Non-archimedean stratifications in T-convex fields

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To my family

The space of night is infinite,

The blackness and emptiness

Crossed only by thin bright fences

Of logic.

K. Rexroth

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Abstract

We prove that whenever T is a power-bounded o-minimal theory, *t-stratifications* exist for definable maps and sets in *T-convex fields*. To this effect, a thorough analysis of definability in T-convex fields is carried out. One of the conditions required for the result above is *the Jacobian property*, whose proof in this work is a long and technical argument based on an earlier proof of this property for *valued fields with analytic structure*. An example is given to illustrate that t-stratifications do not exist in general when T is not power-bounded. We also show that if T is power-bounded, the theory of all T-convex fields is *b-minimal with centres*.

We also address several applications of t-stratifications. For this we exclusively work with a power-bounded T. The first application establishes that a t-stratification of a definable set X in a T-convex field *induces* t-stratifications on the tangent cones of X. This is a contribution to local geometry and singularity theory. Regarding $\mathbb{R}$ as a model of T, the remaining applications are derived by considering the stratifications induced on $\mathbb{R}$ by t-stratifications in non-standard models. We prove that each such induced stratification is a *C^1-Whitney stratification*; this in turn leads to a new proof of the existence of Whitney stratifications for definable sets in $\mathbb{R}$. We also deal with interactions between tangent cones of definable sets in $\mathbb{R}$ and stratifications.

Keywords: Model theory, valued fields, t-stratifications, o-minimality, T-convex fields, weak o-minimality, b-minimality, tangent cones, Jacobian property, archimedean t-stratifications, Whitney stratifications.

Contents

Mathematical notation/conventions

We explain a few notational and mathematical conventions used throughout this work.

- The *characteristic function* $\chi_X : K \longrightarrow \{0,1\}$ of the set $X \subseteq K$, is defined as $\chi_X(x) := 1$ if $x \in X$ and $\chi_X(x) := 0$ otherwise.
- A *non-archimedean* field is an ordered field in which the set $\mathbb{Z}$ is bounded. Equivalently, the field contains *infinitesimal elements*, elements x such that $0 < x < 1/n$ for all $n \in \mathbb{Z}^+$.
- If S is a linear order, $X \subseteq S$ and $x \in S$, we write $x < X$ to mean that $x < y$ for all $y \in X$. The expressions $x \leq X$, $x > X$ and $x \geq X$ are defined similarly.
- If $(G, +, 0, <)$ is an ordered group, $G_{>0}$ and $G_{\geq 0}$ denote the sets $\{x \in G \mid x > 0\}$ and $\{x \in G \mid x \geq 0\}$, respectively.
- If S is a ring, $S^\times$ denotes the set of *units* of S, the invertible elements in S. Sometimes, when S lacks a ring structure, $S^\times$ simply denotes $S \setminus \{0\}$.
- If S is a ring, $\mathrm{Mat}_n(S)$ denotes the ring of all $(n \times n)$-matrices with entries in S. If S is a subring of a field K, by $\mathrm{GL}_n(S)$ we denote the group of all invertible matrices M in $\mathrm{Mat}_n(K)$ such that both M and M^{-1} have coefficients in S.
- If X is a subset of a topological space, $\mathrm{cl}(X)$ will denote the closure of X in such topology.
- We frequently and freely identify a singleton $\{x\}$ with the point x itself. Thus often we write ‘point’ for ‘singleton containing the point’.
- By ‘definable’ we always mean ‘definable with parameters’. If A is a set of parameters, we write ‘A-definable’ to imply that *all* the parameters needed come from A. We often work with a few languages concurrently and thus write ‘L-definable’ to mean ‘definable with parameters by a formula in the language L’.
- By ‘definable partition’ we mean that each of the parts is definable.

Introduction

This thesis is a contribution to the model theory of valued fields and to the applications of model theory to geometry.

Stratifications are an important tool for the analysis of singularities of sets. With a stratification one aims at classifying the points of a set according to their degree of singularity. The interest on stratifications started with the work of Whitney [53], in which the following partition of an analytic variety $X \subseteq \mathbb{C}^n$ was suggested: X is partitioned into the varieties X_{rg}, $(X_{sg})_{rg}$, $((X_{sg})_{sg})_{rg}$, etcetera, where Y_{rg} denotes the subvariety of all the regular points of Y, and Y_{sg} denotes the subvariety of the singular points of Y. The dimension of the sets in the partition above decreases, so the obtained partition is in fact finite. After some slight refinements, this partition satisfies the *Whitney (regularity) conditions*; these are geometrical requirements on the interaction between the pieces of the partition. Nowadays, any partition of X satisfying the Whitney conditions is known as a *Whitney stratification* of X. These stratifications were later shown to indeed provide a useful classification of the points of X, with Thom [46] proving that points in the same strata are *normally equi-singular*. A now classic exposition of this result and its consequences is in Mather [36]. A few of the many papers studying and deriving applications from Whitney stratifications in semi-algebraic and sub-analytic contexts are [42, 45, 47, 51]. Strengthenings of Whitney stratifications have been also explored; among these are *Kuo* and *Verdier stratifications* (Kuo [33], Verdier [52]), and *Lipschitz stratifications* (Mostowski [39]; see also Parusiński [43]).

Whitney stratifications—and stratifications in general—have enjoyed attention from model theorists. As a generalisation of the semi-algebraic situation, Loi [34] proved that Whitney (and Verdier) stratifications exist for every definable set in an o-minimal structure on the real field $\mathbb{R}$. A new geometric proof of this result has been recently offered in

Nguyen et al. [41]. Even earlier investigations of this kind—on stratifications, triangulations, etcetera, in general o-minimal structures over the real field—were carried out by van den Dries and Miller [18]. Additionally, there have been efforts to generalise known applications of Whitney stratifications to the wider o-minimal setting, see e.g. Trotman and Valette [49].

Other model-theoretic approaches to stratifications have instead looked for analogous notions of stratifications in contexts beyond o-minimality. The paper by Cluckers et al. [6] contains a notion of *regular stratifications* for definable sets in the p-adic field $\mathbb{Q}_p$. This notion is essentially a literal translation of Whitney stratifications, with several of the results and proofs finding clear analogues in the setting of the real field. It was proved that these regular stratifications exist for all definable sets in $\mathbb{Q}_p$ in both the p-adic semi-algebraic and the p-adic sub-analytic languages. The cited paper also includes an application of these stratifications to local density problems. These regular stratifications were the first introduced in a valuational context.

This thesis is devoted to a notion of stratifications in Henselian valued fields of both field and residue characteristic 0. The definition of these stratifications is due to Halupczok [26], and they are known as *t-stratifications*. T-stratifications were introduced following the efforts to classify the definable sets in the ring of p-adic integers $\mathbb{Z}_p$. This approach reached a classification of said sets up-to definable *risometries* for big enough p—where a risometry is an isometry with an extra rigidity property. It was soon realised that t-stratifications possess further potential both for applications to geometry and in the research on (model-theoretic) tameness of valued fields. This work provides some advancements in the theory and the applications of t-stratifications.

In [26], t-stratifications were proved to exist for definable sets in *valued fields with analytic structure* (introduced in Cluckers and Lipshitz [8]). Basic but relevant examples of such valued fields are pure algebraically closed valued fields and pure real closed valued fields[1]. A large part of the investigations reported in this thesis aims at proving

[1]By pure we mean that no extra structure is assumed apart from that of a field and an ordered field,

that t-stratifications exist in T-*convex fields* whenever T is power-bounded, these being particular expansions of real closed valued fields that are not covered in the setting of valued fields with analytic structure.

Let K be a Henselian valued field of both field and residue characteristic 0. Following the driving idea of classifying the singularities of sets, a t-stratification of a subset X of K^n is a definable partition $(S_i)_{i\leq n} := (S_0, \dots S_n)$ of R^n satisfying that:

1. for each $d \leq n$, the dimension of $S_0 \cup \cdots \cup S_d$ is less than or equal to d;
2. for each $d \leq n$ and valuative, open or closed ball $B \subseteq K^n$ with $B \subseteq S_d \cup \cdots \cup S_n$, the sets $S_d \cap B, \dots, S_n \cap B$ and $X \cap B$ are *almost translation invariant in the direction of a* d*-dimensional vector subspace of* K^n.

The meaning of $Y \subseteq K^n$ being almost translation invariant in the direction of a vector space $\overline{V} \subseteq K^n$ is that there exists a definable *risometry* φ such that $\varphi(Y)$ is invariant under translations by elements of $\overline{V}$. This is equivalent to $\varphi(Y)$ being a union of cosets of $\overline{V}$. Condition (2) is regarded as the regularity requirement for t-stratifications, and is analogous to the *Whitney conditions* mentioned earlier for Whitney stratifications.

We now describe the main results in this thesis. We first introduce T-convex fields. Let L be a language containing the language of ordered rings $L_{\text{or}} := \{+, -, \cdot, 0, 1, <\}$, and let T be an o-minimal L-theory containing the L_{or}-theory of real closed fields. If R is a model of T, a T-*convex subring* of R is a convex proper subring $\mathcal{O}$ of R satisfying that $f(\mathcal{O}) \subseteq \mathcal{O}$ for any 0-definable continuous function $f : R \longrightarrow R$ (van den Dries and Lewenberg [16]). The pair $(R, \mathcal{O})$ is called a T-*convex field*. It is easy to see that $\mathcal{O}$ must then be a valuation ring of R, so $(R, \mathcal{O})$ is from now on regarded as a valued field naturally expanding a real closed valued field. The value group of $(R, \mathcal{O})$ is denoted by Γ and the associated valuation map is denoted by $\mathrm{v} : R^\times \longrightarrow \Gamma$ (where $R^\times := R \setminus \{0\}$). The residue field of $(R, \mathcal{O})$ is denoted with $\overline{R}$, while the associated residue map is $\mathrm{res} : \mathcal{O} \longrightarrow \overline{R}$. A way of enriching $(R, \mathcal{O})$ is by introducing

respectively.

the structure $\mathrm{RV}^\times := R^\times/(1+M)$, where M is the unique maximal ideal of $\mathcal{O}$. The use of $\mathrm{RV}^\times$ started with the work of Basarab [2], and it has since then featured in several projects in model theory—for example, its presence is now standard in model-theoretic approaches to motivic integration (see [10, 31, 56]). The function taking each $x \in R^\times$ to its class in $\mathrm{RV}^\times$ is denoted by rv. It is easy to see that $\overline{R} \setminus \{0\}$ embeds into $\mathrm{RV}^\times$ via the map i defined by $i(\mathrm{res}(x)) := \mathrm{rv}(x)$ for all $x \in \mathcal{O} \setminus \{0\}$, and that defining $\mathrm{v}_{\mathrm{RV}}(\mathrm{rv}(x)) := \mathrm{v}(x)$ for each $x \in R^\times$ provides a well defined map from $\mathrm{RV}^\times$ onto Γ. Intuitively, $\mathrm{RV}^\times$ combines $\overline{R}$ and Γ into a single structure; this is supported by the fact that $1 \longrightarrow \overline{R} \setminus \{0\} \xrightarrow{i} \mathrm{RV}^\times \xrightarrow{\mathrm{v}_{\mathrm{RV}}} \Gamma \longrightarrow 0$ is a short exact sequence.

The language $L_{\mathrm{RV^{eq}}}$ in which t-stratifications are defined is multi-sorted. As first sort $L_{\mathrm{RV^{eq}}}$ has the L-structure R; the remaining sorts are all the imaginary sorts defined from $\mathrm{RV} := \mathrm{RV}^\times \cup \{0\}$ (the natural maps between RV and such sorts are added too). A T-convex field $(R, \mathcal{O})$ is then turned naturally into an $L_{\mathrm{RV^{eq}}}$-structure $(R, \mathrm{RV^{eq}})$. The following is one of the main result in this thesis.

Theorem A. If T is power-bounded, then every $L_{\mathrm{RV^{eq}}}$-definable set $X \subseteq R^n$ admits an $L_{\mathrm{RV^{eq}}}$-definable t-stratification.

This result is achieved by verifying that the conditions in [26] for the existence of t-stratifications hold for $(R, \mathrm{RV^{eq}})$. Crucial results towards proving those conditions for T-convex fields come from van den Dries [14], Halupczok *op.cit.*, Holly [30] and Yin [57]. The most difficult of the conditions—and indeed the one occupying us for longer in this work—is the *Jacobian property*.

For $(x_1, \dots x_n) \in R^n$ we set $\hat{\mathrm{v}}((x_1, \dots, x_n)) := \min\{\mathrm{v}(x_i) \mid 1 \leq i \leq n\}$. The usual inner product on powers of R is denoted by $\langle \cdot, \cdot \rangle$. If $f : R^n \longrightarrow R$, we say that f has *the Jacobian property* if there exist a sort $S \subseteq \mathrm{RV^{eq}}$ and an $L_{\mathrm{RV^{eq}}}$-definable map $\chi : R^n \longrightarrow S$ such that whenever $q \in \chi(R^n)$ satisfies that $\dim(\chi^{-1}(q)) = n$, either

$f|_{\chi^{-1}(q)}$ is constant or there exists $z \in R^n \setminus \{0\}$ such that for all distinct $x, x' \in \chi^{-1}(q)$,

$$\mathrm{v}(f(x) - f(x') - \langle z, x - x' \rangle) > \hat{\mathrm{v}}(z) + \hat{\mathrm{v}}(x - x').$$

When $n = 1$, the property above is a variant of conditions employed in axiomatic approaches to motivic integration (particularly in the style developed by Cluckers and Loeser [10]). In fact, a similar property appears in Yin [56], where investigations on motivic integration in T-convex fields are pursued.

The difficult result needed for Theorem A is the following.

Theorem B. Suppose that T is power-bounded and let $n \geq 1$. Then every $L_{\mathrm{RV}^{\mathrm{eq}}}$-definable function $f : R^n \longrightarrow R$ has the Jacobian property.

On other kinds of stratifications in T-convex fields, Halupczok and Yin [28] introduced *valuative Lipschitz stratifications* with the aim of proving that if T is power-bounded and $\mathbb{R}$ is a model of T, then every closed L-definable subset of $\mathbb{R}^n$ admits a *Lipschitz stratification*. Valuative Lipschitz stratifications are also stratifications in a T-convex field $(R, \mathcal{O})$ but their existence was proved only for L-definable subsets of R^n when T is power-bounded. It is unknown whether they exists for all definable sets in the valuational language (for example, for $L \cup \{\mathcal{O}\}$-definable sets). It is also open how they relate to t-stratifications, but there are intuitive reasons to believe that valuative Lipschitz stratifications are stronger (i.e. their regularity conditions are more restrictive than the translatability discussed earlier for t-stratifications).

On our path to prove Theorems A and B, we obtain several results on general definability in T-convex fields. The following is of interest, as it connects this work to research on minimality conditions for valued fields.

Theorem C. If T is power-bounded, then the $L_{\mathrm{RV}^{\mathrm{eq}}}$-theory of all T-convex fields is b-*minimal with centres* over the sorts $\mathrm{RV}^{\mathrm{eq}}$.

The notion of b-*minimality* was introduced by Cluckers and Loeser [8]; among its consequences is, for example, a cell-decomposition theorem. Additionally, b-minimality is comparable to other minimality conditions for valued fields, e.g. p-*minimality* (Haskell and Macpherson [29]) and v-*minimality* (Hrushovski and Kazhdan [31]).

This thesis also deals with applications of t-stratifications. The *tangent cone of* $X \subseteq R^n$ *at* $p \in R^n$, denoted as $\mathcal{C}_p(X)$, is the set of all $y \in R^n$ satisfying that for every $\gamma \in \Gamma$, there exist $x \in X$ and positive $r \in R$ such that $\hat{\mathrm{v}}(x - p) > \gamma$ and $\hat{\mathrm{v}}(r(x - p) - y) > \gamma$. In other words, $\mathcal{C}_p(X)$ is the cone formed by all the tangent (half-)rays to X at p. For example, when p is a non-singular point of X, $\mathcal{C}_p(X)$ equals the usual tangent space of X at p. Like tangent spaces for smooth manifolds, *tangent cones* are important in the analysis of the local geometry of sets, see for instance [3, 20, 21, 24]. Research exploiting the interactions of stratifications and tangent cones can be found in [6, 40]. The following is another main result in this thesis.

Theorem D. Suppose that T is power-bounded. If $(S_i)_{i \le n}$ is a t-stratification of $X \subseteq R^n$ and $p \in R^n$, then the sets $\mathcal{C}_{p,0} := \mathcal{C}_p(S_0)$ and $\mathcal{C}_{p,i} := \mathcal{C}_p(S_0 \cup \cdots \cup S_i) \setminus \mathcal{C}_p(S_0 \cup \cdots \cup S_{i-1})$ for $1 \le i \le n$, constitute a t-stratification of $\mathcal{C}_p(X)$.

In this way we say that t-stratifications *induce* t-stratifications on tangent cones. It is hoped that Theorem D could offer a practical application in the study of singularities, perhaps after establishing the right notion of equi-singularity between points in the same stratum of a t-stratification.

With a view towards further applications of t-stratifications, we prove that the stratifications obtained through Theorem A can be replaced by t-stratifications with L-definable strata. The proof follows the strategy for a similar result when $L = L_{\text{or}}$ in [26]. Making t-stratifications L-definable motivates exploring applications in a purely L-definable context. If T admits $\mathbb{R}$ as a model, then an (non-principal) ultrapower ${}^*\mathbb{R}$ of $\mathbb{R}$ can be readily made into a T-convex field. The *transfer principle* of non-standard analysis helps to fruitfully transport stratifications and tangent cones between ${}^*\mathbb{R}$ and $\mathbb{R}$. The following results are obtained through this method. In their statements we use the usual *-notation of non-standard analysis. As global hypotheses we suppose that T is power-bounded and admits $\mathbb{R}$ as model.

Theorem E. If $X \subseteq \mathbb{R}^n$ is L-definable and $(S_i)_{i \leq n}$ is an L-definable partition of $\mathbb{R}^n$ such that $({}^*S_i)_{i \leq n}$ is a t-stratification of ${}^*X$ in ${}^*\mathbb{R}$, then $(S_i)_{i \leq n}$ is a C^1-Whitney stratification of X.

When $L = L_{\text{or}}$, Theorem E was proved in [26], and the same is true for the next one.

Theorem F. Every L-definable set $X \subseteq \mathbb{R}^n$ admits a C^1-Whitney stratification.

Theorem F is superseded by the much earlier result of Loi [34] mentioned previously (in Loi's result power-boundedness of $\mathbb{R}$ as an o-minimal L-structure is not required).

One last result new result is the following.

Theorem G. If $X \subseteq \mathbb{R}^n$ and $(S_i)_{i \leq n}$ are as in the hypotheses of Theorem E, then $(S_i)_{i \leq n}$ *induces* C^1-Whitney stratifications on the tangent cones of X.

The notions of tangent cones and *inducing* in this theorem are analogous to the earlier ones in the context of $(R, \mathrm{RV}^{\mathrm{eq}})$. Theorem G reveals a property of $(S_i)_{i \leq n}$ that Whitney stratifications do not have: it is not difficult to see that the latter do not in general induce Whitney stratifications on tangent cones (see Example 5.3.5). Therefore, stratifications like $(S_i)_{i \leq n}$ in Theorem E constitute a genuine new kind of stratification for subsets of $\mathbb{R}^n$.

To end this introduction, we describe the organisation of this thesis. Chapter 1 provides the preliminaries needed for t-stratifications. Several examples are presented and enlightening results are described. The reader is also directed to Halupczok [27] for a short informal introduction to t-stratifications. Chapter 2 contains the preliminaries needed on T-convex fields. In Sections 2.2 and 2.4 new results essential for later chapters are presented. In Chapter 3, Theorems A, B and C are proved. In Section 3.3 it is proved that t-stratifications can be made L-definable, and in Section 3.4 an example shows that t-stratifications cannot exist in general when T is not power-bounded. In Chapter 4, Theorem D and a few other results on tangent cones are presented. The last chapter, Chapter 5, contains the applications of t-stratifications to the setting of $\mathbb{R}$. There, Theorems E, F and G are proved. An afterword and two appendices finish this thesis. The afterword is a list of open problems around the topics in this thesis. Appendix A provides a brief introduction to o-minimality and weak o-minimality—notions that are prominent throughout this work. Appendix B presents the proof of a result in [57] fundamental for the main theorem in Section 2.4.

Chapter 1

T-stratifications

This chapter is an introduction to t-stratifications accompanied by preliminary material needed in later chapters. It contains no original results. Nevertheless, some of the proofs and most of the examples presented are new. The convention is that all those results/proofs with no cited reference are new. The material in this chapter comes mainly from the founding work by Halupczok [26] on t-stratifications. Another, more informal introduction to the subject is [27].

1.1 The setting and Hypotheses 1.1.9

We let K be a Henselian valued field of characteristic 0. We denote the valuation ring of K by O_K, the unique maximal ideal of O_K by M_K and the residue field O_K/M_K of K by k. We assume that k is of characteristic 0 too; in this case we say K is a Henselian valued field *of equi-characteristic* 0. The residue map will be denoted by $\mathrm{res} : O_K \longrightarrow k$. Furthermore, we also apply the map res to elements of O_K^n coordinate-wise and to matrices (with entries in O_K) entry-wise.

The valuation map on K will be denoted by $\mathrm{v} : K \longrightarrow \Gamma_\infty$, where $\Gamma_\infty := \Gamma \cup \{\infty\}$ and $\Gamma := K^\times/O_K^\times$ is the value group of K. We extend v to cartesian powers of K by defining $\hat{\mathrm{v}}(x) := \min\{\mathrm{v}(x_i) \mid 1 \le i \le n\}$, for each $x = (x_1, \ldots, x_n) \in K^n$. Accordingly, we define the valuation of an $n \times n$-matrix M with entries in K by $\hat{\mathrm{v}}(M) := \min\{\mathrm{v}(M_{ij}) \mid 1 \le i, j \le n\}$. Observe that if $x \in K^n$, then $\hat{\mathrm{v}}(Mx) \ge \hat{\mathrm{v}}(M) + \hat{\mathrm{v}}(x)$.

The *valuative topology* on K^n is the one generated by the *open (valuative) balls* $B(x, > \gamma) := \{y \in K^n \mid \hat{\mathrm{v}}(x - y) > \gamma\}$, where $x \in K^n$ and $\gamma \in \Gamma$. A *closed ball* is a set of the form $B(x, \ge \gamma) := \{y \in K^n \mid \hat{\mathrm{v}}(x - y) \ge \gamma\}$ with $x \in K^n$ and $\gamma \in \Gamma$. If B is an open ball, the *radius* of B, denoted by $\mathrm{rad}(B)$, is the element $\gamma \in \Gamma$ such that $B = B(x, > \gamma)$. If B is a closed ball we similarly define $\mathrm{rad}(B) \in \Gamma$. It could happen that B is both open and closed (for instance when Γ is discrete), and in such case we let $\mathrm{rad}(B)$ be the radius of B as an open ball; this is typically enough for arguments involving $\mathrm{rad}(B)$.

The structure $\mathrm{RV}^\times := K^\times/(1 + M_K)$ combines the residue field and the value group, and plays an important role in the contemporary research on valued fields. If rv denotes the canonical map from $K^\times$ to $\mathrm{RV}^\times$, the valuation map factors through $\mathrm{RV}^\times$ as follows:

$$\begin{tikzcd} K^\times \arrow[rr, "\mathrm{v}"] \arrow[dr, "\mathrm{rv}"'] & & \Gamma \\ & \mathrm{RV}^\times \arrow[ur, "\mathrm{v}_{\mathrm{RV}}"'] & \end{tikzcd}$$

where $\mathrm{v}_{\mathrm{RV}} : \mathrm{RV}^\times \longrightarrow \Gamma$ is defined as $\mathrm{v}_{\mathrm{RV}}(\xi) := \mathrm{v}(x)$ for some (eq. any) $x \in \mathrm{rv}^{-1}(\xi)$, for all $\xi \in \mathrm{RV}^\times$. Furthermore, the residue field embeds into $\mathrm{RV}^\times$ via the map $i : k^\times \longrightarrow \mathrm{RV}^\times$ given as $i(\mathrm{res}(x)) := \mathrm{rv}(x)$ for any $x \in O_K^\times$. The importance of $\mathrm{RV}^\times$ is thus succinctly resumed in the short exact sequence

$$1 \longrightarrow k^\times \xrightarrow{i} \mathrm{RV}^\times \xrightarrow{\mathrm{v}_{\mathrm{RV}}} \Gamma \longrightarrow 0.$$

We also add an element 0 to $\mathrm{RV}^\times$ to obtain $\mathrm{RV} := \mathrm{RV}^\times \cup \{0\}$, and we then extend the maps above setting $\mathrm{rv}(0) := 0$, $i(0) := 0$ and $\hat{\mathrm{v}}_{\mathrm{RV}}(0) := \infty$. Note that the properties of $0 \in \mathrm{RV}$ suit the intuition around the nature of $\mathrm{RV}^\times$. In this work we will employ a multi-dimensional version of RV.

Definition 1.1.1. We define $\mathrm{RV}^{(n)}$ as the quotient of K^n by the equivalence relation $x \sim y$ if and only if either $x = y = 0$ or $\hat{\mathrm{v}}(x-y) > \hat{\mathrm{v}}(x)$. We use $\hat{\mathrm{rv}} : K^n \longrightarrow \mathrm{RV}^{(n)}$ for the map taking $x \in K^n$ to its class in $\mathrm{RV}^{(n)}$.

Note that $\mathrm{RV}^{(1)} = \mathrm{RV}$ and in that case $\hat{\mathrm{rv}} = \mathrm{rv}$. When $n = 1$, we thus keep the notation RV for $\mathrm{RV}^{(n)}$ and rv for $\hat{\mathrm{rv}}$.

A non-singleton fibre of $\hat{\mathrm{rv}}$ will be called an RV-*ball*. Note that RV-balls are open valuative balls for if $\xi \in \mathrm{RV}^{(n)}$ and $x \in \hat{\mathrm{rv}}^{-1}(\xi)$, then $\hat{\mathrm{rv}}^{-1}(\xi) = B(x, > \hat{\mathrm{v}}(x))$. Also, we generalise the map v_{RV} to $\mathrm{RV}^{(n)}$ with $n > 1$ by putting for each $\xi \in \mathrm{RV}^{(n)}$, $\hat{\mathrm{v}}_{\mathrm{RV}}(\xi) := \hat{\mathrm{v}}(x)$ for some (eq. any) $x \in \hat{\mathrm{rv}}^{-1}(\xi)$.

Each $\mathrm{RV}^{(n)}$ can also be seen as a quotient of K^n, generalising the case of $\mathrm{RV}^{\times}$. To see this, first notice that res applied to matrices (see the first paragraph of this section) induces an epimorhism of groups $\mathrm{GL}_n(O_K) \longrightarrow \mathrm{GL}_n(k)$. We let U_n denote the kernel of this epimorphism, and let U_n act on K^n by usual matrix-vector multiplication (for example, when $n = 1$, we have that $U_n = 1 + M$ and the action is simple multiplication in K). We then obtain that $K^n/U_n \simeq \mathrm{RV}^{(n)}$. Indeed, if the orbit of $x \in K^n$ under the action of U_n is denoted by xU_n, then the map $xU_n \mapsto \hat{\mathrm{rv}}(x)$ is an isomorphism from K^n/U_n to $\mathrm{RV}^{(n)}$.

We next introduce an important class of maps, ubiquitous throughout this work. These maps are rigid (isometric) and *preserve the direction* of vectors (see Remark 1.2.9).

Definition 1.1.2. Let X and Y be subsets of K^n. A bijection $\varphi : X \longrightarrow Y$ is called a *risometry*[1] if for all $x, x' \in X$, we have that $\hat{\mathrm{rv}}(\varphi(x) - \varphi(x')) = \hat{\mathrm{rv}}(x - x')$.

It is clear that the composition of two risometries is a risometry. The inverse of a risometry $\varphi : X \longrightarrow Y$ is a risometry too, for if $x, x' \in X$, then

$$\hat{\mathrm{rv}}(x - x') = \hat{\mathrm{rv}}(\varphi \circ \varphi^{-1}(x) - \varphi \circ \varphi^{-1}(x')) = \hat{\mathrm{rv}}(\varphi^{-1}(x) - \varphi^{-1}(x')).$$

[1] In personal communication, Halupczok explained that "risometry" stands for "residue field isometry".

Additionally, as indicated earlier, any risometry φ must be an *isometry*, i.e. for all $x, x' \in X$, we have that $\hat{\mathrm{v}}(\varphi(x) - \varphi(x')) = \hat{\mathrm{v}}(x - x')$. Indeed, for $x, x' \in X$,

$$\hat{\mathrm{v}}(\varphi(x) - \varphi(x')) \geq \min\{\hat{\mathrm{v}}(\varphi(x) - \varphi(x') - (x - x')), \hat{\mathrm{v}}(x - x')\} = \hat{\mathrm{v}}(x - x'),$$

and if the strict inequality held, then $\hat{\mathrm{v}}(\varphi(x)-\varphi(x')-(x-x')) = \hat{\mathrm{v}}(x-x')$, contradicting that $\mathrm{r\hat{v}}(\varphi(x) - \varphi(x')) = \mathrm{r\hat{v}}(x - x')$. The following can be seen as a further rigidity property of risometries.

Lemma 1.1.3. *Let* $\varphi : O_K^n \longrightarrow O_K^n$ *be a risometry. Then for all* $x, x' \in O_K^n$ *we have that* $\mathrm{res}(\varphi(x) - \varphi(x')) = \mathrm{res}(x - x')$. *In particular, for any* $Y \subseteq O_K^n$, $\mathrm{res}(\varphi(Y)) = \mathrm{res}(Y) + \mathrm{res}(\varphi(0))$.

Proof. For $x, x' \in O_K^n$, $\hat{\mathrm{v}}(\varphi(x) - \varphi(x') - (x - x')) > \hat{\mathrm{v}}(x - x') \geq 0$, and the desired equation follows. For the second part, if $x \in Y$, by putting $x' = 0$ we have that $\mathrm{res}(\varphi(x)) - \mathrm{res}(\varphi(0)) = \mathrm{res}(\varphi(x) - \varphi(0)) = \mathrm{res}(x)$. ❒

Examples 1.1.4. A translation $x \mapsto x+a$ (for some $a \in K^n$) is clearly a risometry. Other examples are given by multiplication by matrices M in $\mathrm{GL}_n(O_K)$ for which $\mathrm{res}(M) := (\mathrm{res}(M_{ij})) = Id_n$. Indeed, if $x, x' \in K^n$, then

$$\hat{\mathrm{v}}(M(x - x') - (x - x')) \geq \hat{\mathrm{v}}(M - I) + \hat{\mathrm{v}}(x - x') > \hat{\mathrm{v}}(x - x').$$

So $\mathrm{r\hat{v}}(Mx - Mx') = \mathrm{r\hat{v}}(x - x')$, as required. The subset of $\mathrm{GL}_n(O_K)$ of all such matrices will be denoted by U_n.

By definition, if $M \in \mathrm{GL}_n(O_K)$, then $\hat{\mathrm{v}}(M) \geq 0$ and also $\hat{\mathrm{v}}(M^{-1}) \geq 0$. In fact, we have that $0 = \hat{\mathrm{v}}(M^{-1} \circ M) \geq \hat{\mathrm{v}}(M^{-1}) + \hat{\mathrm{v}}(M) \geq 0$, so $\hat{\mathrm{v}}(M) = \hat{\mathrm{v}}(M^{-1}) = 0$. We use this fact in the following lemma, which helps us in constructing new risometries from a given one.

Lemma 1.1.5. *Let* $M \in GL_n(O_K)$. *The following hold.*

(a) *If* $x \in K^n$*, then* $\hat{\mathrm{v}}(Mx) = \hat{\mathrm{v}}(M) + \hat{\mathrm{v}}(x)$.

(b) M *induces a map on* $\mathrm{RV}^{(n)}$ *given by* $M(\mathrm{r\hat{v}}(x)) := \mathrm{r\hat{v}}(Mx)$ *for all* $x \in K^n$.

(c) *If* $X, Y \subseteq K^n$ *and* $\varphi : X \longrightarrow Y$ *is a risometry, then* $\varphi^M := M \circ \varphi \circ M^{-1}$ *is a risometry.*

Proof. (a) If $x \in K^n$, the equation $\hat{\mathrm{v}}(Mx) = \hat{\mathrm{v}}(M) + \hat{\mathrm{v}}(x)$ follows from the inequalities: $\hat{\mathrm{v}}(Mx) \geq \hat{\mathrm{v}}(M) + \hat{\mathrm{v}}(x) = \hat{\mathrm{v}}(x) = \hat{\mathrm{v}}(M^{-1} \circ Mx) \geq \hat{\mathrm{v}}(M^{-1}) + \hat{\mathrm{v}}(Mx) = \hat{\mathrm{v}}(Mx)$.
(b) We need to show that $\mathrm{r\hat{v}}(x) = \mathrm{r\hat{v}}(x')$ implies $\mathrm{r\hat{v}}(Mx) = \mathrm{r\hat{v}}(Mx')$, for all $x, x' \in K^n$. Certainly, $\hat{\mathrm{v}}(Mx - Mx') = \hat{\mathrm{v}}(M) + \hat{\mathrm{v}}(x - x') > \hat{\mathrm{v}}(M) + \hat{\mathrm{v}}(x) = \hat{\mathrm{v}}(Mx)$ hold for any $x, x' \in K^n$.
(c) Since both M and M^{-1} are in $\mathrm{GL}_n(O_K)$, this follows easily from (b): for $x, x' \in X$ we have that

$$\begin{aligned} \mathrm{r\hat{v}}(\varphi^M(x) - \varphi^M(x')) &= \mathrm{r\hat{v}}(M \circ \varphi \circ M^{-1}(x) - M \circ \varphi \circ M^{-1}(x')) \\ &= \mathrm{r\hat{v}}(\varphi \circ M^{-1}(x) - \varphi \circ M^{-1}(x')) \\ &= \mathrm{r\hat{v}}(M^{-1}(x) - M^{-1}(x')) = \mathrm{r\hat{v}}(x - x'). \end{aligned}$$ ❒

If $\varphi : X \longrightarrow Y$ is a risometry and $A \subseteq X$ and $B \subseteq Y$, we say that φ *takes* A *to* B (or *sends* A *to* B) if for any $x \in X$, we have that $x \in A$ if and only if $\varphi(x) \in B$. Furthermore, if χ is any map whose domain is contained in $X \cup Y$, we say that φ *respects* χ if $\chi|_X = \chi|_Y \circ \varphi$. Also, when $X = Y$, we say that φ respects a set $A \subseteq K^n$ if it respects the characteristic function $\chi_{A \cap X} : X \longrightarrow \{0, 1\}$.

The following are two technical lemmas used frequently later.

Lemma 1.1.6 (Part of [26, Lemma 2.15])**.** *Let* F *be a finite non-empty subset of* K^n*. Then each fibre of the map* $x \mapsto \mathrm{r\hat{v}}(x - F) := \{\mathrm{r\hat{v}}(x - a) \mid a \in F\}$ *is either a singleton* $\{a\}$*, with* $a \in F$*, or a maximal ball disjoint from* F*.*

Proof. We first claim that for all distinct $x, x' \in K^n$, $\mathrm{r\hat{v}}(x - F) = \mathrm{r\hat{v}}(x' - F)$ if and only if $B(x, \geq \hat{\mathrm{v}}(x - x')) \cap F = \varnothing$. Fix distinct $x, x' \in K^n$ and set $B := B(x, \geq \hat{\mathrm{v}}(x - x'))$.

