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5. Physical properties of clusters

5.1 Physical properties of the hot gas in clusters

5.1.1 Mean free path

[S:5.4.1]

→ The mean free path for electron-electron Coulomb collisions is

$$\lambda_e \propto \frac{T_e^2}{n_e \ln \Lambda}, \tag{5.2}$$

where $\ln \Lambda$ is the Coulomb logarithm (ln of the ratio between largest and smallest impact parameter), weakly dependent of T_e and n_e:

$$\ln \Lambda = 37.8 \ln \left[\left(\frac{T_e}{10^8 K} \right) \left(\frac{n_e}{10^{-3} cm^{-3}} \right)^{-1/2} \right]. \tag{5.3}$$

→ For typical T_e and n_e of Intracluster Medium (ICM):

$$\lambda_e \simeq 23 \left(\frac{T_e}{10^8 K} \right)^2 \left(\frac{n_e}{10^{-3} cm^{-3}} \right)^{-1} kpc \tag{5.4}$$

$\implies \lambda_e \ll$ cluster size $\implies$ ICM is a collisional fluid satisfying the hydrodynamic equations.

→ Electrons achieve an isotropic Maxwellian velocity distribution in a timescale

$$t_{ee} \equiv \frac{\lambda_e}{\sigma_e} \simeq 3 \times 10^5 \left(\frac{T_e}{10^8 K} \right)^{3/2} \left(\frac{n_e}{10^{-3} cm^{-3}} \right)^{-1} yr, \tag{5.5}$$

where σ_e is the r.m.s. electron veocity, given by

$$\frac{1}{2} m_e \sigma_e^2 = \frac{3}{2} k_B T_e. \tag{5.6}$$

→ For protons $t_{pp} \simeq \sqrt{m_p/m_e} t_{ee}$, and for equipartition $t_{ep} \simeq (m_p/m_e) t_{ee} \simeq 6 \times 10^8 yr \ll$ cluster age $\implies$ ICM is a plasma at $T_{gas} \sim T_e \sim T_p$.

5.1.2 Hydrostatic models of the ICM

[S:5.5, S:5.5.1, S:5.5.2, V.II.B.2]

→ The sound crossing time $t_s = D/c_s \propto D/\sqrt{T_e}$ (where c_s is sound speed and D is the cluster size) is shorter than the cluster age and cooling time $\Longrightarrow$ hydrostatic equilibrium

$$\nabla P = -\rho_{gas} \nabla \Phi, \tag{5.7}$$

where Φ is the total gravitational potential.

If spherically symmetric

$$\frac{dP}{dr} = -\rho_{gas} \frac{d\Phi}{dr} = -\rho_{gas} \frac{GM(r)}{r^2}, \tag{5.8}$$

where $M(r)$ is the total mass (DM+gas+galaxies) within r.

Isothermal distributions

→ Consider isothermal gas at temperature T_e in equilibrium $\Longrightarrow$ using $P = \rho_{gas} k_{\rm B} T_e/\mu m_{\rm p} = n_{gas} k_B T_e$ we get

$$\frac{d \ln n_{gas}}{d \ln r} = -\frac{\mu m_p r}{k_B T_e} \frac{d\Phi}{dr} = -\frac{\mu m_p}{k_B T_e} \frac{GM(r)}{r}. \tag{5.9}$$

→ For example if the total mass distribution is a singular isothermal sphere (SIS) $\rho(r) = \sigma^2/2\pi G r^2$, $M(r) = 2\sigma^2 r/G$ ($\sigma = const$ is 1D velocity dispersion), the density distribution of an isothermal gas in equilibrium is a power law $n_{gas} \propto r^{-\alpha}$ with

$$\alpha = -\frac{d \ln n_{gas}}{d \ln r} = \frac{2\mu m_p \sigma^2}{k_B T_e}, \tag{5.10}$$

dependent on the gas temperature.

→ Introducing the virial temperature of the cluster (SIS) potential $T_{vir} \equiv \mu m_p \sigma^2/k_B$, the density slope is $\alpha = 2T_{vir}/T_e$.

→ Observed cluster temperature profiles are almost (but not exactly) isothermal. Typically the temperature slightly increases with radius in the central regions and decreases with radius in the outer regions.

