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EOQ under exogenous and periodic demand

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Abstract. In this paper we give a sufficient condition for the existence of the economic batch to a Wilson-type inventory model loaded by a fully exogenous continuous demand function of time. After some cases solvable in closed form, the computational problem is introduced of inverting the reordering time versus the ordered quantity as necessary step to obtain the cost function to be minimized. Such a mixed (theoretical/numerical) approach is applied to a demand consisting of three different behaviors: growth, decrease and prolonged zero. Such a wave-form is assumed to iterate itself periodically and the relevant seasonal demand is expanded in a Fourier series of time. Performing the integration and reverting the reordering time, the cost function is computed and its minimizing EOQ detected. Finally an example shows that the above conditions guarantee the existence but not uniqueness to solution.

Introduction

Economic Order Quantity (EOQ) is a set of microeconomic models defining the optimal quantity of a good to be ordered for minimizing the total variable costs required to make orders and to hold inventory. They went in existence long before the computer, the first having been developed by [Harris, 1913], though [Wilson, 1934] is credited for his early in-depth analysis.

Main underlying assumptions are:

i) the unit time demand δ for the goods is known, and deterministic;

ii) no lead time (between order and arrivals) is taken into account;

iii) the receipt of the order occurs in a single instant and immediately after ordering it;

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iv) delivery, $A > 0$, and holding $h > 0$, *specific* costs are constant.

Several extensions have been made to the EOQ model: the demand can change with the amount itself or with time; the model can include backordering costs and multiple items, their perishability and so on, see for instance, we do not claim to be exaustive, [Giri and Chaudhuri, 1998], [Goh, 1994], [Weiss, 1982], [Mingari Scarpello and Ritelli, 2008].

Recall that, in the basic Wilson model, [Wilson, 1934], the inventory is assumed to be loaded under a demand of constant variation rate. Accordingly, let $q(t)$ be the amount of goods stored at time t; we have:

$$\begin{cases} q'(t) = -\delta, \\ q(0) = Q, \end{cases}$$

whence the blow-down law will be: $q(t) = Q - \delta\, t$ and $T = Q/\delta$ is the time, called *reordering time*, after which the demand clears out all the inventory. Coming to the costs, the delivery A and holding h are *specific*, namely for unity of item, and time-invariant, then the *total* cost, i.e. delivering plus holding, for a whichever $Q > 0$ amount of goods can be formed as:

$$C(Q) = \frac{A}{T} + \frac{h}{T}\int_0^t q(\tau)\mathrm{d}\tau = \frac{\delta\, A}{Q} + \frac{h}{2}\, Q.$$

1. Our model

This article will treat some EOQ models having the peculiarity that demand is a exogenous function of time. When $\delta = \delta(t)$ is a given and positive and continuous function of time, the inventory dynamics is ruled by the differential equation:

$$\begin{cases} q'(t) = -\delta(t), \\ q(0) = Q, \end{cases} \quad \Longrightarrow \quad q(t) = Q - \int_0^t \delta(\tau)\,\mathrm{d}\tau. \tag{1}$$

The reordering time $T = T(Q)$ is such that $q(T) = 0$, or, it will solve the equation:

$$Q = \int_0^T \delta(\tau)\,\mathrm{d}\tau. \tag{2}$$

Let μ the mean value of the stock on hand $q(t)$ between the times $t = 0$ and $t = T$ of full and empty:

$$\mu = \frac{1}{T}\int_0^T q(t)\,\mathrm{d}t = \frac{1}{T}\int_0^T \left\{ Q - \int_0^t \delta(\tau)\mathrm{d}\tau \right\}\,\mathrm{d}t.$$

Minding the Q definition, one gets:

$$\mu = \frac{1}{T}\int_0^T \left\{ \int_0^T \delta(\tau)\mathrm{d}\tau - \int_0^t \delta(\tau)\mathrm{d}\tau \right\}\,\mathrm{d}t = \frac{1}{T}\int_0^T \left\{ \int_t^T \delta(\tau)\mathrm{d}\tau \right\}\,\mathrm{d}t,$$

or, changing the integrations' order:

$$\mu = \frac{1}{T}\int_0^T \tau\delta(\tau)\mathrm{d}\tau,$$

so that the total cost function becomes:

$$C(Q) = \frac{1}{T(Q)}\left\{ A + h\int_0^{T(Q)} \tau\delta(\tau)\mathrm{d}\tau \right\}, \tag{3}$$

which, once that (2) has been solved to T, shows that the total cost depends on the order Q. The aim of all EOQ analysis is to detect the best order, namely that special Q-value, say Q^*, which minimizes $C(Q)$.