Assume that $\hat{\mathrm{rv}}(x - F) = \hat{\mathrm{rv}}(x' - F)$ but $B \cap F \neq \varnothing$. As a first case, we will suppose that $\hat{\mathrm{v}}(x - x') = 0$. Without loss, we may also assume that $x, x' \in O_K^n$ (if this fails, apply the argument below to 0 and $(x' - x)$, replacing F with $(-x + F) := \{-x + a \mid a \in F\}$). Observe that for any $a \in B \cap F$ there exists $a' \in B \cap F$, such that $\hat{\mathrm{rv}}(x - a) = \hat{\mathrm{rv}}(x' - a')$; indeed, from the assumptions, there is $a' \in F$ such that $\hat{\mathrm{rv}}(x - a) = \hat{\mathrm{rv}}(x' - a')$, so $\hat{\mathrm{v}}(x' - a') = \hat{\mathrm{v}}(x - a) \geq \mathrm{rad}(B)$; thus, $a' \in B$. Therefore $\hat{\mathrm{rv}}(x - (B \cap F)) = \hat{\mathrm{rv}}(x' - (B \cap F))$, and $\mathrm{res}(x - (B \cap F)) = \mathrm{res}(x' - (B \cap F))$. Taking the sum on both sides of the last equation we get that $\sum_{a \in B \cap F} \mathrm{res}(x - a) = \sum_{a \in B \cap F} \mathrm{res}(x' - a)$, and, using that res is a ring morphism, we have that

$$\sum_{a \in B \cap F} \mathrm{res}(x) - \mathrm{res}(a) = \sum_{a \in B \cap F} \mathrm{res}(x') - \mathrm{res}(a).$$

Simplifying further,

$$|B \cap F| \,\mathrm{res}(x) - \sum_{a \in B \cap F} \mathrm{res}(a) = |B \cap F| \,\mathrm{res}(x') - \sum_{a \in B \cap F} \mathrm{res}(a).$$

We conclude that $\mathrm{res}(x) = \mathrm{res}(x')$, contradicting that $\hat{\mathrm{v}}(x - x') = 0$.

In the general case, when $\hat{\mathrm{v}}(x - x') \neq 0$, take $r \in K$ such that $\hat{\mathrm{v}}(rx - rx') = 0$. Clearly, $\hat{\mathrm{rv}}(x - F) = \hat{\mathrm{rv}}(x' - F)$ implies that $\hat{\mathrm{rv}}(rx - rF) = \hat{\mathrm{rv}}(rx' - rF)$. By applying the previous argument to rx and rx' (with rF as F), we find that $B(rx, \geq 0) \cap rF = \varnothing$. It easily follows that $B(x, \geq \hat{\mathrm{v}}(x - x')) \cap F = \varnothing$. For the remaining direction of the claim, assume that $\hat{\mathrm{rv}}(x - F) \neq \hat{\mathrm{rv}}(x' - F)$. Then there is $a \in F$ such that $\hat{\mathrm{rv}}(x - a) \neq \hat{\mathrm{rv}}(x' - a)$, so $a \in B \cap F \neq \varnothing$.

We now address the lemma. Let A be a fibre of the map $x \mapsto \hat{\mathrm{rv}}(x - F)$. Suppose that $A \cap F \neq \varnothing$ and take a in that intersection. If $x \in A$ and $x \neq a$, the claim above implies that $B(a, \geq \hat{\mathrm{v}}(x - a)) \cap F = \varnothing$, but a is clearly in this intersection, a contradiction. Thus $A = \{a\}$. Additionally, an argument similar to the one below shows that if A is a singleton, it must equal $\{a\}$ for some $a \in F$.

Now we assume that $A \cap F = \varnothing$, and take $x \in A$. If $\lambda := \max\{\hat{\mathrm{v}}(x-a) \mid a \in F\}$, it is clear that $B(x, > \lambda)$ is a maximal ball disjoint from F. We claim that $A = B(x, > \lambda)$. Let $x' \in A$. By the claim in the first part of the proof, $B(x, \geq \hat{\mathrm{v}}(x-x')) \cap F = \varnothing$, so $x' \in B(x, \geq \hat{\mathrm{v}}(x-x')) \subseteq B(x, > \lambda)$. We have proved that $A \subseteq B(x, > \lambda)$. Now, if $x' \in B(x, > \lambda)$ but $x' \notin A$, by the claim in the first part of the proof there is some $b \in B(x, \geq \hat{\mathrm{v}}(x-x')) \cap F$. It follows that $\hat{\mathrm{v}}(x-b) \geq \hat{\mathrm{v}}(x-x') > \lambda$, and so $b \in B(x, > \lambda) \cap F$, a contradiction. We conclude that $A = B(x, > \lambda)$. ❒

Lemma 1.1.7. *Let F be a finite non-empty subset of K^n. Then any maximal ball disjoint from F is of the form $a + \hat{\mathrm{rv}}^{-1}(\xi)$ for some $a \in F$ and some $\xi \in \mathrm{RV}^{(n)}$.*

Proof. Let B be a maximal ball disjoint from F and take $a \in F$. Since $a \notin B$, we have that for any $x, x' \in B$, $\hat{\mathrm{v}}(x-x') > \hat{\mathrm{v}}(x-a)$, so $\hat{\mathrm{rv}}(x-a) = \hat{\mathrm{rv}}(x'-a)$. It follows that $B \subseteq a + \hat{\mathrm{rv}}^{-1}(\hat{\mathrm{rv}}(x-a))$ for any $x \in B$ and any $a \in F$. Fix $x_0 \in B$, let $a \in F$ be such that $\hat{\mathrm{v}}(x_0-a) = \max\{\hat{\mathrm{v}}(x_0-y) \mid y \in F\}$, and set $\xi := \hat{\mathrm{rv}}(x-a)$. Notice that if $b \in F \cap (a + \hat{\mathrm{rv}}^{-1}(\xi))$, then $\hat{\mathrm{rv}}(b-a) = \xi$, and so $\hat{\mathrm{v}}(x_0-b) > \hat{\mathrm{v}}(x_0-a)$, contradicting the choice of a. We conclude that the ball $a + \hat{\mathrm{rv}}^{-1}(\xi)$ is disjoint from F; the maximality of B then implies that $B = a + \hat{\mathrm{rv}}^{-1}(\xi)$. ❒

By combining the two previous lemmas we deduce that, for $F \subseteq K^n$ finite and non-empty, each fibre of the map $x \mapsto \hat{\mathrm{rv}}(x-F)$ is either a singleton $\{a\}$ with $a \in F$, or a ball of the form $a + \hat{\mathrm{rv}}^{-1}(\xi)$ for some $a \in F$ and some $\xi \in \mathrm{RV}^{(n)}$.

1.1.1 The base language

Several languages are suitable for the model-theoretic study of valued fields. Among these are the language $\{+, -, \cdot, 0, 1, \mathcal{O}\}$, where $\mathcal{O}$ is a predicate for the valuation ring of K, and the three-sorted (self-explanatory) language $(K, \Gamma, k, \mathrm{v}, \mathrm{res})$. In our approach we allow many more sorts, the first of which is K as a structure in a language L containing the language of rings $L_{\mathrm{r}} := \{+, -, \cdot, 0, 1\}$.

We wish to introduce the structure of RV to the language. Note that RV can be made into a structure in the language $L_{\mathrm{g}} := \{*, ^{-1}, 1, 0\}$, by interpreting $*$ and $^{-1}$ as the natural operations associated to the group $\mathrm{RV}^{\times}$, extended by $0 * \xi = 0$, for all $\xi \in \mathrm{RV}$, and $0^{-1} = 0$; 1 and 0 are constant symbols for $\mathrm{rv}(1) \in \mathrm{RV}$ and $0 \in \mathrm{RV}$, respectively. We then consider the two-sorted, self-explanatory structure (K, RV), where the only connecting map between the sorts is $\mathrm{rv} : K \longrightarrow \mathrm{RV}$. The definitive language is obtained by considering $\mathrm{RV}^{\mathrm{eq}}$, the collection of all imaginaries defined from RV in (K, RV); that is, $\mathrm{RV}^{\mathrm{eq}}$ is the set

$$\{\mathrm{RV}^n/{\sim} \mid n \in \mathbb{Z}^+ \text{ and } \sim \text{ is an equivalence relation on } \mathrm{RV}^n \text{ 0-definable in } (K, \mathrm{RV})\}.$$

To (K, RV) we add the elements of this set as new sorts, along with all their canonical maps $\mathrm{RV}^n \longrightarrow \mathrm{RV}^n/{\sim}$. The resulting language is denoted by $\mathcal{L}_{\mathrm{Hen}}$. The sorts in $\mathrm{RV}^{\mathrm{eq}}$ are called *auxiliary*. Notationally, we think of $\mathrm{RV}^{\mathrm{eq}}$ as the union of all auxiliary sorts, and whenever we refer to a function $f : X \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ we mean that f is a function from X to some auxiliary sort S in $\mathrm{RV}^{\mathrm{eq}}$. We say that Q is an *auxiliary set* if it is a subset of a finite product of auxiliary sorts.

Examples 1.1.8. (a) Each set $\mathrm{RV}^{(n)}$ is an auxiliary sort, for the following is a factorisation of $\hat{\mathrm{rv}}$ through RV^n,

$$\begin{tikzcd} K^n \arrow[rr, "\hat{\mathrm{rv}}"] \arrow[dr, "j"'] & & \mathrm{RV}^{(n)} \\ & \mathrm{RV}^n \arrow[ur, "p"'] & \end{tikzcd}$$

where $j(x_1, \ldots, x_n) := (\mathrm{rv}(x_1), \ldots, \mathrm{rv}(x_n))$ and $p(\xi_1, \ldots, \xi_n) := \hat{\mathrm{rv}}(y_1, \ldots, y_n)$, for some (eq. any) $y_i \in \mathrm{rv}^{-1}(\xi_i)$, with $1 \leq i \leq n$.

(b) The value group Γ is an auxiliary sort. The factorisation of v through RV presented in page 2 proves this.

(c) The residue field k is an auxiliary sort. Note that maximal ideal M_K of O_K is defined by the $\mathcal{L}_{\text{Hen}}$-formula $\text{rv}(x+1) = \text{rv}(1)$. On RV we define $\hat{\text{rv}}(x) E \hat{\text{rv}}(y)$ if and only if either (1) ($x \in M_K$ or $x^{-1} \in M_K$) and ($y \in M_K$ or $y^{-1} \in M_K$) or (2) $(x, y, x^{-1}, y^{-1} \notin M_K)$ and $(x - y \in M_K)$. Then E is a 0-definable equivalence relation on RV. If we denote the E-class of $\text{rv}(x)$ by $\text{rv}(x)_E$ then the quotient RV$/E$ can be identified with k via the map $\text{rv}(x)_E \mapsto \text{res}(x)$ if $x \in O_K$, and $\text{rv}(x)_E \mapsto \text{res}(x) = 0$ otherwise.

Let S and Q be finite products of sorts and let $X \subseteq S$ be defined by the $\mathcal{L}_{\text{Hen}}$-formula $\phi(x, q)$, for some $q \in Q$. We define the *code* $\ulcorner X \urcorner$ of X as the class of q under the 0-definable equivalence relation given by $q' \sim q''$ if and only if $\phi(x, q')$ and $\phi(x, q'')$ define the same subset of S. In the literature, $\ulcorner X \urcorner$ is also called the *canonical parameter* of X. Note that even when S is a product of only auxiliary sorts, $\ulcorner X \urcorner$ could be an imaginary not in RV$^{\text{eq}}$ (thus not considered in $\mathcal{L}_{\text{Hen}}$). Nevertheless, one of the model-theoretic conditions we impose later (*stable embeddedness* of RV) will helps us to ensure that all the codes we work with are already in some auxiliary sort or K itself.

For a definable family of sets $(X_q)_{q \in Q}$ with Q a (subset of a) finite product of sorts, we may furthermore define codes uniformly through a definable function, i.e. we can find a product of sorts Q' and a definable map $f : Q \longrightarrow Q'$ and then set $\ulcorner X_q \urcorner := f(q)$ for each $q \in Q$. Notice that Q' could require imaginary sorts from K^{eq}, but we usually make sure that Q' can be taken as a product of the sorts in $\mathcal{L}_{\text{Hen}}$. In fact, Q is typically taken as an auxiliary set, so the above occurs automatically after assuming *stable embeddedness* of RV.

1.1.2 Model-theoretic assumptions and dimension

We let $\mathcal{T}_{\text{Hen}}$ be the $\mathcal{L}_{\text{Hen}}$-theory of K. In models of $\mathcal{T}_{\text{Hen}}$ we can already introduce the notion of stratifications that concerns us but in search of generality we instead work on models of an expansion $\mathcal{T}$ of $\mathcal{T}_{\text{Hen}}$ in a language $\mathcal{L} \supseteq \mathcal{L}_{\text{Hen}}$ with the same sorts as $\mathcal{L}_{\text{Hen}}$. In this section we discuss the model-theoretic conditions we impose on $\mathcal{T}$ to guarantee a tame behaviour in its models, ensuring, for example, a good dimension theory. In the rest of the chapter *definable* will always mean $\mathcal{L}$-definable with parameters. Models of $\mathcal{T}$ will be denoted as $(K, \mathrm{RV}^{\mathrm{eq}})$.

Hypotheses 1.1.9. ([26, Section 2.5]) Let $(K, \mathrm{RV}^{\mathrm{eq}}) \models \mathcal{T}$ and $A \subseteq K \cup \mathrm{RV}^{\mathrm{eq}}$.

(1) The sort RV is *stably embedded* in K, i.e. every definable subset of RV^n is definable using only parameters from RV.

(2) Every definable function from RV into K has finite image.

(3) For every A-definable set $X \subseteq K$ there exists a finite A-definable set $S_0 \subseteq K$ such that for every ball $B \subseteq K \setminus S_0$, either $B \cap X = \varnothing$ or $B \subseteq X$.

(4) For every A-definable $X \subseteq K$ and A-definable function $f : X \longrightarrow K$ there exists an A-definable map $\chi : X \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ such that for each $q \in \chi(X)$, $f|_{\chi^{-1}(q)}$ is either constant or injective.

These conditions imply *b-minimality* for $\mathcal{T}$ as introduced in [8].

Proposition 1.1.10. *Hypotheses 1.1.9 imply that $\mathcal{T}$ is b-minimal over* $\mathrm{RV}^{\mathrm{eq}}$*; that is, the following hold for any model $(K, \mathrm{RV}^{\mathrm{eq}})$ of $\mathcal{T}$, any set of parameters $A \subseteq K \cup \mathrm{RV}^{\mathrm{eq}}$, every A-definable set $X \subseteq K$ and every A-definable function $f : X \longrightarrow K$.*

(b_1) *There exists an A-definable function $\rho : X \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ such that for each $q \in \rho(X)$, $\rho^{-1}(q)$ is either an open ball or a point;*

(b_2) *there is no definable surjection from an auxiliary set to an open ball in K;*

(b_3) *there exists an A-definable function $\chi : X \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ such that for each $q \in \chi(X)$, $f|_{\chi^{-1}(q)}$ is either constant or injective.*

Proof. [26, Lemma 2.26]. Alternatively, see the proofs of Theorems 3.1.1 and 3.1.2. ❐

Several important consequences of b-minimality are proved in [8], including a *cell-decomposition theorem*. The most important consequence for us is the existence of a good dimension theory in models of $\mathcal{T}$. From now on we assume that $\mathcal{T}$ satisfies Hypotheses 1.1.9 and we let $(K, \mathrm{RV}^{\mathrm{eq}})$ be a model of $\mathcal{T}$.

Definition 1.1.11. If $X \subseteq K^n$ is non-empty and definable, the *dimension* $\dim(X)$ of X is the maximal $d \leq n$ for which there exists a coordinate projection $\pi : K^n \longrightarrow K^d$ such that $\pi(X)$ contains an open ball. Conventionally, we set $\dim(\varnothing) = -\infty$.

So defined, $\dim(\_)$ is easily seen to satisfy many of the basic properties of a dimension function. For instance, if $X \subseteq Y \subseteq K^n$ are definable, then $\dim(X) \leq \dim(Y)$, and $\dim(\pi(X)) \leq \dim(X)$ whenever π is a coordinate projection. Following [8, Section 4] and [26, Subsection 2.6], the following less obvious properties of dimension hold.

Proposition 1.1.12. *Dimension has the following properties.*

(a) *If $X \subseteq K^n$ is non-empty and definable, X is 0-dimensional if and only if it is finite.*

(b) *If Q is a subset of a product of sorts and $(X_q)_{q\in Q}$ is a definable family of subsets of K^n, the set $Q_d := \{q \in Q \mid \dim(X_q) = d\}$ is definable for each $d \geq 0$.*

(c) *For any definable sets $X \subseteq K^n$ and $Y \subseteq K^m$,* $\dim(X \times Y) = \dim(X) + \dim(Y)$.

(d) *If Q is an auxiliary set and $(X_q)_{q\in Q}$ is a definable family of subsets of K^n, then* $\dim(\bigcup_{q\in Q} X_q) = \max\{\dim(X_q) \mid q \in Q\}$.

(e) *If $X \subseteq K^n$ and $Y \subseteq K^m$ are definable and $f : X \longrightarrow Y$ is a definable function for which every fibre $f^{-1}(y)$ has dimension d, then* $\dim(X) = \dim(Y) + d$. *In particular, dimension is preserved under definable bijections.*

(f) *The* local dimension *of X at a point $x \in X$, denoted by* $\dim_x(X)$, *is defined as* $\min\{\dim(X \cap B(x, > \gamma)) \mid \gamma \in \Gamma\}$. *If $X \subseteq K^n$ is definable of dimension d and Y is the set of points of X at which X has local dimension $< d$, then Y is definable and* $\dim(Y) < d$.

(g) *If $X \subseteq K^n$ is definable, then* $\dim(\mathrm{cl}(X) \setminus X) < \dim(X)$.

(h) *Under the assumptions of* (g), $\dim(\mathrm{cl}(X)) = \dim(X)$.

Proof. (a) Suppose first that $X \subseteq K$, and let S_0 be as in Hypotheses 1.1.9 (3). We show that if X is infinite, then $\dim(X) \neq 0$. Assume that X is infinite and take $y \in X \setminus S_0$. If $S_0 \neq \varnothing$, set $\gamma := \max\{\mathrm{v}(y - s) \mid s \in S_0\}$, otherwise, set $\gamma := \mathrm{v}(y)$. It follows that $B(y, > \gamma) \cap S_0 = \varnothing$, and hence $B(y, > \gamma) \subseteq X$ by the property of S_0. Thus, $\dim(X) = 1$. For a 0-dimensional $X \subseteq K^n$ (with $n > 1$) it is enough to apply the previous case to the image of X under coordinate projections $K^n \longrightarrow K$.
For (b), notice that $\dim(X_q) = d$ is expressible by a first order formula with variable q and parameter d. Item (c) is clear, while (d) and (e) are proved via cell-decomposition in b-minimal theories, [8, Proposition 4.3 (4)-(5)]. For (f), the conditions to apply [22, Theorem 3.1] hold by previous items. Item (g) follows from [13, 2.3 Proposition], since K is a Henselian valued field of characteristic 0. To prove (h) suppose for the sake of a contradiction that $\dim(X) < \dim(\mathrm{cl}(X))$. Then $\dim(\mathrm{cl}(X)) = \dim(\mathrm{cl}(X) \setminus X)$, and using (g) we get that $\dim(X) < \dim(\mathrm{cl}(X)) = \dim(\mathrm{cl}(X) \setminus X) < \dim(X)$, a contradiction. Without the need of (g), item (h) follows from [22, Proposition 2.1]. ❐

1.2 Translatability

In this section we introduce *translatability*, and discuss equivalent, frequently more useful definitions of it. This notion is meant to capture a *geometric regularity* condition of sets. If a set $X \subseteq K^n$ is translatable on a ball B, it means that $X \cap B$ is 'almost translation invariant' in $\dim(X \cap B)$-many directions. The almost part comes from allowing the application of risometries, after which one does get translation invariance.

We assume that $\mathcal{T}$ satisfies Hypotheses 1.1.9 and that $(K, \mathrm{RV}^{\mathrm{eq}}) \models \mathcal{T}$.

The following convention will allow us to apply the upcoming definitions to sets and tuples of maps and sets.

Convention 1.2.1. Let $B_0 \subseteq K^n$ and let P be a property of maps $B_0 \longrightarrow \mathrm{RV}^{\mathrm{eq}}$. If $X \subseteq B_0$, we say that X *has property* P if and only if the characteristic function of X,

$\chi_X : B_0 \longrightarrow \{0, 1\}$ has the property P. Furthermore, if $\rho_i : B_0 \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ $(i \leq r)$ is a collection of functions and $X_i \subseteq B_0$ $(i \leq s)$ is a collection of sets, we say that the tuple $(\rho_1, \ldots, \rho_r, X_1, \ldots, X_s)$ *has property* P if and only if the map $\chi : B_0 \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ given by $\chi(x) := (\rho_1(x), \ldots, \rho_r(x), \chi_{X_1}(x), \ldots, \chi_{X_s}(x))$ has the property P.

From now on 'subspace' will stand for 'vector subspace'. In the following definition, *a lift* of the subspace V of k^n is any subspace $\overline{V} \subseteq K^n$ such that $V = \{\mathrm{res}(x) \mid x \in \overline{V} \cap O_K^n\}$.

Definition 1.2.2. Let $B_0 \subseteq K^n$ be a definable set. Suppose that $B \subseteq B_0$ is a ball (open or closed) and that $\chi : B_0 \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ is definable.

(i) If $\overline{V}$ is a subspace of K^n, we say that χ is $\overline{V}$*-translation invariant* on B if for all $x, x' \in B$, $x - x' \in \overline{V}$ implies $\chi(x) = \chi'(x)$.

(ii) If V is a subspace of k^n, we say that χ is V*-translatable* on B if there exist a lift $\overline{V} \subseteq K^n$ of V and a definable risometry $\varphi : B \longrightarrow B$ such that $\chi \circ \varphi$ is $\overline{V}$-translation invariant on B.

(iii) For an integer d with $0 \leq d \leq n$, we say that χ is d*-translatable* on B if there is a d-dimensional subspace V of k^n such that χ is V-translatable on B.

By Lemma 1.2.12, X cannot be more than $\dim(X \cap B)$-translatable on a ball B, so we say that X is *translatable* on B if it is $\dim(X \cap B)$-translatable on B. The risometry φ in (ii) is called a *straightener* of χ on B. The choice of the lift $\overline{V}$ of V in (ii) is not fundamentally important. Once there is a straightener associated with the lift $\overline{V}$, we can always find another straightener for any other given lift of V. We prove this below with help from the next claim.

Claim 1.2.3. Let $\overline{V}$ and $\overline{V'}$ be lifts of $V \subseteq k^n$. Then there exists $M \in U_n$ (see Examples 1.1.4) such that $M(\overline{V}) = \overline{V'}$.

Proof. (Part of the proof of [26, Lemma 2.8]) Set $d := \dim(V)$ and let $\{v_i\}_{i \leq d}$ be a basis of V. Extend $\{v_i\}_{i \leq d}$ to a full basis $\{v_i\}_{i \leq n}$ of k^n. For each $i \leq n$, pick elements

$x_i \in \overline{V} \cap O_K^n$ and $x_i' \in \overline{V'} \cap O_K^n$ such that $\mathrm{res}(x_i) = v_i = \mathrm{res}(x_i')$. We set $M \in \mathrm{GL}_n(O_K)$ to be the change-of-basis matrix taking x_i to x_i' for each $i \leq n$; so, $M(\overline{V}) = \overline{V'}$. To verify that $M \in U_n$, notice that for every $i \leq n$, $\mathrm{res}(M)(v_i) = \mathrm{res}(M(x_i)) = \mathrm{res}(x_i') = v_i$, so $\mathrm{res}(M) = Id_n$ on k^n. ❒

With the previous notation, suppose that $\varphi : B \longrightarrow B$ is a definable risometry for which $\chi \circ \varphi$ is $\overline{V}$-translation invariant. Let $\overline{V'}$ be another lift of V and $M \in U_n$ be the matrix in Claim 1.2.3. By Examples 1.1.4, M is a risometry. Then $\varphi \circ M^{-1}$ is a definable risometry and $\chi \circ (\varphi \circ M^{-1})$ is $\overline{V'}$-translation invariant.

Remark 1.2.4. The following statements hold.

(a) The set $X \subseteq B_0$ is $\overline{V}$-translation invariant on B if and only if there is $A \subseteq K^n$ such that

$$X \cap B = \bigcup_{a \in A} (a + \overline{V}) \cap B.$$

(b) If $\chi : B_0 \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ is d-translatable on a ball $B \subseteq B_0$, then χ is d-translatable on any subball $B' \subseteq B$.

Proof. (a) First suppose that $X \cap B = \bigcup_{a \in A}(a + \overline{V}) \cap B$ for some $A \subseteq K^n$, and let $x, x' \in B$ be such that $x' = x + v$ for some $v \in \overline{V}$. If, say, $x \in X \cap B$, then $x = a + u$ for some $a \in A$ and $u \in \overline{V}$. Hence $x' = (a + u) + v \in (a + \overline{V}) \cap B \subseteq X \cap B$. So, $\chi_X(x) = \chi_X(x')$. Second, assume that $X \cap B$ is not a union of cosets of V. Then there is a coset $a + \overline{V}$ such that $(a + \overline{V}) \cap X \cap B \neq \varnothing$ but nevertheless $(a + \overline{V}) \cap B \nsubseteq X \cap B$. Take x in the former intersection and $x' \in ((a + \overline{V}) \cap B) \setminus (X \cap B)$. Then clearly, $x - x' \in \overline{V}$ but $\chi_X(x) \neq \chi_X(x')$.
(b) This is clear. ❒

Item (a) above helps to visualise translatability. If $X \subseteq K^2$ is 1-translatable on a ball B, then there are a line $\overline{V} \subseteq K^2$ through 0 and a definable risometry $\varphi : B \longrightarrow B$ such that $\varphi(X)$ is a bunch of lines in K^2 with direction (i.e. parallel to) $\overline{V}$. This is a strong geometric requirement on X; it forbids spiralling and other non-tame behaviour.

Examples 1.2.5. The following are examples on translatability.

(1) 0-translatability is a trivial condition, in that any definable function $\chi : B_0 \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ is 0-translatable on any ball $B \subseteq B_0$.

(2) If $X \subseteq B_0 \subseteq K^n$ and $B \subseteq B_0$ is a ball, then X is n-translatable on B if and only if either $X \cap B = \varnothing$ or $B \subseteq X$. From right to left the implication is clear. For the other direction assume that X is n-translatable on B but $X \cap B \neq \varnothing \neq B \setminus X$. Note that the only possible choices for V and $\overline{V}$ are k^n and K^n respectively. Let $\varphi : B \longrightarrow B$ be a straightener of X. The bijectivity of φ implies that there are $x, x' \in B$ such that $\varphi(x) \in X \cap B$ and $\varphi(x') \in B \setminus X$, but this is a contradiction to $\overline{V}$-translation invariance of $X \cap B$ since obviously $x - x' \in \overline{V}$.

(3) Let $T \subseteq K^n$ be finite and non-empty, and define $\chi : K^n \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ as the map $x \mapsto \ulcorner \hat{\mathrm{rv}}(x-T) \urcorner$, where $\hat{\mathrm{rv}}(x-T) := \{\hat{\mathrm{rv}}(x-t) \mid t \in T\}$. By Lemma 1.1.6 each fiber of χ is either a point $t \in T$, or a maximal ball disjoint from T. We claim that for $0 < d \leq n$ and any ball $B \subseteq K^n$, χ is d-translatable on B if and only if B is disjoint from T. The claim from right to left is trivial (for each d, we can take $V := k^d$, $\overline{V} := K^d$ and the risometry $\varphi := Id_B$). In the other direction, fix $0 < d \leq n$ and suppose that $B \cap T \neq \varnothing$. For the sake of a contradiction, assume that there is a d-dimensional subspace $V \subseteq k^d$, a lift $\overline{V}$ of V, and a risometry $\varphi : B \longrightarrow B$ such that $\chi \circ \varphi$ is $\overline{V}$-translation invariant on B. Fix $t \in B \cap T$. For all $x \in \varphi^{-1}(t) + \overline{V}$, we have that $\chi \circ \varphi(x) = \chi \circ \varphi(\varphi^{-1}(t)) = \chi(t)$, so for all such x, $0 \in \hat{\mathrm{rv}}(t - T) = \hat{\mathrm{rv}}(\varphi(x) - T)$ and $\varphi(x) \in T$. We have thus reached the absurd conclusion that the infinite set $\varphi(\varphi^{-1}(t) + \overline{V})$ is contained in T.

(4) Consider the set $X := \{(x, y) \in K^2 \mid xy = 0\}$. For a ball $B \subseteq K^2$, we claim that X is 1-translatable on B if and only if $0 \notin B$. First assume that B is a ball not containing 0. If $X \cap B = \varnothing$, there is nothing to check. We thus suppose otherwise. Necessarily, B intersects only one of the axes making up X, for if $(x, 0)$ and $(0, y)$ were elements of $X \cap B$ with $x \neq 0$ and, say, $B = B((x, 0), \geq \gamma)$, then $\gamma \leq \hat{\mathrm{v}}(x, -y) = \min\{\mathrm{v}(x), \mathrm{v}(y)\} \leq \mathrm{v}(x)$, so $\hat{\mathrm{v}}((x, 0) - (0, 0)) = \mathrm{v}(x) \geq \gamma$, contradict-

ing that $0 \notin B$. The 1-translatability of X on B is obvious now: if B intersects the axis $K \times \{0\}$, we take $V := k \times \{0\}$, $\overline{V} := K \times \{0\}$ and the identity map on B as straightener. The choices when B intersects $\{0\} \times K$ are clear too.

Now we suppose that B is a ball containing 0. For the sake of a contradiction, suppose that there are a 1-dimensional subspace V of k^2, a lift $\overline{V}$ of V and a definable risometry $\varphi : B \longrightarrow B$ such that $\varphi(X)$ is $\overline{V}$-translation invariant on B. Intuitively, on the one hand $\varphi(X \cap B)$ is a set of lines parallel to $\overline{V}$, and on the other, the rigidity of a risometry tells us that $\varphi(X \cap B)$ should look roughly like $X \cap B$, which clearly does not look like a set of lines parallel to $\overline{V}$ (no matter what $\overline{V}$ actually is); a contradiction. We formalise this argument below.

To take advantage of Lemma 1.1.3, for now we assume that $B = O_K^2$. On one hand, Lemma 1.1.3 implies that $\operatorname{res}(\varphi(X \cap B)) = \operatorname{res}(X \cap B) + a$, where $a := \operatorname{res}(\varphi(0))$. So $\operatorname{res}(\varphi(X \cap B))$ is a translation of the set $\operatorname{res}(X \cap B) = \{(x, y) \in k^2 \mid xy = 0\}$, the 'axes cross' in k^2. On the other hand, by $\overline{V}$-translation invariance, Remark 1.2.4 (a) implies that there exists $A \subseteq k^2$ such that $\operatorname{res}(\varphi(X \cap B)) = \bigcup_{a \in A}(a + V)$; so $\operatorname{res}(\varphi(X \cap B))$ is a union of lines parallel to V in k^2. This is a contradiction, for the cross $(k \times \{0\}) \cup (\{0\} \times k)$ does not equal a set of lines parallel to a fixed line in k^2. Lastly, a similar argument applies for any other closed ball $B = B(0, \geq \gamma)$, by scaling up or down to O_K^2, i.e. by taking an element $r \in \mathrm{v}^{-1}(\gamma)$ and then applying the bijection $x \mapsto r^{-1}x$ from B to O_K^2. If B is open, we can take a closed subball containing 0 and follow the previous case; this is enough to imply the non-translatability on B by Remark 1.2.4 (b).

Example (4) incidentally shows that d-translatability on a ball B does not imply d-translatability on bigger balls. In that example, if B does not contain 0, we have 1-translatability, but 1-translatability fails on any bigger ball containing both 0 and B.

1.2.1 Equivalences of translatability

In this subsection we discuss two equivalences of translatability; both have the advantage of eliminating the reference to a lift in Definition 1.2.2 (ii) (but have the disadvantage of being more technical).

By the coordinate-wise definition of res on O_K^n, a coordinate projection $\pi : K^n \longrightarrow K^d$ induces a coordinate projection $\tilde{\pi} : k^n \longrightarrow k^d$ satisfying that $\tilde{\pi}(\mathrm{res}(x)) = \mathrm{res}(\pi(x))$ for all $x \in O_K^n$.

Definition 1.2.6. (1) Let V be a d-dimensional subspace of k^n. We say that the coordinate projection $\pi : K^n \longrightarrow K^d$ *exhibits* V, or that π is an *exhibition* of V, if the corresponding projection $\tilde{\pi}$ is an isomorphism between V and k^d.

(2) If $x \in K^n \setminus \{0\}$, the *direction* of x is the 1-dimensional subspace $\mathrm{res}((K \cdot x) \cap O_K^n)$ of k^n, which we denote by $\mathrm{dir}(x)$. In practice $\mathrm{dir}(x)$ will be treated as a representative $y \in \mathrm{res}((K \cdot x) \cap O_K^n)$ of the direction, i.e. $\mathrm{dir}(x)$ will be taken as a generator of $\mathrm{res}((K \cdot x) \cap O_K^n)$.

Remark 1.2.7. If $\pi : K^n \longrightarrow K^d$ exhibits V, then for any lift $\overline{V}$ of V and $x \in K^d$, the set $\pi^{-1}(x) \cap \overline{V}$ must be a singleton. Otherwise, suppose that $z, z' \in \pi^{-1}(x) \cap \overline{V}$ are distinct and let $r \in K$ be such that $\mathrm{v}(r) = -\hat{\mathrm{v}}(z - z')$. Then $0 \neq \mathrm{res}(r(z - z')) \in V$ but $\tilde{\pi}(\mathrm{res}(r(z - z'))) = \mathrm{res}(r\pi(z - z)) = \mathrm{res}(r(x - x)) = 0$, contradicting that $\pi|_V$ is an isomorphism.

We now prove a couple of technical results used frequently throughout this work. With the exception of (b), these are part of [26, Lemma 2.10]. We denote the usual inner product on K^n and k^n by $\langle \cdot, \cdot \rangle$.

Lemma 1.2.8. *Let $\pi : K^n \longrightarrow K^d$ be a coordinate projection.*

(a) *If $x, x' \in K^n$ satisfy that $\hat{\mathrm{v}}(x + x') = \min\{\hat{\mathrm{v}}(x), \hat{\mathrm{v}}(x')\}$, then $\mathrm{r\hat{v}}(x + x')$ is completely determined by $\mathrm{r\hat{v}}(x)$ and $\mathrm{r\hat{v}}(x')$, i.e. for any other $y, y' \in K^n$ with $\mathrm{r\hat{v}}(x) = \mathrm{r\hat{v}}(y)$ and $\mathrm{r\hat{v}}(x') = \mathrm{r\hat{v}}(y')$, we have $\mathrm{r\hat{v}}(x + x') = \mathrm{r\hat{v}}(y + y')$.*

(b) *If $x, x' \in K^n$ are such that* $\mathrm{r\hat{v}}(x) = \mathrm{r\hat{v}}(x')$*, then* $\mathrm{dir}(x) = \mathrm{dir}(x')$.

