ... + \beta_r^* L^r)$ where $\beta_i^* = \beta_i / \beta_0, i = 1,2,..., r$.

4. Concluding Remarks

This paper has provided a modeling approach which allows variables observed at different frequencies to be framed within a single model without converting the higher frequency variable into a lower frequency via systematic sampling or temporal aggregation. This approach entails a number of advantages[3]. Firstly, we can recover the impulse responses or multipliers at the high frequency time units. This information is totally lost if one were to use the standard systematic sampling or temporal aggregation approach. Secondly, this approach is likely to provide better forecasts compared to those based on the standard approach. Thirdly, forecast updating can easily be done as and when the high frequency data become available.

The cases that we cover are mostly suitable for macroeconomic analysis, where data are usually available in monthly, quarterly or annual frequencies. An extension to other combinations of frequencies may be fruitful, especially for areas like finance. Other possible avenues for future research include the extension of our methods to vector autoregression models and for causality testing.

[3] For an illustrative application see Abeysinghe (1998).

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Appendix A1

Proof of Theorem 1

By multiplying $(1+\phi_1 L+\phi_2 L^2+...+\phi_p L^p)(1+\lambda_1 L+\lambda_2 L^2+...+\lambda_{2p} L^{2p})$, and substituting the expressions for λ_i from the theorem, we see that π_k takes the form

$$\begin{aligned}\pi_k &= \sum_{i=0}^{k}\lambda_i\phi_{k-i} \\ &= -\frac{1}{2}\sum_{i=0}^{k}\sum_{j=0}^{i}c_{i,j}\phi_j\phi_{i-j}\phi_{k-i}\end{aligned}$$

where $c_{i,j} = \begin{cases} -2 & if\ \ rem\left(\dfrac{i-2j}{3}\right)=0 \\ 1 & otherwise \end{cases}$.

Note that the subscripts of ϕ_j, ϕ_{i-j} and ϕ_{k-i} add up to m. Note also that for π_k to be zero, all terms in the double summation containing the same set of ϕ's must sum to zero, e.g., all terms of the form, say, $\phi_1\phi_3\phi_4$, must sum to zero, likewise all $\phi_2\phi_5{}^2$ terms must sum to zero, and so on.

Consider any one term in the summation in π_k containing, ϕ_a, ϕ_b and ϕ_{k-a-b} (where ϕ_a, ϕ_b and ϕ_{k-a-b} are not necessarily distinct). ϕ_a, ϕ_b and ϕ_{k-a-b} may appear because $j=a$, $i-j=b$ and $k-i=k-a-b$. There are six possibilities, with the corresponding values for *i* and $i-2j$, as follows

j	$i - j$	$k - i$	i	$i - 2j$
a	b	$k - a - b$	$a + b$	$b - a$
a	$k - a - b$	b	$k - b$	$k - 2a - b$
b	a	$k - a - b$	$a + b$	$a - b$
b	$k - a - b$	a	$k - a$	$k - a - 2b$
$k - a - b$	a	b	$k - b$	$2a + b - k$
$k - a - b$	b	a	$k - a$	$2b + a - k$

We now show that $\pi_k = 0$ for each of these 6 cases, when k takes the form $3j-1$ or $3j-2$ for any positive integer value j. This amounts to showing that $\sum_{i=0}^{k}\sum_{j=0}^{i} c_{i,j} = 0$ in each case.

For these 6 cases, we have to divide the problem into 18 sub-cases, 9 each for the cases where k takes the form $3j-1$ and $3j-2$, and depending on whether a takes the form $3m$, $3m - 1$ or $3m - 2$, and whether b takes the form $3n$, $3n - 1$ or $3n - 2$, where m and n are arbitrary integer values. The label the eighteen sub-cases as follows

case	k	a	b	case	k	a	b
1			3n	10			3n
2		3m	3n – 1	11		3m	3n – 1
3			3n – 2	12			3n – 2
4			3n	13			3n
5	3j – 1	3m – 1	3n – 1	14	3j – 2	3m – 2	3n – 1
6			3n – 2	15			3n – 2
7			3n	16			3n
8		3m – 2	3n – 1	17		3m – 2	3n – 1
9			3n – 2	18			3n – 2