It has been worked hitherto rather formally: now we are going to define which assumptions can secure that for each $Q > 0$ equation (2) can be actually solved to T. For the purpose we will assume that waiting for an infinite time, the market will ask for an unlimited amount of goods:

$$\lim_{t\to\infty} \int_0^t \delta(\tau)\,\mathrm{d}\tau = \infty. \tag{4}$$

Well-posedness of the model: a sufficient condition for the minimum

Theorem 1. *Suppose that there exist $c > 0$ and $\alpha \leq 1$ so that:*

$$\delta(\tau) \geq \frac{c}{\tau^\alpha}, \tag{5}$$

furthermore assume that there exist $M > 1$ and $N > 0$ such that for any $t > N$ we have:

$$t\delta(t) > \frac{M}{t} \int_0^t \tau\delta(\tau)\mathrm{d}\tau, \tag{6}$$

then there exists $Q^* > 0$ such that:

$$\inf_{Q>0} C(Q) = C(Q^*).$$

Proof. By the definition we get that:

$$\lim_{Q\to 0^+} C(Q) = \infty. \tag{7}$$

In fact, after seeing that $Q \to 0 \implies T(Q) \to 0$, then de l'Hospital-Bernoulli rule gives:

$$\lim_{Q\to 0^+} \frac{1}{T(Q)} \int_0^T \tau\delta(\tau)\mathrm{d}\tau = 0,$$

so that (7) holds. Passing to study $C(Q)$ for $Q \to \infty$ we get that (5) allows that:

$$\inf_{Q>0} C(Q) > 0.$$

In fact (5) implies that for large Q-values:

$$C(Q) \geq \frac{A}{T} + \frac{ch}{2-\alpha} T^{1-\alpha}$$

what is preventing that $C(Q)$ goes to zero when $Q \to \infty$ and guarantees the lower bound inf is greater than zero. Then our thesis will be achieved if proving there exist a real root of $C'(Q)$. We have:

$$C'(Q) = \frac{T'(Q)}{T(Q)}\, h \left(T(Q)\delta(T(Q)) - \frac{A}{hT(Q)} - \frac{1}{T(Q)} \int_0^{T(Q)} \tau\delta(\tau)\,\mathrm{d}\tau \right). \tag{8}$$

If $Q \to 0$ the bracket expression in (8) is negative, while (6) grants the same expression is greater than zero for large Q values. $\square$

2. Some effective computations

After having fixed a general sufficient condition capable of ensuring the cost function really attains a minimum, now we pass to detail three different exmples.

Closed form solution

Some few problems can be solved in explicit closed form. For instance, if $\delta(t) = (1+t)^{-1}$ one finds:

$$C(Q) = \frac{A - hQ}{e^Q - 1} + h,$$

and then:

$$C'(Q) = 0 \iff N(Q) := h - e^Q(A - hQ + h) = 0.$$

Recalling Lemma 2.1 in [Mingari Scarpello and Ritelli, 2007], we get the required EOQ is given by:

$$Q^* = W_0\left(-\exp\left(-\frac{A+h}{h}\right)\right) + \frac{A}{h} + 1$$

where W_0 is the Lambert[1] function. The conclusion is slightly different if $\delta(t) = a\,(b+ct)^{-1}$:

$$C(Q) = \frac{Ac - bhQ}{b\left(\exp\left(\frac{cQ}{a}\right) - 1\right)} + \frac{ah}{c}, \quad Q^* = \frac{a}{c}\,W_0\left(-\exp\left(-\frac{Ac^2}{abh} - 1\right)\right) + \frac{a}{c} + \frac{Ac}{bh}.$$

Anyway, the detection of closed form solutions to this type of EOQ problem through the special functions is rather rare: in the majority of practical cases a numerical treatment is required.

The numerical treatment

In the practice just the evalutation of the re-ordering time $T(Q)$, see equation (2), can lead to a severe computational problem. Neverheless the help of Mathematica® can be conclusive. We used the command:

```
In[1]:= inv[f_, s_] := Function[{t}, s /. FindRoot[f - t, {s, 1}]]
```

to revert numerically a function whose explicit expression is not available. So, if for example, $\delta(t) = 2 + \sin t$, (2) becomes $1 + 2T - \cos T = Q$. Herefrom $T(Q)$ shall be numerically pulled out, so that the cost function $C(Q)$ can be implemented (assuming the practical values $A = 1$ and $h = 1$) through the instructions:

```
In[2]:= einv = inv[1 + 2 T - Cos[T], T]
        c[Q_] := 1/einv[Q] (1 + (NIntegrate[τ (2 + Sin[τ]), {τ, 0, einv[Q]}]));

Out[2]= Function[{t$}, T /. FindRoot[(1 + 2 T - Cos[T]) - t$, {T, 1}]]
```

getting a cost plot versus the order, see Figure 1.