(c) *For $x \in K^n$, $\hat{\mathrm{v}}(\pi(x)) = \hat{\mathrm{v}}(x)$ if and only if $\tilde{\pi}(\mathrm{dir}(x)) \neq 0$. In that case, $\tilde{\pi}(\mathrm{dir}(x)) = \mathrm{dir}(\pi(x))$, and if $x' \in K^n$ is such that $\pi(x') = \pi(x)$ and $\mathrm{dir}(x') = \mathrm{dir}(x)$, then $\mathrm{r\hat{v}}(x') = \mathrm{r\hat{v}}(x)$.*

(d) *For any $x, x' \in K^n$, $\mathrm{v}(\langle x, x'\rangle) \geq \hat{\mathrm{v}}(x) + \hat{\mathrm{v}}(x')$; and, moreover, if x and x' are non-zero, the strict inequality holds if and only if $\langle \mathrm{dir}(x), \mathrm{dir}(x')\rangle = 0$.*

Proof. The proof of these facts is easy but technical.

(a) Notice that

$$\hat{\mathrm{v}}(x + x' - (y + y')) \geq \min\{\hat{\mathrm{v}}(x - y), \hat{\mathrm{v}}(x' - y')\} > \min\{\hat{\mathrm{v}}(x), \hat{\mathrm{v}}(x')\} = \hat{\mathrm{v}}(x + x').$$

(b) If $x = 0$ or $x' = 0$, the result is obvious. From now we assume that $x, x' \in K^n \setminus \{0\}$. If $\mathrm{res}(\lambda x) \in \mathrm{dir}(x)$, with $\lambda \in K^\times$ and $\lambda x \in O_K^n$, then $\hat{\mathrm{v}}(\lambda x') = \mathrm{v}(\lambda) + \hat{\mathrm{v}}(x) \geq 0$, so $\lambda x' \in O_K^n$ too. Since $\hat{\mathrm{v}}(\lambda x - \lambda x') > \hat{\mathrm{v}}(\lambda x) \geq 0$, $\mathrm{res}(\lambda x) = \mathrm{res}(\lambda x') \in \mathrm{dir}(x')$. This proves that $\mathrm{dir}(x) \subseteq \mathrm{dir}(x')$. A symmetrical argument allows us to conclude that $\mathrm{dir}(x) = \mathrm{dir}(x')$.

(c) Clearly, $\hat{\mathrm{v}}(\pi(x)) \geq \hat{\mathrm{v}}(x)$ always holds. Now we assume that $x \neq 0$ and let $r \in K$ be such that $\mathrm{v}(rx) = 0$. If $\hat{\mathrm{v}}(\pi(x)) = \hat{\mathrm{v}}(x)$ then $\hat{\mathrm{v}}(\pi(rx)) = \hat{\mathrm{v}}(rx) = 0$, and it follows that $\tilde{\pi}(\mathrm{dir}(x)) = \tilde{\pi}(\mathrm{res}(rx)) = \mathrm{res}(\pi(rx)) \neq 0$. On the other hand, if $\hat{\mathrm{v}}(\pi(x)) > \hat{\mathrm{v}}(x)$, then $\hat{\mathrm{v}}(\pi(rx)) > \hat{\mathrm{v}}(rx) \geq 0$, so $\tilde{\pi}(\mathrm{dir}(x)) = \tilde{\pi}(\mathrm{res}(rx)) = 0$.

To prove the second assertion we assume that $\tilde{\pi}(\mathrm{dir}(x)) \neq 0$. Then for all $r \in K$ such that $\mathrm{v}(rx) = 0$, we have that $\tilde{\pi}(\mathrm{res}(rx)) = \mathrm{res}(\pi(rx))$, so indeed $\tilde{\pi}(\mathrm{dir}(x)) = \mathrm{dir}(\pi(x))$. For the last part of (c) we let $x' \in K^n$ be such that $\pi(x') = \pi(x)$ and $\mathrm{dir}(x') = \mathrm{dir}(x)$. For simplicity of notation, we assume that π is the projection to the first d coordinates and let $\pi^\perp$ denote the complementary projection to the last $n - d$ coordinates. Observe that $\mathrm{dir}(\pi(x), \pi^\perp(x)) = \mathrm{dir}(\pi(x), \pi^\perp(x'))$, so

$$\mathrm{dir}(\pi(x), \pi^\perp(x) - \pi^\perp(x')) = \mathrm{dir}(x) \in k^d \times \{0\}^{n-d}.$$

We thus have that $\hat{\mathrm{v}}(\pi^{\perp}(x) - \pi^{\perp}(x')) > \hat{\mathrm{v}}(\pi(x)) = \hat{\mathrm{v}}(x)$, and $\mathrm{r\hat{v}}(x') = \mathrm{r\hat{v}}(x)$ follows.

(d) If either $x = 0$ or $x' = 0$, $\mathrm{v}(\langle x, x' \rangle) = \infty = \hat{\mathrm{v}}(x) + \hat{\mathrm{v}}(x')$. So we assume that $x \neq 0 \neq x'$. Suppose that $x = (x_1, \ldots, x_n)$ and $x' = (x'_1, \ldots, x'_n)$, and let x_{i_0} and x'_{j_0} be such that $\hat{\mathrm{v}}(x) = \mathrm{v}(x_{i_0})$ and $\hat{\mathrm{v}}(x') = \mathrm{v}(x'_{j_0})$. We thus have that $\hat{\mathrm{v}}(x_i x'_i / x_{i_0} x'_{j_0}) \geq 0$, for all $i \in \{1, \ldots, n\}$. Hence,

$$\mathrm{v}\left(\frac{\langle x, x' \rangle}{x_{i_0} x'_{j_0}}\right) = \mathrm{v}\left(\sum_{i=1}^{n} \frac{x_i x'_i}{x_{i_0} x'_{j_0}}\right) \geq \min\left\{\mathrm{v}\left(\frac{x_i x'_i}{x_{i_0} x'_{j_0}}\right) \mid i \in \{1, \ldots, n\}\right\} \geq 0.$$

It follows that $\mathrm{v}(\langle x, x' \rangle) \geq \mathrm{v}(x_{i_0} x'_{j_0}) = \hat{\mathrm{v}}(x) + \hat{\mathrm{v}}(x')$, as desired.

Before addressing the second part, notice that in general, if $x, x' \in O_K^n$, then

$$\langle \mathrm{res}(x), \mathrm{res}(x') \rangle = \sum_{i=1}^{n} \mathrm{res}(x_i) \,\mathrm{res}(x'_i) = \mathrm{res}\left(\sum_{i=1}^{n} x_i x'_i\right) = \mathrm{res}(\langle x, x' \rangle)$$

With this observation made, we now prove the second part. Let $x, x' \in K^n \setminus \{0\}$ and let $r, r' \in K$ be such that $\mathrm{v}(r) = -\hat{\mathrm{v}}(x)$ and $\mathrm{v}(r') = -\hat{\mathrm{v}}(x')$. Then $\mathrm{dir}(x) = \mathrm{res}(rx)$ and $\mathrm{dir}(x') = \mathrm{res}(r'x')$. It follows that $\langle \mathrm{dir}(x), \mathrm{dir}(x') \rangle = 0$ if and only if $\mathrm{res}(\langle rx, r'x' \rangle) = 0$ by the observation above. In turn, the latter equality is equivalent to $\mathrm{v}(\langle rx, r'x' \rangle) > 0$, which clearly holds if and only if $\mathrm{v}(\langle x, x' \rangle) > -\mathrm{v}(rr') = \hat{\mathrm{v}}(x) + \hat{\mathrm{v}}(x')$. ❐

The following is yet another property of risometries.

Remark 1.2.9. If $\varphi : X \longrightarrow Y$ is a risometry and $x, x' \in X$, we know by definition that $\mathrm{r\hat{v}}(\varphi(x) - \varphi(x')) = \mathrm{r\hat{v}}(x - x')$, so also $\mathrm{dir}(\varphi(x) - \varphi(x')) = \mathrm{dir}(x - x')$ by (b) above.

In the definition below, we let $B_0 \subseteq K^n$ and $\chi : B_0 \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ be definable and $B \subseteq B_0$ be a ball. Furthermore, V will be a d-dimensional subspace of k^n and $\pi : K^n \longrightarrow K^d$ an exhibition of V. Note that the set $B - B := \{x - y \mid x, y \in B\}$ is the ball of the same radius as B containing 0.

Definition 1.2.10. A definable family of risometries $(\alpha_x : B \longrightarrow B)_{x \in \pi(B-B)}$ is said to be a *translater* of χ on B (with respect to π) if the following properties hold for all

$x, x' \in \pi(B - B)$ and $z \in B$,

(i) $\chi \circ \alpha_x = \chi$;[2]

(ii) $\alpha_x \circ \alpha_{x'} = \alpha_{x+x'}$;

(iii) $\pi(\alpha_x(z) - z) = x$;

(iv) if $x \neq 0$, $\mathrm{dir}(\alpha_x(z) - z) \in V$.

The proposition below shows that the existence of a translater is equivalent to translatability. The proposition also presents the third announced equivalence. This one transforms translatability into the existence of a definable equivalence relation on the fibres of a projection.

Proposition 1.2.11. *Suppose that $B_0 \subseteq K^n$ and $\chi : B_0 \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ are definable and let $B \subseteq B_0$ be a ball. Fix $d \leq n$. The following are equivalent.*

(a) *χ is d-translatable on B;*

(b) *there exist a d-dimensional subspace V of k^d, an exhibition π of V and a translater $(\alpha_x)_{x \in \pi(B-B)}$ of χ on B (with respect to π);*

(c) *there exist a d-dimensional subspace V of k^d, an exhibition π of V and a definable equivalence relation $\sim$ on B with the following properties,*

(c.i) *$\sim$ refines the fibres of χ, i.e. for all $z, z' \in B$, $z \sim z'$ implies $\chi(z) = \chi(z')$;*

(c.ii) *for each equivalence class E of $\sim$, $\pi : E \longrightarrow \pi(B)$ is a bijection;*

(c.iii) *for all distinct $z, z' \in B$, $z \sim z'$ implies $\mathrm{dir}(z - z') \in V$;*

(c.iv) *for all $z, z', w \in B$ with $z \sim z'$, there is $w' \in B$ such that $w' \sim w$ and $\hat{\mathrm{rv}}(z' - w') = \hat{\mathrm{rv}}(z - w)$.*

Proof. (a) implies (b) (Part of the proof of [26, Lemma 3.7]). Let V be a d-dimensional subspace of k^n such that χ is V-translatable on B. Fix a lift $\overline{V}$ and an exhibition π of V. By [26, Lemma 3.6 (1)], there is a straightener $\varphi : B \longrightarrow B$ of χ on B respecting

[2] In other words, each α_x *respects* χ.

π, i.e. $\pi \circ \varphi = \pi$. Heeding Remark 1.2.7, for each $x \in \pi(B - B)$, let ξ_x be the sole element of $\pi^{-1}(x) \cap \overline{V}$, and define $\alpha'_x : B \longrightarrow B$ as the translation $z \mapsto z + \xi_x$. We set $\alpha_x : B \longrightarrow B$ to be the composition $\varphi \circ \alpha'_x \circ \varphi^{-1}$. Clearly α_x is a risometry, for it is a composition of risometries.

Now it is routine to prove that $(\alpha_x)_{x \in \pi(B-B)}$ satisfies the conditions (i)-(iv) in Definition 1.2.10. Let $x, x' \in \pi(B - B)$. Since φ is a straightener of χ, we know that $\chi \circ \varphi(z) = \chi \circ \varphi(z')$ holds whenever $z - z' \in \overline{V}$. Applying this to z and $\alpha'_x(z)$ for each $z \in B$, we get that $\chi \circ \varphi \circ \alpha'_x = \chi \circ \varphi$ (because $\alpha'_x(z) - z = \xi_x \in \overline{V}$ for all $z \in B$). It follows that $\chi \circ \alpha_x = \chi \circ (\varphi \circ \alpha'_x \circ \varphi^{-1}) = (\chi \circ \varphi) \circ \varphi^{-1} = \chi$, proving (i). For (ii), we have that

$$\alpha_x \circ \alpha'_x = \varphi \circ \alpha'_x \circ \alpha'_{x'} \circ \varphi^{-1} = \varphi \circ \alpha'_{x+x'} \circ \varphi^{-1} = \alpha_{x+x'},$$

because $\xi_{x+x'} = \xi_x + \xi_{x'}$.

For (iii), notice that if $z \in B$, then $\pi \circ \alpha'_x(z) = \pi(z + \xi_x) = \pi(z) + x$. Using that $\pi \circ \varphi = \pi = \pi \circ \varphi^{-1}$, it follows that for all $z \in B$,

$$\pi(\alpha_x(z) - z) = \pi(\varphi \circ \alpha'_x \circ \varphi^{-1}(z) - z) = \pi(z) + x - \pi(z) = x.$$

Lastly, we assume that $x \neq 0$ and we aim to prove (iv). Notice that for every $w \in B$, $\operatorname{dir}(\alpha'_x(w) - w) = \operatorname{dir}(\xi_x) \in V$. Using that φ is a risometry and the definition of α_x, we have that $\hat{\mathrm{rv}}(\alpha_x(z) - z) = \hat{\mathrm{rv}}(\alpha'_x \circ \varphi^{-1}(z) - \varphi^{-1}(z))$; so, by Remark 1.2.9 we conclude that $\operatorname{dir}(\alpha_x(z) - z) = \operatorname{dir}(\alpha'_x(\varphi^{-1}(z)) - \varphi^{-1}(z)) \in V$.

(b) implies (c). For $z, z' \in B$ we define $z \sim z'$ if and only if there exists $x \in \pi(B - B)$ such that $\alpha_x(z) = z'$. Clearly, $\sim$ is definable. Since $0 \in \pi(B - B)$, $\sim$ is reflexive. Notice that $\alpha_x^{-1} = \alpha_{-x}$ for all $x \in \pi(B - B)$; this implies the symmetry of $\sim$. The transitivity of $\sim$ clearly follows from (ii). We now show that $\sim$ satisfies (c.i)-(c.iv). If $z, z' \in B$ and $\alpha_x(z) = z'$ for some $x \in \pi(B - B)$, we have that $\chi(z) = \chi \circ \alpha_x(z) = \chi(z')$, proving (c.i). For (c.ii), note that if $\alpha_x(z) = z'$ and $\pi(z) = \pi(z')$, then by (iii), $x = \pi(z' - z) = 0$. So $\alpha_x = \alpha_0$ is the identity on B and $z = z'$. This shows that $\pi : E \longrightarrow \pi(B)$ is injective;

its surjectivity is obvious. Note that (c.iii) follows immediately from (iv). For (c.iv), if $\alpha_x(z) = z'$, we let $w' := \alpha_x(w)$. Then clearly $w' \sim w$, and the required equation holds because α_x is a risometry.

(c) implies (a). To simplify notation, we assume that $0 \in B$, $V = k^d \times \{0\}^{n-d}$ and that π is the projection to the first d coordinates. We fix the lift $\overline{V} := K^d \times \{0\}^{n-d}$ of V. Points in B will be written as pairs (x, y) where $x \in K^d$ and $y \in K^{n-d}$. For $(x, y) \in B$, we set $\varphi(x, y)$ as the unique element of $\pi^{-1}(x) \cap E_{(0,y)}$, where $E_{(0,y)}$ denotes the $\sim$-equivalence class of $(0, y)$. We show that φ is a straightener of χ on B. Clearly, φ is definable and—by (c.ii)—bijective. Now assume that $(x, y), (x', y') \in B$ are such that $(x, y) - (x', y') \in \overline{V}$. Then $y = y'$ and this immediately implies that $\varphi(x, y) \sim \varphi(x', y')$; so, from (c.i) we deduce that $\chi \circ \varphi(x, y) = \chi \circ \varphi(x', y')$. Lastly, we prove that for any $(x, y), (x', y') \in B$, we have that $\hat{\mathrm{rv}}(\varphi(x, y) - \varphi(x', y')) = \hat{\mathrm{rv}}((x, y) - (x', y'))$. Heeding Lemma 1.2.8 (c), it is enough to prove the equations

$$\hat{\mathrm{rv}}(\varphi(x, y) - \varphi(x', y)) = \hat{\mathrm{rv}}((x, y) - (x', y)) \tag{1.1}$$

and

$$\hat{\mathrm{rv}}(\varphi(x', y) - \varphi(x', y')) = \hat{\mathrm{rv}}((x', y) - (x', y')). \tag{1.2}$$

Let $\pi^{\perp} : K^n \longrightarrow K^{n-d}$ be the projection to the last $n - d$ coordinates. By definition, (1.1) is equivalent to $\hat{\mathrm{v}}((0, \pi^{\perp}(\varphi(x, y)) - \pi^{\perp}(\varphi(x', y)))) > \hat{\mathrm{v}}((x - x', 0))$, which reduces to $\hat{\mathrm{v}}(\pi^{\perp}(\varphi(x, y)) - \pi^{\perp}(\varphi(x', y))) > \hat{\mathrm{v}}(x - x')$. We thus prove this inequality. Let $r \in K$ be such that $\mathrm{v}(r) = -\hat{\mathrm{v}}(\pi^{\perp}(\varphi(x, y)) - \pi^{\perp}(\varphi(x', y)))$. If the desired inequality failed, we would have that $\hat{\mathrm{v}}(r(x - x')) \geq 0$ and $\hat{\mathrm{v}}(r(\pi^{\perp}(\varphi(x, y)) - \pi^{\perp}(\varphi(x', y)))) = 0$, so $v := \mathrm{dir}(r(x - x'), r(\pi^{\perp}(\varphi(x, y)) - \pi^{\perp}(\varphi(x', y)))) \notin k^d \times \{0\}^{n-d}$. However, notice that $v = \mathrm{dir}(x - x', \pi^{\perp}(\varphi(x, y)) - \pi^{\perp}(\varphi(x', y)))$ and that by definition of φ, $\varphi(x, y) \sim (0, y) \sim \varphi(x', y)$, so by (c.iii), $v = \mathrm{dir}(x - x', \pi^{\perp}(\varphi(x, y)) - \pi^{\perp}(\varphi(x', y))) = \mathrm{dir}(\varphi(x, y) - \varphi(x', y)) \in k^d \times \{0\}^{n-d}$, a contradiction. We have proved (1.1).

We now prove (1.2). Since $(0, y) \sim \varphi(x', y)$, (c.iv) implies that there is $(p_1, p_2) \in B$ such that $(0, y') \sim (p_1, p_2)$ and $\hat{\mathrm{rv}}(\varphi(x', y) - (p_1, p_2)) = \hat{\mathrm{rv}}((0, y - y')) = \hat{\mathrm{rv}}((x', y) - (x', y'))$. For (1.2) it is hence enough to show that $\hat{\mathrm{rv}}(\varphi(x', y) - \varphi(x', y')) = \hat{\mathrm{rv}}(\varphi(x', y) - (p_1, p_2))$. Notice that $(p_1, p_2) \sim (0, y') \sim \varphi(x', y')$, so $\mathrm{dir}(\varphi(x', y') - (p_1, p_2)) \in k^d \times \{0\}^{n-d}$ by (c.iii). This in particular implies that $\hat{\mathrm{v}}(\pi^\perp(\varphi(x', y')) - p_2) > \hat{\mathrm{v}}(x' - p_1)$. Additionally, since $\hat{\mathrm{rv}}(\varphi(x', y) - (p_1, p_2)) = \hat{\mathrm{rv}}((x' - x', y - y'))$, we have that

$$\hat{\mathrm{v}}(x' - p_1) \geq \hat{\mathrm{v}}(x' - p_1, \pi^\perp(\varphi(x', y)) - p_2 - y + y')\} > \hat{\mathrm{v}}(x' - p_1, \pi^\perp(\varphi(x', y)) - p_2);$$

from which it follows that $\hat{\mathrm{v}}(x' - p_1) > \hat{\mathrm{v}}(\pi^\perp(\varphi(x', y)) - p_2)$. We conclude that

$$\hat{\mathrm{v}}(\pi^\perp(\varphi(x', y)) - p_2) < \hat{\mathrm{v}}(x' - p_1) < \hat{\mathrm{v}}(\pi^\perp(\varphi(x', y')) - p_2). \tag{1.3}$$

Finally, the desired equation $\hat{\mathrm{rv}}(\varphi(x', y) - (p_1, p_2)) = \hat{\mathrm{rv}}(\varphi(x', y) - \varphi(x', y'))$ is equivalent to $\hat{\mathrm{v}}(\varphi(x', y') - (p_1, p_2)) > \hat{\mathrm{v}}(\varphi(x', y) - (p_1, p_2))$, and—expanding further—to

$$\hat{\mathrm{v}}(x' - p_1, \pi^\perp(\varphi(x', y')) - p_2) > \hat{\mathrm{v}}(x' - p_1, \pi^\perp(\varphi(x', y)) - p_2). \tag{1.4}$$

By (1.3), the left-hand-side of (1.4) equals $\hat{\mathrm{v}}(x' - p_1)$, and the right-hand-side equals $\hat{\mathrm{v}}(\pi^\perp(\varphi(x', y)) - p_2)$. Again by (1.3), (1.4) is clear. ❒

Consequently, for a given V, a translater $(\alpha_x)_{x \in \pi(B-B)}$ of χ on B is said to *witness* the V-translatability of χ on B, or that $(\alpha_x)_{x \in \pi(B-B)}$ is a translater *witnessing* V-translatability of χ on B. We work extensively with translaters in Chapter 4.

The next result is used implicitly throughout this work and is [26, Lemmas 3.3 and 3.10].

Lemma 1.2.12. *Let $B_0 \subseteq K^n$ be definable and $B \subseteq B_0$ be a ball.*

(a) *Let V_1 and V_2 be subspaces of k^n and $\chi : B_0 \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ be definable. If χ is V_i-translatable on B for $i = 1, 2$, then χ is $V_1 + V_2$-translatable on B.*

(b) *Let V be a subspace of k^n and $X \subseteq K^n$ be definable. If X is V-translatable on B, then* $\dim(V) \leq \dim(X)$. *In fact, if $\pi : K^n \longrightarrow K^{\dim(V)}$ is an exhibition of V, then for each $q \in \pi(V)$, we have that* $\dim(X) = \dim(X \cap \pi^{-1}(q)) + \dim(V)$.

Proof. (a) [26, Lemma 3.3]; (b) Set $d := \dim(V)$ and let $(\alpha_x)_{x\in\pi(B-B)}$ be a translater of X on B. Notice that for every $q, q' \in \pi(B)$ the risometry $\alpha_{q-q'}$ restricted to $X \cap \pi^{-1}(q)$ is a definable bijection to $X \cap \pi^{-1}(q')$, so by the second part of Proposition 1.1.12 (e), dimension is constant on the sets $X \cap \pi^{-1}(q)$ with $q \in \pi(B)$. By Proposition 1.1.12 (e) again we conclude that for any $q \in \pi(B)$, $\dim(X) = \dim(X \cap \pi^{-1}(q)) + d$, as required. ❐

Notation 1.2.13. By (a) above, there always exists a maximal subspace V of k^n for which χ is V-translatable on B. Such space is called the *translatability space* of χ on B, and will be denoted by $\mathrm{tsp}_B(\chi)$.

Observe that for $d \leq n$, χ is d-translatable on B if and only if $d \leq \dim(\mathrm{tsp}_B(\chi))$. And if V is a subspace of k^n, then χ is V-translatable on B if and only if $V \subseteq \mathrm{tsp}_B(\chi)$.

1.3 T-stratifications

In [53], H. Whitney introduced the stratifications now known as *Whitney stratifications* for analytic varieties in $\mathbb{C}^n$ and $\mathbb{R}^n$. The driving idea in Whitney's work was to classify the singularities of a variety. The original construction of a Whitney stratification for an analytic variety $X \subseteq \mathbb{C}^n$ is as follows. We first split X into the subvariety of regular points of X, X_{rg}, and the subvariety of singular points of X, X_{sg}. We then split X_{sg} again into the subvarieties $(X_{sg})_{rg}$ and $(X_{sg})_{sg}$. By continuing in this fashion, we obtain the sequence of varieties, $X,\ X_{rg},\ X_{sg},\ (X_{sg})_{rg},\ (X_{sg})_{sg},\ ((X_{sg})_{sg})_{rg},\ \ldots$. A partition of X is then given by the sets:

$$X_{rg}, \quad (X_{sg})_{rg}, \quad ((X_{sg})_{sg})_{rg}, \quad (((X_{sg})_{sg})_{sg})_{rg}, \ldots.$$

In [53, Theorem 19.2] Whitney describes how to refine this partition to obtain a *regular*—nowadays known as Whitney—stratification of X. It was later proved by R. Thom

([46]) that points in the same strata are *normally equi-singular*, thus showing that a Whitney stratification indeed provides a classification of the points of X in terms of how singular they are ([36] is a fine exposition of these results by J. Mather). To this day, Whitney stratifications and their variations have played, and play, an important role in the study of singularities.

The first model-theoretic approach to Whitney stratifications consisted of an investigation of whether they exist for definable sets in arbitrary o-minimal expansions of the real field. This was answered positively by T. L. Loi in [34][3]. More recently, model theorists have looked into analogues to Whitney stratifications in very different settings. In this thesis we are interested in an analogue in valued fields.

In [6], R. Cluckers, G. Comte and F. Loeser introduced a notion of regular stratification for definable sets in the p-adic field $\mathbb{Q}_p$. Their approach is so far the only one known to work in some instances of the mixed characteristic case. The stratifications we are interested in were introduced by I. Halupczok in [26], and their definition makes them suitable only in valued fields of equi-characteristic 0. Finding an appropriate analogue in the mixed characteristic setting remains as an important open problem.

A tuple of sets $(S_0, \dots, S_n)$ will be denoted by $(S_i)_{i\leq n}$, and for each $d \leq n$, the sets $\bigcup_{i\leq d} S_i$ and $\bigcup_{i\geq d} S_i$ will be denoted by $S_{\leq d}$ and $S_{\geq d}$, respectively.

Definition 1.3.1. Let $B_0 \subseteq K^n$ be a definable set. A definable partition $(S_i)_{i\leq n}$ of B_0 is said to be a *t-stratification* if the following hold.

(1) For each $d \leq n$, $\dim(S_{\leq d}) \leq d$;

(2) for each $d \leq n$ and ball $B \subseteq S_{\geq d}$, the tuple $(S_i)_{i\leq n}$ is d-translatable on B.

Furthermore, if $\chi : B_0 \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ is a definable function, we say that the t-stratification $(S_i)_{i\leq n}$ *reflects* χ if the following strengthening of (2) holds.

(2') For any $d \leq n$ and ball $B \subseteq S_{\geq d}$, the tuple $((S_i)_{i\leq n}, \chi)$ is d-translatable on B.

[3] In an earlier paper, L. van den Dries and C. Miller claimed this result but their proof was later retracted.

For $X \subseteq B_0$, we say that $(S_i)_{i\leq n}$ is a *t-stratification of* X if $(S_i)_{i\leq n}$ is a t-stratification reflecting χ_X. We may also say that $(S_i)_{i\leq n}$ is a t-stratification of a tuple of maps and sets heeding Convention 1.2.1.

Notice that if $(S_i)_{i\leq n}$ is a t-stratification reflecting a map $\chi : R^n \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ and $(S'_i)_{i\leq n}$ is in turn a t-stratification reflecting $(S_i)_{i\leq n}$, then $(S'_i)_{i\leq n}$ reflects χ too. We use this fact later without further mentioning.

Examples 1.3.2. (1) Let $X \subseteq K$ be definable and let S_0 be the finite set obtained from Hypotheses 1.1.9 (3). We claim that $(S_0, S_1 := K \setminus S_0)$ is a t-stratification of X. For a ball $B \subseteq K \setminus S_0$, either $X \subseteq B$ or $X \cap B = \varnothing$; hence, 1-translatability of (S_0, S_1, X) on B holds by Examples 1.2.5 (2). This example tells us that Hypotheses 1.1.9 (3) plainly states the existence of t-stratifications of definable subsets of K.

(2) Let $T \subseteq K^n$ be finite and let $\chi : K^n \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ be the map $x \mapsto \ulcorner \hat{\mathrm{rv}}(x - T)\urcorner$. By Examples 1.2.5 (3), the sets $S_0 := T$, $S_i := \varnothing$ for $1 \leq i < n$, and $S_n := K^n \setminus T$, form a t-stratification of K^n reflecting χ.

(3) Consider the set $X = \{(x, y) \in K^2 \mid xy = 0\}$ and define $S_0 := \{0\}$, $S_1 := X \setminus \{0\}$ and $S_2 := K^2 \setminus X$. From Examples 1.2.5 (4) it follows that $((S_i)_{i\leq 2}, X)$ is 1-translatable on any ball $B \subseteq S_{\geq 1} = K^2 \setminus S_0$. By Examples 1.2.5 (2), 2-translatability of $((S_i)_{i\leq 2}, X)$ on a ball $B \subseteq S_2$ is immediate as $B \cap X = \varnothing$ by definition of S_2. Therefore, $(S_i)_{i\leq 2}$ is a t-stratification of X.

(4) Let $\xi \in \mathrm{RV} \setminus \{0\}$ and set $X := \mathrm{rv}^{-1}(\xi) \subseteq K$. Using that X is an open ball, we can see that for any point $x_0 \in X$, the sets $S_0 = \{x_0\}$ and $S_1 = K \setminus \{x_0\}$ form a t-stratification of X.

The choice of $x_0 \in X$ was arbitrary and this raises the question of whether the t-stratification (S_0, S_1) is definable over the same parameters used to define X—in this case only ξ. The next lemma will help us to obtain a ξ-definable t-stratification of X.

Lemma 1.3.3. *Let* $\xi \in \mathrm{RV}^{\times}$. *If* $a \in K$, *then* $S_0 = \{a\}$ *and* $S_1 = K \setminus \{a\}$ *constitute a t-stratification of* $X := \mathrm{rv}^{-1}(\xi)$ *if and only if* $\mathrm{v}(a) \geq \mathrm{v}_{\mathrm{RV}}(\xi)$.

Proof. Suppose that $a \in K$ is such that $\mathrm{v}(a) \geq \mathrm{v}_{\mathrm{RV}}(\xi)$. For 1-translatability on a ball $B \subseteq K \setminus \{a\}$, it is enough to show that either $B \cap X = \varnothing$ or $B \subseteq X$. Assume that $B \cap X \neq \varnothing$ but $B \nsubseteq X$. Let $x \in B \cap X$ and $y \in B \setminus X$. Using that $\mathrm{rv}(x) \neq \mathrm{rv}(y)$, $\mathrm{v}(a) \geq \mathrm{v}_{\mathrm{RV}}(\xi) = \mathrm{v}(x) \geq \mathrm{v}(x-y) > \mathrm{rad}(B)$. So $0 \in B$ and, consequently, $a \in B$ too, contradicting the choice of B. Thus 1-translatability holds on B. For the other direction, assume that $\mathrm{v}(a) < \mathrm{v}_{\mathrm{RV}}(\xi)$. Then $a \notin B := B(0, \geq \mathrm{v}_{\mathrm{RV}}(\xi))$ and $X \subsetneq B$, so X is not 1-translatable on B. ❐

Therefore, $S_0 := \{0\}$ and $S_1 := K \setminus \{0\}$ constitute a 0-definable t-stratification of $X := \mathrm{rv}^{-1}(\xi)$, for any $\xi \in \mathrm{RV}^{\times}$. A similar statement—and lemma—can be proved for RV-balls in K^n for $n > 1$.

Recall that the classical driving idea about stratifications is to classify the points of a set according to how singular they are. Under this idea, the points in the first (lowest dimensional) stratum are deemed as the worst, most singular points of the set; while the elements of the last (highest dimensional) stratum are the most regular (or irrelevant) points of the set. This seems to change with t-stratifications. Frequently, one is forced to put points into S_0 which would be otherwise considered unproblematic (non-singular). This happened in the last example, with 0 being apparently irrelevant to $\mathrm{rv}^{-1}(\xi)$. This is a loss in intuition but a gain in power: points that we may have ordinarily seen as regular could become singularities in the residue field (or other quotients). A t-stratification is strong enough to capture these 'quasi-singularities'.

(5) To visualise this example, we assume that K expands a real closed valued field. Let $X := \{(x, y) \in K^2 \mid y = x^2\}$. It is surprising that the sets $\varnothing, X$ and $K^2 \setminus X$ do not form a t-stratification of X (we would expect $S_0 = \varnothing$ as X does not have singularities). The problem is that in some balls containing 0 (e.g., in O_K^2) the *direction* of X (which for now can be thought of as the tangent at some point of X, see the definition of $\mathrm{affdir}(X)$ in page 33) is significantly different to the right and left of 0, leaving no suitable choice of a 1-dimensional space for 1-translatability on B. We avoid this issue by putting a point

in S_0, which automatically bans balls as above. We set $S_0 := \{0\}, S_1 := X \setminus \{0\}$ and $S_2 := K^2 \setminus X$, and we claim that $(S_i)_{i\leq 2}$ is a t-stratification of X. This can be visualised without difficulty: a ball $B \subseteq K^2 \setminus \{0\}$ could either (a) intersect a single branch of the parabola; hence, is enough to straighten $X \cap B$ down to the tangent of $X \cap B$ at some (arbitrary) point $x_0 \in X \cap B$, or (b) intersect both branches of the parabola, implying that B is big and considerably far away from 0, and that $X \cap B$ is 'almost' parallel to the axis $\{0\} \times K$; in this case $X \cap B$ is straightened to two vertical lines (one per 'branch'). We formalise part of these ideas below. The reader could skip this argument as it is entirely and heavily technical; we provide it with the aim of exemplifying more specific techniques when dealing with translatability. For instance, we define some risometries explicitly.

Let $B \subseteq K^2 \setminus \{0\}$. First assume that B contains a point (x_0, x_0^2) but not the point $(-x_0, x_0^2)$. Let $\overline{V}$ be the line with slope $2x_0$ containing the origin (so $\overline{V}$ is parallel to the tangent line of X at (x_0, x_0^2)). We set $V := \operatorname{res}(\overline{V} \cap O_K^2)$, and claim that X is V-translatable on B. To define a straightener of X on B, we need to consider whether $\mathrm{v}(2x_0) \geq 0$.

Case I. Suppose that $\mathrm{v}(2x_0) \geq 0$. The function $\psi(x, y) := (x, y + (x^2 - 2x_0x + x_0^2))$ moves the points of B vertically taking the tangent line of X at (x_0, x_0^2) to X. To show that ψ is a risometry we may first notice that $B \subsetneq O_K^2$. Indeed, since $x_0 \in B \cap O_K^2$, either $B \subsetneq O_K^2$ or $O_K^2 \subseteq B$, but the latter is impossible because $0 \notin B$, so the former holds. If $(x, y), (x', y') \in B$, the equation $\hat{\mathrm{rv}}(\psi(x, y) - \psi(x', y')) = \hat{\mathrm{rv}}((x, y) - (x', y'))$ is equivalent to

$$\mathrm{v}(x - x') + \mathrm{v}(x + x' - 2x_0) > \hat{\mathrm{v}}(x - x', y - y').$$

This inequality is clearly implied by $\mathrm{v}(x + x' - 2x_0) > 0$, which follows from $B \subsetneq O_K^2$: we have that $\mathrm{v}(x + x' - 2x_0) \geq \min\{\mathrm{v}(x - x_0), \mathrm{v}(x' - x_0)\} \geq \operatorname{rad}(B) > 0$.