Polytropic distributions

→ Isothermal distributions are special cases of polytropic distributions, in which $p \propto \rho^{\gamma_{\rm pol}}$, where $\gamma_{\rm pol}$ is the polytropic index. If $\gamma_{\rm pol} = 1$ $\Longrightarrow$ isothermal distribution; if $\gamma_{\rm pol} = \gamma = 5/3$ $\Longrightarrow$ adiabatic distribution. We will consider $1 \leq \gamma_{\rm pol} \leq 5/3$.

→ In the adiabatic case $\gamma_{\rm pol} = \gamma = 5/3$ the entropy K of the gas is constant throughout the cluster $\Longrightarrow$ distributions with $\gamma_{\rm pol} > 5/3$ are convectively unstable. If $\gamma_{\rm pol} < 5/3$ entropy increases outwards.

→ In polytropic models $p/p_0 = (\rho/\rho_0)^{\gamma_{\rm pol}}$, $T_e/T_{e,0} = (\rho/\rho_0)^{\gamma_{\rm pol}-1}$ or $p/p_0 = (T_e/T_{e,0})^{\gamma_{\rm pol}/(\gamma_{\rm pol}-1)}$, where subscript 0 indicates quantities evaluated at a reference radius r_0 $\Longrightarrow$

$$\frac{1}{\rho}\nabla P = \frac{\gamma_{\rm pol}}{\gamma_{\rm pol}-1}\frac{k_B}{\mu m_p}\nabla T_e. \tag{5.11}$$

→ From the hydrostatic equation we get the solution

$$\frac{T_e}{T_{e,0}} = 1 - \frac{\mu m_p}{k_B T_{e,0}}\frac{\gamma_{\rm pol}-1}{\gamma_{\rm pol}}(\Phi - \Phi_0) \tag{5.12}$$

and

$$\frac{\rho}{\rho_0} = \left(\frac{T_e}{T_{e,0}}\right)^{\frac{1}{\gamma_{\rm pol}-1}}. \tag{5.13}$$

The gas distribution can be truncated or not depending on the value of $T_{e,0}$.

5.1.3 Intracluster entropy

[V:IV.A.1]

→ A quantity often used in the study of the ICM is the "entropy"

$$K \equiv \frac{P}{\rho^\gamma} = \frac{k_B T}{\mu m_p \rho_{gas}^{2/3}}, \tag{5.14}$$

in units of $erg\, cm^2\, g^{-5/3}$ This quantity is related to the thermodynamic entropy per particle by

$$s = k_B \ln K^{3/2} + s_0. \tag{5.15}$$

→ Sometimes also the following quantity is called "entropy"

$$K_e = k_B T n_e^{-2/3} \tag{5.16}$$

in units of $keV\, cm^2$. K_e is often indicated also as S.

→ Independent of the details of the definition entropy is a quantity such that it is conserved in adiabatic transformations $p = K\rho^{5/3} \implies$ in a polytropic distribution with $\gamma_{\rm pol} = \gamma = 5/3$ (= adiabatic distribution) all gas particles have the same entropy.

→ Heating $\implies K$ increases; cooling $\implies K$ decreases.

→ Observed ISM density, temperature, and entropy profiles: entropy increases outwards.

5.1.4 Intracluster metallicity

→ Metallicity or metal abundance is the fraction of metals with respect to H and He (Metals are all elements other than H and He). Metallicity Z usually measured in units of solar metallicty $Z_\odot$

→ X, Y, Z are the H, He and metal mass fraction. For the sun: $X \simeq 0.74$, $Y \simeq 0.24$ and $Z \simeq 0.02$.

→ ICM abundances are relatively easy to measure from intensity of emission lines of X-ray spectra. In particular 7 kev Fe line. $\Longrightarrow$ on average $Z \sim 0.3Z_{\odot}$.

→ Metallicity gradients: metallicity decreases outward in clusters (with central dominant galaxy)

→ No evidence of variation of ISM metallicity with redshift (at least up to $z \sim 0.5$.

→ Iron in ICM > iron in all stars in the cluster galaxies $\Longrightarrow$ galaxies lose more metals than they retain. It is not clear whether supernova feedback is enough.

5.2 Mass of galaxy clusters

5.2.1 Baryons and dark matter in clusters

→ Total mass with different methods, using galaxies, gas, gravitational lensing. Not always agreement among estimates with different methods, but improving.

→ Total mass profile not far from isothermal ($\rho_{tot} \propto r^{-2}$) at intermediate radii: typically shallower at small radii and steeper at large radii. Total masses $\sim 10^{14} - 10^{15} M_{\odot}$.

