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# EOQ when holding costs grow with the stock level: well-posedness and solutions[1]

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**Abstract.** An existence-uniqueness theorem is proved about a minimum cost order for a class of inventory models whose holding costs grow, following a power law, with the stock level. The theorem requires to perform a check of convergence of some improper integrals, and constitutes the article's main theoretical contribution to the subject. As application, several cases of demand are considered as functions of the stock level.

## 1. Background and motivation

Economic Order Quantity (EOQ) is a set of models defining the optimal quantity of a single item which shall be ordered for minimizing the total cost: ordering and inventory holding. These mathematical models have been in existence long before the computer, going their origin back in time to [Harris, 1913] even though [Wilson, 1934] is credited for his early in-depth analysis, on the subject. Basic underlying assumptions:

1. the monthly (annual or, generally: relevant to unit time) demand for the item is known, and deterministic;
2. no lead time (between order and arrivals) is taken into account;
3. the receipt of the order occurs in a single instant and immediately after ordering it;
4. quantity discounts are not calculated as part of the model;
5. the ordering cost $A$ is a constant.

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Several extensions can be made to EOQ model: the items deterministic demand can change with the instantaneous stock level or with time; the model can include backordering costs and multiple items. Should they undergo deterioration, the perishability can be modelled either constant or variable with the stock level. Finally, the above determinism could be released, leading to a probabilistic view, which we will keep out of.

The approach followed hereinafter is of "geometrical" nature in the sense that quadrature relationships are obtained providing via definite integrals: the reordering time, the global cost function and the minimum cost (optimality) condition. In such a way no previous approximation is inserted, such that of [Giri and Chaudhuri, 1998] where *at the beginning* a linearization is done by a truncated series development.

In our treatment numerical approximations arise *at the end*, in order to evaluate the economic order quantity $Q^*$.

Finally, it shall be highlighted we mark out the foundations to all the subject obtaining-in a rather general frame- some sufficient conditions ensuring the inventory cost function attains a minimum and that it is unique, namely: the EOQ-problem well-posedness.

## The MAB models generalization: variable inventory costs

Let it be $q$ a continuous function describing the instantaneous stock level of our inventory at time $t$. Such a $q$-level is ruled by a blowdown dynamics:

$$\begin{cases} \dot{q}(t) = -f(q(t)) \\ q(0) = Q > 0 \end{cases} \tag{1}$$

where $f : [0, \infty[ \to \mathbb{R}$ is a continuous and positive function describing whatever can get empty the store. We chose, for shortness of mathematics language, to describe the inventory blowdown through only one function of the instantaneous stock level $q$. Such a depletion (which can depend upon multiple causes) is here assumed to occur partly to meet the market demand, and partly for items deterioration during the period of positive inventory. The solution to (1) meets $0 < q(t) \leq Q$ for each $t \geq 0$, and, due to the autonomous (1) nature, we can solve by quadratures. If:

$$F(q) := \int_q^Q \frac{1}{f(u)}\, \mathrm{d}u = t \tag{2}$$

then $q(t) = F^{-1}(t)$ solves (1). We will treat models with the assumptions of the previous section, plus the $q$-monotonic blowdown ($\dot{q} < 0$ for each $t$) and then named MAB (Monotonic Autonomous Blowdown) for the absence of any imposed exogenous time-depending forcing. This paper comes out after a previous one (which is outstanding) and releases the past assumption of fixed inventory costs, assuming they can change as a growing function of $q$ itself, as seen in the practice. The MAB version which will be analyzed here has: invariant *specific* delivery costs ($A > 0$), and holding cost considered as a power function of the on-hand inventory:

$$h(q) = \hat{h} q^{\alpha},\ \alpha > 0,\ \hat{h} > 0$$

following [Giri and Chaudhuri, 1998], subsection 3.2 (Model B) page 471, and will be referred as *generalized MAB model*.