That $\psi^{-1}(X \cap B)$ is $\overline{V}$-translation invariant is clear, as it equals the tangent line to $B \cap X$

at (x_0, x_0^2).

Case II. Suppose that $\mathrm{v}(2x_0) < 0$. We define ψ as the map $(x, y) \mapsto \left(x_0 + \frac{x^2 - x_0^2}{2x_0}, y\right)$ on B. Then ψ moves the points of B horizontally, taking X to the tangent line of X at (x_0, x_0^2). We now verify that ψ is a risometry. To avoid dealing with a large number of cases, we assume that B is closed; this is without loss because if B is open, we can apply the argument below to the closed ball of radius $\mathrm{rad}(B)$ and then return to B by restricting the map. The point of taking B as a closed ball is that $\mathrm{rad}(B) \leq \mathrm{v}(2x_0)$ is then immediately impossible. Indeed, note that $\hat{\mathrm{v}}((x_0, x_0^2) - (-x_0, x_0^2)) = \mathrm{v}(2x_0)$, so if $\mathrm{rad}(B) \leq \mathrm{v}(2x_0)$, then $(-x_0, x_0^2) \in B$, contradicting our initial assumption. Thus, $\mathrm{rad}(B) > \mathrm{v}(2x_0)$

As a first step, we prove that if $(x, y), (x', y) \in B$, then $\mathrm{r\hat{v}}(\psi(x, y) - \psi(x', y)) = \mathrm{r\hat{v}}((x, y) - (x', y))$. Indeed, this equation is equivalent to

$$\hat{\mathrm{v}}\left(\frac{x^2 - x_0^2}{2x_0} - \frac{x'^2 - x_0^2}{2x_0} - (x - x'), y - y\right) > \hat{\mathrm{v}}(x - x', y - y), \tag{1.5}$$

which in turn reduces to $\mathrm{v}((x - x_0)^2 - (x' - x_0)^2) > \mathrm{v}(x - x') + \mathrm{v}(2x_0)$. Further, since $(x - x_0)^2 - (x' - x_0)^2 = ((x + x') - 2x_0)(x - x')$, to prove (1.5) it is enough to show that

$$\mathrm{v}(x + x' - 2x_0) > \mathrm{v}(2x_0). \tag{1.6}$$

We have that $\mathrm{v}(x + x' - 2x_0) \geq \min\{\mathrm{v}(x - x_0), \mathrm{v}(x' - x_0)\} \geq \mathrm{rad}(B) > \mathrm{v}(2x_0)$, which proves (1.6) and finalises this step.

As second step, we prove that if $(x, y), (x, y') \in B$, then $\mathrm{r\hat{v}}(\psi(x, y) - \psi(x, y')) = \mathrm{r\hat{v}}((x, y) - (x, y'))$. This is in fact obvious because $\psi(x, y) - \psi(x, y') = (0, y - y')$ by definition of ψ.

As third and last step, we take arbitrary $(x, y), (x', y') \in B$ and apply the first step to (x, y) and (x', y), and then apply the second step to (x', y) and (x', y'). After that we conclude that $\mathrm{r\hat{v}}(\psi(x, y) - \psi(x', y')) = \mathrm{r\hat{v}}((x, y) - (x', y'))$ by Lemma 1.2.8 (a).

To end Case II, notice that $\psi(X \cap B)$ is $\overline{V}$-translation invariant because it is the tangent line to $X \cap B$ at (x_0, x_0^2).

In the situation when B intersects both of the arms of X, i.e. B contains both (x_0, x_0^2) and $(-x_0, x_0^2)$ for some $x_0 \in K^\times$, we can show that X is $V := (\{0\} \times k)$-translatable on B. Taking $\overline{V}$ to be simply $\{0\} \times K$ we can then define a straightener φ of $B \cap X$ on B by moving the points of B horizontally in such a way that $X \cap B$ is mapped to two vertical lines, implying $\overline{V}$-translation invariance of $\varphi(X)$. We omit the details.

We now present some general facts about t-stratifications.

Lemma 1.3.4. *Let $(S_i)_{i \leq n}$ be a t-stratification.*

(a) *For each $d \leq n$ and $x \in S_{\geq d+1}$, there exists a maximal ball B containing x and such that $B \cap S_{\leq d} = \varnothing$. In particular, each S_d is a closed set.*

(b) *For each $d \leq n$, S_d has local dimension d at each point $x \in S_d$. In particular, either $\dim(S_d) = d$ or $S_d = \varnothing$.*

(c) *If $(S_i)_{i \leq n}$ is a t-stratification of $X \subseteq K^n$, then $X \subseteq S_{\leq \dim(X)}$.*

Proof. (a) and (b), [26, Lemma 3.17].
(c) Set $d := \dim(X)$ and suppose that $X \cap S_{\geq d+1} \neq \varnothing$. Take $B \subseteq S_{\geq d+1}$ such that $B \cap X \neq \varnothing$. By Definition 1.3.1 (2'), $X \cap B$ is $(d+1)$-translatable on B; so, by Proposition 1.2.12 (b) we have that $d+1 \leq \dim(X \cap B) = \dim(X)$, a contradiction. ❐

The next lemma will help us to construct new t-stratifications from a given one.

Lemma 1.3.5 ([26, Lemma 4.20])**.** *Suppose that $X \subseteq K^n$ is a definable set of dimension d and let $\chi : X \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ be definable. We extend χ to the whole of K^n by setting $\chi(K^n \setminus B_0) = \{a_0\}$, for some (arbitrary) $a_0 \in \mathrm{RV}^{\mathrm{eq}}$. Let $(T_i)_{i \leq n}$ and $(S_i)_{i \leq n}$ be t-stratifications of K^n such that $(T_i)_{i \leq n}$ reflects $(S_i)_{i \leq n}$ and χ. Define the partition $(S'_i)_{i \leq n}$ of K^n in such a way that $S'_{\leq i} := T_{\leq i}$ if $i < d$, and $S'_{\leq i} := T_{\leq d-1} \cup X \cup S_{\leq i}$ if $i \geq d$. Then $(S'_i)_{i \leq n}$ is a t-stratification reflecting $(S_i)_{i \leq n}$ and χ, and clearly coincides with $(S_i)_{i \leq n}$ outside of $X \cup T_{\leq d-1}$.*

1.3.1 The rainbow of a t-stratification

The function we define below carries information about the maps reflected by a t-stratification. This function allows us to talk about reflection in terms of *refinements* of maps into $\mathrm{RV}^{\mathrm{eq}}$. We fix $B_0 \subseteq K^n$ as a ball or the whole of K^n, and, unless stated otherwise, we implicitly assume that every t-stratification we refer to is a t-stratification of B_0.

Definition 1.3.6. Let $(S_i)_{i\leq n}$ be a t-stratification. The *rainbow* of $(S_i)_{i\leq n}$ is the map $\rho : B_0 \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ defined as $\rho(x) := \ulcorner(\hat{\mathrm{rv}}(x - S_i))_{i\leq n}\urcorner$.

Clearly ρ is definable using the same parameters used for $(S_i)_{i\leq n}$.

We say that $\chi' : B_0 \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ *refines* $\chi : B_0 \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ if the partition of B_0 given by the fibres of χ' refines the partition of B_0 given by the fibres of χ; that is, for all $q \in \chi(B_0)$, there is $A_q \subseteq \chi'(B_0)$ such that $\chi^{-1}(q) = (\chi')^{-1}(A_q)$. Equivalently, there exists a function $f : \chi'(B_0) \longrightarrow \chi(B_0)$ such that $f \circ \chi' = \chi$ (in that case, $\chi^{-1}(q) = (\chi')^{-1}(f^{-1}(q))$ for each $q \in \chi(B_0)$).

Proposition 1.3.7 ([26, Proposition 4.17])**.** *Let $(S_i)_{i\leq n}$ be a t-stratification of B_0 and let $\chi : B_0 \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ be definable. The following are equivalent.*

(a) $(S_i)_{i\leq n}$ *reflects* χ*;*

(b) *the rainbow ρ of $(S_i)_{i\leq n}$ refines χ;*

(c) *any definable risometry $\varphi : B_0 \longrightarrow B_0$ respecting $(S_i)_{i\leq n}$ respects χ.*

It follows that ρ is the finest map reflected by $(S_i)_{i\leq n}$.

Example 1.3.8. Consider the set $X = K \times \{0\} \subseteq K^2$. A t-stratification for X consists of $S_0 = \varnothing, S_1 = X$ and $S_2 = K^2 \setminus X$. Let ρ be the rainbow of this t-stratification. We claim that for any $(x, y), (x', y') \in K^2$, $\rho(x, y) = \rho(x', y')$ if and only if $\mathrm{rv}(y) = \mathrm{rv}(y')$. Hence, each fibre of ρ has the form $K \times \mathrm{rv}^{-1}(\xi)$ for some $\xi \in \mathrm{RV}$. Pictorially, the fibers of ρ are in a rainbow-like arrangement of ever thinner coloured horizontal bands.

We prove the claim above. For $(x, y), (x', y') \in K^2$, $\rho(x, y) = \rho(x', y')$ holds if and only if all the following three equations hold.

1. $\hat{\mathrm{rv}}((x,y) - S_0) = \hat{\mathrm{rv}}((x',y') - S_0)$, but this has no meaning because $S_0 = \varnothing$.

2. $\hat{\mathrm{rv}}(K \times \{y\}) = \hat{\mathrm{rv}}((x,y) - K \times \{0\}) = \hat{\mathrm{rv}}((x',y') - K \times \{0\}) = \hat{\mathrm{rv}}(K \times \{y'\})$. We claim that this equation holds if and only if $\mathrm{rv}(y) = \mathrm{rv}(y')$. Suppose that $\mathrm{rv}(y) = \mathrm{rv}(y')$. If either $y = 0$ or $y' = 0$, actually $y = y' = 0$ and the conclusion is trivial. Hence we assume that $y \neq 0 \neq y'$. Since $\mathrm{v}(y - y') > \mathrm{v}(y)$, for all $z \in K$, we have that $\hat{\mathrm{rv}}(z, y) = \hat{\mathrm{rv}}(z, y')$, so $\hat{\mathrm{rv}}(K \times \{y\}) = \hat{\mathrm{rv}}(K \times \{y'\})$ as claimed.

The remaining direction asks us to show that $\hat{\mathrm{rv}}(K \times \{y\})$ determines $\mathrm{rv}(y)$, for any $y \in K$. Let $j : \mathrm{RV} \longrightarrow \mathrm{RV}^{(2)}$ be the embedding $\mathrm{rv}(y) \mapsto \hat{\mathrm{rv}}(0, y)$. We claim that for each $y \in K$, $\hat{\mathrm{rv}}(K \times \{y\}) \cap j(\mathrm{RV}) = \{\hat{\mathrm{rv}}(0, y)\}$. From right to left the containment is clear. In the other direction, if $\hat{\mathrm{rv}}(0, z) \in \hat{\mathrm{rv}}(K \times \{y\}) \cap j(\mathrm{RV})$, then there is $x \in K$ such that $\hat{\mathrm{rv}}(0, z) = \hat{\mathrm{rv}}(x, y)$. We then have that $\mathrm{v}(y - z) \geq \hat{\mathrm{v}}(x, y - z) > \mathrm{v}(z)$, so $\hat{\mathrm{rv}}(0, z) = \hat{\mathrm{rv}}(0, y)$. This proves our claim. It follows that $\mathrm{rv}(y)$ is the unique element in $j^{-1}(\hat{\mathrm{rv}}(K \times \{y\}) \cap j(\mathrm{RV}))$, so whenever $\hat{\mathrm{rv}}(K \times \{y\}) = \hat{\mathrm{rv}}(K \times \{y'\})$, we deduce that $\mathrm{rv}(y) = \mathrm{rv}(y')$.

3. $\hat{\mathrm{rv}}((x,y) - K^2 \setminus X) = \hat{\mathrm{rv}}((x',y') - K^2 \setminus X)$. This does not add more to the information from 2. Notice that $\hat{\mathrm{rv}}((x,y) - K^2 \setminus X)$ equals $\mathrm{RV}^{(2)}$ when $(x,y) \in K \times \{0\}$, and equals $\mathrm{RV}^{(2)} \setminus \{0\}$ otherwise. So the equation above is equivalent to '$y = 0$ if and only if $y' = 0$', which is already captured in the equation $\mathrm{rv}(y) = \mathrm{rv}(y')$.

This proves our initial claim, and justifies the suggested visualisation of (the fibres of) the rainbow of $(S_i)_{i \leq 2}$.

Remark 1.3.9. A fibre C of the rainbow of a t-stratification $(S_i)_{i \leq n}$ must be contained in some stratum S_d. Indeed, if $C \cap S_d \neq \varnothing$, and a is taken in that intersection, we have that for any $x \in C$, $0 \in \hat{\mathrm{rv}}(a - S_d) = \hat{\mathrm{rv}}(x - S_d)$, so $x \in S_d$.

Last in this subsection we present a result that, like Lemma 1.3.5, will allow us to construct new t-stratifications from a given one. This lemma is particularly useful in an inductive argument in Section 3.2.

Lemma 1.3.10 ([26, Lemmas 3.16 and (part of) 4.21]). *Suppose that $B_0 \subseteq K^n$ and that $(S_i)_{i\leq n}$ is a t-stratification of B_0 reflecting the definable map $\chi : B_0 \longrightarrow \mathrm{RV}^{\mathrm{eq}}$. Let $B \subseteq B_0$ be a ball and let $\pi : K^n \longrightarrow K^d$ be an exhibition of $\operatorname{tsp}_B((S_i)_{i\leq n})$. If $x \in \pi(B)$ and we set $F := \pi^{-1}(x)$, then $(S_i \cap F)_{d\leq i\leq n}$ is a t-stratification of $B \cap F$ reflecting $\chi|_{B\cap F}$. Furthemore, if C is a fiber of the rainbow of $(S_i)_{i\leq n}$, then $C \cap F$ is a fiber of the rainbow of $(S_i \cap F)_{d\leq i\leq n}$.*

More specific results about the rainbow are discussed when needed, for instance, in Section 3.2.

1.4 Existence of t-stratifications and the Jacobian property

In this section we discuss the result in [26, Subsection 4.3] on the existence of t-stratifications. An important ingredient to its proof is that a t-stratification can be refined such that the fibres of its rainbow become 'properly aligned'. To develop the idea of being 'properly aligned' we introduce the concept of being *subaffine* for subsets of K^n.

Definition 1.4.1. For $X \subseteq K^n$ we define the *affine direction* of X to be the subspace $\operatorname{affdir}(X)$ of k^n generated by the set $\{\operatorname{dir}(x - x') \mid x, x' \in X\}$. Notice that for every $x \in X$, $\dim(\operatorname{affdir}(X)) \geq \dim_x(X)$; if the equality holds for every $x \in X$, we say that X is *subaffine*.

Recall that $\dim_x(X)$ is the local dimension of X at x (see Proposition 1.1.12 (f)). The space $\operatorname{affdir}(X)$ is essentially the set of possible directions of translatability of X, and we always have that $\dim(\operatorname{affdir}(X)) \geq \dim_x(X)$, for all $x \in X$. Translatability of X in the direction of a subspace V must imply that V consists only of such possible directions, i.e. $V \subseteq \operatorname{affdir}(X)$. This is what the following lemma establishes.

Lemma 1.4.2 ([26, Lemma 4.4])**.** *Let B be a ball and $X \subseteq B$ a definable set.*

(a) *If X is V-translatable on B for some $V \subseteq k^n$, then $V \subseteq \operatorname{affdir}(X)$.*

(b) *If there is an exhibition $\pi : K^n \longrightarrow K^d$ of $V := \operatorname{affdir}(X)$ with $\pi(X) = \pi(B)$, then X is V-translatable on B.*

As we have seen, the property of being subaffine is closely related to translatability. In fact, in order to build t-stratifications for definable sets, we start by introducing a way to obtain subaffinity for said sets. This is done by postulating a new axiom for $\mathcal{T}$, the *Jacobian property*. This property implies that the graph of a definable function $K^n \longrightarrow K$ must be subaffine after partitioning its domain into at most $|\mathrm{RV}|$-many pieces (i.e. the size of the partition is less than or equal to the cardinality of RV).

Definition 1.4.3. We define when functions and then when theories have the Jacobian property.

(1) Let X be a subset of K^n and $f : X \longrightarrow K$ be a definable function. We say that f has the *Jacobian property* (on X) if either f is constant or there exists $z \in K^n \setminus \{0\}$ such that for all distinct $x, x' \in X$,

$$\mathrm{v}(f(x) - f(x') - \langle z, x - x' \rangle) > \hat{\mathrm{v}}(z) + \hat{\mathrm{v}}(x - x'). \tag{1.7}$$

(2) We say that the theory $\mathcal{T}$ has the Jacobian property at n if for any $(K, \mathrm{RV}^{\mathrm{eq}}) \models \mathcal{T}$, $A \subseteq K \cup \mathrm{RV}^{\mathrm{eq}}$, $X \subseteq K^n$ and A-definable function $f : X \longrightarrow K$, there exists an A-definable function $\chi : X \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ such that for each $q \in \chi(X)$, if $\chi^{-1}(q)$ contains an open ball, then $f|_{\chi^{-1}(q)}$ has the Jacobian property.

(3) We say that the theory $\mathcal{T}$ has the Jacobian property up to n if it has the Jacobian property at all $m \leq n$. If $\mathcal{T}$ has the Jacobian property at all $n \geq 1$, we simply say that $\mathcal{T}$ has the Jacobian property.

Let $X \subseteq K^n$. The reader might have perceived that asking for $f : X \longrightarrow K$ to have the Jacobian property is reminiscent of asking f to be differentiable on X. This is not mislead as if we assume for simplicity that $n = 1$ and f is indeed differentiable at a point $x_0 \in X$ then in a neighbourhood U of x_0 we have that for all distinct $x, x' \in U$,

$$\mathrm{rv}\left(\frac{f(x) - f(x')}{x - x'}\right) = \mathrm{rv}(f'(x_0)), \tag{1.8}$$

from which easy calculations lead to (1.7) with $z := f'(x_0)$. However, differentiability is not enough for the Jacobian property as, even if taken as maximal, U could end up being too small to cover the whole of X. Even in Definition 1.4.3 (2), it could happen that no choice of χ is possible as the neighbourhoods on which analogues of (1.8) hold are too small. Neither does the Jacobian property imply differentiability. The Jacobian property forces some control on the grow of f, and indeed imposes conditions on the approximate direction of the graph of f (see e.g. 1.4.5), but does not determine the derivative (which is the exact direction).

The Jacobian property in Definition 1.4.3 is regarded as a generalised, multi-dimensional version of the Jacobian property employed in other research problems. For instance, the Jacobian property in [7, Definitions 6.3.5 and 6.3.6] in the setting of valued fields with an analytic structure corresponds to Definition 1.4.3 (1)-(2) for $n = 1$. Notice, however, that the condition in [7, Definitions 6.3.5] is stronger than the one in Definition 1.4.3 (1), in the sense that in the former, the *Jacobian* $\mathrm{Jac}(f)$ (in this case, simply the derivative of f) is required to exist and to satisfy that $\mathrm{rv}(\mathrm{Jac}(f))$ is constant. The latter, Definition 1.4.3 (1) with $n = 1$, instead asks simply for $\mathrm{rv}\left(\frac{f(x)-f(x')}{x-x'}\right)$ to be constant over all $x \neq x'$.

The Jacobian property when $n = 1$ in Definition 1.4.3 also appears in axiomatic presentations of motivic integration in valued fields of equicharacteristic 0, see for example [10, 3.3 Definition] and [57, Proposition 3.15].

We make some quick remarks about the Jacobian property.

Remark 1.4.4. (a) In Definition 1.4.3 (1), the choice of $z \in K^n \setminus \{0\}$ is not unique;

indeed, any other $z' \in K^n \setminus \{0\}$ with $\hat{\mathrm{rv}}(z') = \hat{\mathrm{rv}}(z)$ works for (1.7).

(b) Assume that $\mathcal{T}$ only satisfies Hypotheses 1.1.9 (1)-(3). If $\mathcal{T}$ has the Jacobian property at 1, then $\mathcal{T}$ satisfies Hypotheses 1.1.9 (4).

Proof. (a) Using Lemma 1.2.8 (d) and that $\hat{\mathrm{v}}(z - z') > \hat{\mathrm{v}}(z')$, we deduce that $\mathrm{v}(\langle z - z', x - x' \rangle) > \hat{\mathrm{v}}(z') + \hat{\mathrm{v}}(x - x')$. From this inequality and from (1.7) we get that for distinct $x, x' \in X$, $\mathrm{v}(f(x) - f(x') - \langle z', x - x' \rangle)$ is greater than or equal to

$$\min\{\mathrm{v}(f(x) - f(x') - \langle z, x - x' \rangle), \mathrm{v}(\langle z - z', x - x' \rangle)\} > \hat{\mathrm{v}}(z') + \hat{\mathrm{v}}(x - x').$$

(b) Let $A \subseteq K \cup \mathrm{RV}^{\mathrm{eq}}, X \subseteq K$ and $f : X \longrightarrow K$ be as in Hypotheses 1.1.9 (4). Let $\chi : X \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ be the A-definable function associated to f given by the Jacobian property at $n = 1$, and fix $q \in \chi(K)$. If $\chi^{-1}(q)$ does not contain an open ball, then it is a finite set by Proposition 1.1.12 (a). Further refining of χ allows us to assume that whenever $\chi^{-1}(q)$ does not contain an open ball, it is a singleton; so, in that case $f|_{\chi^{-1}(q)}$ is constant. On the other hand, if $\chi^{-1}(q)$ does contain an open ball, then, according to the Jacobian property at $n = 1$, there is $z \in K^\times$ such that (1.7) holds for all distinct $x, x' \in \chi^{-1}(q)$. Since $\mathrm{v}(\langle z, x - x' \rangle) = \mathrm{v}(z \cdot (x - x')) = \mathrm{v}(z) + \mathrm{v}(x - x')$, clearly (1.7) cannot hold if $f(x) = f(x')$ but $x \neq x'$. In this case $f|_{\chi^{-1}(q)}$ is injective. ❐

The following is a crucial result in Halupczok's work towards the existence of t-stratifications.

Proposition 1.4.5 ([26, Lemma 4.6])**.** *If $X \subseteq K^n$ is definable of dimension d and $f : X \longrightarrow K$ is definable and has the Jacobian property on X, then the graph of f is a subaffine subset of K^{n+1}.*

Proposition 1.4.5 is the way subaffinity is obtained, and is the main reason why the Jacobian property was added as an axiom for $\mathcal{T}$. The Jacobian property is typically harder to prove than Hypotheses 1.1.9. In [26, Section 5], Halupczok proves that the theory of valued fields with analytic structure—as developed by R. Cluckers and L.

Lipshitz in [7]—satisfies Hypotheses 1.1.9 and has the Jacobian property, thus providing a big list of valued fields for which t-stratifications exist. In Chapter 3 we extend their existence to certain expansions of real closed valued fields.

For the sake of completeness, we state Halupczok's theorem.

Theorem 1.4.6 ([26, Theorem 4.12]). *Fix $n \geq 1$. Let $\mathcal{T}$ be a theory expanding $\mathcal{T}_{Hen}$, satisfying Hypotheses 1.1.9 and having the Jacobian property at all $m < n$. Fix a model $(K, \mathrm{RV}^{\mathrm{eq}})$ of $\mathcal{T}$ and a set of parameters $A \subseteq K \cup \mathrm{RV}^{\mathrm{eq}}$. Then for every A-definable ball $B_0 \subseteq K^n$ and A-definable map $\chi : B_0 \longrightarrow \mathrm{RV}^{\mathrm{eq}}$, there exists an A-definable t-stratification $(S_i)_{i \leq n}$ of B_0 reflecting χ.*

It follows that if $\mathcal{T}$ has the Jacobian property (Definition 1.4.3 (3)), then any definable map from any cartesian power of K into $\mathrm{RV}^{\mathrm{eq}}$ admits a t-stratification. It is worth mentioning that the following uniform version of the result is available ([26, Corollary 4.13]): let $\mathcal{T}$ satisfy Hypotheses 1.1.9 and have the Jacobian property, and assume that ϕ is a formula in the language of $\mathcal{T}$ that defines a map $\chi_\phi(K) : K^n \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ for every $(K, \mathrm{RV}^{\mathrm{eq}}) \vDash \mathcal{T}$; then there are formulas $\psi_0, \ldots, \psi_n$ in the language of $\mathcal{T}$ such that for every $(K, \mathrm{RV}^{\mathrm{eq}}) \vDash \mathcal{T}$, $(\psi_i(K))_{i \leq n}$ is a t-stratification of K^n reflecting $\chi_\phi(K)$.

Chapter 2

T-convex fields

This is a second chapter on preliminaries. Here we present and develop the particular setting of our forthcoming work. The valued fields we will work with are built from o-minimal expansions of fields, and we thus presume some familiarity with o-minimal structures. Nevertheless, a brief account of o-minimality is offered in Appendix A. The language of ordered rings is $L_{\text{or}} := \{+, -, \cdot, 0, 1, <\}$. The L_{or}-theory of real closed fields is denoted as RCF. For the whole of this chapter, L will be a (single-sorted, first-order) language containing L_{or}, and T will be a complete o-minimal L-theory containing RCF. The chapter contains no new results except for those in Sections 2.2 and 2.4, which are fundamental in Chapters 4 and 3, respectively.

2.1 T-convexity

Most of the content in this section is based on work of L. van den Dries and A. Lewenberg in [14] and [16]. Let R be a model of T. A subset A of R is said to be *convex* if $\forall x, y \in A, z \in R(x < z < y \to z \in A)$. In this section 'definable' means 'L-definable'.

Definition 2.1.1 ([16])**.** We say that $V \subseteq R$ is a T*-convex subring* of R if V is a proper convex subring of R such that $f(V) \subseteq V$, for each 0-definable continuous function

$f : R \longrightarrow R$. The pair (R, V) for such a V is called a T*-convex field*.

Examples 2.1.2. We discuss some examples of T-convex fields.

(1) Let $L = L_{\text{or}}$ and let R be a non-archimedean real closed field (e.g. a non-principal ultrapower of $\mathbb{R}$). Then the ring of *finite numbers* in R—the convex hull of $\mathbb{Z}$ in R—is an RCF-convex subring of R. Furthermore, the next claim is an easy criterion to obtain RCF-convex subrings.

Claim 2.1.3. Let $R \vDash \text{RCF}$ and suppose that $V \subsetneq R$ is a proper convex subring of R. Then V is an RCF-convex subring of R.

Proof. First of all notice that V must contain the set of real algebraic numbers $\mathbb{R}_{\text{alg}}$, and recall that as an L_{or}-structure $\mathbb{R}_{\text{alg}}$ is the prime model of RCF, i.e. it embeds elementarily into any other real closed field. So we may assume that $\mathbb{R}_{\text{alg}}$ is an elementary substructure of R. Now let $f : R \longrightarrow R$ be 0-definable and continuous; we want to show that $f(V) \subseteq V$. Observe that the restriction of f to $\mathbb{R}_{\text{alg}}$ is 0-definable in $\mathbb{R}_{\text{alg}}$, so polynomial boundedness of RCF (see [15, (3.7)]) implies that there is a positive x_0 in $\mathbb{R}_{\text{alg}}$ and $m \in \mathbb{Z}^+$ such that $\mathbb{R}_{\text{alg}} \vDash \forall x(x_0 \leq |x| \rightarrow |f(x)| \leq |x|^m)$. It follows that $R \vDash \forall x(x_0 \leq |x| \rightarrow |f(x)| \leq x^m)$. Moreover, if we define $g(x) := \max\{|f(t)| \mid t \leq x\}$, then $f(V) \subseteq V$ is equivalent to $g(V) \subseteq V$, so we may assume that f is even (i.e. $f(x) = f(-x)$), non-decreasing and non-negative on $R_{\geq 0}$. Take $x \in V$. If $|x| \geq x_0$, the convexity of V and the fact that $-|x|^m \leq f(x) \leq |x|^m$ imply that $f(x) \in V$. If instead $|x| \leq x_0$, then $0 \leq f(x) \leq f(x_0) \leq x_0^m$, so the convexity of V and the fact that $x_0 \in V$ imply that $f(x) \in V$. From these cases we conclude that $f(V) \subseteq V$. ❐

The claim remains true—and an almost identical proof applies—if we substitute RCF by any other polynomially bounded o-minimal theory, e.g. RCF_{an} (this theory is introduced in page 119).

(2) If T is RCF_{exp} or $\text{RCF}_{\text{an,exp}}$ (see page 119), a proper convex subring V of R containing the prime model of T is T-convex if and only if $\exp(V) \subseteq V$. A similar argument to

the one in the proof of Claim 2.1.3 can be applied using that if $R \vDash T$, then any definable function $f : R \longrightarrow R$ is bounded by an iteration of $\exp$ ([17, Proposition 9.2]).

(3) For arbitrary T, if $R_0 \preceq R$ (i.e. R_0 is an elementary substructure of R), then the convex hull $\operatorname{conv}(R_0)$ of R_0 in R is a T-convex subring of R, provided $\operatorname{conv}(R_0) \neq R$. If $f : R \longrightarrow R$ is 0-definable and continuous, $f|_{R_0}$ remains 0-definable in R_0, therefore $f(R_0) \subseteq R_0$. If $x \in \operatorname{conv}(R_0) \setminus R_0$, then $a < x < b$ for some $a, b \in R_0$. By the *Monotonicity Theorem* (Theorem A.1.8) we may assume that $f|_{[a,b]}$ is either constant or strictly monotone; in both cases, clearly $f(x) \in \operatorname{conv}(R_0)$.

Remark 2.1.4. If V is a proper convex subring of R, then V is a valuation ring of R, i.e. for all $x \in R$ either $x \in V$ or $x^{-1} \in V$.

Proof. If $x > 0$ is not in V, then $x > 1$, so $0 < x^{-1} < 1$. ❒

From now on, we give preference to the valuation-theoretic notation O_R for a T-convex subring of R, and we invariably regard (R, O_R) as a valued field. The convexity of O_R ensures that $0 \leq x < y$ implies that $\mathrm{v}(x) \geq \mathrm{v}(y)$, for all $x, y \in R$.

Notation 2.1.5. We mostly adhere to the valuation-theoretic notation in Chapter 1. The residue field of (R, O_R) is denoted by $\overline{R}$.

The following are basic properties of the structures in consideration.

Proposition 2.1.6. *Let (R, O_R) be a T-convex field. The following hold.*

(a) *the value group Γ is divisible;*

(b) *the residue field $\overline{R}$ is real closed; furthermore, $\overline{R}$ can be made into a model of T;*

(c) *the valued field (R, O_R) is Henselian.*

Proof. (a) Take $\lambda \in \Gamma$ and $n \geq 1$. Pick $x \in R_{>0}$ such that $\mathrm{v}(x) = \lambda$. Since R is a real closed field, there is $y \in R$ such that $x = y^n$, so $\lambda = n\,\mathrm{v}(y)$.

For the next item recall that an ordered field L is real closed if and only if the Intermediate Value Theorem holds for polynomials in $L[x]$.

(b) Let $f(x) \in \overline{R}[x]$ and let $\alpha_1, \alpha_2 \in \overline{R}$ be such that $\alpha_1 < \alpha_2$ and $f(\alpha_1) < 0 < f(\alpha_2)$. Take $F(x) \in O_R[x]$ such that $f(x)$ is obtained by applying res to each of the coefficients of $F(x)$. For $a_i \in O_R$ satisfying that $\mathrm{res}(a_i) = \alpha_i$ ($i = 1, 2$), we have that $a_1 < a_2$ and $F(a_1) < 0 < F(a_2)$. Since the Intermediate Value Theorem holds for R, there is $x \in (a_1, a_2)$ such that $F(x) = 0$. Then $f(\mathrm{res}(x)) = \mathrm{res}(F(x)) = 0$ and $\mathrm{res}(x) \in (\alpha_1, \alpha_2)$. This proves that $\overline{R}$ is real closed. The furthermore part of the statement is [16, (2.16) Remark].

(c) (Based on the proof of [5, Theorem 3]) We prove that R equals its henselianisation $H(R)$. The algebraic closure of R is $R(i)$, where $i := \sqrt{-1}$. Let $\tilde{\mathrm{v}}$ be an extension of v to $R(i)$. Note that the separable closure of R is $R(i)$ itself, so $H(R)$ is the fixed field of the group $A_{\tilde{\mathrm{v}}} := \{\sigma \in \mathrm{Aut}(R(i)) \mid \sigma|_R = id_R \text{ and } \tilde{\mathrm{v}} \circ \sigma = \tilde{\mathrm{v}}\}$. It is enough to show that conjugation belongs to $A_{\tilde{\mathrm{v}}}$, as that would force that $H(R) \subseteq R$. We need to prove that $\tilde{\mathrm{v}}(x + iy) = \tilde{\mathrm{v}}(x - iy)$ for all $x, y \in R$.

Suppose that at least one of $x, y \in R$ is non-zero (otherwise, the claim is trivial). Notice that $0 = \tilde{\mathrm{v}}(-1) = \tilde{\mathrm{v}}(i^2) = 2\tilde{\mathrm{v}}(i)$, so $\tilde{\mathrm{v}}(i) = 0$ and $\tilde{\mathrm{v}}(iy) = \tilde{\mathrm{v}}(y)$. If $\mathrm{v}(x) \neq \mathrm{v}(y)$, then $\tilde{\mathrm{v}}(x + iy) = \min\{\mathrm{v}(x), \mathrm{v}(y)\} = \tilde{\mathrm{v}}(x - iy)$. Thus we may assume that $\mathrm{v}(x) = \mathrm{v}(y)$. Set $u := yx^{-1}$ and note that $\mathrm{v}(u) = 0$ and $\tilde{\mathrm{v}}(1 \pm iu) \geq \min\{\mathrm{v}(1), \mathrm{v}(u)\} = 0$. In general, $1 < a$ implies that $0 \geq \mathrm{v}(a)$, so $0 \geq \mathrm{v}(1 + u^2) \geq \min\{\mathrm{v}(1), \mathrm{v}(u^2)\} = 0$. Then,

$$\tilde{\mathrm{v}}(1 + iu) + \tilde{\mathrm{v}}(1 - iu) = \mathrm{v}((1 + iu)(1 - iu)) = \mathrm{v}(1 + u^2) = 0.$$

It follows that $\tilde{\mathrm{v}}(1 + iu) = \tilde{\mathrm{v}}(1 - iu)$, and with this that $\tilde{\mathrm{v}}(x + iy) = \tilde{\mathrm{v}}(x - iy)$. ❐

Remark 2.1.7. There are two topologies on R^n. The first is given by the open valuative balls (see page 2), and the second is the *Euclidean topology* raised by the norm $\|\cdot\|_R$ defined as $\|(x_1, \ldots, x_n)\|_R := \sqrt{\sum_{i=1}^{n} x_i^2}$, for each $(x_1, \ldots, x_n) \in R^n$. These topologies are in fact the same.