On which we observe that in this situation the cost function has many stationary points.

[1] Lambert W function, named after the German mathematician Johann Heinrich Lambert (1728-1777), is the inverse function of $f(w) = we^w$ where e^w is the natural exponential function and w is any complex number. Such a function is denoted by $W : z = W(z)e^{W(z)}$. The standard W function expresses exact solutions to transcendental algebraic $x-$equations like: $e^{-cx} = a_0(x-r)$, where a_0, c and r are real constants: its solution is $x = r + W(ce^{-cr}/a_o)$. Lambert was an eclectic and authoritative mathematician and probably would be astonished for such a paternity credited with him. It is true he first considered the related trinomial transcendental equation in 1758 which led to a paper by Leonhard Euler in 1783 who discussed a special case of we^w. But the inverse of we^w was really first described by Pólya and Szegö in *Aufgaben und Lehrsätze aus der Analysis*, issued as vol. 19 and 20 of Grundlehren der math. Wiss. in 1925.

In[4]:=
```
Plot[c[Q], {Q, 0, 20}, PlotRange → {0, 9},
  PlotStyle → {Black, Thickness[.0029]}, AxesLabel → TraditionalForm /@ {Q, C},
  LabelStyle → {FontFamily → "Times", FontSize → 14}, AxesStyle → Arrowheads[{-0., 0.030}]]
```

Out[4]=

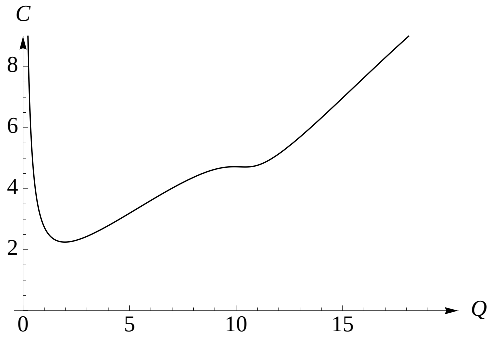

Figure 1: Cost plot versus the order under oscillatory demand $\delta(t) = 2 + \sin t$

Periodic seasonal demand

Let us take into account a time-nonmonotonic demand which is almost everywhere nonzero, but: starts growing at a fixed rate till to its maximum followed by a falling down at a fixed but different rate.

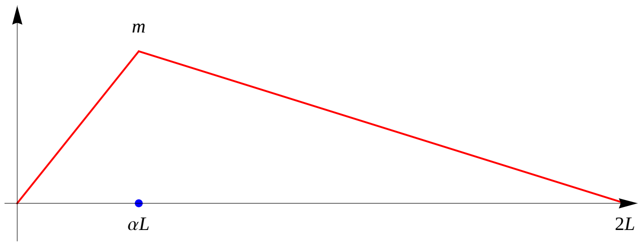

Figure 2: A $2L-$ periodic wave of up-down linear demand versus time.

Let $2L$ be the time-duration of such a wave-form; we have:

$$\delta(t) = \begin{cases} \dfrac{m}{\alpha\, L} t & \text{if } 0 \le t \le \alpha\, L \\ \dfrac{m(t-2L)}{L(\alpha-2)} & \text{if } \alpha\, L < t \le 2\, L \end{cases}$$

where $m > 0$, $0 < \alpha < 2$.

We assume such a demand will iterate its $2L$-behavior, just that of our Figure 2, so generating a $2L$-periodic function of time (indefinite wave-train). In such a way we are meaning to model the seasonal periodicity of demand from the market. We are led in a natural way to expand the demand function in Fourier series. The Fourier series of $\delta(t)$ is

$$\delta(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi t}{L} + b_n \sin \frac{n\pi t}{L} \right)$$

where

$$a_n = \frac{1}{L} \int_0^{2L} \delta(t) \cos \frac{n\pi\, t}{L}\, \mathrm{d}t, \quad b_n = \frac{1}{L} \int_0^{2L} \delta(t) \sin \frac{n\pi\, t}{L}\, \mathrm{d}t.$$

In our case we get:

$$a_0 = \frac{m}{2},\ a_n = \frac{2m\,[1-\cos(\pi n\alpha)]}{\pi^2 n^2(\alpha-2)\alpha},\ b_n = -\frac{2m\sin(\pi n\alpha)}{\pi^2 n^2(\alpha-2)\alpha}.$$