→ Mass fractions (in rich clusters): dark matter 80-87%, baryons (13-20%) [hot gas 11-15%, galaxies (stars) 2-5%]

→ Galaxies and dark matter distribution more concentrated than gas distribution

5.2.2 Virial theorem

[S:2.8]

→ Zeroth-order estimate of cluster mass M_{tot} (Zwicky 1937)

→ Assume cluster is in equilibrium $\Longrightarrow 2T + W = 0$ (virial theorem), where the kinetic energy is

$$T = \frac{1}{2} M_{tot} \sigma_V^2, \tag{5.17}$$

where σ_V is the virial velocity dispersion and the gravitational potential energy is

$$W = -\frac{G M_{tot}^2}{r_g}, \tag{5.18}$$

where r_g is the gravitational radius (dependent on the mass distribution)

→ If one knew σ_V and $r_g \Longrightarrow M_{tot}$ because

$$M_{tot} = \frac{r_g \sigma_V}{G} \tag{5.19}$$

→ Assuming spherical symmetry and isotropic velocity distribution $\Longrightarrow$ $\sigma_V^2 = 3\sigma_{los}^2$, where σ_{los} is the line-of-sight velocity dispersion of cluster galaxies.

→ Assuming also that galaxy distribution traces mass distribution

$$r_g = 2 \left(\sum_i m_i \right)^2 \left(\sum_{i \neq j} \frac{m_i m_j}{r_{ij}} \right)^{-1}, \tag{5.20}$$

where m_i is galaxy mass and r_{ij} galaxy-galaxy separation. In terms of the projected galaxy-galaxy separation R_{ij}, $r_g = (\pi/2) R_g$ (derive) where

$$R_g = 2 \left(\sum_i m_i \right)^2 \left(\sum_{i \neq j} \frac{m_i m_j}{R_{ij}} \right)^{-1}. \tag{5.21}$$

→ Thus

$$M_{tot} = \frac{3 r_g \sigma_{los}^2}{G} = 7 \times 10^{14} \left(\frac{\sigma_{los}}{1000\, km/s} \right)^2 \left(\frac{r_g}{Mpc} \right) \tag{5.22}$$

→ OK order of magnitude, but assumptions not necessarily justified.

5.2.3 Jeans modeling

[Binney & Tremaine (Galactic dynamics):4.8.1]

→ More accurate mass determination using galaxy velocity and density profiles $\Longrightarrow$ Jeans modeling $\Longrightarrow$ determine cluster mass profile.

→ For spherical stationary cluster the radial component $\sigma_r(r)$ of the velocity dispersion tensor is given by solving the Jeans equation

$$\frac{dn_{gal}\sigma_r^2}{dr} + \frac{2\beta n_{gal}\sigma_r^2}{r} = -n_{gal}g_r, \tag{5.23}$$

where $g_r(r) = d\Phi(r)/dr = GM(r)/r^2$, $\Phi(r)$ is the total gravitational potential, $M(r)$ is the total mass profile and

$$\beta(r) \equiv 1 - \frac{\sigma_\vartheta^2 + \sigma_\varphi^2}{2\sigma_r^2} \tag{5.24}$$

is the anisotropy parameter (σ_ϑ and σ_φ are, respectively, the ϑ and φ components of the velocity-dispersion tensor).

The line-of-sight velocity dispersion is

$$\sigma_{los}^2(R) = \frac{2}{\Sigma_{gal}(R)} \int_R^\infty \left[1 - \beta(r)\frac{R^2}{r^2}\right] \frac{n_{gal}(r)\sigma_r^2 r dr}{\sqrt{r^2 - R^2}}, \tag{5.25}$$

where

$$\Sigma_{gal}(R) = 2\int_R^\infty \frac{n_{gal}(r) r dr}{\sqrt{r^2 - R^2}}. \tag{5.26}$$

n_{gal} and Σ_{gal} are the intrinsic and projected number of galaxy density. σ_r and σ_{los} are the intrinsic and projected velocity dispersions of galaxies.

→ It is possible to build model of given $n_{gal}(r)$, $\beta(r)$ and $M(r)$, compute $\Sigma_{gal}(R)$ and $\sigma_{los}(R)$ and compare with observed quantities.