As usually, we mean *reordering time* generated by $Q$, the real value $T(Q) > 0$ capable of getting zero the solution of (1):

$$T(Q) = F(0) = \int_0^Q \frac{1}{f(u)}\, \mathrm{d}u.$$

Minding $A$ and $h$ meanings, then the *total* cost for (delivering + holding) a whichever $Q > 0$ amount of item will be:

$$C(Q) = \frac{A}{T(Q)} + \frac{\hat{h}}{T(Q)} \int_0^{T(Q)} [q(t)]^\alpha \, \mathrm{d}t. \tag{3}$$

The early Wilson model [Wilson, 1934] is found again when the instantaneous stock depletion rate $\dot{q}$ is assumed to have a constant magnitude: $f(q) = \delta > 0$, and $\alpha = 1$. The further ones due to [Goh, 1994] and to [Giri and Chaudhuri, 1998], page 471, will correspond (always for $\alpha = 1$), to $f(q) = \delta q^\beta$ with $0 < \beta < 1$ and to $f(q) = \theta\, q + \delta\, q^\beta$ with $\delta > 0$ and $0 < \theta,\ \beta < 1$.

## 2. The existence of a minimum cost

In our previous paper, [Mingari and Ritelli, to appear], we obtained some conditions sufficient to ensure that a cost function like (3) attains a minimum. The improvement of this article to the previous one is to increase our analysis assuming $h$ not constant any more, but growing, as due, with $q$ itself.

If in the integral at right hand side of (3) one makes the change $t = F(u)$, notice that $t = 0 \Rightarrow u = Q$, $t = T(Q) \Rightarrow u = 0$, and $\mathrm{d}t = -(1/f(u))\mathrm{d}u$, remembering $q(t) = F^{-1}(t)$ one finds:

$$\begin{aligned} C(Q) &= \frac{A}{T(Q)} + \frac{\hat{h}}{T(Q)} \int_0^Q \left[F^{-1}(F(u))\right]^\alpha \frac{\mathrm{d}u}{f(u)} = \\ &= \frac{A + \hat{h} \displaystyle\int_0^Q \frac{u^\alpha}{f(u)} \mathrm{d}u}{\displaystyle\int_0^Q \frac{1}{f(u)}\, \mathrm{d}u}. \end{aligned} \tag{4}$$

The (4) well-posedness requires, if $f(0) = 0$ the integrability at the origin of the functions:

$$\frac{1}{f(u)},\ \frac{u^\alpha}{f(u)}.$$

Formula (4) will allow to infer that $C(Q)$ attains an absolute minimum for some $Q > 0$ at one point only. In fact one can see soon that:

$$\lim_{Q \to 0} C(Q) = \infty.$$

Furthermore, assuming:

$$\int_0^\infty \frac{\mathrm{d}u}{f(u)} = \infty$$

then, the cost function will diverge again if $Q \to \infty$ as one can immediately check through De l'Hospital rule:

$$\lim_{Q \to \infty} C(Q) = \lim_{Q \to \infty} \frac{\dfrac{\hat{h}\, Q^\alpha}{f(Q)}}{\dfrac{1}{f(Q)}} = \infty.$$

The double divergence and the continuity of $C(Q)$ as well, imply that $C(Q)$ is bounded from below and then it shall have somewhere at least a critical point. Then a minimum does exist and, in addition, it has to be unique. In fact, the first derivative of $C(Q)$ becomes zero if and only if $Q$ is a root of the equation:

$$\hat{h}\, Q^\alpha \int_0^Q \frac{\mathrm{d}u}{f(u)} - \left\{ A + \hat{h} \int_0^Q \frac{u^\alpha}{f(u)}\, \mathrm{d}u \right\} = 0. \tag{5}$$

But the function

$$\mathcal{N}(Q) := \hat{h}\, Q^{\alpha} \int_0^Q \frac{\mathrm{d}u}{f(u)} - \left\{ A + \hat{h} \int_0^Q \frac{u^{\alpha}}{f(u)}\,\mathrm{d}u \right\}$$

is a difference of two increasing functions, then the critical point is unique.