By stopping the Fourier expansion of $\delta(t)$ at the N^{th} harmonic, we will manage hereinafter its *truncated Fourier approximation* $\delta_N(t)$, being N our convenient or practical choice: in our simulation we used $N = 40$. We can resort again to the Mathematica® power for:

i) carrying out the integration of (1) which will provide q as a function of t;

ii) performing a numerical inversion of (2) in order to get the reordering time T through the generic order Q;

iii) plugging $T = T(Q)$ in (3) and evaluating there the "first integral moment" of $\delta_N(t)$. In such a way the relevant cost function $C(Q)$ is known and can be plotted Figure 3. We take $L = 2$, $N = 40$, $m = 1$, $\alpha = 0.4$ in our simulation, so that its minimizing order Q^* is promptly obtained.

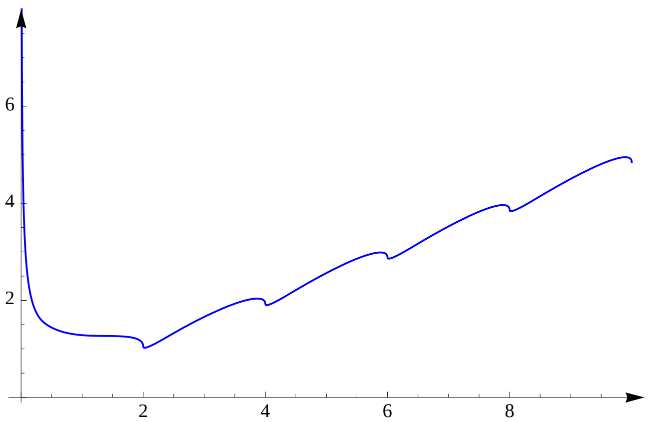

Figure 3: Cost plot versus the order under a demand wave train like Figure 2.

As for the Figure 1, the cost function has again several stationary points: anyhow the minimizing order is unique $(Q^* \simeq 2)$.

A three-fold periodic demand

The particular behavior of this new demand consists of being first growing, after falling and finally zero for a known amount of time (Figure 4). The distribution will depend on m, α, β where $2L$ is again the period of the wave-train. Our analytical description of the whole demand will be:

$$\delta(t) = \begin{cases} \dfrac{m}{\alpha\, L}t & \text{if } 0 \le t \le \alpha\, L \\ \dfrac{m(t-\beta\, L)}{(\alpha-\beta)\, L} & \text{if } \alpha\, L < t < \beta\, L \\ 0 & \text{if } \beta\, L \le t \le 2L \end{cases}$$

where $m > 0$, $0 < \alpha < \beta < 2$,

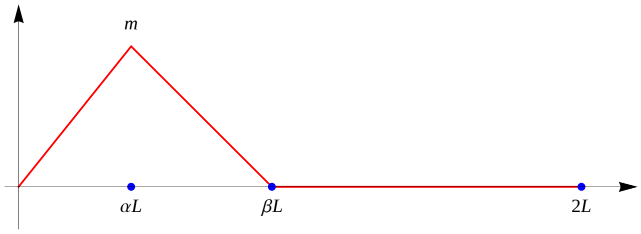

Figure 4: A $2L-$periodic wave of a three-fold linear demand.

The relevant Fourier expansion coefficients are:

$$a_0 = \frac{m\beta}{2},\ a_n = \frac{m\left[\alpha\cos(\pi n\beta) - \beta\cos(\pi n\alpha) - \alpha + \beta\right]}{\pi^2 n^2 \alpha(\alpha - \beta)},$$
$$b_n = \frac{m\left[\alpha\sin(\pi n\beta) - \beta\sin(\pi n\alpha)\right]}{\pi^2 n^2 \alpha(\alpha - \beta)}$$

Approximating $\delta(t)$ by $\delta_N(t)$, where $N = 40$, $m = 1$, $\alpha = 0.4$, $\beta = 0.9$, $L = 24$, we plot the relevant cost function $C(Q)$. We omit all the details and provide directly our Figure 5 with: $N = 40$, $m = 1$, $\alpha = 0.4$, $\beta = 0.9$, $L = 24$