Proof. First of all notice that $\|\cdot\|_R$ is equivalent to the $\max$-*norm* on R, which is defined as $\|(x_1,\dots,x_n)\|_{\max} := \max\{|x_i| \mid i \in \{1,\dots,n\}\}$. So by Euclidean open disc below we mean a set of the form $\{x \in R^n \mid \|x-a\|_{\max} < r\}$ for some $a \in R^n$ and $r \in R_{>0}$.

We need to prove that for all $a \in R^n$ and valuative open ball B containing a there exists an Euclidean open disc D containing a such that $D \subseteq B$, and vice versa. Fix $a \in R^n$ and let $B(b, > \lambda)$ be an open valuative ball containing a. By properties of the valuation map we can take $b = a$ without loss. If we pick $r \in R_{>0}$ such that $\mathrm{v}(r) > \lambda$, we then have that

$$\{x \in R^n \mid \|x-a\|_{\max} < r\} \subseteq B(a, > \lambda).$$

Indeed, if $x \in R^n$ and, say, $|x_1 - a_1| = \|x-a\|_{\max} < r$, then $|x_1 - a_1| \geq |x_i - a_i|$, and so $\mathrm{v}(x_1 - a_1) \leq \mathrm{v}(x_i - a_i)$, for all $i = 1,\dots,n$. It follows that $\hat{\mathrm{v}}(x-a) = \mathrm{v}(x_1 - a_1) \geq \mathrm{v}(r) > \lambda$.

Vice versa, fix $a \in R^n$ and let $D := \{x \in R^n \mid \|x-b\|_{\max} < r\}$ be an Euclidean disc containing a. Pick $s \in R_{>0}$ such that $\{x \in R^n \mid \|x-a\|_{\max} < s\} \subseteq D$. We then get that $a \in B(a, > \mathrm{v}(s))$ and

$$B(a, > \mathrm{v}(s)) \subseteq \{x \in R^n \mid \|x-a\|_{\max} < s\} \subseteq D.$$

Certainly, assume for the sake of a contradiction that there is $x \in B(a, > \mathrm{v}(s))$ with $\|x-a\|_{\max} \geq s$. If $j \in \{1,\dots,n\}$ is such that $|x_j - a_j| = \|x-a\|_{\max}$, it follows that $\mathrm{v}(x_j - a_j) \leq \mathrm{v}(s)$, contradicting that $\mathrm{v}(x_j - a_j) \geq \hat{\mathrm{v}}(x-a) > \mathrm{v}(s)$. ❐

We now present some results on definability.

Let $L_{\text{convex}} := L \cup \{\mathcal{O}\}$, where $\mathcal{O}$ is a unary predicate for O_R. The L_{convex}-theory of all pairs (R, O_R), with $R \models T$ and O_R a T-convex subring of R, will be denoted by T_{convex}. This theory always contains RCVF (= $\mathrm{RCF}_{\text{convex}}$), the theory of real closed fields enriched with a convex valuation ring in the language $L_{\text{or,convex}} := L_{\text{or}} \cup \{\mathcal{O}\}$.

Theorem 2.1.8 ([16, (3.10) Theorem])**.** *If T is universally axiomatisable and admits quantifier elimination in the language L, then T_{convex} admits quantifier elimination in the* *language* L_{convex}.

As stated, the theorem does not apply to—for example—RCF, since this theory does not have a universal axiomatisation. Nevertheless, a result of G. Cherlin and M. A. Dickmann ([5, Section 2]) states that $\mathrm{RCF}_{\mathrm{convex}}$ has quantifier elimination in the language $L_{\mathrm{or}} \cup \{|\}$, where $|$ is interpreted as the relation $a|b$ if and only if $ba^{-1} \in O_R$, for $a \neq 0$ and b in R.

In general, T can be expanded by definitions to a theory $T^{\forall}$ with a universal axiomatisation (see [16, (2.3) and (2.4)]) (also note that quantifier elimination is preserved when expanding by definitions). So we adhere to the following.

Convention 2.1.9. By passing to the expansion mentioned above if needed, we assume that T_{convex} admits quantifier elimination in L_{convex}. (Strictly, this may hold in an expansion of L_{convex}, but the new symbols would be coded by the symbols in L; no harm is done in terms of definability).

The following is a consequence of quantifier elimination.

Theorem 2.1.10. T_{convex} *is complete and* weakly o-minimal *(Definition A.2.14).*

Proof. [16, (3.13) and (3.14) Corollary]. ❐

We come back to quantifier elimination in Section 3—in a different language, however.

By L_Γ we denote the language of ordered groups $\{+, -, <, 0\}$ expanded by predicates for the sets $\mathrm{v}(X) \subseteq \Gamma$, where $X \subseteq (R \setminus \{0\})^n$ is 0-L_{convex}-definable (formally, we add predicates for L_{convex}-formulas $\phi(x)$, in such a way that if $\lambda_i \in \Gamma$, then the predicate for $\phi(x)$ holds for $(\lambda_1, \dots, \lambda_n)$ in Γ if and only if there are $a_i \in \mathrm{v}^{-1}(\lambda_i)$ such that $(R, O_R) \vDash \phi(a_1, \dots, a_n)$; the details are in [16, (3.15)]). The language L_Γ is interpreted in Γ in the obvious way.

If $M := (S_i)_{i\in I}$ is a multi-sorted structure we say that a sort S_i is *stably embedded* in M if any definable set (in principle definable with parameters from all the sorts) is already definable with parameters from S_i.

Proposition 2.1.11. *Let (R, O_R) be a T-convex field. Let $(R, \Gamma, \overline{R})$ denote the three-sorted structure given by: the L_{convex}-structure (R, O_R) as the first sort, the L_Γ-structure Γ as the second sort, the L-structure $\overline{R}$ as the third sort, plus the maps* v *and* res *(the latter extended by $x \mapsto 0$, for $x \in R \setminus O_R$). Let $L_{\Gamma,\overline{R}}$ denote the three-sorted language of $(R, \Gamma, \overline{R})$. The following hold.*

(a) *The sort $\overline{R}$ is* stably embedded *in $(R, \Gamma, \overline{R})$, i.e. any $L_{\Gamma,\overline{R}}$-definable subset of $\overline{R}^n$ is $L_{\Gamma,\overline{R}}$-definable using only parameters from $\overline{R}$.*

(b) *Assume that T is power-bounded. Then the sort Γ is* stably embedded *in $(R, \Gamma, \overline{R})$.*

(c) *As an L_Γ-structure, Γ is weakly o-minimal (Definition A.2.14). Moreover, if T is power-bounded (Definition A.1.12), then Γ is o-minimal. In fact, under power-boundedness, the L_Γ-theory of Γ is simply an expansion by definitions of the theory of non-trivial ordered vector spaces over the* field of exponents *of T (see page 120 for the definition of this field).*

(d) *Assume that T is power-bounded. Any $L_{\Gamma,\overline{R}}$-definable function between the sorts $\overline{R}$ and Γ has finite image.*

Proof. (a) [26, Theorem A]; (b) [26, Theorem B]; (c) The first part is [16, (3.16) Proposition], and the second is [14, (4.3) Proposition]; (d) [14, (5.8) Proposition]. ❒

Remark 2.1.12. The conclusion of (d) above is equivalent to $\overline{R}$ and Γ being *orthogonal*, that is, any $L_{\Gamma,\overline{R}}$-definable subset A of $\overline{R}^l \times \Gamma^k$ is a finite union of rectangles $E \times F$ where $E \subseteq \overline{R}^l$ and $F \subseteq \Gamma^k$ are $L_{\Gamma,\overline{R}}$-definable.

Weak o-minimality of T_{convex} tells us that L_{convex}-definable subsets of R are finite unions of convex sets. The purpose of the last results in this section is to highlight a more precise description of these sets. Below $\langle X \rangle_L$ stands for the L-structure generated by X.

Theorem 2.1.13 (The valuation property for T_{convex})**.** *Suppose that T is power-bounded. Let $(R, O_R) \preceq (R', O_{R'})$ be models of T_{convex} for which there is $a \in R' \setminus R$ such that*

$R' = \langle R \cup \{a\} \rangle_L$ *and* $\mathrm{v}(R') \supsetneq \mathrm{v}(R)$. *Then there exists* $d \in R$ *such that* $\mathrm{v}(a-d) \notin \mathrm{v}(R)$.

Proof. The first approach to this was [19, 9.2 Proposition], where the proof given applies (only) when T is polynomially bounded. In full generality see [50, Theorem 12.10]. ❒

Lemma 2.1.14. *Suppose that T is power-bounded and let x be a single variable. Each L_{convex}-formula $\phi(x)$, is equivalent to a boolean combination of formulas of the form*

$$x = a, \qquad x < a, \qquad \mathrm{v}(x-a) = \mathrm{v}(b), \qquad \mathrm{v}(x-a) < \mathrm{v}(b), \tag{2.1}$$

for some $a, b \in R$.

Proof. This follows from [14, Proposition 7.6] using Theorems 2.1.8 and 2.1.13. (N. B. the last two formulas can be rewritten as L_{convex}-formulas; for instance, $\mathrm{v}(x-a) = \mathrm{v}(b)$ is equivalent to $(x - a = b = 0) \vee (b/(x-a) \in \mathcal{O} \wedge \exists y \in \mathcal{O}(y \cdot b/(x-a) = 1))$. ❒

Assuming that T is power-bounded, it follows that each L_{convex}-definable subset of R is a boolean combination of points, open intervals and balls. This property was first proved for definable sets in real closed valued fields (i.e. in RCF-convex fields) by J. Holly in [30], and it fails if T is not power-bounded, see [14, Observation 7.3]. It follows from Holly's work that L_{convex}-definable subsets of R have a *canonical normal form*, which we describe in the remainder of the section.

Let $B \subseteq R$ be a ball. The *left cut* of B is the set $C_0(B) := \{x \in R \mid x < B\}$, and the *right cut* of B is the set $C_1(B) := \{x \in R \mid \exists y \in B(x \leq y)\}$. The left cut made by a point $a \in R \cup \{\pm\infty\}$ is defined as the set $C_0(a) = \{x \in R \mid x < a\}$, and the corresponding right cut is $C_1(a) = \{x \in R \mid x \leq a\}$. In the rest of the chapter *points* are allowed to be in $R \cup \{\pm\infty\}$.

Definition 2.1.15. Suppose that $B_1, B_2 \subseteq R$ are balls or points, and let $i_1, i_2 \in \{0, 1\}$. If $C_{i_1}(B_1) \subsetneq C_{i_2}(B_2)$, the set $(R \setminus C_{i_1}(B_1)) \cap C_{i_2}(B_2)$ is called a *cut-interval*. We say that two cut-intervals are *properly disjoint* if their union is not a cut-interval (notice that in particular they must be disjoint).

Suppose that $A \subseteq R$. Clearly, if B_1 and B_2 are A-L_{convex}-definable balls or points, then each of the (four possible) cut-intervals defined from B_1 and B_2 is A-L_{convex}-definable too. The converse of this statement is true and follows from the claim below.

Claim 2.1.16. If the cut $C_i(B) \subseteq R$ is A-L_{convex}-definable, so is B.

Proof. First of all we show that the cut determines B, i.e. that if B' is another ball or point with $C_i(B) = C_i(B')$ then $B' = B$. Notice that the cut of a point cannot be made by a ball, and vice versa, so B and B' must be of the same kind. If they are points, the conclusion is obvious, so we assume that they are balls. Necessarily $B \cap B' \neq \varnothing$, otherwise either $B' < B$ or $B < B'$ and clearly the cut $C_i(B')$ would not equal $C_i(B)$. In particular we deduce that either $B \subseteq B'$ or $B' \subseteq B$. If the containment is strict, say $B' \subsetneq B$, then there are $a \in B'$ and $r \in R_{>0}$ such that $a \pm r \in B \setminus B'$. If $i = 0$, then $a - r$ is in $C_0(B') \setminus C_0(B)$, and if $i = 1$, then $a + r$ is in $C_1(B) \setminus C_1(B')$; in each case, a contradiction. Thus $B' = B$.

Now we assume that $C_i(B)$ is A-L_{convex}-definable. We also take $i = 0$; the case when $i = 1$ is treated similarly. Consider a ball $B(x, \square\, \lambda)$, with $\square \in \{>, \geq\}$. We have that $C_0(B(x, \square\, \lambda)) = C_0(B)$ if and only if $\forall x \in C_0(B) \forall x' \in R(\hat{\mathrm{v}}(x' - b) \,\square\, \lambda \longrightarrow x < x')$, so the set $\mathcal{B} := \{(x, \lambda) \in R \times \Gamma \mid C_0(B(x, \square\, \lambda)) = C_0(B)\}$ is A-L_{convex}-definable. By the first part of this proof, we know that only one $\lambda \in \Gamma$ appears in $\mathcal{B}$, so this λ must be A-L_{convex}-definable. It follows that the set $\{x \in R \mid C_0(B(x, \square\, \lambda)) = C_0(B)\}$ is A-L_{convex}-definable, and this set equals B, again by the uniqueness of B in the first part of the proof. ❒

In the context of $\mathrm{RCF}_{\text{convex}}$, the original theorem of Holly ([30, Theorem 4.8]) states that each L_{convex}-definable subset X of R is uniquely a union of finitely many properly disjoint cut-intervals. The proof only uses that each such X is a boolean combination of points, open intervals and balls, so the result is readily extended to T_{convex} for arbitrary power-bounded T (by the comment after Lemma 2.1.14). We reinforce this result by ensuring that whenever $X \subseteq R$ is A-L_{convex}-definable, the balls or points determining the cut-intervals in the partition of X are A-L_{convex}-definable too. This is crucial in Section 2.4. Recall that ∞ and $-\infty$ are allowed as points, and in such case A-L_{convex}-definability is naturally dropped.

Proposition 2.1.17 ([30, Theorem 4.8]). *Let $A \subseteq R$. For every A-L_{convex}-definable set $X \subseteq R$ there are unique A-L_{convex}-definable balls or points $B_1, \ldots, B_m$ and a unique tuple $(i_1, \ldots, i_m) \in \{0,1\}^m$ such that $C_{i_1}(B_1) \subsetneq C_{i_2}(B_2) \subsetneq \cdots \subsetneq C_{i_m}(B_m)$ and X equals the union of pair-wise properly disjoint cut-intervals*

$$\big[(R \setminus C_{i_1}(B_1)) \cap C_{i_2}(B_2)\big] \cup \cdots \cup \big[(R \setminus C_{i_{m-1}}(B_{m-1})) \cap C_{i_m}(B_m)\big]. \tag{2.2}$$

Proof. Using Lemma 2.1.14, the argument by J. Holly for [30, Theorem 4.8] ensures the existence and uniqueness of $B_1, \ldots, B_m$ and the tuple $(i_1, \ldots, i_m)$ such that X has the form in (2.2). By performing some reductions in (2.2), we can also assume that all the cut-intervals are pair-wise properly disjoint.

By Claim 2.1.16, to prove that each B_i is A-L_{convex}-definable, it is enough to show that each cut $C_{i_j}(B_j)$ is so. In turn, the latter follows by showing that each cut-interval in (2.2) is A-L_{convex}-definable; this is what we prove now. That X equals the union in (2.2) implies that there is a finite partition $D_1 < \cdots < D_k$ of R into cut-intervals such that X is either the union of all the D_i's with i odd or the union of all the D_i's with i even. There are four possible shapes for X, depending on whether k is even or odd, and then whether $D_1 \subseteq X$ or not. We only deal with one of the cases, as the rest are handled similarly. We assume that k is even and that $D_1 \subseteq X$. It follows that $X = D_1 \cup D_3 \cup \cdots \cup D_{k-1}$. For each j with $1 \leq j \leq k$, we define the formula $\phi_j(x)$ as:

$$\exists y_1, \ldots, y_j \Big(\bigwedge_{i<i' \leq j} y_i < y_{i'} \wedge \bigwedge_{i \leq j,\, i \text{ odd}} y_i \in X \wedge \bigwedge_{i \leq j,\, i \text{ even}} y_i \notin X \wedge y_j \leq x \Big).$$

For each $j < k$, the A-L_{convex}-formula $\phi_j(x) \wedge \neg\phi_{j+1}(x)$ defines D_j, while D_k is defined by the A-L_{convex}-formula $\phi_k(x)$. This finishes the proof. ❐

2.2 A curve selection lemma

By Remark 2.1.7, the concepts of limit, continuity, derivative, etc. in (R, O_R) can be thought of in terms of either the norm $\|\cdot\|_R$ or the valuation. Recall that 'L_{convex}-definable' means 'definable by an L_{convex}-formula with parameters'; unlike in other sections of this chapter, we do not try to specify precisely the parameters used.

Proposition 2.2.1. *Let $f : R \longrightarrow R$ be an L_{convex}-definable function. Then there are an L_{convex}-definable partition into convex sets $C_1, \ldots, C_m$, and L-definable functions $f_1, \ldots, f_m : R \longrightarrow R$ such that $f|_{C_j} = f_j|_{C_j}$, for each j with $1 \le j \le m$. Moreover, f can be assumed to be continuous, constant or strictly monotone, and even, if desired, differentiable on the interior of each C_j.*

Proof. [14, Lemma 2.6 and Corollary 2.8]. Alternatively, see the proof of Proposition 2.3.3. ❒

A function $f : R^n \longrightarrow R^m$ is said to be *bounded* if there is $M \in R_{>0}$ such that $\|f(x)\|_R \le M$, for all $x \in R^n$.

Lemma 2.2.2. *Let $f : R \longrightarrow R^n$ be a bounded L_{convex}-definable function, and $a \in R$. Then $\lim_{x\to a^+} f(x)$ exists in R^n.*

Proof. For simplicity we assume that $n = 1$. Let $\{C_j\}_{1\le j\le m}$ and $\{f_j\}_{1\le j\le m}$ be as in the statement of Proposition 2.2.1. The limit of f when $x \to a^+$ is the limit of $f_j|_{C_j}$ when $x \to a^+$ for some $j \in \{1, \ldots, m\}$, which exists in $(R \cup \{\pm\infty\})^n$ by the continuity and monotonicity of $f_j|_{C_j}$. Since f is bounded, $\lim_{x\to a^+} f(x)$ must be in R^n. ❒

Definition 2.2.3. A *curve* (in R^n) is an injective continuous function $\gamma : (a, b) \longrightarrow R^n$, where $a < b$ are in R. The obvious meaning is given to *differentiable curve*.

The existence of definable Skolem functions in o-minimal theories implies an *o-minimal Curve Selection Lemma* (Proposition A.1.9). This kind of result helps to turn sequential-convergence (or more generally net-convergence) into *curve-convergence*, where continuous maps witness limits. The following Curve Selection Lemma for T_{convex} is a consequence of the existence of definable Skolem functions as well.

Remark 2.2.4 ([14, (2.7) Remark]). Let $c \in R_{>0}$ be of positive valuation. If we add a constant symbol for c to L_{convex} and the sentence $c > 0 \wedge c \notin \mathcal{O}$ to T_{convex}, the resulting theory, denoted by $T_{\text{convex},c}$, has definable Skolem functions in the language $L_{\text{convex}} \cup \{c\}$.

Below, $\mathrm{cl}(X)$ denotes the topological closure of $X \subseteq R^n$.

Proposition 2.2.5 (Curve selection lemma for T-convex fields). *Suppose that $X \subseteq R^n$ is L_{convex}-definable and that $x \in \mathrm{cl}(X) \setminus X$. Then there exists an L_{convex}-definable curve $\gamma : (0, 1) \longrightarrow X$ such that* $\lim_{t \to 0^+} \gamma(t) = x$.

Proof. Consider the set $A := \{(t, y) \in R_{>0} \times X \mid \|x - y\|_R < t\}$, which is an L_{convex}-definable subset of R^{1+n}. Since $x \in \mathrm{cl}(X)$, for every $t \in R_{>0}$ there exists $y \in X$ such that $(t, y) \in A$. By the existence of definable Skolem functions for $T_{\text{convex},c}$, there exists an L_{convex}-definable function $f : R_{>0} \longrightarrow X$ such that $(t, f(t)) \in A$, for each $t \in R_{>0}$. By Proposition 2.2.1, there is $\varepsilon \in R_{>0}$ such that f is continuous and injective on $(0, \varepsilon)$. We set γ to be the composition $f|_{(0,\varepsilon)} \circ g$, where g is the L_{convex}-definable homeomorphism $x \mapsto \varepsilon x$ from $(0, 1)$ to $(0, \varepsilon)$. ❐

Notice that in the proof of Proposition 2.2.5 we may have added c as a parameter to define γ; this does not matter much in applications.

2.3 The language L_{RV} and further definability

Although a T-convex field can be readily made into a structure in the multi-sorted language described in Section 1.1, we postpone this approach for now and introduce an intermediate language. Let (R, O_R) be a T-convex field.

The order of R induces a linear order on $\mathrm{RV} := R^\times/(1 + M_R) \cup \{0\}$ given by $\xi < \eta$ if and only $\mathrm{rv}^{-1}(\xi) < \mathrm{rv}^{-1}(\eta)$, for all $\xi, \eta \in \mathrm{RV}$. To the natural language $\{*, ^{-1}, 1\}$ of

RV (discussed in page 8) we add a symbol $<$ for the order above, a predicate k for the residue field $\overline{R}$, on which we put the language L (having in mind Proposition 2.1.6 (b)), and a constant symbol 0 for $\mathrm{rv}(0)$. We assume that the symbols 0 and 1 are shared between k and RV. Thus, RV is from now on regarded as a structure in the language $L_{\mathrm{og}}^k := \{*, ^{-1}, 1, <, k, 0\}$, with the obvious interpretation.

Definition 2.3.1. The language L_{RV} consists of:

(a) A sort for the field R with the language L;

(b) A sort for RV with the language L_{og}^k; k carries a copy of the language L, and the symbols 0 and 1 are shared between RV and k.

(c) A map from the first sort to the second, which stands for $\mathrm{rv} : R \longrightarrow \mathrm{RV}$.

The first sort will be called *the field-sort* and the second *the* RV*-sort*. The common L_{RV}-theory of all T-convex fields will be denoted by T_{RV}. A model of T_{RV} will be denoted as a pair (R, RV).

Since, for example, $O_R = \{x \in R \mid \mathrm{rv}(x) \in \overline{R} \vee \mathrm{rv}(x+1) = \mathrm{rv}(1)\}$, L_{RV} and L_{convex} are *interdefinable*. Thus, a subset of R^n is 0-L_{RV}-definable if and only if it is 0-L_{convex}-definable. In general, if $A \subseteq R \cup \mathrm{RV}$ and we set $A' := ((A \cap R) \cup \mathrm{rv}^{-1}(A \cap \mathrm{RV}))$, any A-L_{RV}-definable subset of R^n is A'-L_{convex}-definable. The converse of this fact is in general false, for instance, if $\xi \in \mathrm{RV}^{\times}$, all the elements of $\mathrm{rv}^{-1}(\xi)$ are trivially $\mathrm{rv}^{-1}(\xi)$-L_{convex}-definable but $\mathrm{rv}^{-1}(\xi)$ might not contain any ξ-L_{RV}-definable point (cf. Lemma 2.4.1). Also, a model of T_{RV} can be made naturally into a model of T_{convex}, and vice versa. The language L_{RV} has the advantage of connecting our interests to those of modern research on valued fields, prominently those on notions of minimality (e.g. [8]), motivic integration (e.g. [9] and [31]), and, of course, stratification theory.

The value group Γ is sometimes thought of as a third sort (accompanied by the maps v and v_{RV}) but only to simplify notation; any reference to Γ can be syntactically written out in the language L_{RV}. For example, $\mathrm{v}(x) > \mathrm{v}(y)$ can be recast as $\mathrm{rv}(x+y) = \mathrm{rv}(y)$.

As for T_{convex}, we also have quantifier elimination for T_{RV}. The theorem below was first stated and proved by Y. Yin in [57].

Theorem 2.3.2. *If T is power-bounded, then T_{RV} admits quantifier elimination.*

Proof. [57, Theorem 1.8]; the proof reduces the result to the quantifier elimination of T_{convex} and employs the *Wilkie inequality*. The latter is only available when T is power-bounded, see [14, §5].

Alternatively, Schoenfield's test for quantifier elimination can be performed via an argument similar to the one in the proof of Lemma B.1.18. ❐

To carry out some compactness arguments below, from now on we assume that the model (R, RV) of T_{RV} is sufficiently saturated. The following sharper version of Proposition 2.2.1, and its proof below, appeared first as [57, Lemma 2.1 and Corollary 2.2].

Proposition 2.3.3. *Let $A \subseteq R \cup \mathrm{RV}$ and suppose that $f : R \longrightarrow R$ is A-L_{RV}-definable. Then there exist an A-L_{RV}-definable partition of R into cut-intervals $C_1, \ldots, C_m$, and A-L-definable functions $f_1, \ldots, f_m : R \longrightarrow R$ such that for each $j \in \{1, \ldots, n\}$, $f|_{C_j} = f_j|_{C_j}$. Moreover, f can be assumed to be continuous, constant or strictly monotone, and even, if desired, differentiable on the interior of each C_j.*

For the proof of this lemma we first point out a convenient simplification. Let x be a tuple of field-sort variables. Notice that whenever $t_1(x)$ and $t_2(x)$ are L-terms, the formula $t_1(x) = t_2(x)$ is equivalent to $\mathrm{rv}(t_1(x) - t_2(x)) = \mathrm{rv}(0)$, whilst $t_1(x) < t_2(x)$ is equivalent to $\mathrm{rv}(t_1(x) - t_2(x)) < \mathrm{rv}(0)$. If L contains no relation symbols other than $=$ and $<$, this observation allows us to assume that each L-term in an L_{RV}-formula appears under the scope of rv. By combining this with quantifier elimination we deduce that each A-L_{RV}-formula with (only) x as free variables is equivalent to a quantifier-free A-L^k_{og}-formula $\phi(\mathrm{rv}(t_1(x)), \ldots, \mathrm{rv}(t_l(x)), \eta)$, where each $t_i(x)$ is an $(A \cap R)$-L-term and $\eta \in A \cap \mathrm{RV}$. The condition of L having no relation symbols is easily circumvented: we replace each relation symbol U in L by a symbol for its characteristic function $\chi_U : R \longrightarrow \{0, 1\} \subseteq \mathrm{RV}$; hence, '$x \in U$' is simply equivalent to '$\mathrm{rv}(\chi_U(x)) = 1$'.

Thus, regardless of whether L contains new relation symbols, still each A-L_{RV}-formula is equivalent to a quantifier-free L^k_{og}-formula $\phi(\mathrm{rv}(t_1(x)), \ldots, \mathrm{rv}(t_l(x)), \eta)$ as above. (Incidentally, according to [57, Remark 2.3]—which in turn refers to a non-published argument by van den Dries—, the condition of L having no relation symbols other than $=$ and $<$ can be removed with greater advantage: it is always possible to find a language $L' \supseteq \{=, <\}$ that defines exactly the same sets as L in models of T and such that all the symbols in $L' \setminus \{=, <\}$ are interpreted as total continuous functions on models of T. We have not verified this process—particularly the mysterious claim about continuity—but we have kept the spirit of this idea with our modification above.)

Proof of Proposition 2.3.3. Let $\phi(x, y) = \phi(\mathrm{rv}(t_1(x, y)), \ldots, \mathrm{rv}(t_l(x, y)))$ be an A-L_{RV}-formula defining f. Fix $a \in R$. For each i with $1 \leq i \leq l$, we have that $t_i(a, \cdot)$ is an L-definable function from R to itself. By the Monotonicity Theorem (Proposition A.1.8), there is a finite $(A \cap R)$-L-definable collection of points $B^-_{a,i} \subseteq R$ such that between any two order-consecutive elements of $B^-_{a,i}$ the function $t_i(a, \cdot)$ is continuous and either constant or strictly monotone. We add to $B^-_{a,i}$ all the points in $f^{-1}(0)$ that are between two order-consecutive elements of $B_{a,i}$ between which f is injective. Notice that each $B_{a,i}$ remains $(A \cap R)$-L-definable and finite. We claim that $f(a) \in B_a := \bigcup_{i \leq l} B_{a,i}$. Otherwise, $f(a)$ is contained in an open interval I on which each $t_i(a, \cdot)$ for $i = 1, \ldots, l$ is continuous and either constant or strictly monotone. Hence there is $b \in I \setminus \{f(a)\}$ such that $\mathrm{rv}(t_i(a, b)) = \mathrm{rv}(t_i(a, f(a)))$. By the form of $\phi(x, y)$ at the beginning of the proof, $\phi(a, b)$ holds too, contradicting that f is a function and proving our claim.

We have proved that for each $a \in R$, $f(a)$ belongs to a finite $(A \cap R)$-L-definable set $B_a \subseteq R$. If for each $a \in R$ we take a formula defining B_a, by compactness it follows that finitely many of these formulas define all the sets B_a, i.e. there are $(A \cap R)$-L-formulas $\psi_1(x, \cdot), \ldots \psi_h(x, \cdot)$ such that for any $a \in R$, there is $j \in \{1, \ldots, h\}$ for which $B_a = \{b \in R \mid \psi_j(b, a)\}$. Furthermore, by *uniform finiteness* in o-minimal structures [15, page 53] there is $m \in \mathbb{Z}^+$ such that all the sets B_a have at most m elements. In fact, without loss of generality, we assume that all B_a have exactly m

elements (for example, if B_a has less elements than $B_{a'}$, add to B_a some of the elements of $B_{a'}$ until they have the same size; this procedure preserves the properties of B_a). Now, for each j with $1 \leq j \leq m$ we define $f_j : R \longrightarrow R$ as the function sending $a \in R$ to the j-th element of B_a according to the order $<$ on R. Since the $(A \cap R)$-L-formulas $\psi_1(x, \cdot), \ldots, \psi_h(x, \cdot)$ define all the sets B_a, each f_j is $(A \cap R)$-L-definable. Now, since $f(a) \in B_a$ for all $a \in R$, we have that for each $a \in R$ there is $j \in \{1, \ldots, m\}$ such that $f(a) = f_j(a)$. This implies that the sets $X_j := \{x \in R \mid f_j(x) = f(x)\}$ with $1 \leq j \leq m$ cover the whole of R, and it is obvious that $f|_{X_j} = f_j|_{X_j}$ for all $j \in \{1, \ldots, m\}$. Each X_j is clearly A-L_{RV}-definable, so we can partition these sets into A-L_{RV}-definable cut-intervals using Proposition 2.1.17.

The last part of the statement follows from o-minimal monotonicity (Proposition A.1.8) applied to each f_j. The claim on differentiability follows from [15, Chapter 7, (3.2) Theorem]. ❒

The following is a trivial consequence of the previous proposition; our intention with it is to move towards Hypotheses 1.1.9 (4).

Corollary 2.3.4 ([57, Corollary 2.3]). *Suppose that $A \subseteq R \cup \mathrm{RV}$ and let $f : R \longrightarrow R$ be A-L_{RV}-definable. Then there is an A-L_{RV}-definable map $\chi : R \longrightarrow \mathrm{RV}$ such that for each $q \in \chi(R)$, $f|_{\chi^{-1}(q)}$ is either constant or injective.*

Proof. By Proposition 2.3.3, there is a finite A-L_{RV}-definable partition $C_1, \ldots, C_m$ of R such that for each $1 \leq j \leq m$, $f|_{C_j}$ is either constant or strictly monotone. Observing that for any two positive integers $i \neq j$, we have that $\mathrm{v}(i - j) = 0$ and so $\mathrm{rv}(i) \neq \mathrm{rv}(j)$, it is enough to put $\chi(x) := \mathrm{rv}(j) \in \mathrm{RV}$, whenever $x \in C_j$. ❒

The following is a laxer version of Proposition 2.3.3 that applies to functions of more than one variable; it is laxer because the precise description of the sets in the partition is lost.

Lemma 2.3.5. *Suppose that $A \subseteq R \cup \mathrm{RV}$ and that $f : R^n \longrightarrow R$ is A-L_{RV}-definable.*

Then there are an A-L_{RV}-definable partition $X_1, \dots, X_m$ of R^n and A-L-definable functions $f_1, \dots, f_m : R^n \longrightarrow R$ such that $f|_{X_j} = f_j|_{X_j}$, for each $1 \in \{1, \dots, m\}$.

Proof. The proof consists of a routine compactness argument. We sketch the argument for $n = 2$; for $n > 2$ the result follows by induction and a similar argument (by partitioning the tuple of variables $(x_1, \dots, x_n)$ into $(x_1, \dots, x_{n-1})$ and x_n). Let $\phi((x, y), z)$ be an A-L_{RV}-formula defining $f : R^2 \longrightarrow R$. For each $a \in R$, by Proposition 2.3.3 the formula $\phi((a, y), z)$ (which defines the function $y \mapsto f(a, y)$) is equivalent to a finite disjunction of (pair-wise inconsistent) L-formulas. By compactness, there are finitely many of these L-formulas that work for all $a \in R$. So $\phi((x, y), z)$ is equivalent to a finite disjunction of (without loss of generality, pair-wise inconsistent) L-formulas. The result follows. ❐

The corollary below will be crucial in our later treatment of the Jacobian property in T-convex fields. As is usual, we say that $f : R^n \longrightarrow R$ is *differentiable* on X if $\mathrm{Jac}(f)(x) := \big(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x)\big)$ exists for all $x \in X$ (note that this only makes sense if X is topologically open).

Corollary 2.3.6. *Let f be as in the lemma above. Then there is $l \in \mathbb{Z}^+$ and an A-L_{RV}-definable function $\chi : R^n \longrightarrow \mathrm{RV}^l$ such that if $q \in \chi(R^n)$ and $\dim(\chi^{-1}(q)) = n$, then $\chi^{-1}(q)$ is open, f is differentiable on $\chi^{-1}(q)$, and $\hat{\mathrm{rv}} \circ \mathrm{Jac}(f)$ is constant on $\chi^{-1}(q)$.*

Proof. Let $X_1, \dots, X_m$ and $f_1, \dots, f_m$ be as in Lemma 2.3.5. Fix j such that $1 \leq j \leq n$. By L-definability of f_j, the set X_j can be further A-L_{RV}-definably partitioned in such a way that on the interior of each new piece, f_j is differentiable (see [15, Chapter 7 §3]). Additionally, if $S \subseteq X_j$ consists of the points at which $\mathrm{Jac}(f_j)$ does not exist, then S has empty interior, so $\dim(S) < n$. This allows us to further partition X_j into A-L_{RV}-definable sets $X_j^1, \dots, X_j^{l_j}$ such that each X_j^i is either open or has empty interior, and in the former case $\mathrm{Jac}(f)(x)$ exists for all $x \in X_j^i$. Define the map $\chi_j : X_j \longrightarrow \mathrm{RV}$ by setting $\chi_j(x) = \mathrm{rv}(i)$, whenever $x \in X_j^i$. We combine all the maps χ_j for $j \in \{1, \dots, n\}$ into a suitable single map $\chi' : R^n \longrightarrow \mathrm{RV}^m$ (e.g., by sending

$x \in R^n$ to $(0, \ldots, 0, \chi_j(x), 0 \ldots, 0)$ if $x \in X_j$). Finally, set $\chi : R^n \longrightarrow \mathrm{RV}^{m+n}$ as the map $x \mapsto (\chi'(x), \hat{\mathrm{rv}}(\mathrm{Jac}(f)(x)))$, with the convention that, say, $\mathrm{Jac}(f)(x) := 0$ if $\mathrm{Jac}(f)$ does not exist at x. Then χ is as we want. ❐

Because of the order on both sorts, we have that the algebraic closure and the definable closure (with respect to L_{RV}) on (R, RV) coincide, so we only deal with the latter. By dcl we denote the definable closure operator with respect to L_{RV}, and with dcl_L we denote the definable closure operator with respect to L (which is naturally only applied to subsets of R). For $X \subseteq R$ we have that $\mathrm{dcl}_L(X) = \mathrm{dcl}(X) \cap R$. The containment from left to right is obvious. In the other direction, if $y \in \mathrm{dcl}(X) \cap R$, then there is an L_{RV}-definable function $f : R \longrightarrow R$ and $x \in X$ such that $f(x) = y$. From Proposition 2.3.3, there exists a cut-interval C containing x and an L-definable function $\bar{f} : R \longrightarrow R$ such that $\bar{f}|_C = f|_C$. We have that $\bar{f}(x) = y$, so $y \in \mathrm{dcl}_L(X)$.