→ Viceversa, assuming $\beta(r)$ it is possible to derive $M(r)$ if we know $n_{gal}(r)$ and $\sigma_r(r)$. By deprojecting Σ_{gal} one obtains the corresponding intrinsic density distribution (Abel inversion)

$$n_{gal}(r) = -\frac{1}{\pi}\int_r^\infty \frac{d\Sigma_{gal}}{dR}\frac{dR}{\sqrt{R^2-r^2}} \tag{5.27}$$

Simlarly, deprojecting $\sigma_{los}(R)$ one can obtain $\sigma_r(r)$.

5.2.4 Hydrostatic equilibrium of hot gas

[S:5.5.5,V:II.B.1,V:II.B.2]

→ Assume hot gas in hydrostatc equilibrium in the cluster gravitational potential:

$$\nabla P = -\rho_{gas}\nabla\Phi. \tag{5.28}$$

If spherically symmetric

$$\frac{dP}{dr} = -\rho_{gas}\frac{d\Phi}{dr} = -\rho_{gas}\frac{GM(r)}{r^2}. \tag{5.29}$$

→ Using $P = \rho_{gas}k_BT_e/\mu m_p$ we get

$$M(r) = -\frac{k_BrT_e(r)}{\mu m_pG}\left(\frac{d\ln\rho_{gas}}{d\ln r} + \frac{d\ln T_e}{d\ln r}\right) \tag{5.30}$$

→ If we know ρ_{gas} and $T_e \implies M(r)$. But the observables are X-ray SB profiles and spectrum. When spectra and intensity at different annuli are available it is possible to obtain T_e and n_e profiles by deprojection, assuming spherical symmetry (not easy! different techniques).

→ Recall that X-ray emissivity depends on temperature as well as on density $\epsilon \propto n_e^2\Lambda(T)$

→ If gas is isothermal at temperature T_e:

$$M(r) = -\frac{k_BrT_e}{\mu m_pG}\frac{d\ln\rho_{gas}}{d\ln r} \tag{5.31}$$

If β model (derive)

$$M(r) = \frac{3\beta r k_B T_e}{\mu m_p G} \frac{r^2}{r^2 + r_c^2}. \tag{5.32}$$

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6. Gravitational lensing by cluster of galaxies

6.0.2 Introduction

[V:II.A.3]

→ Gravitational lensing is sensitive to mass within a projected radius R, which deflects light from background galaxies. Large deflection angle $\Longrightarrow$ strong lensing; small deflection angle $\Longrightarrow$ weak lensing

→ Strong lensing produces gravitational arcs (typically tangential). Needs high surface mass density $\Longrightarrow$ central regions of clusters $\Longrightarrow$ measures of the central mass distribution

→ Weak lensing produces small distortion of shape and orientation of background galaxies $\Longrightarrow$ background galaxies result tangentially stretched w.r.t. cluster mass distribution. Intrinsic orientation of galaxies uncorrelated. Weak lensing does not need high surface density $\Longrightarrow$ mass distribution also in outer regions of cluster

6.1 Bending of light by a point-like deflector

[MR:2.1.6, 2.2.1]

→ Gravitational lensing is an effect predicted by general relativity. Consider spacetime interval $ds^2 = g_{ij}dx^i dx^j$, where g_{ij} is the metric tensor.

→ Schwarzschild metric (solution of Einstein equations outside a spherical distribution of total mass M):

$$ds^2 = \left(1 - \frac{r_S}{r}\right) c^2 dt^2 - \left(1 - \frac{r_S}{r}\right)^{-1} dr^2 - r^2(\sin^2\theta d\phi^2 + d\theta^2) \quad (6.2)$$

where

$$r_S \equiv \frac{2GM}{c^2} \quad (6.3)$$

is the Schwarzschild radius.

→ Consider trajectory of photon close to a point-like object. Trajectory is in a plane (say $\theta = \pi/2$) $\Longrightarrow$

$$ds^2 = \left(1 - \frac{r_S}{r}\right) c^2 dt^2 - \left(1 - \frac{r_S}{r}\right)^{-1} dr^2 - r^2 d\phi^2 \quad (6.4)$$

Photons follow null geodesics

$$g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} = 0, \quad (6.5)$$

where λ is a parameter used to parameterize the trajectory. $\Longrightarrow$

$$d\phi = \frac{J}{r^2}\frac{dr}{\sqrt{1 - \frac{J^2}{r^2}\left(1 - \frac{r_s}{r}\right)}}. \quad (6.6)$$

where $J = r^2(d\phi/d\lambda) = const$ is an integral of motion.