At the same conclusion one arrives through analogous argument, whenever:

$$\int_0^{\infty} \frac{\mathrm{d}u}{f(u)} \in \mathbb{R}, \quad \int_0^{\infty} \frac{u}{f(u)}\,\mathrm{d}u = \infty$$

and:

$$\int_0^{\infty} \frac{\mathrm{d}u}{f(u)} \in \mathbb{R}, \quad \int_0^{\infty} \frac{u}{f(u)}\,\mathrm{d}u \in \mathbb{R}.$$

What above is not due to the $h$ growth law like $q^{\alpha}$ : in fact, assuming for it an arbitrary function $k : [0, \infty[ \to \mathbb{R}$ continuous, positive, and so that $k(0) = 0$ , one arrives at:

$$\begin{aligned} C(Q) &= \frac{A}{T(Q)} + \frac{\hat{h}}{T(Q)} \int_0^Q k\left(F^{-1}(F(u))\right) \frac{\mathrm{d}u}{f(u)} \\ &= \frac{A}{T(Q)} + \frac{\hat{h}}{T(Q)} \int_0^Q \frac{k(u)}{f(u)}\mathrm{d}u \end{aligned} \tag{6}$$

whose relevant condition becomes:

$$\hat{h}\, k(Q) \int_0^Q \frac{\mathrm{d}u}{f(u)} - \left\{ A + \hat{h} \int_0^Q \frac{k(u)}{f(u)}\,\mathrm{d}u \right\} = 0. \tag{7}$$

# 3. Some $f(q)$ laws of interest

The absence of a direct $Q-$ formula took in the literature the effect that a more long path shall be followed in performing an EOQ analysis: a $f(q)$ is assumed, the autonomous ODE to $q(t)$ is solved with the initial condition $q(0) = Q$, the reordering time $T$ so that $q(T) = 0$ is computed. Successively the cost function $C(Q)$ (3) is formed, and, putting $dC/dQ$ to zero, the transcendental $Q$-equation is finally written, to be numerically solved to $Q^*$. The existence- uniqueness for this kind of models defines which conditions are sufficient to ensure that a minimum cost exists and is unique for a (generalized) MAB model. Conversely, let us now present several cases of $f(q) > 0$ which can be considered, writing down (5) immediately. Notice that $f(q)$ could also be known not analytically. In fact experimental data set could be fitted in some reliable analytical expression: this explains the theoretical $f(q)$ demand laws we are going to study. We will show hereinafter some of them with the holding cost $\alpha$-power law, and with all the following (rather obvious) assumptions:

$$\begin{aligned} &\hat{h} > 0,\ \alpha > 0,\ Q > 0,\ A > 0; \\ &0 < \beta < 1,\ 0 < \varepsilon < 1,\ 0 < \delta < 1, a > 0,\ b > 0,\ p < 0,\ r < 0. \end{aligned}$$

For each of the $f(q)$, at least one of the integral sufficient conditions is met, and being this possible for more than one, we decided of omitting at all such a elementary sequence of checks: anyway we are sure the Economic Order Quantity does exist and is unique. In order to know it, we provide the transcendental equations, more complicated than in [Mingari and Ritelli, to appear], and then to be solved numerically in any case. For shortness, only the transcendental optimality condition will be displayed, omitting at all the re-ordering time and the global cost function $C(Q)$.

### Wilson

For the basic model we have $f(q) = \delta$, so that (7) gives:

$$\frac{\delta A}{\hat{h}} = -\frac{1}{2}Q^2 + Q^{1+\alpha},$$

which, nothwithstanding the basic $f(q)$ expression, requires a numerical treatment, unless $\alpha = 1$. If $\alpha = 2$ cardanic formulae can be an alternative tool.