The following table shows $c_{i,j}$ for each of the 18 x 6 cases, and computes $\sum_{i=0}^{k}\sum_{j=0}^{i} c_{i,j}$ for each case :

i – 2j \ case	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
b – a	-2	1	1	1	-2	1	1	1	-2	-2	1	1	1	-2	1	1	1	-2
k – 2a – b	1	-2	1	1	1	-2	-2	1	1	1	1	-2	-2	1	1	1	-2	1
a – b	-2	1	1	1	-2	1	1	1	-2	-2	1	1	1	-2	1	1	1	-2
k – a – 2b	1	1	-2	-2	1	1	1	-2	1	1	-2	1	1	1	-2	-2	1	1
2a + b – k	1	-2	1	1	1	-2	-2	1	1	1	1	-2	-2	1	1	1	-2	1
2b + a – k	1	1	-2	-2	1	1	1	-2	1	1	-2	1	1	1	-2	-2	1	1
$\sum_{n=0}^{m}\sum_{j=0}^{n} c_{n,j}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

In all cases $\sum_{i=0}^{k}\sum_{j=0}^{i} c_{i,j} = 0$, hence $\pi_k = 0$. A similar exercise for *k* of the form 3*j* will show that in that case $\pi_k \neq 0$ in general. Q.E.D.

Proof of Theorem 2

By multiplying $(1+\phi_1 L+\phi_2 L^2+...+\phi_p L^p)(1+\lambda_1 L+\lambda_2 L^2+...+\lambda_p L^p)$, we see that π_k takes the form,

$$\begin{aligned}\pi_k &= \sum_{i=0}^{k}\lambda_i\phi_{k-i} \\ &= -\sum_{i=0}^{k}(-1)^i\phi_i\phi_{k-i}\end{aligned}$$

for $k = 1, 2, ..., p$. If *k* of the form *2j – 1* then there is an even number of terms in the summation, with the $\phi_a\phi_{k-a}$ terms canceling out the $\phi_{k-a}\phi_a$ terms, therefore $\pi_k = 0$ if *k* is odd. Q.E.D.

Appendix A2

The following table provides the coefficients of the transformation polynomial $(1+\lambda_1 L+\lambda_2 L^2+...+\lambda_{2p} L^{2p})$ for the ARX(p,.) case where the dependent variable is observed quarterly and the independent variable is observed monthly. $\lambda_{i,j}$ refers to the coefficient in the ijth cell indicated by row λ_i and column ARX(j,.).

	ARX(1,.)	ARX(2,.)	ARX(3,.)	ARX(4,.)	ARX(5,.)
λ_1	$-\phi_1$	$\lambda_{1,1}$	$\lambda_{1,1}$	$\lambda_{1,1}$	$\lambda_{1,1}$
λ_2	ϕ_1^2	$\lambda_{2,1}-\phi_2$	$\lambda_{2,2}$	$\lambda_{2,2}$	$\lambda_{2,2}$
λ_3		$-\phi_1\phi_2$	$\lambda_{3,2}+2\phi_3$	$\lambda_{3,3}$	$\lambda_{3,3}$
λ_4		ϕ_2^2	$\lambda_{4,2}-\phi_1\phi_3$	$\lambda_{4,3}-\phi_4$	$\lambda_{4,4}$
λ_5			$-\phi_2\phi_3$	$\lambda_{5,3}+2\phi_1\phi_4$	$\lambda_{5,4}-\phi_5$
λ_6			ϕ_3^2	$\lambda_{6,3}-\phi_2\phi_4$	$\lambda_{6,4}-\phi_1\phi_5$
λ_7				$-\phi_3\phi_4$	$\lambda_{7,4}+2\phi_2\phi_5$
λ_8				ϕ_4^2	$\lambda_{8,4}-\phi_3\phi_5$
λ_9					$-\phi_4\phi_5$
λ_{10}					ϕ_5^2

Figure 1 Autocorrelation in the Quarterly-Monthly Flow Dependent Variable ARX(1,*r*) Case