→ At closest approach $r = r_m$, $dr/d\phi = 0$ $\Longrightarrow$

$$J = \frac{r_m}{\sqrt{1 - \frac{r_s}{r_m}}}. \quad (6.7)$$

→ Combining the two equations above and integrating we get

$$\phi_m - \phi_\infty = \int_0^1 \frac{dx}{\sqrt{1 - x^2 - \frac{r_S}{r_m}(1 - x^3)}}, \quad (6.8)$$

where $x \equiv r_m/r$. If $r_m \gg r_S$ $\Longrightarrow$

$$\phi_m - \phi_\infty = \frac{\pi}{2} + \frac{r_S}{r_m}, \quad (6.9)$$

$\Longrightarrow$ deflection angle for a point-like object of mass M

$$\hat{\alpha} = 2(\phi_m - \phi_\infty) - \pi = 2\frac{r_S}{r_m} = \frac{4GM}{r_m c^2} \quad (6.10)$$

6.2 Bending of light by extended mass distribution

[MR:2.2.3]

→ Consider mass density distribution $\rho(\mathbf{x}) \Longrightarrow$ surface mass density

$$\Sigma(\boldsymbol{\xi}) = \int dz \rho(\mathbf{x}), \tag{6.11}$$

where $\mathbf{x} = (x, y, z)$, $\boldsymbol{\xi} = (x, y)$ and z is the line of sight.

→ Thin-lens approximation: all mass distributed in the lens plane

→ In weak field limit we can sum contributions from mass elements $\Longrightarrow$

$$\hat{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = \frac{4G}{c^2} \int d^2\boldsymbol{\xi}' \Sigma(\boldsymbol{\xi}') \frac{\boldsymbol{\xi} - \boldsymbol{\xi}'}{|\boldsymbol{\xi} - \boldsymbol{\xi}'|^2} \tag{6.12}$$

→ Define projected gravitational potential:

$$\psi(\boldsymbol{\xi}) = \int dz \Phi(\mathbf{x}), \tag{6.13}$$

$\Longrightarrow$ from Poisson equation

$$\nabla^2_\xi \psi(\boldsymbol{\xi}) = 4\pi G \Sigma(\boldsymbol{\xi}) \tag{6.14}$$

or

$$\psi(\boldsymbol{\xi}) = 2G \int \Sigma(\boldsymbol{\xi}') \ln |\boldsymbol{\xi} - \boldsymbol{\xi}'| d^2\boldsymbol{\xi}'. \tag{6.15}$$

$\Longrightarrow$

$$\hat{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = \frac{2}{c^2} \boldsymbol{\nabla}_\xi \psi(\boldsymbol{\xi}) \tag{6.16}$$

→ Consider spherical mass distribution. From Gauss theorem

$$\int \boldsymbol{\nabla} \cdot \hat{\boldsymbol{\alpha}} dA = \oint \hat{\boldsymbol{\alpha}} \cdot \frac{\boldsymbol{\xi}}{\xi} dl; \tag{6.17}$$

but $\boldsymbol{\nabla} \cdot \hat{\boldsymbol{\alpha}} = 8\pi G \Sigma / c^2$ and $\oint \hat{\boldsymbol{\alpha}} \cdot \frac{\boldsymbol{\xi}}{\xi} dl = 2\pi \xi \hat{\boldsymbol{\alpha}} \Longrightarrow$

$$\hat{\alpha}(\xi) = \frac{4GM(\xi)}{c^2 \xi}, \tag{6.18}$$

where

$$M(\xi) = 2\pi \int_0^{\xi} d\xi' \xi' \Sigma(\xi') \tag{6.19}$$

→ For a singular isothermal sphere with projevted mass $M(\xi) = \pi\sigma^2\xi/G$ (σ is 1D velocity dispersion)

$$\hat{\alpha} = \frac{4\pi\sigma^2}{c^2} \tag{6.20}$$

independent of radius.