It is well known that the exchange principle holds for dcl_L because R is an o-minimal L-structure (see, e.g., [44, Theorem 4.1]; though an easier proof via Theorem A.1.8 is possible). Below, Aa denotes the set $A \cup \{a\}$.

Proposition 2.3.7. *Let (R, RV) be a model of T_{RV}. Then the exchange principle holds in the field-sort R: for any set $A \subseteq R \cup \mathrm{RV}$ and $a, b \in R$, if $b \in \mathrm{dcl}(Aa) \setminus \mathrm{dcl}(A)$, then $a \in \mathrm{dcl}(Ab)$.*

Proof. If $b \in \mathrm{dcl}(Aa) \setminus \mathrm{dcl}(A)$, then there is an A-L_{RV}-definable function $f : R \longrightarrow R$ sending a to b. By Proposition 2.3.3, there is a finite partition of X into A-L_{RV}-definable cut-intervals such that f is constant or injective on each piece. Let D be the cut-interval in the partition containing b. If D is not a point and f is constant on D, then $\{b\} = f(C)$ is an A-L_{RV}-definable set, contradicting that $b \notin \mathrm{dcl}(A)$. Thus either C is a point or f is injective on C. In either case, $a \in \mathrm{dcl}(Ab)$ (in the second case we have that $(f|_C)^{-1}$ is A-L_{RV}-definable and takes b to a). ❐

Below we describe important characteristics of the sort RV. Most of them were first stated and proved by Y. Yin in [55] and/or [57], and their proofs follow to some extent those in said papers.

The exchange principle holds in the RV-sort too. We are downplaying this fact because all the model-theoretic conditions on the RV-sort needed to work with t-stratifications are described in Hypotheses 1.1.9, and those do not involve the exchange principle for RV. The importance of this observation is that we are allowed to put extra structure on RV—as long as Hypotheses 1.1.9 remain true—and this extra structure could make the exchange principle fail for the RV-sort. Having said this, the exchange principle in the RV-sort is not used at all in this work.

Lemma 2.3.8 ([57, Lemma 2.4]). *Given $\xi \in \mathrm{RV}^n$ and $A \subseteq R^m$, if $a \in R$ is $(A\xi)$-L_{RV}-definable, then a is A-L_{RV}-definable.*

Proof. We apply induction. If $\xi = (\xi_1, \dots, \xi_n) \in \mathrm{RV}^n$ and $a \in \mathrm{dcl}(A\xi)$ we get that $a \in \mathrm{dcl}(Ab(\xi_1, \dots, \xi_{n-1}))$ for all $b \in \mathrm{rv}^{-1}(\xi_n)$. By induction we obtain that $a \in \mathrm{dcl}(Ab)$ for all $b \in \mathrm{rv}^{-1}(\xi_n)$. If $a \notin \mathrm{dcl}(A)$, Proposition 2.3.7 implies that $b \in \mathrm{dcl}(Aa)$ for all $b \in \mathrm{rv}^{-1}(\xi_n)$, i.e. $\mathrm{rv}^{-1}(\xi_n) \subseteq \mathrm{dcl}(A)$. However, this is absurd since $\mathrm{rv}^{-1}(\xi_n)$ is infinite and definable. Thus indeed, $a \in \mathrm{dcl}(A)$. ❒

Corollary 2.3.9 ([57, Corollary 2.5]). *Suppose that $\Xi \subseteq \mathrm{RV}$ and let $p : \Xi \longrightarrow R$ be an L_{RV}-definable function. Then $p(\Xi) \subseteq R$ is finite.*

Proof. Evidently, for each $\xi \in \Xi$, $p(\xi)$ is ξ-L_{RV}-definable. By Lemma 2.3.8, each element of $p(\Xi)$ is then $\varnothing$-L_{RV}-definable. The result follows by compactness. ❒

Remark 2.3.10. Clearly, the conclusion of Corollary 2.3.9 remains true for any L_{RV}-definable function $p : \Xi \longrightarrow R^n$, where $\Xi \subseteq \mathrm{RV}^m$, for all $n, m \geq 1$.

The following was proved in the context of algebraically closed valued fields in [55, Lemma 4.9].

Lemma 2.3.11. *Suppose that $A \subseteq R \cup \mathrm{RV}$. If $X \subseteq R$ is an A-L_{RV}-definable finite set, then there is an A-L_{RV}-definable injection $j : X \longrightarrow \mathrm{RV}^m$, for some $m \geq 1$.*

Proof. If X is a singleton, the conclusion is trivial. Let $X = \{b_1, \ldots, b_l\}$. By considering $b'_i := b_i - \frac{1}{l}\sum_{j=1}^{l} b_j$ for each $j \in \{1, \ldots, l\}$ we can assume that the average of X is 0. We claim that rv is not constant on X. This claim is obvious if v is not constant on X, so we assume that $\mathrm{v}(X) = \{\alpha\}$ for some $\alpha \in \Gamma$. To show that rv is not constant on X it suffices to find $b_i \neq b_j$ such that $\mathrm{v}(b_i - b_j) = \alpha$. If such two elements did not exist, we would have that

$$\alpha = \mathrm{v}(lb_1) = \mathrm{v}((l-1)b_1 - (b_2 + \cdots + b_l)) \geq \min\{\mathrm{v}(b_1 - b_i) \mid i \neq 1\} > \alpha,$$

a contradiction. Thus, b_i and b_j exist and rv is not constant on X. It follows that for each $\xi \in \mathrm{rv}(X)$, $1 \leq |\,\mathrm{rv}^{-1}(\xi) \cap X| < m$. By induction (on the size of X), for some $l' \geq 1$ and for each $\xi \in \mathrm{rv}(X)$ there is an $(A\xi)$-L_{RV}-definable injection $j_\xi : \mathrm{rv}^{-1}(\xi) \cap X \longrightarrow \mathrm{RV}^{l'}$. Notice that each $\xi \in \mathrm{rv}(X)$ is A-L_{RV}-definable, so each j_ξ is in fact A-L_{RV}-definable. The function $j : X \longrightarrow \mathrm{RV}^{l'+1}$ given by $j(x) = (j_{\mathrm{rv}(x)}(x), \mathrm{rv}(x))$ is the desired injection. ❒

The next result—a version of [55, Lemma 4.4]—is another application of the exchange principle in the field-sort.

Lemma 2.3.12. *Suppose that $X \subseteq R$ and let $f : X \longrightarrow R$ be an L_{RV}-definable function. Then there are L_{RV}-definable disjoint sets B_1 and B_2 such that:*

(i) $B_1 \cup B_2 = f(X)$;

(ii) B_1 *is finite and for each* $b \in B_1$, $f^{-1}(b)$ *is infinite;*

(iii) $f|_{f^{-1}(B_2)}$ *is finite-to-one.*

Proof. For $b \in f(X)$, if $f^{-1}(b)$ is infinite then $f^{-1}(b) \not\subseteq \mathrm{dcl}(b)$. Otherwise, for each $a \in f^{-1}(b)$ there is an algebraic L_{RV}-formula $\psi_a(x, b)$ such that $\psi_a(a, b)$ holds. By compactness, there are finitely many L_{RV}-formulas $\psi_{a_1}, \ldots, \psi_{a_l}$ that work for all $a \in f^{-1}(b)$. Each of these finitely many formulas has only finitely many solutions in R, leading to a contradiction with $f^{-1}(b)$ being infinite. Now, for fixed $b \in f(X)$ with $f^{-1}(b)$ infinite, let $a \in f^{-1}(b) \setminus \mathrm{dcl}(b)$; clearly $b \in \mathrm{dcl}(a)$. By Proposition 2.3.7, $b \in \mathrm{dcl}(\varnothing)$. Another compactness argument ensures that each such b satisfies one of only finitely many parameter-free L_{RV}-formulas that have only finitely many solutions in R. Let B_1' be the (finite) set defined by the disjunction of such finitely many formulas. Let B_1 be the set obtained from B_1' by removing all those elements b in B_1' for which $f^{-1}(b)$ is finite; B_1 is then finite and L_{RV}-definable. Putting $B_2 := f(X) \setminus B_1$, the result is proved. ❐

Lemmas 2.3.11 and 2.3.12 can be used to give another proof of Corollary 2.3.4 (in principle, with the insubstantial difference that we may have to consider RV^m, for some $m > 1$, as the codomain of χ in Corollary 2.3.4).

2.4 Towards Hypotheses 1.1.9 (3)

In this last section we prove a preliminary version of Hypotheses 1.1.9 (3). If $X \subseteq R$ is L_{RV}-definable, the existence of a finite definable set S with the required property follows by taking a point from each ball B_i in the statement of Proposition 2.1.17. However, whether we can take S to be definable over the same parameters as X requires a further argument. The following result of Y. Yin in [57] will help us in that task. Yin's proof of this lemma is offered in Appendix B.

Lemma 2.4.1 ([57, Lemma 2.19])**.** *Suppose that $A \subseteq R \cup \mathrm{RV}$. Then every A-L_{RV}-definable closed ball $B \subseteq R$ contains an A-L_{RV}-definable point.*

We can then take a first explicit step towards Hypotheses 1.1.9 (3).

Proposition 2.4.2. *Let $A \subseteq R \cup \mathrm{RV}$ and let $X \subseteq R$ be A-L_{RV}-definable. Then there is a finite A-L_{RV}-definable set $S \subseteq R$ such that for every ball $B \subseteq R \setminus S$ either $B \cap X = \varnothing$ or $B \subseteq X$.*

Proof. By taking $A' \subseteq R$ to be the union of $A \cap R$ and all $\hat{\mathrm{rv}}^{-1}(\xi)$ with $\xi \in A$, we obtain that X is A'-L_{convex}-definable. Applying Proposition 2.1.17 to X, we have that X equals

$$X = \big[(R \setminus C_{i_1}(B_1)) \cap C_{i_2}(B_2)\big] \cup \cdots \cup \big[(R \setminus C_{i_{m-1}}(B_{m-1})) \cap C_{i_m}(B_m)\big], \tag{2.3}$$

where $(i_1, \ldots, i_m) \in \{0,1\}^m$ and each B_i is an A'-L_{convex}-definable ball or point (recall that $\pm\infty$ are allowed as points; in such case the A'-definability is waived). It follows that each cut-interval above is A-L_{RV}-definable and thus, as done in the proof of Proposition 2.1.17, we conclude that each B_i is A-L_{RV}-definable.

We now collect the elements of S from each B_i. If B_i is a point in R, we put this point in S (N.B. when B_i is ∞ or $-\infty$ we do not put any corresponding point in S). If B_i is a closed ball, then by Lemma 2.4.1 there is an A-L_{RV}-definable point in B_i, which we put in S. Lastly, if B_i is an open ball $B(a, > \gamma)$, then the closed ball $B' := B(a, \geq \gamma)$ is A-L_{RV}-definable too; by Lemma 2.4.1, B' contains an A-L_{RV}-definable point, wich we put in S too. We claim that the set S so defined is as we want.

First of all, by construction, S is A-L_{RV}-definable. Now let $B \subseteq R \setminus S$ be a ball and suppose that $B \cap X \neq \varnothing$. It follows that $B \cap \big[(R \setminus C_{i_j}(B_j)) \cap C_{i_{j+1}}(B_{j+1})\big] \neq \varnothing$ for some $1 \leq j < m$. If $B \nsubseteq X$, then B must intersect either $C_{i_j}(B_j)$ or $R \setminus C_{i_{j+1}}(B_{j+1})$. Say the first case holds. If B_j is a point, then by the convexity of B and the fact that $B \cap X \neq \varnothing$ we get that the point in B_j is in $S \cap B$, a contradiction. If B_j is a closed ball, then either $B \subseteq B_j$ or $B_j \subseteq B$. We claim that the former containment cannot hold. When $i_j = 0$, $B \subseteq B_j$ implies that $B \cap C_0(B_j) = \varnothing$, and when $i_j = 1$, $B \subseteq B_j$ implies that $B \subseteq C_1(B_j)$, and so that $B \cap X = \varnothing$; a contradiction is reached in both cases. We must then have that $B_j \subseteq B$. It follows that the point in S corresponding to B_j is in B, a contradiction. Lastly, suppose that B_j is an open ball, and let B'_j be the closed ball with the same centre and radius of B_j. We have that either $B \subseteq B_j$ or $B_j \subsetneq B$, and the first containment cannot hold by an argument similar to the one when B_j was closed. Hence we have that $B_j \subsetneq B$. Clearly then $B'_j \subseteq B$, so the point in $B'_j \cap S$ is in B, a contradiction. Therefore, in any case, $B \subseteq X$. ❐

Chapter 3

T-stratifications in power-bounded T-convex fields

This chapter contains our first major new result. We prove that whenever T is a power-bounded o-minimal L-theory expanding RCF, the theory of all T-convex fields satisfies Hypotheses 1.1.9 and has the Jacobian property. This proves consequently that t-stratifications exist in such valued fields. A priori, these t-stratifications are $L_{\mathrm{RV}^{\mathrm{eq}}}$-definable, however, we also show in this chapter that they can be made L-definable. For the full picture on t-stratifications in T-convex fields, we give an example at the end of the chapter to show that power-boundedness of T is in fact necessary for their existence.

Throughout the chapter we assume that L is a language containing L_{or}, and, except for Section 3.4, we assume that T is a power-bounded o-minimal L-theory. We work with two languages for T-convex fields, L_{RV} described on page 51 and its expansion $L_{\mathrm{RV}^{\mathrm{eq}}}$ described on page 7. If (R, O_R) is a T-convex field, we write (R, RV) when working with L_{RV} and $(R, \mathrm{RV}^{\mathrm{eq}})$ when working with $L_{\mathrm{RV}^{\mathrm{eq}}}$.

3.1 Hypotheses 1.1.9 and b-minimality for $T_{\mathrm{RV}^{\mathrm{eq}}}$.

We prove that Hypotheses 1.1.9 hold for $T_{\mathrm{RV}^{\mathrm{eq}}}$, the common $L_{\mathrm{RV}^{\mathrm{eq}}}$-theory of all T-convex fields. We need to work in the language $L_{\mathrm{RV}^{\mathrm{eq}}}$ but we take advantage of the results in the last section by reducing $L_{\mathrm{RV}^{\mathrm{eq}}}$-definability to L_{RV}-definability, for if $A \subseteq R \cup \mathrm{RV}^{\mathrm{eq}}$ and X is an A-$L_{\mathrm{RV}^{\mathrm{eq}}}$-definable subset of R^n, then there exists a tuple η of elements of RV such that X is $((A \cap R) \cup \{\eta\})$-$L_{\mathrm{RV}}$-definable. The similar process of changing an $L_{\mathrm{RV}^{\mathrm{eq}}}$-formula with only field-sort variables into an L_{RV}-formula is frequently performed implicitly. In the whole of this chapter, by 'definable' we mean '$L_{\mathrm{RV}^{\mathrm{eq}}}$-definable'; all references to L_{RV}-definability will be made explicit.

Theorem 3.1.1. *Assume that T is power-bounded. Then the theory $T_{\mathrm{RV}^{eq}}$ satisfies Hypotheses 1.1.9. That is, if $(R, \mathrm{RV}^{\mathrm{eq}}) \models T_{\mathrm{RV}^{eq}}$ and $A \subseteq R \cup \mathrm{RV}^{\mathrm{eq}}$, then,*

(1) RV *is stably embedded in $(R, \mathrm{RV}^{\mathrm{eq}})$;*

(2) *every A-definable function $g : \mathrm{RV} \longrightarrow R$ has finite image;*

(3) *for every A-definable set $X \subseteq R$ there exists a finite A-definable set $S_0 \subseteq R$ such that every ball $B \subseteq R \setminus S_0$ is either contained in X or disjoint from X;*

(4) *for every A-definable function $f : R \longrightarrow R$ there exists an A-definable function $\chi : R \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ such that for each $q \in \chi(R)$, $f|_{\chi^{-1}(q)}$ is either constant or injective.*

Proof. (1) This is simply a consequence of rv being the only connection between R and RV, and quantifier elimination of T_{RV}. By a similar argument to the one in the discussion just before the proof of Proposition 2.3.3, a definable subset Q of RV is defined by a quantifier-free L^k_{og}-formula $\phi(z, \mathrm{rv}(t_1(a)), \ldots, \mathrm{rv}(t_m(a)), \eta)$ where z is a tuple of RV-sort variables, and a and η are tuples of elements of R and RV, respectively. If we set $\eta' := (\mathrm{rv}(t_1(a)), \ldots, \mathrm{rv}(t_m(a))) \in \mathrm{RV}^m$, then clearly Q is defined by the formula $\phi(z, \eta', \eta)$.

(2) Set $A_0 := A \cap R$ and let η be a tuple of elements of RV such that g is $(A_0 \cup \{\eta\})$-

definable. The result follows from Corollary 2.3.9.

(3) Let A_0 and η be as above. Let S_0 be the finite $(A_0 \cup \{\eta\})$-L_{RV}-definable set given by Proposition 2.4.2. Automatically, S_0 has the required property with respect to all balls $B \subseteq R \setminus S_0$. Finally, since S_0 is finite, using Lemma 2.3.8 we conclude that S_0 is A-definable.

(4) Let A_0 and η be as before. By Corollary 2.3.4, there is an $(A_0 \cup \{\eta\})$-L_{RV}-definable map $\chi' : R \longrightarrow \mathrm{RV}^m$ for which the desired conclusion holds. Let $\phi(x, y, z)$ be an A_0-L_{RV}-formula such that $\phi(x, y, \eta)$ defines χ'. We define $\chi'' : R \times \mathrm{RV}^l \longrightarrow \mathrm{RV}^m \subseteq \mathrm{RV}^{\mathrm{eq}}$ by declaring $\chi''(x, z) = y$ if and only if $\phi(x, y, z)$. Then χ'' is A_0-definable and for all $x \in R$, we have that $\chi''(x, \eta) = \chi'(x)$. For $x \in R$, we set $p_x : \mathrm{RV}^l \longrightarrow \mathrm{RV}^m$ as the map $z \mapsto \chi''(x, z)$ and then consider the code $\ulcorner p_x \urcorner$ of p_x. Notice that the map $x \mapsto \ulcorner p_x \urcorner$ is A-definable. Furthermore, since RV is stably embedded, we may assume that $\ulcorner p_x \urcorner$ is an element of $\mathrm{RV}^{\mathrm{eq}}$ (as opposed to it being an imaginary from the field-sort R). Therefore, if $\chi : R \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ is defined as $\chi(x) := \ulcorner p_x \urcorner$ for each $x \in R$, then χ is A-definable and, since each fibre of χ is contained in a fibre of χ', χ has the required property. ❒

We know from Proposition 1.1.10 that $T_{\mathrm{RV}^{\mathrm{eq}}}$ is b-minimal over $\mathrm{RV}^{\mathrm{eq}}$. The following is a stronger notion of b-minimality introduced in [8, Section 6].

Theorem 3.1.2. *Assume that T is power bounded. Then $T_{\mathrm{RV}^{eq}}$ is b-minimal with centres over $\mathrm{RV}^{\mathrm{eq}}$. That is, if $(R, \mathrm{RV}^{\mathrm{eq}})$ is a model of $T_{\mathrm{RV}^{eq}}$ and $A \subseteq R \cup \mathrm{RV}^{\mathrm{eq}}$, then*

- (b_1') *for every A-definable set $X \subseteq R$, there are A-definable functions $\chi : X \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ and $c : \chi(X) \longrightarrow R$ such that for each $q \in \chi(X)$, there is $\xi \in \mathrm{RV}$ for which $\chi^{-1}(q) = \mathrm{rv}^{-1}(\xi) + c(q)$;*
- (b_2) *there is no definable surjection from an auxiliary set to a ball in R;*
- (b_3) *for every $X \subseteq R$ and A-definable function $f : X \longrightarrow R$ there exists an A-definable function $\chi : X \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ such that for each $q \in \chi(X)$, $f|_{\chi^{-1}(q)}$ is either constant or injective.*

Proof. We only need to prove (b_1'). Let S_0 be the finite A-definable set obtained through

Theorem 3.1.1 (3). We let $\chi : X \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ be given by $\chi(x) = \ulcorner \mathrm{rv}(x - S_0) \urcorner$, for each $x \in X$. By Lemma 1.1.6, for every $q \in \chi(X)$ either $\chi^{-1}(q) = \{s\}$ with $s \in S_0$, or $\chi^{-1}(q)$ is a maximal ball disjoint from S_0. By compactness we obtain an A-definable function $c : \chi(X) \longrightarrow R$ such that $\hat{\mathrm{v}}(b - c(\chi(b))) = \max\{\hat{\mathrm{v}}(b - t) \mid t \in S_0\}$ for all $b \in X$. If $\chi^{-1}(q)$ is a singleton, then clearly $\chi^{-1}(q) - c(q) = \{0\} = \mathrm{rv}^{-1}(0)$. If instead $\chi^{-1}(q)$ is an open ball, where, say, $q = \chi(b)$, we have that

$$\begin{aligned}\chi^{-1}(q) = B(b, > \hat{\mathrm{v}}(b - c(q))) &= \{x \in R \mid \mathrm{rv}(x - c(q)) = \mathrm{rv}(b - c(q))\} \\ &= c(q) + \mathrm{rv}^{-1}(\mathrm{rv}(b - c(q))).\end{aligned}$$ ❒

From b-minimality with centres we obtain a theorem on *cell-decomposition with centres* ([8, Theorem 6.4]). Although it could be interesting to explore this and other consequences of b-minimality with centres we do not do so in this work.

We now make a remark of general interest. In the proof of Theorem 3.1.2 we used Theorem 3.1.1 (3) to obtain Theorem 3.1.2 (b_1'). The remark below shows that this works the other way around too. The importance of this fact is that proving Theorem 3.1.2 (b_1') could be a strategy to obtain Theorem 3.1.1 (3) in other contexts.

Remark 3.1.3. Assume that T is power bounded. Then the following argument shows that Theorem 3.1.2 (b_1') can be used to deduce Theorem 3.1.1 (3).

Proof. Let $X \subseteq R$ be A-definable, and suppose that χ and c are as in (b_1'). The set $S_0 := c(\chi(X))$ is A-definable and, by Corollary 2.3.9, finite. Let $B \subseteq R$ be any ball disjoint from S_0 and suppose that $B \cap X \neq \varnothing$. We want to show that $B \subseteq X$. Fix $b \in B$. We first claim that for each $x \in B$, we have that $\mathrm{rv}(x - c(\chi(b))) = \mathrm{rv}(b - c(\chi(b)))$. If this failed, there would be $x \in B$ such that $\hat{\mathrm{v}}(b - c(\chi(b))) \geq \hat{\mathrm{v}}(x - b)$ and so $c(\chi(b)) \in B \cap S_0$, a contradiction. To finish, notice that $\chi^{-1}(\chi(b)) - c(\chi(b)) = \mathrm{rv}^{-1}(\mathrm{rv}(b - c(\chi(b))))$ because $b \in \chi^{-1}(\chi((b))$; so, the previous claim implies that $B \subseteq \chi^{-1}(\chi(b)) \subseteq X$. ❒

3.2 The Jacobian property in power-bounded T-convex fields

In this section we prove that the Jacobian property holds for $T_{\mathrm{RV}^{\mathrm{eq}}}$. The proof is inspired by that of I. Halupczok of the Jacobian property for valued fields with analytic structure, [26, Subsection 5.3]. The strategy is as follows. We start an inductive argument on n. In the inductive step we assume that $n \geq 1$ and, if $n > 1$, that $T_{\mathrm{RV}^{\mathrm{eq}}}$ has the Jacobian property up to $n-1$ (in Halupczok's original presentation the Jacobian property at $n = 0$ is simply Theorem 3.1.1 (4)). This assumption and Theorem 3.1.1 allow us to apply Theorem 1.4.6 to ensure that for any $m \leq n$, t-stratifications exist for every definable map from R^m to RV$^{\mathrm{eq}}$. Using results from Chapter 2, if $f : R^n \longrightarrow R$ is an A-definable function, where $A \subseteq R \cup \mathrm{RV}^{\mathrm{eq}}$, we obtain an A-definable map $\rho : R^n \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ such that on each fibre $\rho^{-1}(q)$, f equals the restriction of an L-definable function, the Jacobian $\mathrm{Jac}(f)$ of f exists and $\mathrm{rv}(\mathrm{Jac}(f))$ is constant. We take a t-stratification for ρ and improve it in such a way that the fibres of its rainbow χ have a particular form. The final step is a calculation showing that f has the Jacobian property on each fibre of χ containing an open ball.

The following easy lemma, an analogue of [26, Lemma 5.8], helps us to make the Jacobian property rest on properties of L-definable functions and will be used in the final calculation.

Lemma 3.2.1. *Let $g : O_R \longrightarrow O_R$ be an L-definable differentiable function such that for all $x \in O_R$, $g'(x) \in O_R$ and $\mathrm{res}(g'(x))$ is constant. Then for all distinct $x, x' \in O_R$,*

$$\mathrm{v}(g(x) - g(x') - g'(0)(x - x')) > \mathrm{v}(x - x').$$

Proof. We assume that $g'(0) = 0$, as otherwise we may replace g by $g(x) - g'(0)x$. It follows that $\mathrm{v}(g'(x)) > 0$ for all $x \in O_R$. By the Mean Value Theorem for L-definable functions (see [15, Chapter 7 §2]), for distinct x and x' in O_R, there exists $x'' \in O_R$ such that $g(x) - g(x') = g'(x'')(x - x')$. Hence $\mathrm{v}(g(x) - g(x')) > \mathrm{v}(x - x')$ as required. ❐

We now show how to improve a t-stratification so that the fibres of its rainbow have a desired specific shape. We fix a set of parameters $A \subseteq R \cup \mathrm{RV}^{\mathrm{eq}}$ and an A-definable set $B_0 \subseteq R^n$ (typically a ball or the whole of R^n). When not specified otherwise, by $(S_i)_{i \leq n}$ being a t-stratification we mean that $(S_i)_{i \leq n}$ is a t-stratification of the fixed set B_0.

The following lemma is essentially [26, Lemma 4.3 and Lemma 4.22] and sheds more light on the nature of the rainbow of a t-stratification.

Lemma 3.2.2. *Let $(S_i)_{i \leq n}$ be an A-definable t-stratification and C a fibre of its rainbow. The following hold.*

(a) *C either consists of a single point in S_0 or it is entirely contained in a ball $B \subseteq S_{\geq 1}$;*

(b) *if $C \subseteq S_d$ and $B \subseteq S_{\geq d}$ intersects C, then $\operatorname{affdir}(C) = \operatorname{tsp}_B((S_i)_{i \leq n})$; in particular, C is subaffine;*

(c) *Let $\pi : K^n \longrightarrow K^d$ be an exhibition of $\operatorname{affdir}(C)$. Then there exists an A-definable function $c : \pi(C) \longrightarrow R^{n-d}$ such that $C = \operatorname{graph}(c)$;*

(d) *Let $d := \dim(C)$ and for simplicity assume that π above is the projection to the first d coordinates. The function $\hat{c} : \pi(C) \times R^{n-d} \longrightarrow \pi(C) \times R^{n-d}$ defined as $\hat{c}(x, y) := (x, y + c(x))$ is A-definable and can be written as $\varphi \circ M$, where φ is a risometry, $M \in GL_n(O_R)$, and $\pi \circ \varphi = \pi \circ M = \pi$.*

Proof. (a) Since $\hat{\mathrm{rv}}(x - S_0) = \hat{\mathrm{rv}}(x' - S_0)$ for all $x, x' \in C$, C is contained in a fibre of the map $x \mapsto \hat{\mathrm{rv}}(x - S_0)$. Using that S_0 is finite, the result follows immediately from Lemma 1.1.6.

(b) Assume that $C \subseteq S_d$ and let $B \subseteq S_{\geq d}$ be a ball intersecting C. By Proposition 1.3.7 $(S_i)_{i \leq n}$ reflects C, so $((S_i)_{i \leq n}, C)$ is d-translatable on B. Lemma 1.2.12 (b) implies that $\dim(C) \geq d$; and we also have that $\dim(C) \leq \dim(S_d) = d$, so $\dim(C) = d$. We set $V := \operatorname{tsp}_B(S_i)_{i \leq n}$ and claim that $\operatorname{affdir}(C) \subseteq V$. When $d = 0$, by an argument like the one for (a), C must consist of a point in S_0 and so $\operatorname{affdir}(C) = \{0\} \subseteq V$ trivially. As inductive hypothesis we assume that (b) holds for all the less-than-d-dimensional fibres of rainbows of t-stratifications. By (a), there is a ball $B' \subseteq S_{\geq 1}$ containing C. Let ρ be an exhibition of $V' := \operatorname{tsp}_{B'}((S_i)_{i\leq n})$. Let $q \in \rho(B)$ and set $F_q := \rho^{-1}(q) \cap B'$. Lemma 1.3.10 implies that $(S_i \cap F_q)_{d\leq i\leq n}$ is a t-stratification of B' reflecting $C \cap F_q$, and, moreover, $C \cap F_q$ is a fibre of the rainbow of $(S_i \cap F_q)_{d\leq i\leq n}$. By the inductive hypothesis, for any ball $B'' \subseteq F_q$ intersecting $C \cap F_q$, we have that

$$\operatorname{affdir}(C \cap F_q) = \operatorname{tsp}_{B''}((S_i \cap F_q)_{d\leq i\leq n}).$$

Since $((S_i)_{i\leq n}, C)$ is V'-translatable on B' and ρ exhibits V', $\operatorname{tsp}_{B''}((S_i \cap F_q)_{d\leq i\leq n})$ does not depend on $q \in \rho(B)$ and B'' as above; we denote this space by V''. We pick $q \in \rho(B)$ such that $F := F_q = \rho^{-1}(q)$ intersects $C \cap B$, and set $B'' := B \cap F$. It follows that that $V = \operatorname{tsp}_B((S_i)_{i\leq n}) = V' + V''$. We are finally ready to show that $\operatorname{affdir}(C) \subseteq V$. Let $(\alpha_w : B \longrightarrow B)_{w\in\pi(B-B)}$ be a translater witnessing V-translatability of $((S_i)_{i\leq n}, C)$ on B. For distinct $x, x' \in C$ we have that $\alpha_{\pi(x-x')}$ takes $\pi^{-1}(\pi(x)) \cap B$ to $\pi^{-1}(\pi(x')) \cap B$, so $\alpha_{\pi(x-x')}(x) = x'$, and it follows that $\operatorname{dir}(x - x') \in V$ by Definition 1.2.10 (iii).

(c) If C consists of a single point in S_0 the result is trivial. We thus assume that C is entirely contained in a ball $B \subseteq S_{\geq 1}$. It is enough to show that for each $q \in \pi(C)$, $C \cap B \cap \pi^{-1}(q)$ is a singleton. Indeed, if x, x' are in that intersection but $x \neq x'$, then $\operatorname{dir}(x - x') \neq 0$ but $\tilde{\pi}(\operatorname{dir}(x - x')) = 0$ because $\pi(x) = \pi(x')$; this contradicts that π exhibits $\operatorname{affdir}(C)$.

(d) If $d = 0$, then C consists of a single point in S_0, and in such case the result is trivial, so we assume that $d > 0$. It follows that C is contained in a ball $B \subseteq S_{\geq d}$ with $B \cap C \neq \varnothing$. We set $V := \operatorname{affdir}(C)$; recall that by (b), $V = \operatorname{tsp}_B((S_i)_{i\leq n})$. The idea to prove (d) is that mapping $\pi(C)$ to a lift of V is a linear map, and then V-translatability implies that a risometry takes (a coset of) such lift to C. We fix a lift $\overline{V} \subseteq R^n$ of V. We denote the projection to the last $n - d$ coordinates by $\pi^\perp$. Since π exhibits V, for all $x \in \pi(B)$ the set $\pi^{-1}(x) \cap \overline{V}$ consists of a single point (x, v_x), with $\hat{\mathrm{v}}(v_x) \geq \hat{\mathrm{v}}(x)$ (see Remark 1.2.7). For $(x, y) \in R^d \times R^{n-d}$ define $M(x, y) := (x, y + \pi^{\perp}(v_x))$. Then M is clearly linear and invertible, and, since the inequalities $\hat{\mathrm{v}}(M(x, y)) \geq \min\{\hat{\mathrm{v}}(x), \hat{\mathrm{v}}(y), \hat{\mathrm{v}}(\pi^{\perp}(v_x))\} \geq \hat{\mathrm{v}}(x, y)$ hold for all $(x, y) \in R^d \times R^{n-d}$, we have that $M \in \mathrm{GL}_n(O_R)$. It only remains to show that $\varphi := \hat{c} \circ M^{-1}$ is a risometry. Let $(x, y), (x', y') \in R^d \times R^{n-d}$. Note that $\hat{\mathrm{rv}}(\varphi(x, y) - \varphi(x', y')) = \hat{\mathrm{rv}}((x, y) - (x', y'))$ is implied by $\hat{\mathrm{v}}(-v_x + v_{x'} + c(x) - c(x')) > \hat{\mathrm{v}}(x - x')$, which in turn follows from

$$\hat{\mathrm{rv}}(x - x', v_x - v_{x'}) = \hat{\mathrm{rv}}(x - x', c(x) - c(x')), \tag{3.1}$$

using that $\hat{\mathrm{v}}(v_x - v_{x'}) \geq \hat{\mathrm{v}}(x - x')$. We thus only need to prove (3.1). Obviously, $\pi(x - x', v_x - v_{x'}) = \pi(x - x', c(x) - c(x'))$, and $\mathrm{dir}(x - x', v_x - v_{x'}) = \mathrm{dir}(x - x', 0) = \mathrm{dir}(x - x', c(x) - c(x'))$. Also notice that $\hat{\mathrm{v}}(\pi(x - x', v_x - v_{x'})) = \hat{\mathrm{v}}(x - x')$, so (3.1) follows by Lemma 1.2.8 (c). ❐

Definition 3.2.3. Let $(S_i)_{i \leq n}$ be a t-stratification and let $d \leq n$. We say that $(S_i)_{i \leq n}$ has the property $(*)_d$ if for any $j \geq d$ and fibre $C \subseteq S_j$ of the rainbow of $(S_i)_{i \leq n}$, the corresponding function $c : \pi(C) \longrightarrow R^{n-j}$ is the restriction of an L-definable differentiable function to $\pi(C)$.