### Goh's model

In [Goh, 1994] being $f(q) = \delta q^\beta$ we obtain:

$$\frac{\delta A}{\hat{h}} = \frac{Q^{1+\alpha-\beta}}{1-\beta} - \frac{Q^{2-\beta}}{2-\beta}.$$

### Giri-Chaudhuri model

Setting $f(q) = \theta q + \delta q^\beta$ as in [Giri and Chaudhuri, 1998], where $\theta$ is the rate of deterioration, the optimality condition leads to an integral not expressible through known functions. We provide a numerical simulation, obtained via Mathematica® taking: $A = 1$, $\hat{h} = 3$, $\beta = 1/3$. The picture below shows the global cost $Q-$function, for which we get the minimum $Q^* = 0.591744$.

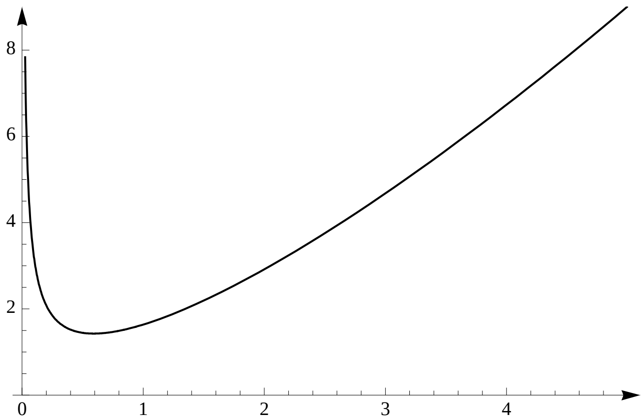


Figure 1: The global cost function of Giri-Chaudhuri with $A = 1$, $\hat{h} = 3$, $\beta = 1/3, \theta = 0.1$, $\delta = 1$

### Unnamed model

Setting $f(q) = \delta + \varepsilon q^\beta$, the slightly less complicated nature of this problem leads to a tractable optimality condition:

$$\frac{\delta A}{\hat{h}} = -\frac{1}{2}Q^2 \, {}_2\mathrm{F}_1\left(\begin{matrix} 1, \frac{2}{\beta} \\ \frac{2}{\beta}+1 \end{matrix} \middle| -\frac{\varepsilon}{\delta}Q^\beta\right) + Q^{1+\alpha} \, {}_2\mathrm{F}_1\left(\begin{matrix} \frac{1}{\beta}, 1 \\ 1+\frac{1}{\beta} \end{matrix} \middle| -\frac{\varepsilon}{\delta}Q^\beta\right)$$

which will provide $Q^*$ through a numerical approach. Notice that ${}_2\mathrm{F}_1$ is the Gauss hypergeometric $x$ -power series, $|x| < 1$:

$${}_2\mathrm{F}_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| x\right) = \sum_{n=0}^{\infty} \frac{(a)_n \, (b)_n}{(c)_n} \frac{x^n}{n!},$$

where $(a)_k$ is a Pochhammer symbol: $(a)_k = a(a+1)\cdots(a+k-1)$. For ${}_2\mathrm{F}_1$ a powerful integral representation theorem is available:

$$ {}_2\mathrm{F}_1\left(\begin{matrix} a,\, b \\ c \end{matrix}\middle|\, x\right) = \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)}\int_0^1 \frac{t^{a-1}(1-t)^{c-a-1}}{(1-xt)^b}\,\mathrm{d}t, $$

whose validity ranges are: $\mathrm{Re}\, a > \mathrm{Re}\, c > 0$, $|x| < 1$.

### Affine

If $f(q) = \delta + \varepsilon q$ we obtain:

$$ \varepsilon\frac{A}{\hat{h}} = -Q + \left(\frac{\delta}{\varepsilon} + Q\alpha\right)\ln\left(1 + \frac{Q\varepsilon}{\varepsilon}\right), $$

which shall be numerically solved in spite of the $f(q)$ easiness.

### Rational (first)

When the inventory depletion is rational:

$$ f(q) = \frac{a}{b+q}, $$

the optimum condition leads to the $Q-$equation:

$$ \frac{aA}{\hat{h}} = -\left(\frac{b}{2} + \frac{Q}{3}\right)Q^2 + \left(b + \frac{Q}{2}\right)Q^{1+\alpha}, $$

to be solved through a numerical approach.