6.3 The lens equation for a point-mass lens

[MR:3.1.1]

→ Consider point-mass lens of mass M where $D_{\rm OS}$, $D_{\rm OL}$ and $D_{\rm LS}$ are the angular diameter distances from the observer to the source, from the observer to the lens and from the lens to the source, respectively. $\Longrightarrow$

$$\theta D_{\rm OS} = \beta D_{\rm OS} + \hat{\alpha} D_{\rm LS}, \tag{6.21}$$

where θ is the angular position of the image, β is the angular position of the source and $\hat{\alpha}$ is the deflection angle. $\Longrightarrow$

$$\beta = \theta - \alpha \qquad \text{(lens equation)}, \tag{6.22}$$

where

$$\alpha \equiv \hat{\alpha} \frac{D_{\rm LS}}{D_{\rm OS}} \tag{6.23}$$

is the reduced deflection angle.

→ The closest approach distance $r_m = \theta D_{\rm OL}$ $\Longrightarrow$

$$\alpha = \frac{4GM}{c^2\theta} \frac{D_{\rm LS}}{D_{\rm OL} D_{\rm OS}} \tag{6.24}$$

→ Lens equation can be written as

$$\theta^2 - \beta\theta - {\theta_{\rm E}}^2 = 0, \tag{6.25}$$

where

$$\theta_{\rm E} \equiv \sqrt{\frac{4GM}{c^2}\frac{D_{\rm LS}}{D_{\rm OL}D_{\rm OS}}} \qquad \text{(Einstein angle)}. \tag{6.26}$$

→ When $\beta = 0$ (observer, lens and source on a straight line) $\Longrightarrow$ the image is a ring of angular radius $\theta_{\rm E}$ (Einstein ring). Einstein radius $r_{\rm E} \equiv D_{\rm OL}\theta_{\rm E}$.

6.4 The lens equation for an extended lens

[MR:3.1.2; MR:3.3; MR:3.4]

→ Thin-lens approximation $\Longrightarrow$

$$\boldsymbol{\beta} = \boldsymbol{\theta} - \boldsymbol{\alpha}, \tag{6.27}$$

where $\boldsymbol{\alpha} = \hat{\boldsymbol{\alpha}} D_{\rm LS}/D_{\rm OS}$ and $\boldsymbol{\theta} = \boldsymbol{\xi}/D_{\rm OL}$. $\Longrightarrow$

$$\boldsymbol{\beta} = \boldsymbol{\theta} - \boldsymbol{\nabla}_\theta \Psi(\boldsymbol{\theta}), \tag{6.28}$$

where

$$\Psi(\boldsymbol{\theta}) \equiv \frac{2}{c^2}\frac{D_{\rm LS}}{D_{\rm OS}D_{\rm OL}}\psi(\boldsymbol{\xi}). \tag{6.29}$$

→ For given $\boldsymbol{\beta}$ we can have different $\boldsymbol{\theta}$ $\Longrightarrow$ possible multiple images (strong lensing)

→ The lens equation can be seen as a 2D mapping between the positions of the images $\boldsymbol{\theta}$ and the positions of the sources $\boldsymbol{\beta}$. Consider the Jacobian of the trasformation

$$J = \det \frac{\partial \boldsymbol{\beta}}{\partial \boldsymbol{\theta}}. \tag{6.30}$$

→ Let us define the matrix of the mapping

$$\mathcal{T}_{ij} \equiv \frac{\partial \beta_i}{\partial \theta_j}, \tag{6.31}$$

which can be written as

$$\mathcal{T}_{ij} = \delta_{ij} - \frac{\partial^2 \Psi}{\partial \theta_i \partial \theta_j}. \tag{6.32}$$

$\Longrightarrow$

$$\mathrm{Tr} \frac{\partial^2 \Psi}{\partial \theta_i \partial \theta_j} = \nabla_\theta^2 \Psi = 2 \frac{\Sigma(\boldsymbol{\theta})}{\Sigma_{cr}} \equiv 2\kappa(\boldsymbol{\theta}), \tag{6.33}$$

where

$$\Sigma_{cr} \equiv \frac{c^2 D_{\mathrm{OS}}}{4\pi G D_{\mathrm{OL}} D_{\mathrm{LS}}} \tag{6.34}$$

is the critical surface density and

$$\kappa(\boldsymbol{\theta}) \equiv \frac{\Sigma(\boldsymbol{\theta})}{\Sigma_{cr}} \tag{6.35}$$

is the convergence.