The *product* of $\chi_1, \chi_2 : B_0 \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ is defined as the map $x \mapsto (\chi_1(x), \chi_2(x))$. To make the compactness arguments below to work, we assume that $(R, \mathrm{RV}^{\mathrm{eq}})$ is a sufficiently saturated model of $T_{\mathrm{RV}^{\mathrm{eq}}}$.

Lemma 3.2.4. *Suppose that $T_{\mathrm{RV}^{eq}}$ has the Jacobian property up to $n - 1$. If the A-definable t-stratification $(S_i)_{i \leq n}$ has property $(*)_{d+1}$ for some $0 \leq d < n$, then there is an A-definable t-stratification $(S'_i)_{i \leq n}$ reflecting $(S_i)_{i \leq n}$ and having property $(*)_d$.*

Proof. Let $C \subseteq S_d$ be a fibre of the rainbow of $(S_i)_{i \leq n}$. Note that if $d = 0$, then there is nothing else to do, so we assume that $d > 0$. We let $\pi : R^n \longrightarrow R^d$ be an exhibition of $\mathrm{affdir}(C)$ and $c : \pi(C) \longrightarrow R^{n-d}$ be as in Lemma 3.2.2. By Proposition 2.3.3 there is a $\ulcorner C \urcorner$-definable map $\chi_\pi : \pi(C) \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ (in fact with finite image) such that on any fibre of χ_π, c is the restriction of an L-definable differentiable function. By composing with $\pi : C \longrightarrow \pi(C)$ we obtain a definable map $C \longrightarrow \mathrm{RV}^{\mathrm{eq}}$. Now we do this for all exhibitions of $\mathrm{affdir}(C)$, obtaining a collection of maps. We let $\chi_C : C \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ be the product of all the maps in said collection. Note that χ_C is $\ulcorner C \urcorner$-definable. We repeat this construction for all the fibres of the rainbow of $(S_i)_{i \leq n}$ contained in S_d. Using that these fibres cover the whole of S_d, and through a compactness argument, we obtain a single A-definable map $\chi : S_d \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ such that for any C, π and c as above and fibre $X \subseteq C$ of χ, the map $c|_{\pi(X)}$ equals the restriction of an L-definable differentiable function to $\pi(X)$.

By Theorem 3.1.1 and the assumption that $T_{\mathrm{RV}^{\mathrm{eq}}}$ has the Jacobian property up to $n-1$, we can apply Theorem 1.4.6 to obtain an A-definable t-stratification $(T_i)_{i \leq n}$ reflecting both $(S_i)_{i \leq n}$ and χ. By Lemma 1.3.5, setting $S'_{\leq j} := T_{\leq j}$ for $j < d$ and $S'_{\leq j} := S_{\leq j} \cup T_{\leq d-1}$ for $j \geq d$, defines an A-definable t-stratification reflecting both $(S_i)_{i \leq n}$ and χ, and satisfying that $S'_j \subseteq S_j$ whenever $j \geq d$ (and of course $S'_{\leq n} = S_{\leq n} = B_0$). We claim that $(S'_i)_{i \leq n}$ has property $(*)_d$.

Let $C' \subseteq S'_j$ be a fibre of the rainbow of $(S'_i)_{i \leq n}$, for some $j \geq d$. Then there is a fibre $C \subseteq S_j$ of the rainbow of $(S_i)_{i \leq n}$ such that $C' \subseteq C$. It then follows that $\mathrm{affdir}(C') = \mathrm{affdir}(C)$ and, if π is an exhibition of this subspace, the corresponding map $c' : \pi(C') \longrightarrow R^{n-j}$ is the restriction of the map $c : \pi(C) \longrightarrow R^{n-j}$ to $\pi(C')$. If $j > d$, by property $(*)_{d+1}$ for $(S_i)_{i \leq n}$, the map c is the restriction to $\pi(C)$ of an L-definable differentiable function, and this obviously implies a similar conclusion for c'. Now we let instead $j = d$. Since $(S'_i)_{i \leq n}$ reflects χ, Proposition 1.3.7 tells us that the rainbow of $(S'_i)_{i \leq n}$ refines χ, so χ is constant on C' (because said rainbow is constant on C'). The construction of χ then ensures that c' is the restriction to $\pi(C')$ of an L-definable differentiable function. ❐

Incidentally, notice that by the proof of Lemma 3.2.4 $(S'_i)_{i \leq n}$ can be taken such that $S'_{\leq j} \subseteq S_{\leq j}$ for each $j \leq n$. Iterating Lemma 3.2.4 we obtain the following corollary.

Corollary 3.2.5. *Under the assumptions of the last lemma, for any A-definable t-stratification $(S_i)_{i\leq n}$ there exists an A-definable t-stratification $(S'_i)_{i\leq n}$ reflecting $(S_i)_{i\leq n}$, satisfying that $S'_{\leq j} \subseteq S_{\leq j}$ for all $j \leq n$ and fulfilling $(*)_0$; that is, for any $j \leq n$, any fibre $C \subseteq S'_j$ of the rainbow of $(S'_i)_{i\leq n}$, and any exhibition π of $\operatorname{affdir}(C)$, the map $c : \pi(C) \longrightarrow R^{n-j}$ is the restriction of an L-definable differentiable function to $\pi(C)$.*

The following remark is used in the proof of our next result. Recall that $\mathrm{GL}_n(O_R)$ stands for the set of all invertible $(n \times n)$-matrices M such that both M and M^{-1} have entries in O_R.

Remark 3.2.6. Let $X, Y, Z \subseteq R^n$ and suppose that $h : X \longrightarrow Y$ equals the composition of a risometry and a matrix in $\mathrm{GL}_n(O_R)$. Then the order of composition does not matter; that is, there are risometries φ_i and matrices $M_i \in \mathrm{GL}_n(O_R)$ for $i = 1, 2$, such that $M_1 \circ \varphi_1 = h = \varphi_2 \circ M_2$. Moreover, if $h' : Y \longrightarrow Z$ is also the composition of a risometry and a matrix in $\mathrm{GL}_n(O_R)$, then $h' \circ h$ can be written as the composition of a risometry and a matrix in $\mathrm{GL}_n(O_R)$.

Proof. We assume that $h = M \circ \varphi$, with φ a risometry and $M \in \mathrm{GL}_n(O_R)$. By Lemma 1.1.5 (c), $\varphi^M := M \circ \varphi \circ M^{-1}$ is a risometry. Trivially, $\varphi^M \circ M = M \circ \varphi = h$, showing that the order of composition in h does not matter. The second part of the statement is then clear, using the trivial fact that the composition of two risometries is a risometry. ❐

The lemma below is an adaptation of [26, Lemma 5.9] and provides a particular description of the fibres of the rainbow of $(S'_i)_{i\leq n}$ in Corollary 3.2.5. In the proof we make use of balls in different powers of R and frequently write that a 'ball $B \subseteq R^d$ is contained in a ball $B' \subseteq R^{d'}$', with $d < d'$, to mean that $B \subseteq \pi(B')$ where $\pi : R^{d'} \longrightarrow R^d$ is a coordinate projection. Most of the time we use this when B and B' have the same radius (and are either both open or both closed).

Lemma 3.2.7. *Suppose that $T_{\mathrm{RV}^{eq}}$ has the Jacobian property up to $n-1$ and let $(S_i)_{i\leq n}$*

be an A-definable t-stratification of R^n. Then there exists an A-definable t-stratification $(S'_i)_{i\leq n}$ satisfying that $S'_{\leq j} \subseteq S_{\leq j}$ for all $j \leq n$, and such that whenever $C \subseteq S'_n$ is a fibre of the rainbow of $(S'_i)_{i\leq n}$, then there are open balls $B_1, \ldots, B_n \subseteq R$ and an L-definable differentiable bijection $h : B_1 \times \cdots \times B_n \longrightarrow C$ which equals the composition of a risometry and a matrix in $GL_n(O_R)$.

N.B. Recall that by simply 'L-definable' we mean 'L-definable with parameters'. In particular, h above is L-definable with some parameters from R; however, we did not make an effort to pin down what parameters are used (and their relation to A). This is not important as the only place at which we use h later is to apply the technical Lemma 3.2.1.

Proof of Lemma 3.2.7. Let $(S'_i)_{i\leq n}$ be as in Corollary 3.2.5, and let $C \subseteq S'_n$ be a fibre of its rainbow. To be able to start an inductive argument below from 0, we suppose that $S'_0 \neq \varnothing$ (otherwise, we apply the process below after putting $S'_0 = \{0\}$). Note that $C \cap S'_0 = \varnothing$ because $S'_n \cap S'_0 = \varnothing$, so Lemma 3.2.2 (a) implies that C is entirely contained in a ball $B' \subseteq R^n \setminus S'_0$. Once we know this, we take $x_0 \in S'_0$ and let B_0 be a ball containing both x_0 and B' (it is enough to set $B_0 := B(a, \geq \hat{\mathrm{v}}(x_0 - a))$ for some $a \in B'$). Since $B_0 \cap S'_0 \neq \varnothing$, we have that $(S'_i)_{i\leq n}$ is not d-translatable on B_0 for any $d > 0$, so $\mathrm{tsp}_{B_0}((S'_i)_{i\leq n}) = \{0\}$. This situation corresponds to the case $d = 0$, with $\pi_0 : R^n \longrightarrow R^0$ $(= \{0\})$ and $\lambda := \mathrm{rad}(B_0)$, in the following set of conditions, which we aim to prove for $d = n$.

For a coordinate projection $\pi : R^n \longrightarrow R^d$ and $\lambda \in \Gamma$:

(1_d) λ is maximal such that: for every $q \in \pi(C)$, there exists an open ball $B_q \subseteq \pi^{-1}(q)$ of radius λ satisfying that $C \cap \pi^{-1}(q) \subseteq B_q$;

(2_d) for every $q, q' \in \pi(C)$, there exists a risometry between B_q and $B_{q'}$ respecting the rainbow of $(S'_i)_{i\leq n}$;

(3_d) for every $q \in \pi(C)$, if $B \subseteq R^n$ is the open ball of radius λ containing B_q, then π exhibits $\mathrm{tsp}_B((S'_i)_{i\leq n})$;

(4_d) there are open balls $B_1, \ldots, B_d \subseteq R$ and an L-definable differentiable bijection

$h_d : B_1 \times \cdots \times B_d \longrightarrow \pi(C)$ which can be written as the composition of a risometry and a matrix in $\mathrm{GL}_d(O_R)$.

Clearly the result follows from (4_n). We pursue an inductive argument to show that indeed (4_n), along with (1_n)-(3_n), holds. We argued earlier that (1_d)-(4_d) hold when $d = 0$, and below we show that whenever $d < n$ and there are π and λ as above satisfying conditions (1_d)-(4_d), then there is $d' > d$ and corresponding π' and λ' such that ($1_{d'}$)-($4_{d'}$) hold.

Suppose that $d < n$ and that there are $\lambda \in \Gamma$ and $\pi : R^n \longrightarrow R^d$ that make (1_d)-(4_d) hold. We first propose $d' > d$, $\lambda' \in \Gamma$ and $\pi' : R^n \longrightarrow R^{d'}$. To simplify notation we assume that π is the projection to the first d coordinates. Fix $q \in \pi(C)$ and let B_q and B be as in (2_d) and (3_d). If $S'_d \cap B_q = \varnothing$, then $S'_d \cap B = \varnothing$ and $(S'_i)_{i \leq n}$ would be $(d+1)$-translatable on B, so $\dim(\mathrm{tsp}_B((S'_i)_{i\leq n})) \geq d + 1$. But this is absurd because from (3_d), the space $\mathrm{tsp}_B((S'_i)_{i\leq n})$ is exhibited by $\pi : R^n \longrightarrow R^d$, so $\dim(\mathrm{tsp}_B((S'_i)_{i\leq n})) = d$ by Lemma 1.2.12 (b). We conclude that $S'_d \cap B_q$ is non-empty. Again because π exhibits $\mathrm{tsp}_B((S'_i)_{i\leq n})$, Lemma 1.3.10 implies that $(S'_i \cap B_q)_{d\leq i\leq n}$ is a t-stratification of B_q and that $C \cap B_q$ is a fibre of its rainbow. Clearly $(C \cap B_q) \cap S'_d = \varnothing$ (because $C \cap S'_d = \varnothing$ by Remark 1.3.9), so Lemma 3.2.2 (a) implies that $C \cap B_q$ is contained in a ball $D_q \subseteq B_q \setminus S'_d$. Since $S'_d \cap B_q$ is the first stratum of a t-stratification, it must be finite; so, if we further assume that D_q above is maximal, then Lemma 1.1.7 implies that there is $s_q \in S'_d \cap B_q$ and $\xi_q \in \mathrm{RV}^{(n-d)}$ such that $D_q = s_q + (\{0\}^d \times \hat{\mathrm{rv}}^{-1}(\xi_q))$. For any other $q' \in \pi(C)$, applying the risometry in (2_d) from B_q to $B_{q'}$ provides us with $D_{q'}$ and $s'_q \in S'_d \cap B_{q'}$ with analogous properties. For the balls $D_{q'}$ the corresponding $\xi_{q'}$ equals ξ_q by construction, so all of these balls have radius $\lambda' := \hat{\mathrm{v}}_{\mathrm{RV}}(\xi_q)$. We set $V := \mathrm{tsp}_{D_q}((S'_i \cap B_q)_{d\leq i\leq n})$, and we claim that for any other $q' \in \pi(C)$, $\mathrm{tsp}_{D_{q'}}((S'_i \cap B_{q'})_{d\leq i\leq n}) = V$. This can be easily seen by composing a translater witnessing V-translatability of $(S'_i \cap B_q)_{d\leq i\leq n}$ on D_q with the risometry in (2_d) from B_q to $B_{q'}$; the result is a translater witnessing V-translatability of $(S'_i \cap B_{q'})_{d\leq i\leq n}$ on $D_{q'}$. This establishes our claim. Now, since $S'_d \cap D_q = \varnothing$, the collection $(S'_i \cap B_q)_{d\leq i\leq n}$ is at least

1-translatable on D_q, so $\dim(V) \geq 1$. We set $d' := d + \dim(V)$ and $\pi' : R^n \longrightarrow R^{d'}$ as the map $x \mapsto \pi(x) \oplus \rho(x - \pi(x))$, where $\rho : R^{n-d} \longrightarrow R^{d'-d}$ is an exhibition of V. We show below that d', λ' and π' validate ($1_{d'}$)-($4_{d'}$).

Before starting the proof of ($1_{d'}$)-($4_{d'}$), we point out that, if $q \in \pi(C)$ and the corresponding $s_q \in S'_d \cap B_q$ is in a fibre $\widetilde{C}$ of the rainbow of $(S'_i)_{i \leq n}$, then in fact the whole collection $\{s_q\}_{q \in \pi(C)}$ is contained in $\widetilde{C}$, for the risometries in (2_d) respect the rainbow of $(S'_i)_{i \leq n}$. Fixing such $\widetilde{C}$, it is also clear that $\pi(\widetilde{C}) = \pi(C)$ (by construction, each $s_q \in \widetilde{C}$ correspond to the fibre $\pi^{-1}(q)$). We also have that $\operatorname{affdir}(\widetilde{C}) = \operatorname{tsp}_B((S'_i)_{i \leq n})$, so π exhibits $\operatorname{affdir}(C)$ according to (3_d).

To simplify notation we assume that $\rho : R^{n-d} \longrightarrow R^{d'-d}$ is the projection to the first $d' - d$ coordinates. So π' is the projection to the first d' coordinates on R^n. We also apply π to elements of $R^{d'}$ in the obvious way, namely, $\pi(x_1, \ldots, x_d, \ldots, x_{d'}) := (x_1, \ldots, x_d)$.
($1_{d'}$) For $q \in \pi'(C)$ we set $B'_q := D_{\pi(q)} \cap \pi'^{-1}(q)$. Since $C \cap \pi^{-1}(\pi(q)) \subseteq D_{\pi(q)}$, clearly $C \cap \pi'^{-1}(q) \subseteq B'_q \subseteq \pi'^{-1}(q)$.
($2_{d'}$) Let $q, q' \in \pi'(C)$ and let $\varphi := \varphi_{\pi(q),\pi(q')} : B_{\pi(q)} \longrightarrow B_{\pi(q')}$ be the risometry given by (2_d). Also, let $(\alpha_r^{q'} : D_{\pi(q')} \longrightarrow D_{\pi(q')})_{r \in \rho(D_{\pi(q')} - D_{\pi(q')})}$ be a translater witnessing V-translatability of $(S'_i)_{i \leq n}$ on $D_{\pi(q')}$. The needed risometry $\psi_{q,q'} : B'_q \longrightarrow B'_{q'}$ is obtained as follows: if $x \in B'_q$, we have that $x' := \varphi(x) \in B_{\pi(q')}$, and since $\alpha_{\rho(x'-q')}$ takes $\rho^{-1}(x') \cap D_{\pi(q')}$ to $\rho^{-1}(q') \cap D_{\pi(q')}$, we have that $x'' := \alpha_{\rho(x'-q')}(x') \in B'_{q'}$; it is then enough to put $\psi_{q,q'}(x) := x''$. From the properties of φ and the maps $\alpha_r^{q'}$ we get that $\psi_{q,q'}$ is a bijection from B'_q to $B'_{q'}$. Using Lemma 1.2.8 (a), showing that $\psi_{q,q'}$ satisfies that $\hat{\operatorname{rv}}(\psi_{q,q'}(x_1) - \psi_{q,q'}(x_2)) = \hat{\operatorname{rv}}(x_1 - x_2)$ for all $x_1, x_2 \in B_q$, is a straightforward, technical argument analogous to the one in the last part of the proof of Lemma 4.3.2.
($3_{d'}$) Fix $q \in \pi'(C)$ and let $B' \subseteq R^n$ be the open ball of radius λ' containing B'_q. Following (3_d), we let $B \subseteq R^n$ be the open ball of radius λ containing $B_{\pi(q)}$. Since π exhibits $\operatorname{tsp}_B((S'_i)_{i \leq n})$ and ρ exhibits $V = \operatorname{tsp}_{D_{\pi(q)}}((S'_i \cap B_{\pi(q)})_{d \leq i \leq n})$, to see that π' exhibits $\operatorname{tsp}_{B'}((S'_i)_{i \leq n}) \subseteq k^{d'}$ it is enough to notice that $\operatorname{tsp}_{B'}((S'_i)_{i \leq n}) = \operatorname{tsp}_B((S'_i)_{i \leq n}) \oplus V$, which holds by construction.

($4_{d'}$) Since $\widetilde{C} \subseteq S'_d$ is a fibre of the rainbow of $(S'_i)_{i\leq n}$ and π exhibits $\mathrm{affdir}(\widetilde{C})$, the properties of $(S'_i)_{i\leq n}$ described in Corollary 3.2.5 imply that the corresponding function $\tilde{c} : \pi(\widetilde{C}) \longrightarrow R^{n-d}$ (see Lemma 3.2.2 (c)) is the restriction to $\pi(\widetilde{C})$ of an L-definable differentiable function. Recall that $\pi(\widetilde{C}) = \pi(C)$; we use this fact below without further mention. We let U be the open ball $\rho(\hat{\mathrm{rv}}^{-1}(\xi_{\pi(q)})) \subseteq R^{d'-d}$. By Lemma 3.2.2 (d), the bijection $g : \pi(C) \times U \longrightarrow \pi'(C)$ defined as $g(x, y) := (x, \rho \circ \tilde{c}(x) + y)$ is the composition of a risometry and a matrix in $\mathrm{GL}_{d'}(O_R)$. We define $h_{d'} : B_1 \times \cdots \times B_d \times U \longrightarrow \pi'(C)$ as the composition $g \circ (h_d \times id_U)$, that is, for any $x \in B_1 \times \cdots \times B_n$ and any $y \in U$, $h_{d'}(x, y) := (h_d(x), \rho \circ \tilde{c} \circ h_d(x) + y)$. Using the property of h_d in (4_d), $h_d \times id_U$ is the composition of a risometry and a matrix in $\mathrm{GL}_{d'}$, so the same holds for $h_{d'}$ by Remark 3.2.6. Lastly, since both h_d and $\tilde{c}$ are L-definable and differentiable, so is $h_{d'}$. Thus ($4_{d'}$) holds, and the proof is finished. ❒

We finally prove the main result of the section, which is Theorem B in the Introduction.

Theorem 3.2.8. *The theory $T_{\mathrm{RV}^{eq}}$ has the Jacobian property. That is, for any $n \geq 1$, any $(R, \mathrm{RV^{eq}}) \vDash T_{\mathrm{RV}^{eq}}$ and any $A \subseteq R \cup \mathrm{RV^{eq}}$, if $X \subseteq R^n$ and $f : X \longrightarrow R$ are A-definable, then there exist an A-definable map $\chi : X \longrightarrow \mathrm{RV^{eq}}$ such f has the Jacobian property on each n-dimensional fibre of χ, i.e. whenever $q \in \chi(X)$ is such that $\dim(\chi^{-1}(q)) = n$, either $f|_{\chi^{-1}(q)}$ is constant or there exists $z \in R^n \setminus \{0\}$ satisfying that for all distinct $x, x' \in \chi^{-1}(q)$,*

$$\mathrm{v}(f(x) - f(x') - \langle z, x - x' \rangle) > \hat{\mathrm{v}}(z) + \hat{\mathrm{v}}(x - x'), \tag{3.2}$$

Proof. We let $n \geq 1$ and we assume as inductive hypothesis that $T_{\mathrm{RV^{eq}}}$ has the Jacobian property up to $n - 1$ (no extra assumption is considered when $n - 1 = 0$). Let A, X and f be as in the statement of the theorem. As usual, after picking parameters in RV, we make f to be L_{RV}-definable. By Lemma 2.3.6, there is an A-definable function $\chi' : R^n \longrightarrow \mathrm{RV^{eq}}$ such that if $q \in \chi'(R^n)$ and $\dim(\chi'^{-1}(q)) = n$, then $\chi'^{-1}(q)$ is open, $\mathrm{Jac}(f)$ exists on $\chi'^{-1}(q)$ and $\hat{\mathrm{rv}} \circ \mathrm{Jac}(f)$ is constant on $\chi'^{-1}(q)$.

By the inductive hypothesis and Theorem 1.4.6, there exists an A-definable t-stratification $(S_i)_{i\leq n}$ reflecting χ'. Let $(S'_i)_{i\leq n}$ be the t-stratification obtained by applying Lemma 3.2.7 to $(S_i)_{i\leq n}$. By Proposition 1.3.7 the rainbow χ of $(S'_i)_{i\leq n}$ refines χ', so on each n-dimensional fibre of χ, $\mathrm{Jac}(f)$ exists and $\hat{\mathrm{rv}} \circ \mathrm{Jac}(f)$ is constant. Using this, we prove below that (3.3) holds on all the n-dimensional fibres of χ.

We fix an n-dimensional fibre C of χ. It follows that $C \subseteq S_n$ (by Remark 1.3.9 we know that $C \subseteq S_d$ for some $d \leq n$, and if $d < n$, then $\dim(C) \leq \dim(S_d) \leq d < n$, a contradiction); we then let $B_1, \dots, B_n$ and $h : B_1 \times \cdots \times B_n \longrightarrow C$ be as in the conclusion of Lemma 3.2.7. Let φ be a risometry and M a matrix in $\mathrm{GL}_n(O_R)$ such that $h = \varphi \circ M$. Let ξ be the (unique) value taken by $\hat{\mathrm{rv}} \circ \mathrm{Jac}(f|_C)$. If $\xi = 0$, then $\mathrm{Jac}(f|_C)(x) = 0$ for all $x \in C$, so $f(x)$ is constant on C and there is nothing left to prove. We thus suppose that $\xi \neq 0$. Fix $x_0 \in C$ and set $z := \mathrm{Jac}(f|_C)(x_0)$. We use Lemma 3.2.1 and properties of h to prove that for all $x \in C \setminus \{x_0\}$,

$$\mathrm{v}(f(x) - f(x_0) - \langle z, x - x_0 \rangle) > \hat{\mathrm{v}}(z) + \hat{\mathrm{v}}(x - x_0). \tag{3.3}$$

Let $x \in C \setminus \{x_0\}$. We set $\eta : O_R \longrightarrow B_1 \times \cdots \times B_n$ as $\eta(t) := th^{-1}(x) + (1-t)h^{-1}(x_0)$ for each $t \in O_R$. Clearly the function $\theta := h \circ \eta : O_R \longrightarrow C$ is L-definable and differentiable. Moreover, for any distinct $t, t' \in O_R$, we have that $M \circ \eta(t) - M \circ \eta(t') = (t-t')(M \circ h^{-1}(x) - M \circ h^{-1}(x_0)) = (t-t')(\varphi^{-1}(x) - \varphi^{-1}(x_0))$, so in turn, $\hat{\mathrm{rv}}\left(\frac{\theta(t)-\theta(t')}{t-t'}\right)$ equals

$$\begin{aligned} \hat{\mathrm{rv}}\left(\frac{\varphi \circ M \circ \eta(t) - \varphi \circ M \circ \eta(t')}{t - t'}\right) &= \hat{\mathrm{rv}}\left(\frac{M \circ \eta(t) - M \circ \eta(t')}{t - t'}\right) \\ &= \hat{\mathrm{rv}}(\varphi^{-1}(x) - \varphi^{-1}(x_0)) = \hat{\mathrm{rv}}(x - x_0). \end{aligned}$$

This means that $\hat{\mathrm{rv}}\left(\frac{\theta(t)-\theta(t')}{t-t'}\right)$ is constant for different $t, t' \in O_R$ (and equals $\hat{\mathrm{rv}}(x - x_0)$).

It follows that $\hat{\mathrm{rv}} \circ \mathrm{Jac}(\theta)$ is constant on O_R, and that for all $t, t', s \in O_R$ with $t \neq t'$,

$$\hat{\mathrm{rv}}(\mathrm{Jac}(\theta)(s)) = \hat{\mathrm{rv}}\left(\frac{\theta(t) - \theta(t')}{t - t'}\right) = \hat{\mathrm{rv}}(x - x_0). \tag{3.4}$$

Set $g := f|_C \circ \theta : O_R \longrightarrow R$. Then g is (the restriction of) an L-definable differentiable function and by the chain rule, $g'(t) = \langle \mathrm{Jac}(f|_C)(\theta(t)), \mathrm{Jac}(\theta)(t)\rangle$ for each $t \in O_R$. From Lemma 1.2.8 (d) it follows that for each $t \in O_R$,

$$\mathrm{v}(g'(t)) = \mathrm{v}(\langle \mathrm{Jac}(f|_C)(\theta(t)), \mathrm{Jac}(\theta)(t)\rangle) \geq \hat{\mathrm{v}}(\mathrm{Jac}(f|_C)) + \hat{\mathrm{v}}(\mathrm{Jac}(\theta)), \tag{3.5}$$

where $\hat{\mathrm{v}}(\mathrm{Jac}(f|_C)) := \hat{\mathrm{v}}(\mathrm{Jac}(f|_C)(x))$ for some $x \in C$, and $\hat{\mathrm{v}}(\mathrm{Jac}(\theta)) := \hat{\mathrm{v}}(\mathrm{Jac}(\theta)(t))$ for some $t \in O_R$ (these definitions do not depend on the choices of x and t because both $\hat{\mathrm{rv}} \circ \mathrm{Jac}(f|_C)$ and $\hat{\mathrm{rv}} \circ \mathrm{Jac}(\theta)$ are constant). Using again both Lemma 1.2.8 (d) and the fact that $\hat{\mathrm{rv}} \circ \mathrm{Jac}(f|_C)$ and $\hat{\mathrm{rv}} \circ \mathrm{Jac}(\theta)$ are constant, we have that for $t \neq t'$ in O_R,

$$\begin{aligned} \mathrm{v}(g'(t) - g'(t')) &\geq \hat{\mathrm{v}}(\mathrm{Jac}(f|_C)(\theta(t)) - \mathrm{Jac}(f|_C)(\theta(t'))) + \hat{\mathrm{v}}(\mathrm{Jac}(\theta)(t) - \mathrm{Jac}(\theta)(t')) \\ &> \hat{\mathrm{v}}(\mathrm{Jac}(f|_C)) + \hat{\mathrm{v}}(\mathrm{Jac}(\theta)). \end{aligned} \tag{3.6}$$

Let $r \in R^\times$ be such that $\mathrm{v}(r) = \hat{\mathrm{v}}(\mathrm{Jac}(f|_C)) + \hat{\mathrm{v}}(\mathrm{Jac}(\theta))$. By (3.5) and (3.6), the function $t \mapsto g(t)/r$ from O_R to itself satisfies the hypotheses of Lemma 3.2.1; thus, for all distinct $t, t' \in O_R$, $\mathrm{v}(g(t) - g(t') - g'(0)(t - t')) > \mathrm{v}(t - t') + \hat{\mathrm{v}}(\mathrm{Jac}(f|_C)) + \hat{\mathrm{v}}(\mathrm{Jac}(\theta))$. By considering $t = 1$ and $t' = 0$ we obtain that

$$\mathrm{v}(f(x) - f(x_0) - g'(0)) > \hat{\mathrm{v}}(\mathrm{Jac}(f|_C)) + \hat{\mathrm{v}}(\mathrm{Jac}(\theta)). \tag{3.7}$$

Since $z := \mathrm{Jac}(f|_C)(x_0)$ and $\theta(0) = x_0$, we have that $g'(0) = \langle z, \mathrm{Jac}(\theta)(0)\rangle$. Using that $\mathrm{rv} \circ \mathrm{Jac}(f|_C)$ is constant we also get that $\hat{\mathrm{v}}(z) = \hat{\mathrm{v}}(\mathrm{Jac}(f|_C))$, and from (3.4) we have that $\hat{\mathrm{v}}(\mathrm{Jac}(\theta)) = \hat{\mathrm{v}}(x - x_0)$. Then (3.7) becomes

$$\mathrm{v}(f(x) - f(x_0) - \langle z, \mathrm{Jac}(\theta)(0)\rangle) > \hat{\mathrm{v}}(z) + \hat{\mathrm{v}}(x - x_0). \tag{3.8}$$

Putting $s = 0$ in (3.4) tells us that $\hat{\mathrm{v}}(\mathrm{Jac}(\theta)(0) - (x - x_0)) > \hat{\mathrm{v}}(x - x_0)$, and hence we get that $\mathrm{v}(\langle z, \mathrm{Jac}(\theta)(0)\rangle - \langle z, x - x_0\rangle) > \hat{\mathrm{v}}(z) + \hat{\mathrm{v}}(x - x_0)$. Combining this inequality with (3.8) proves (3.3).

The last part of the proof consists of showing that $z := \mathrm{Jac}(f|_C)(x_0)$ makes (3.2) hold for all $x \neq x'$ in C (not just for x and x_0). Let x and x' be distinct arbitrary elements of C. Notice that (3.3) holds after replacing x_0 by x' and z by $z' := \mathrm{Jac}(f|_C)(x')$. Using this and that $\mathrm{r}\hat{\mathrm{v}}(z) = \mathrm{r}\hat{\mathrm{v}}(z')$ (because $\mathrm{r}\hat{\mathrm{v}} \circ \mathrm{Jac}(f|_C)$ is constant on C) we conclude that $\mathrm{v}(f(x) - f(x') - \langle z, x - x'\rangle) \geq \min\{\mathrm{v}(f(x) - f(x') - \langle z', x - x'\rangle), \mathrm{v}(\langle z - z', x - x'\rangle)\}$ $> \hat{\mathrm{v}}(z) + \hat{\mathrm{v}}(x - x')$, as required. ❐

Therefore, t-stratifications exist for any definable map $B_0 \longrightarrow \mathrm{RV}^{\mathrm{eq}}$. We write this explicitly for later reference.

Corollary 3.2.9. *Let ψ and ϕ be $L_{\mathrm{RV}^{eq}}$-formulas such that ψ has only field-sort free variables and ϕ defines a map $\chi_\phi(R) : \psi(R) \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ in each model $(R, \mathrm{RV}^{\mathrm{eq}})$ of $T_{\mathrm{RV}^{eq}}$. Then there are $L_{\mathrm{RV}^{eq}}$-formulas $\psi_0, \ldots, \psi_n$ such that in every model $(R, \mathrm{RV}^{\mathrm{eq}})$ of $T_{\mathrm{RV}^{eq}}$, the collection $(\psi_i(R))_{i \leq n}$ is a t-stratification reflecting $\chi_\phi(R)$.*

Proof. We have proved that $T_{\mathrm{RV}^{\mathrm{eq}}}$ satisfies Hypotheses 1.1.9 and has the Jacobian property, the result then follows from the discussion after Theorem 1.4.6 (cf. [26, Corollary 4.13]) ❐

3.3 O-minimally definable t-stratifications

We have established the existence of t-stratifications definable in the language $L_{\mathrm{RV}^{\mathrm{eq}}}$. In this section we prove that we can further assume that t-stratifications are L-definable.

The meaning of this claim is that for any given t-stratification $(S_i)_{i\leq n}$ (say, of B_0) there exists a t-stratification $(S'_i)_{i\leq n}$ reflecting $(S_i)_{i\leq n}$ and for which each set S'_i is the intersection of B_0 with an L-definable set. In the case when $L = L_{\text{or}}$ (i.e. when (R, O_R) is a real closed valued field with no extra structure) this was proved in [26, Section 6]. We follow an analogous argument; however, this time we use deeper structural results about definable sets in R, such as cell-decomposition.

Definition 3.3.1. Let x be a tuple of field-sort variables and $\phi(x)$ an $L_{\mathrm{RV}^{\mathrm{eq}}}$-formula. The *dimension of* $\phi(x)$ is defined as $\dim(\phi(x)) := \max\{\dim(\phi(R)) \mid (R, \mathrm{RV}^{\mathrm{eq}}) \models T_{\mathrm{RV}^{\mathrm{eq}}}\}$.

The following proposition is an improvement on [26, Proposition 6.2] (in the latter, the formula ϕ^{Δ} in (2) below is required to be minimal with respect to implication).

Proposition 3.3.2. *Let Δ be a set of $L_{\mathrm{RV}^{eq}}$-formulas such that*

(1) *Δ is closed under disjunctions and contains $\perp$;*

(2) *for each $L_{\mathrm{RV}^{eq}}$-formula ϕ there is a formula $\phi^{\Delta} \in \Delta$ satisfying that $\phi \to \phi^{\Delta}$ and* $\dim(\phi) = \dim(\phi^{\Delta})$.

Also assume that $(\phi_i)_{i\leq n}$ is a tuple of $L_{\mathrm{RV}^{eq}}$-formulas defining a t-stratification in all models of $T_{\mathrm{RV}^{eq}}$. Then there exists a tuple of formulas $(\psi_i)_{i\leq n}$ in Δ that defines a t-stratification reflecting the t-stratification defined by $(\phi_i)_{i\leq n}$ in all models of $T_{\mathrm{RV}^{eq}}$.