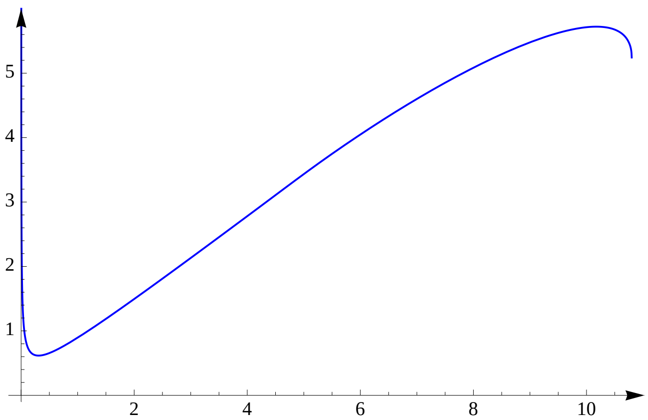

Figure 5: Cost plot versus the order under a demand wave train likr Figure 4

Non uniqueness

Instead of writing $C(Q)$ as in (3) set $C(Q) = E(y)$ evaluated when $y = T(Q)$ so that

$$E(y) = \frac{A + h\displaystyle\int_0^y \tau\delta(\tau)\mathrm{d}\tau}{y}$$

Assume in the following that $A = h = 1$. Then choose $\delta(t) = t^2 - (9/2)t + 13/2$. Conditions (4), (5) and (6) are are clearly verified. It is easy too see that

$$E(y) = \frac{1}{4}y^3 - \frac{3}{2}y^2 + \frac{13}{4}y + \frac{1}{y}$$

We see that $E'(1) = E'(2) = 0$ and that $E(1) = E(2) = 3$. Here below the short Mathematica code confirms the non-uniqueness of the minimum: see Figure 6.

```
In[32]:= einv = inv[13 T/2 - 9 T^2/4 + T^3/3, T]
         c[Q_] := 1/einv[Q] (1 + (NIntegrate[τ (τ^2 - 9/2 τ + 13/2), {τ, 0, einv[Q]}]));

Out[32]= Function[{t$}, T /. FindRoot[(13 T/2 - 9 T^2/4 + T^3/3) - t$, {T, 1}]]

In[34]:= pary2 = Line[{{0, 3}, {10, 3}}];
         ispi2 = Graphics[{Dashed, Red, pary2}];

In[36]:= gr = Plot[c[Q], {Q, 0, 11}, PlotStyle → {Black, Thickness[.0029]}, PlotRange → {2.8, 3.2},
           AxesLabel → TraditionalForm /@ {Q, C}, LabelStyle → {FontFamily → "Times", FontSize → 14},
           AxesStyle → Arrowheads[{-0., 0.030}]];

In[37]:= Show[gr, ispi2]

Out[37]=
```

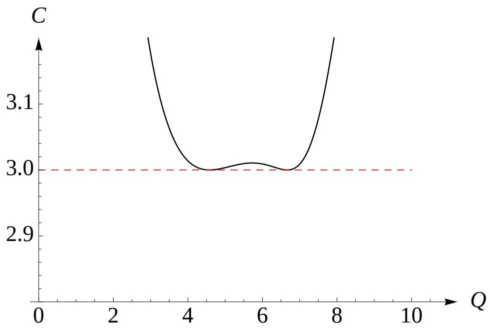

Figure 6: Cost plot versus the order: non uniqueness of the minimizing batch.

Conclusions

We found a sufficient condition, Theorem 1, for the existence of the economic batch to a Wilson-type inventory model whose demand $\delta(t)$ is a positive and continuous function of time. After having treated a case which can be solved in closed form, the computational problem is described of inverting the reordering time versus the ordered quantity, what is a necessary step for obtain the cost function to be minimized. The above theoretical/numerical procedure is detailed assuming a non monotonic demand formed of three different behaviors: growth, decrease and lasting zero (double "light" and "dark"). Such a wave-form is assumed to iterate itself periodically: the relevant seasonal demand is then expanded in a Fourier series of time. The cost function is computed and the relevant EOQ is detected. In tackling the possible wave-forms building the periodic train, we intentionally omitted those with jumps, like the saw-tooth or the square wave. First, for avoiding to enter the field of nondifferential optimization. Afterwards, the Fourier series of the periodic demand formed of the above wave-forms, would be affected by the Gibbs phenomenon[2] (1899). Namely, the N^{th} partial sum of the Fourier series of a continuously differentiable periodic function shows large oscillations near the jump, so that "wiggles" appear around the discontinuities, and even if $n \to \infty$, they never disappear so that the overshoot approaches a finite limit. Some of our plots (see Figures 1 and 3) displayed many stationary points, but always only one

[2]Named after the American physicist Josiah Willard Gibbs (1839-1903)

minimizing batch Q^*. A parabolic demand has been finally introduced without best batch uniqueness, what is reasonably due to the lack of constraints on $\delta(t)$.

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