→ The matrix of the mapping can be written as

$$\mathcal{T} = \begin{pmatrix} 1 - \kappa - \gamma_1 & -\gamma_2 \\ -\gamma_2 & 1 - \kappa + \gamma_1 \end{pmatrix}, \tag{6.36}$$

where

$$\gamma_1 \equiv \frac{1}{2} \left(\frac{\partial^2 \Psi}{\partial \theta_1^2} - \frac{\partial^2 \Psi}{\partial \theta_2^2} \right) \tag{6.37}$$

and

$$\gamma_2 \equiv \frac{\partial^2 \Psi}{\partial \theta_1 \partial \theta_2} \tag{6.38}$$

are the component of the shear $\boldsymbol{\gamma}$.

→ The amplification is

$$A = J^{-1} = (\det \mathcal{T})^{-1} = \frac{1}{(\kappa - 1)^2 - \gamma^2}, \tag{6.39}$$

where $\gamma = \|\boldsymbol{\gamma}\| = \sqrt{\gamma_1^2 + \gamma_2^2}$.

→ The convergence κ changes the size of the image, but not the shape. The shear $\boldsymbol{\gamma}$ is responsible for the distortion of the image.

→ A sufficient condition to produce multiple images is that at some point in the lens plane $\Sigma(\boldsymbol{\theta}) > \Sigma_{cr}$ (i.e. $\kappa(\boldsymbol{\theta}) > 1$)

→ Caustics are positions in the source plane in which $A \to \infty$. Critical lines are the corresponding positions in the lens plane in which $A \to \infty$.

6.5 Gravitational lensing by galaxy clusters

[MR:4.4, 4.5]

→ Lens: galaxy cluster. Sources: background galaxies.

6.5.1 Strong lensing

→ Strong gravitational lensing: multiple images, strong distortion $\Longrightarrow$ image shape cannot be accounted for without lensing ($\Longrightarrow$ arcs).

→ Arcs detected in $\sim 1/3$ of X-ray selected clusters.

→ Arcs are formed in correspondence of critical lines: $A \to \infty$, i.e. $\kappa \sim 1 - \gamma$ ($\Longrightarrow \Sigma \lesssim \Sigma_{cr}$ in the presence of shear).

→ Spherical lens $\Longrightarrow$ arcs form on the Einstein ring at an angular distance from the centre $\theta_{\rm E}$ $\Longrightarrow$ total projected mass within Einstein angle

$$M(\theta_{\rm E}) = \frac{c^2}{4G}\frac{{r_{\rm E}}^2 D_{\rm OS}}{D_{\rm OL} D_{\rm LS}} = \pi(D_{\rm OL}\theta_{\rm E})^2\Sigma_{cr} = \pi {r_{\rm E}}^2\Sigma_{cr} \qquad (6.40)$$

→ From location of tangential arcs $\Longrightarrow$ integrated mass within Einstein radius $r_{\rm E}$

→ From radial structure (radial arcs or width of tangential arcs) $\Longrightarrow$ mass density profile.

→ Strong lensing $\Longrightarrow$ central parts of clusters (high Σ)

6.5.2 Weak lensing

→ Weak gravitational lensing: no multiple images, weak distortion, $\Sigma < \Sigma_{cr}$ ($\kappa < 1 - \gamma$).

→ Amplification components:

$$A_1 = \frac{1}{1 - \kappa + \gamma}, \tag{6.41}$$

$$A_2 = \frac{1}{1 - \kappa - \gamma}. \tag{6.42}$$

$\Longrightarrow$ shear induces tangential deformations.

→ Average ellipticity $< (a - b)/(a + b) >= 0$ for non-lensed galaxies.

→ A circular source is deformed in an ellispe of axis ratio $b/a = A_1/A_2$. $\Longrightarrow$ Average ellipticity $< (a - b)/(a + b) >= \gamma/(1 - \kappa)$ for lensed galaxies. (also statistical lensing)

→ When $\kappa \ll 1$ ellipticity map $\Longrightarrow$ shear map.

→ Using inversion techniques it is possible to derive κ from shear maps of $< \epsilon >$. From $\kappa \Longrightarrow \Sigma = \kappa\Sigma_{cr} \Longrightarrow$ surface mass density profile

→ Mass-sheet degeneracy: from shear alone we cannot detect the presence of a uniform Σ

→ To break mass-sheet degeneracy one can estimate κ using estimate of the amplification (e.g. if one knows intrinsic luminosity/size of sources).

→ Weak lensing $\Longrightarrow$ outer parts of clusters (low Σ)