### Rational (second)

If

$$ f(q) = \frac{a}{b^2+q^2}, $$

then:

$$ \frac{aA}{\hat{h}} = -\left(b^2 + \frac{Q^2}{2}\right)\frac{Q^2}{2} + \left(b^2 + \frac{Q^2}{3}\right)Q^{1+\alpha}, $$

which can be solved algebraically if $\alpha = 1$, numerically for $\alpha \neq 1$

### Quadratic

Let the instantaneous inventory stock level be ruled by (1) with $0 < f(q) = (q-p)(q-r)$. In such a way the optimum condition (5) will specialize in:

$$ \frac{A}{\hat{h}} = \frac{Q^\alpha}{p-r}\ln\left(\frac{r(Q-p)}{p(Q-r)}\right) - \frac{1}{p-r}p\ln\left(1 - \frac{Q}{p}\right) + r\ln\left(\frac{-r}{Q-r}\right). $$

### Exponential

The inventory manager is faced with an aperiodic demand which either is always increasing, or decreasing, namely $f(q) = a\,e^q$, or $f(q) = a\,e^{-q}$ . Even if the integrals in (5) are all elementary for the exponential situation, the relevant $Q$-equations:

$$ \begin{aligned} f(q) &= ae^{-q} \Longrightarrow \frac{\left(-1+e^Q\right)\hat{h}Q^\alpha}{a} - A - \frac{\hat{h}\left(e^Q(Q-1)+1\right)}{a} = 0 \\ f(q) &= ae^{q} \Longrightarrow \frac{\left(1-e^{-Q}\right)\hat{h}Q^\alpha}{a} - A - \frac{\hat{h}\left(1-e^{-Q}(Q+1)\right)}{a} = 0 \end{aligned} $$

are transcendental yet. In addition the $f(q)$ exponential nature is not an analytical oddness, but has a deep market meaning. In this last situation, we provide a numerical simulation, obtained using Mathematica® taking: $A = 1$, $\hat{h} = 2$, $a = 1$, $\alpha = 1/3$. The figure below gives the inventory global cost as a function of $Q$: we get $Q^* = 2.64317$.

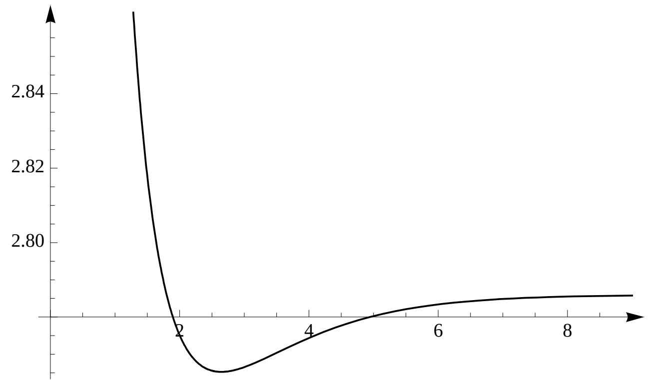


Figure 2: The global cost function to exponential increasing demand with $A = 9$, $\hat{h} = 5$, $a = 1$, $\alpha = 1/3$

# 4. Conclusions

An existence-uniqueness theorem is proved about a minimum cost batch for a class of inventory MAB models, leading to a set of sufficient conditions. This article enlarges our previous analysis assuming $h$ not constant any more, but growing, as due, with $q$ itself. The analyzed MAB version has therefore invariant *specific* delivery costs ($A > 0$) and holding costs variable as:

$$h(q) = \hat{h}q^{\alpha}, \alpha > 0, \hat{h} > 0.$$

The sufficient conditions require to check the convergence of some improper integrals, and form the article's main theoretical effort. As application, several cases have been considered of demand $f(q)$ assumed as continuous function of the stock level $q$. Being one of the sufficient conditions met in any case, the economic order quantity is unique, and the relevant computations lead to transcendental equations. In some cases the plot of the global costs is given, and, even if the optimality condition can be written in closed (but transcendental) form, its solution shall in (almost) any case be faced numerically.

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