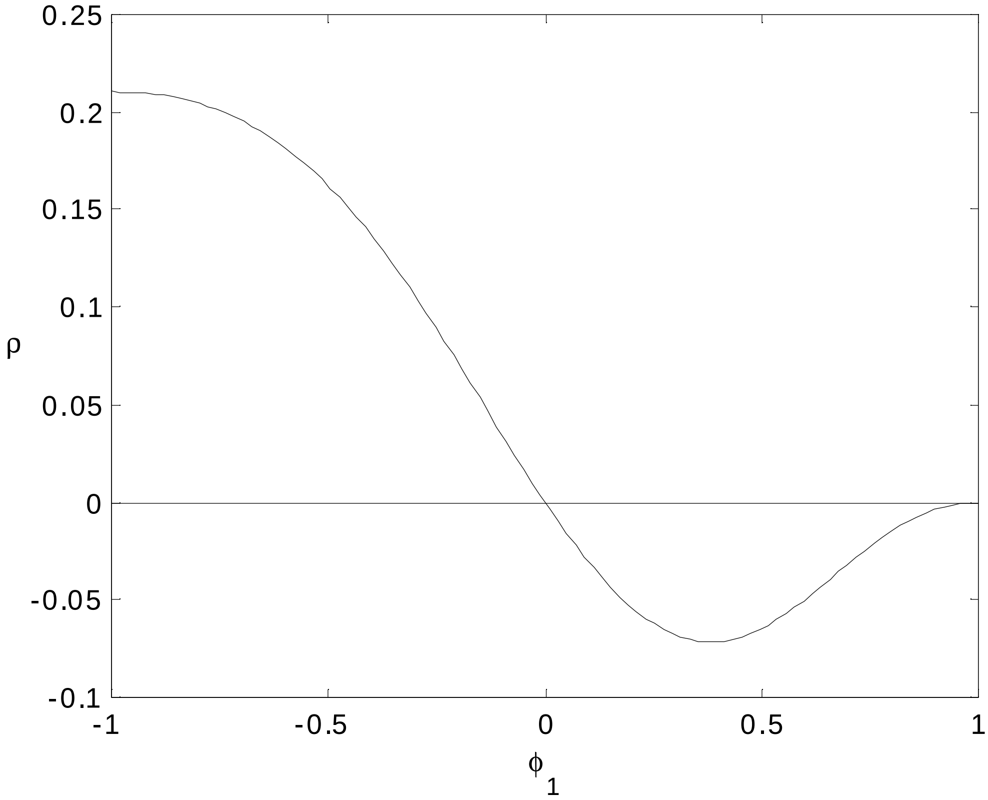
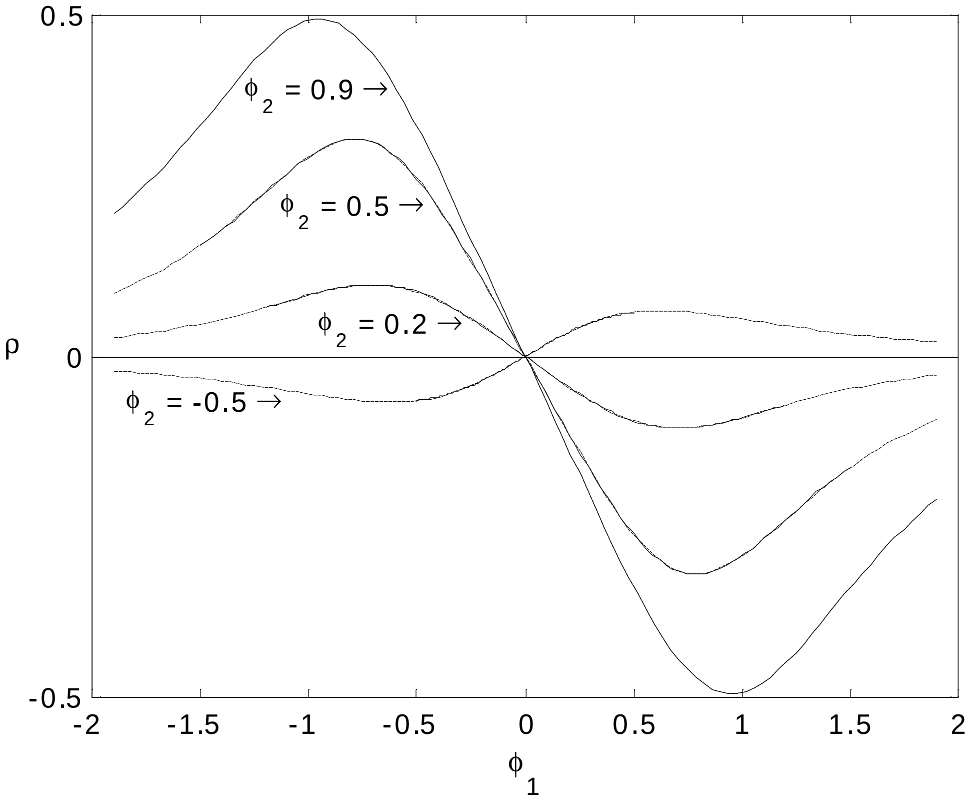


Figure 2 Autocorrelation in the Quarterly-Monthly Stock Dependent Variable ARX(2,*r*) Case



Notes: The dashed portions of the graphs show values of ρ in the non-stationary range of ϕ_1 and ϕ_2.