Proof. Let $(\phi_i)_{i\leq n}$ be a tuple of formulas as in the hypotheses and pick ϕ_i^{Δ} for each ϕ_i as in (2). For each $i \leq n$, we set $\phi_{\leq i} := \phi_0 \vee \cdots \vee \phi_i$ and similarly $\phi_{\leq i}^{\Delta} := \phi_0^{\Delta} \vee \cdots \vee \phi_i^{\Delta}$. Using (1) and Proposition 1.1.12 (d), we have that for each $i \leq n$, $\phi_{\leq i}^{\Delta} \in \Delta$, $\phi_{\leq i} \to \phi_{\leq i}^{\Delta}$ and $\dim(\phi_{\leq i}^{\Delta}) = \dim(\phi_{\leq i})$. Conventionally, for a formula ϕ with only field-sort free variables, $\dim(\phi) \leq -1$ simply expresses that ϕ has no solutions in any model of $T_{\mathrm{RV}^{\mathrm{eq}}}$ (i.e. $T_{\mathrm{RV}^{\mathrm{eq}}} \vdash \neg\exists x \phi(x)$). Let $d \in \{-1, 0, \ldots, n\}$ be such that $\dim(\phi_{\leq i}^{\Delta} \wedge \neg(\phi_{\leq i})) \leq d$, for all $i \leq n$. If $d = -1$, we have finished, as in that case each ϕ_i is already equivalent to the formula $\phi_i^{\Delta} \in \Delta$. So we assume that $d \geq 0$. As an abuse of terminology, we call a tuple of formulas defining a t-stratification in all models of $T_{\mathrm{RV}^{\mathrm{eq}}}$ simply a t-stratification.

We also talk about reflection of formulas with the obvious meaning. Applying the next claim inductively takes us back to the case when $d = -1$, finishing the proof.

Claim. There is a t-stratification $(\phi'_i)_{i\leq n}$ reflecting $(\phi_i)_{i\leq n}$, and formulas $\phi^*_0, \dots, \phi^*_n \in \Delta$ such that for each $i \leq n$, $\phi'_i \to \phi^*_i$, and $\dim(\phi^*_{\leq i} \wedge \neg(\phi'_{\leq i})) \leq d-1$, where $\phi^*_{\leq i} := \phi^*_0 \vee \dots \vee \phi^*_i$ and similary for $\phi'_{\leq i}$.
Proof of Claim. Set $\delta_n := \phi^{\Delta}_{\leq n} \wedge \neg(\phi_{\leq n})$ and pick $\delta^{\Delta}_n \in \Delta$ as in (2). Clearly, $\dim(\delta^{\Delta}_n) = \dim(\delta_n) \leq d$. Fix $i < n$ and suppose that we have defined δ_{i+1} with $\dim(\delta_{i+1}) \leq d$ and taken $\delta^{\Delta}_{i+1} \in \Delta$ as in (2). We set

$$\delta_i := (\phi^{\Delta}_{\leq i} \wedge \neg(\phi_{\leq i})) \vee (\delta^{\Delta}_{i+1} \wedge \neg\delta_{i+1}) \vee \dots \vee (\delta^{\Delta}_n \wedge \neg\delta_n),$$

and we then fix a choice of $\delta^{\Delta}_i \in \Delta$ as in (2). This process defines δ_i and δ^{Δ}_i recursively for all $0 \leq i \leq n$. Notice that $\dim(\delta_i) \leq d$ for all $i \leq n$, so the set $\delta := \bigvee_{i\leq n} \delta_i$ is of dimension no greater than d by Proposition 1.1.12 (d). This condition is important to apply Lemma 1.3.5 further down in the proof.

We claim that for each $i \leq n$ the formula $\phi_{\leq i} \vee \delta$ is equivalent to a disjunction of formulas in Δ; indeed, using $\equiv$ to denote logical equivalence, for each $i \leq n$ we have that

$$\begin{aligned}
&\phi_{\leq i} \vee \delta \equiv \bigvee_{j\leq i} (\phi_{\leq j} \vee \delta_j \vee \dots \vee \delta_n) \\
&\equiv \bigvee_{j\leq i} \big(\phi_{\leq j} \vee (\phi^{\Delta}_{\leq j} \wedge \neg(\phi_{\leq j})) \vee (\delta^{\Delta}_{j+1} \wedge \neg\delta_{j+1}) \vee \dots \vee (\delta^{\Delta}_n \wedge \neg\delta_n) \vee \delta_{j+1} \vee \dots \vee \delta_n\big) \\
&\equiv \bigvee_{j\leq i} \big((\phi_{\leq j} \vee (\phi^{\Delta}_{\leq j} \wedge \neg(\phi_{\leq j}))) \vee ((\delta^{\Delta}_{j+1} \wedge \neg\delta_{j+1}) \vee \delta_{j+1}) \vee \dots \vee ((\delta^{\Delta}_n \wedge \neg\delta_n) \vee \delta_n)\big) \\
&\equiv \bigvee_{j\leq i} (\phi^{\Delta}_{\leq j} \vee \delta^{\Delta}_{j+1} \vee \dots \vee \delta^{\Delta}_n) \in \Delta.
\end{aligned}$$

Applying Corollary 3.2.9, we let $(\psi_i)_{i\leq n}$ be a t-stratification reflecting $((\phi_i)_{i\leq n}, \delta)$. Recalling that $\dim(\delta) \leq d$, Lemma 1.3.5 implies that setting

$$\phi'_{\leq i} := \begin{cases} \psi_{\leq i} & \text{if } i \leq d-1, \\ \phi_{\leq i} \vee \delta \vee \psi_{\leq d-1} & \text{if } i \geq d, \end{cases}$$

(and implicitly, $\phi'_0 := \phi'_{\leq 0}$ and $\phi'_i := (\phi'_{\leq i} \wedge \neg(\phi'_{\leq i-1}))$ for $0 < 1 \leq n$; when $d = 0$, $\phi'_{\leq -1} := \perp$) defines a t-stratification $(\phi'_i)_{i \leq n}$ that reflects $(\phi_i)_{i \leq n}$ and—with the natural meaning—coincides with $(\psi_i)_{i \leq n}$ outside of $\phi_{\leq d} \vee \delta$.

To define the formulas ϕ^*_i we first fix formulas $\psi^\Delta_i \in \Delta$ as in (2) for each ψ_i (putting $\psi^\Delta_{-1} := \perp$ if $d = 0$). We use the notation $\psi^\Delta_{\leq i}$ for $\psi^\Delta_0 \vee \cdots \vee \psi^\Delta_i$, for each $i \leq n$. Below we implicitly replace $\phi_{\leq i} \vee \delta$ by $\bigvee_{j \leq i}(\phi^\Delta_{\leq j} \vee \delta^\Delta_{j+1} \vee \cdots \vee \delta^\Delta_n)$ for each $i \leq n$. We put

$$\phi^*_{\leq i} := \begin{cases} \psi^\Delta_{\leq i} & \text{if } i \leq d-1, \\ \bigvee_{j \leq i}(\phi^\Delta_{\leq j} \vee \delta^\Delta_{j+1} \vee \cdots \vee \delta^\Delta_n) \vee \psi^\Delta_{\leq d-1} & \text{if } i \geq d. \end{cases}$$

(Again, when $d = 0$ we put $\phi^*_{\leq -1} := \perp$.) Clearly $\phi^*_{\leq i} \in \Delta$ and $\phi'_{\leq i} \to \phi^*_{\leq i}$ for all $i \leq n$. Lastly, we check that for each $i \leq n$ we have that $\dim(\phi^*_{\leq i} \wedge \neg(\phi'_{\leq i})) \leq d-1$. To this effect we show that for all $i \leq n$ the following implication holds

$$(\phi^*_{\leq i} \wedge \neg(\phi'_{\leq i})) \to \psi^\Delta_{\leq d-1}. \tag{3.9}$$

By definition, $\phi^*_{\leq i} = \psi^\Delta_{\leq i}$ for all $i \leq d-1$, so (3.9) is trivial for such i. If instead $i \geq d$, we have that

$$\phi^*_{\leq i} \wedge \neg(\phi'_{\leq i}) \equiv (\phi_{\leq i} \vee \delta \vee \psi^\Delta_{\leq d-1}) \wedge \neg(\phi_{\leq i} \vee \delta \vee \psi_{\leq d-1}) \to (\psi^\Delta_{\leq d-1} \wedge \neg(\psi_{\leq d-1})) \to \psi^\Delta_{\leq d-1}.$$

We have showed that (3.9) holds for all $i \leq n$. Finally, since $(\psi_i)_{i \leq n}$ is a t-stratification, we have that $\dim(\psi^\Delta_{\leq d-1}) = \dim(\psi_{\leq d-1}) \leq d-1$ (also $\dim(\psi^\Delta_{\leq -1}) = \dim(\perp) \leq -1$ when $d = 0$), so it follows from (3.9) that for each $i \leq n$, $\dim(\phi^*_{\leq i} \wedge \neg(\phi'_{\leq i})) \leq d-1$, finishing the proof of the claim. ❐

We aim to apply Proposition 3.3.2 to the set Δ of all L-formulas. Clearly such a set satisfies condition (1) above. The rest of the section is devoted to proving that (2) holds. Proving (2) translates to showing that any $L_{\mathrm{RV}^{\mathrm{eq}}}$-definable set is contained in an L-definable set of the same dimension. This is enough by the completeness of $T_{\mathrm{RV}^{\mathrm{eq}}}$. Furthermore, it is enough to show the statement is true for L_{RV}-definable sets, and to this end we digress slightly to talk about cell decomposition. We work with two kinds of cells available for subsets of powers of R, the first kind coming from the o-minimal setting of R as an L-structure, and the second coming from the weakly o-minimality of R as the field-sort of (R, RV).

Cells in o-minimal structures were first introduced in [32].

Definition 3.3.3 (*o-minimal cells*)**.** A 1-*L-cell* is simply a point or an open interval in R (for intervals, we allow $\pm\infty$ as endpoints). We say that $C \subseteq R^{n+1}$ is an $(n+1)$-*L-cell* if there exists an n-L-cell $C' \subseteq R^n$ such that either

(i) $C = \{(a, f(a)) \in R^{n+1} \mid a \in C'\}$, for some L-definable function $f : R^n \longrightarrow R$;

or

(ii) $C = \{(a, b) \in R^{n+1} \mid a \in C' \text{ and } f(a) < b < g(a)\}$, for some L-definable functions $f, g : R^n \longrightarrow R$ satisfying that $f(x) < g(x)$ for all $x \in C'$.

We say that X is an *L-cell* if it is an n-L-cell for some $n \geq 1$.

The second kind of cells was introduced in [35, Subsection 4.2], where a cell decomposition theorem was proved for weakly o-minimal theories. Below $\widehat{R}$ denotes the Dedekind completion of R as an ordered field. By 'L_{RV}-definable function $f : R^n \longrightarrow \widehat{R}$' we really mean an L_{RV}-definable family of cuts $(C_x)_{x\in R^n}$ of R; so, f is simply the map $x \mapsto C_x$. Accordingly, $b < f(x)$ means that $b \in C_x$, and so on.

Definition 3.3.4. A 1-*L_{RV}-cell* is a convex L_{RV}-definable subset of R (note that points are 1-L_{RV}-cells). We say that $D \subseteq R^{n+1}$ is an $(n+1)$-*L_{RV}-cell* if there exists an n-L_{RV}-cell $D' \subseteq R^n$ such that either

(i) $D = \{(a, f(a)) \in R^{n+1} \mid a \in D'\}$, for some L_{RV}-definable function $f : R^n \longrightarrow R$;

or

(ii) $D = \{(a, b) \in R^{n+1} \mid a \in D' \text{ and } f(a) < b < g(a)\}$, for some L_{RV}-definable functions $f, g : R^n \longrightarrow \widehat{R}$ satisfying that $f(x) < g(x)$ for all $x \in D'$.

We say that X is an L_{RV}-*cell* if it is an n-L_{RV}-cell for some $n \geq 1$.

Unlike the usual definition (e.g. [15, (2.3) Definition]), we did not ask for continuity for the functions appearing in Definition 3.3.3; this has the purpose of easing the compatibility between L-cells and L_{RV}-cells—see the proof of Proposition 3.3.7.

Theorem 3.3.5 ([35]). *Every L_{RV}-definable subset of R^n admits a partition into finitely many L_{RV}-cells.*

Proof. Every such subset is L_{convex}-definable, so the conclusion follows from [35, Theorem 4.6] by the weakly o-minimality of T_{convex}. ❐

The following is a corollary of this theorem and Proposition 2.3.3.

Corollary 3.3.6. *Let $f : X \longrightarrow R$ be an L_{RV}-definable function with $X \subseteq R^n$. Then there are a partition of X into finitely many L_{RV}-cells $D_1, \ldots, D_m$, and L-definable functions $f_1, \ldots, f_m : R^n \longrightarrow R$ such that $f|_{D_i} = f_i|_{D_i}$, for each $i \in \{1, \ldots, m\}$.*

The following crucial result states that L_{RV}-cells *are suitably covered* by L-cells.

Proposition 3.3.7. *If D is an L_{RV}-cell, then there are finitely many L-cells $C_1, \ldots, C_m$ such that $D \subseteq C_1 \cup \cdots \cup C_m$ and $\dim(D) = \dim(C_1 \cup \cdots \cup C_m)$.*

Proof. Let $D \subseteq R$ be a 1-L_{RV}-cell. If D is a point, it is a 1-L_{RV}-cell already. If D is a convex set of dimension 1, we may take R as the L-cell required. We now let D be an $(n+1)$-L_{RV}-cell and we assume that D' is an n-L_{RV}-cell from which D is obtained through either (i) or (ii) in Definition 3.3.4. As inductive hypothesis we assume that there are finitely many L-cells C'_i such that $D' \subseteq \bigcup_i C'_i$ and $\dim(D') = \dim(\bigcup_i C'_i)$.
Case (i). $D = \{(a, f(a)) \in R^{n+1} \mid a \in D'\}$, where $f : R^n \longrightarrow R$ is an L_{RV}-definable function. By applying Corollary 3.3.6, let $\{A_j\}_j$ be a decomposition of $\bigcup_i C'_i$ into L_{RV}-

cells and $\{f_j : R^n \longrightarrow R\}_j$ L-definable functions such that $f|_{A_j} = f_j|_{A_j}$, for each j. Define for all i and j, $C_{i,j} := \{(a, f_j(a)) \mid a \in C_i'\}$. Then clearly each $C_{i,j}$ is an L-cell and $X \subseteq \bigcup_{i,j} C_{i,j}$. Moreover, for any i and j, $\dim(C_{i,j}) = \dim(C_i')$, so

$$\dim\left(\bigcup_{i,j} C_{i,j}\right) = \dim\left(\bigcup_i C_i'\right) = \dim(D') = \dim(X),$$

as required.

Case (ii). $D = \{(a,b) \in R^{n+1} \mid a \in D' \text{ and } f(a) < b < g(a)\}$, where $f, g : R^n \longrightarrow \widehat{R}$ are L_{RV}-definable functions satisfying $f(x) < g(x)$ for all $x \in D'$. Following the idea of covering an L_{RV}-definable convex set of dimension 1 by the whole of R, we simply put $C_i := C_i' \times R$. Clearly each C_i is an L-cell and $X \subseteq \bigcup_i C_i$. Lastly, notice that for every i, $\dim(C_i) = \dim(C_i') + 1$, so

$$\dim\left(\bigcup_i C_i\right) = \dim\left(\bigcup_i C_i'\right) + 1 = \dim(D') + 1 = \dim(D),$$

finishing this case and the whole proof. ❐

By Theorem 3.3.5 and the last proposition, for every L_{RV}-definable set $X \subseteq R^n$ there exists an L-definable set $Y \subseteq R^n$ such that $X \subseteq Y$ and $\dim(Y) = \dim(X)$, as postulated earlier. We conclude with the claim at the beginning of the section.

Theorem 3.3.8. *Let $(\phi_i)_{i \le n}$ be a tuple of $L_{\text{RV}^{eq}}$-formulas defining a t-stratification in all models of $T_{\text{RV}^{eq}}$. Then there exists a tuple of L-formulas $(\phi_i')_{i \le n}$ such that for each model $(R, \text{RV}^{\text{eq}})$ of $T_{\text{RV}^{eq}}$, $(\phi_i'(R))_{i \le n}$ is a t-stratification reflecting $(\phi_i(R))_{i \le n}$.*

Proof. By passing from $L_{\text{RV}^{\text{eq}}}$ to L_{RV} as usual, it follows from the previous paragraph that each $L_{\text{RV}^{\text{eq}}}$-definable set $X \subseteq R^n$ is contained in an L-definable $Y \subseteq R^n$ such that $\dim(X) = \dim(Y)$. By the completeness of $T_{\text{RV}^{\text{eq}}}$, we conclude that the set Δ of all L-formulas satisfies the hypotheses (1) and (2) of Proposition 3.3.2; the result follows. ❐

We present an application of this theorem in Chapter 5.

3.4 No t-stratifications in arbitrary T-convex fields

The following example shows that the hypothesis of power-boundedness on T is essential for the existence of t-stratifications. We consider the theory $T = \mathrm{RCF}_{\exp}$ (see page 119), which is an o-minimal theory and is not power-bounded. We let R be a non-principal ultrapower of the real field $\mathbb{R}$ as a model of $\mathrm{RCF}_{\exp}$, and let O_R be an $\mathrm{RCF}_{\exp}$-convex subring of R (say, O_R is the convex hull of $\mathbb{R}$ in R). We consider the model $(R, \mathrm{RV}^{\mathrm{eq}})$ of the theory $(\mathrm{RCF}_{\exp})_{\mathrm{RV}^{\mathrm{eq}}}$. The following simple example was suggested by I. Halupczok in private communication.

Recall that M denotes the maximal ideal of O_R. Note that the continuity of the logarithm $\log : \mathbb{R}_{>0} \longrightarrow \mathbb{R}$ implies that $\log(1+M) \subseteq M$. Indeed, on $\mathbb{R}$ it is true that $\forall \varepsilon \in \mathbb{R}_{>0} \exists \delta \in \mathbb{R}_{>0}(\|1-x\| < \delta \rightarrow \|\log(x)\| < \varepsilon)$. Fixing ε and consequently δ, it is clear that any element $x \in M$ satisfies that $\|1-(1+x)\|_R < \delta$, so using the transfer principle (see Fact 5.1.2) we have that $\|\log(1+x)\|_R < \varepsilon$. Since this holds for arbitrary $\varepsilon \in \mathbb{R}_{>0}$, we conclude that $\log(1+x) \in M$ whenever $x \in M$.

Proposition 3.4.1. *The definable map $\chi : R \longrightarrow \mathrm{RV}$ given as $\chi(x) := \mathrm{rv}(e^x)$ does not admit a t-stratification.*

Proof. First of all we claim that χ cannot be 1-translatable on any closed ball $B \subseteq R$ with $\mathrm{rad}(B) \leq 0$. Assume the contrary, and let B be as said and let $\varphi : B \longrightarrow B$ be a straightener of χ on B; note that the direction of translatability must be $V = \overline{R}$, so a lift of V is simply R. Let $x, x' \in B$ be such that $\mathrm{v}(x-x') = 0$. Since trivially $x - x' \in R$, we have that $\chi \circ \varphi(x) = \chi \circ \varphi(x')$. Hence, $\mathrm{rv}(e^{\varphi(x)}) = \mathrm{rv}(e^{\varphi(x')})$ and this is equivalent to $\varphi(x) - \varphi(x') = \log(e^{\varphi(x)-\varphi(x')}) \in \log(1+M)$. By the comment preceding the proposition, $\varphi(x)-\varphi(x') \in M$, and this contradicts that $\mathrm{v}(\varphi(x)-\varphi(x')) = \mathrm{v}(x-x') = 0$ (since $\mathrm{rv}(\varphi(x)-\varphi(x')) = \mathrm{rv}(x-x')$). The claim is proved.

We now address the proposition. Since the exponential map e is trivially definable in any model of $\mathrm{RCF}_{\exp}$, χ is clearly definable in $(R, \mathrm{RV}^{\mathrm{eq}})$. Now we assume for the sake of a contradiction that (S_0, S_1) is a t-stratification of R reflecting χ. According to this, χ must be 1-translatable on any ball disjoint from S_0. So, by the first part of the proof it is enough to find a closed ball $B \subseteq R \setminus S_0$ with $\mathrm{rad}(B) \leq 0$. This follows easily from the finiteness of S_0: if $a_0 < \cdots < a_m$ lists all of S_0, then either one of the balls $B(b, \geq \hat{\mathrm{v}}(a_{i+1} - a_i))$ with $b \in (a_i, a_{i+1})$ and $i \in \{0, \ldots, m-1\}$ is as we want or $S_0 \subseteq B(a_0, > 0)$, which clearly also implies that the needed ball B exists (take $x \notin B(a_0, \geq 0)$ and set $B := B(x, \geq 0)$). ❒

Remark 3.4.2. The concrete choices of $\mathrm{RCF}_{\mathrm{exp}}$ and R were made with the purpose of easing some of the arguments. The example is pertinent to any arbitrary not power-bounded o-minimal theory T for Theorem A.1.13 ensures that an *exponential* is always definable in any model of such T.

Chapter 4

T-stratifications of tangent cones

Throughout this chapter, L will be a language containing L_{or} and T will be a power-bounded o-minimal L-theory expanding RCF. We work in models $(R, \mathrm{RV}^{\mathrm{eq}})$ of $T_{\mathrm{RV}^{\mathrm{eq}}}$, and by 'definable', we will always mean '$L_{\mathrm{RV}^{\mathrm{eq}}}$-definable'. We prove that a t-stratification of a definable set $X \subseteq R^n$ induces a t-stratification on each of the tangent cones of X. This chapter corresponds to [23].

4.1 Tangent cones

Following the work of H. Whitney [53], the *tangent cone* of a set $X \subseteq \mathbb{C}^n$ at a point $p \in \mathbb{C}^n$ is the union of all the *limiting secant lines* to X at p. This definition makes sense for subsets of $\mathbb{R}^n$ but loses the tight relation with the local geometry of the set. This can be seen in the cusp curve, the set $X \subseteq \mathbb{R}^2$ defined by the equation $x^3 - y^2 = 0$. At 0, X has only one limiting secant line, the horizontal axis, so this line would be the tangent cone of X at 0. Clearly the negative part of the axis conveys little information about the set. In order to recover the tighter relation of the tangent cone with the local geometry of the set, it is preferred to define the tangent cone as the union of all the *limiting secant rays* to X at p. A *ray* is a set of points of the form tx, with fixed unitary $x \in \mathbb{R}^n$ (the *direction* of the ray) and t ranging in $[0, \infty)$. The propriety of this definition is exemplified by the applications of these tangent cones in the research on the local geometry of subsets of $\mathbb{R}^n$, see for instance [21] and [20] on matching prescribed tangent cones to algebraic subsets of $\mathbb{R}^n$. We adopt this definition for subsets of R^n. Recall that by $\|\cdot\|_R$ we denote the Euclidean norm on R (see page 42).

Definition 4.1.1. Let $X \subseteq R^n$ and $p \in R^n$. The *tangent cone of X at p*, denoted as $\mathcal{C}_p(X)$, is the set

$$\{y \in R^n \mid \forall \varepsilon \in R_{>0} \exists x \in X, r \in R_{>0}(\|x - p\|_R < \varepsilon \wedge \|r(x - p) - y\|_R < \varepsilon)\}.$$

A set $C \subseteq R^n$ is called a *cone* if for all $x \in C$ and $r \in R_{>0}$, $rx \in C$. It is clear that $\mathcal{C}_p(X)$ is always a cone. Some authors prefer the term *semicone* for what we call a cone to stress that we only consider positive scalars r. We believe there is no chance of confusion in our setting.

The following are further immediate properties of tangent cones.

Proposition 4.1.2. *Let $X, Y \subseteq R^n$ and $p \in R^n$. The following hold.*

(a) *If $X \subseteq Y$, $\mathcal{C}_p(X) \subseteq \mathcal{C}_p(Y)$;*

(b) $\mathcal{C}_p(X \cup Y) = \mathcal{C}_p(X) \cup \mathcal{C}_p(Y)$*;*

(c) $\mathcal{C}_p(X \cap Y) \subseteq \mathcal{C}_p(X) \cap \mathcal{C}_p(Y)$*, and the strict containment may hold;*

(d) $\mathcal{C}_p(X)$ *is a closed set;*

(e) *if X is definable in $(R, \mathrm{RV}^{\mathrm{eq}})$, so is $\mathcal{C}_p(X)$;*

(f) $\mathcal{C}_p(X)$ *equals the set*

$$\{y \in R^n \mid \forall \lambda \in \Gamma \exists x \in X, r \in R_{>0}(\hat{\mathrm{v}}(x - p) > \lambda \wedge \hat{\mathrm{v}}(r(x - p) - y) > \lambda\}.$$

Proof. (a), (b) and (e) are clear, while (f) follows from Remark 2.1.7. In (c), it is clear that $\mathcal{C}_p(X \cap Y) \subseteq \mathcal{C}_p(X) \cap \mathcal{C}_p(Y)$. To see that the strict containment could hold, consider

$X := \{(x,y) \in R_{\geq 0} \times R_{\geq 0} \mid y^2 = x^3\}$ and $Y := \{(x,y) \in R_{\geq 0} \times R_{<0} \mid y^2 = x^3\}$. Then $\mathcal{C}_0(X \cap Y) = \{0\}$ but $\mathcal{C}_0(X) \cap \mathcal{C}_0(Y) = R_{\geq 0} \times \{0\}$. For (d), suppose that $z \in \mathrm{cl}(\mathcal{C}_p(X))$ and take $\lambda \in \Gamma$. Then there is $y \in \mathcal{C}_p(X)$ such that $\hat{\mathrm{v}}(y - z) > \lambda$. Also, by (f), there are $x \in X$ and $r \in R_{>0}$ for which $\hat{\mathrm{v}}(x - p) > \lambda$ and $\hat{\mathrm{v}}(r(x-p) - y) > \lambda$. If follows that $\hat{\mathrm{v}}(x - p) > \lambda$ and $\hat{\mathrm{v}}(r(x-p) - z) \geq \min\{\hat{\mathrm{v}}(r(x-p) - y), \hat{\mathrm{v}}(y - z)\} > \lambda$. Thus $z \in \mathcal{C}_p(X)$. ❒

If $X \subseteq \mathbb{R}^n$, the *tangent cone* of X at $p \in \mathbb{R}^n$ is defined by replacing R with $\mathbb{R}$ in Definition 4.1.1 and employing the usual norm $\|\cdot\|$ on $\mathbb{R}^n$. We denote this tangent cone by $C_p(X)$. The motivation for our upcoming result about tangent cones in R is the following proposition about tangent cones in $\mathbb{R}$. The spirit of both results goes back to [53, Theorem 11.8], where a similar statement is proved for complex analytic varieties.

Proposition 4.1.3. *Let X be an L-definable subset of $\mathbb{R}^n$ and p a non-isolated point of X. For $u \in \mathbb{R}^n$, $u \in C_p(X)$ if and only if there exists an L-definable differentiable curve $\eta : (0,1) \longrightarrow X$ such that $\lim_{t\to 0^+} \eta(t) = p$ and*

$$\lim_{t\to 0^+} \eta'(t) = \lim_{t\to 0^+} \tfrac{\eta(t)-p}{t} = u.$$

Proof. See the proof of the next theorem; it is essentially the same, replacing Proposition 2.2.5 with Proposition A.1.9. ❒

Using Proposition 2.2.5 we obtain the same result for definable subsets of R^n.

Theorem 4.1.4. *Let $X \subseteq R^n$ be definable and p a non-isolated point of X. For $y \in R^n$, $y \in C_p(X)$ if and only if there exists a definable differentiable curve $\eta : (0,1) \longrightarrow X$ such that $\lim_{t\to 0^+} \eta(t) = p$ and*

$$\lim_{t\to 0^+} \eta'(t) = \lim_{t\to 0^+} \tfrac{\eta(t)-p}{t} = y.$$

Proof. To simplify notation we put $p = 0$. We employ the *deformation to the tangent* *cone* described in [6, Subsection 3.5]. Set $D(X) := \{(x,r) \in R^n \times R_{>0} \mid rx \in X\}$. Then $D(X)$ is definable and $\mathrm{cl}(D(X)) \cap (R^n \times \{0\}) = \mathcal{C}_0(X) \times \{0\}$. If $y \in \mathcal{C}_0(X)$, then $(y,0) \in \mathrm{cl}(D(X))$ and by Proposition 2.2.5, there exists a definable differentiable curve $\gamma : (0,1) \longrightarrow D(X)$ such that $\lim_{t\to 0^+} \gamma(t) = (y,0)$. Let $\gamma_1 : (0,1) \longrightarrow R^n$ and $\gamma_2 : (0,1) \longrightarrow R_{>0}$ be the definable curves satisfying $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ for all $t \in (0,1)$. Then clearly $y \in \mathcal{C}_0(\gamma_1 \cdot \gamma_2)$. The required curve η is then obtained by reparametrising $\gamma_1 \cdot \gamma_2$, if necessary. ❐

4.2 Risometries and tangent cones

In the following proposition and its proof, B_0 will denote the ball $B(0, > 0)$. The following indicates a strong relation between sets and their tangent cones.

Proposition 4.2.1. *Let $X, Y \subseteq R^n$ be definable sets and p be a non-isolated point of both X and Y. If $\varphi : B(p, > 0) \longrightarrow B(p, > 0)$ is a definable risometry sending $X \cap B(p, > 0)$ onto $Y \cap B(p, > 0)$ and fixing p, then there exists a definable risometry $\psi : B_0 \longrightarrow B_0$ taking $\mathcal{C}_p(X) \cap B_0$ onto $\mathcal{C}_p(Y) \cap B_0$ and fixing 0.*

Proof. We first construct ψ without referring explicitly to tangent cones; at the end of the proof we indicate how the proposition follows from the construction. We assume $p = 0$ for simplicity. Recall that a *curve* is an injective continuous function with an interval $(a,b) \subseteq R$ as domain.

Consider $x \in B_0$ and a definable curve $\gamma : (0,1) \longrightarrow B_0$ such that $\lim_{t\to 0^+} t^{-1}\gamma(t) = x$. Notice that $\varphi \circ \gamma$ is definable and we can therefore regard it as a definable curve itself. Also note that $t^{-1}\varphi \circ \gamma(t)$ is bounded. By Lemma 2.2.2, the limit of $t^{-1}\varphi \circ \gamma(t)$ as $t \to 0^+$ exists in B_0; we set $\psi(x) := \lim_{t\to 0^+} t^{-1}\varphi \circ \gamma(t)$. We claim that so defined, $\psi : B_0 \longrightarrow B_0$ is a definable risometry.

First of all, that ψ is well-defined is a consequence of φ being an isometry. Indeed, if $\eta : (0,1) \longrightarrow B_0$ is another definable curve such that $t^{-1}\eta(t)$ converges to x as $t \to 0^+$, then for each $t \in (0,1)$ it holds that,

$$\hat{\mathrm{v}}\left(t^{-1}\varphi\circ\gamma(t) - t^{-1}\varphi\circ\eta(t)\right) = \hat{\mathrm{v}}\left(t^{-1}\gamma(t) - t^{-1}\eta(t)\right).$$

It is then clear that $t^{-1}\varphi\circ\gamma(t)$ and $t^{-1}\varphi\circ\eta(t)$ have the same limit when $t \to 0^+$. To show that ψ is bijective, we consider the map ψ' built from φ^{-1} in the analogous way as ψ was built from φ. For $x \in B_0$, let γ be a definable curve such that $t^{-1}\gamma(t)$ converges to x as $t \to 0^+$. Without loss of generality, we assume that $\varphi\circ\gamma$ is already a definable curve. Then

$$\psi'\circ\psi(x) = \psi'\big(\lim_{t\to 0^+} t^{-1}\varphi\circ\gamma(t)\big) = \lim_{t\to 0^+} t^{-1}\varphi^{-1}\circ\varphi\circ\gamma(t) = \lim_{t\to 0^+} t^{-1}\gamma(t) = x,$$

where we have used that both maps ψ and ψ' are well defined. Similarly, we can check that $\psi\circ\psi'$ is also the identity on B_0. Finally, we show that $\mathrm{rv}(\psi(x)-\psi(y)) = \mathrm{rv}(x-y)$ for all $x, y \in B_0$. First notice that if $f : R \longrightarrow R$ is an injective function with $\lim_{t\to 0^+} f(t) = x \neq 0$, then for $t > 0$ sufficiently small, we have that $\mathrm{rv}(f(t)) = \mathrm{rv}(x)$. Let $x \neq y$ be arbitrary elements of B_0 and let $\gamma, \eta : (0,1) \longrightarrow B_0$ be definable curves such that $\lim_{t\to 0^+} t^{-1}\gamma(t) = x$ and $\lim_{t\to 0^+} t^{-1}\eta(t) = y$. Moreover, we assume that $\gamma(t) \neq \eta(t)$ for all sufficiently small $t \in (0,1)$ (there is no loss of generality here as we can always reparametrise one of the curves). Then, for t sufficiently small,

$$\begin{aligned}\mathrm{rv}(\psi(x)-\psi(y)) &= \mathrm{rv}\left(t^{-1}\varphi\circ\gamma(t) - t^{-1}\varphi\circ\eta(t)\right)\\ &= \mathrm{rv}\left(t^{-1}\gamma(t) - t^{-1}\eta(t)\right) = \mathrm{rv}(x-y).\end{aligned}$$

Now we deduce the proposition. Notice that $\psi(0) = 0$ by construction. By Theorem 4.1.4, if $x \in \mathcal{C}_0(X)\cap B_0$, then there is a definable curve $\gamma : (0,1) \longrightarrow X$ such that $x = \lim_{t\to 0^+} t^{-1}\gamma(t)$. We can further assume that $\gamma((0,1)) \subseteq X \cap B_0$, and so $\varphi\circ\gamma((0,1)) \subseteq Y \cap B_0$. Clearly then $\psi(x) = \lim_{t\to 0^+} t^{-1}\varphi\circ\gamma(t) \in \mathcal{C}_0(Y)\cap B_0$. In this way, the function ψ constructed above maps $\mathcal{C}_0(X)\cap B_0$ bijectively to $\mathcal{C}_0(Y)\cap B_0$. ❐

4.3 T-stratifications induced on tangent cones

In this section we prove the main result of the chapter. We start by stating what is meant by *inducing a t-stratification* on a tangent cone. Let $(S_i)_{i\leq n}$ be a t-stratification of R^n.

Definition 4.3.1. For fixed $p \in R^n$, the partition $(\mathcal{C}_{p,i})_{i\leq n}$ of R^n is defined as follows.

$$\mathcal{C}_{p,i} := \begin{cases} \mathcal{C}_p(S_0) & \text{if } i = 0, \\ \mathcal{C}_p(S_{\leq i}) \setminus \mathcal{C}_p(S_{\leq i-1}) & \text{if } 0 < i \leq n. \end{cases}$$

Recall that by definition $(S_i)_{i\leq n}$ is a definable partition, so $(\mathcal{C}_{p,i})_{i\leq n}$ is definable too. If $X \subseteq R^n$ and $(S_i)_{i\leq n}$ is a t-stratification of X, we say that $(S_i)_{i\leq n}$ *induces* a t-stratification on $\mathcal{C}_p(X)$ if $(\mathcal{C}_{p,i})_{i\leq n}$ turns out to be a t-stratification of $\mathcal{C}_p(X)$. We prove below that this is always the case whenever X is definable.

The following lemma is a variant of [26, Corollary 3.22]. This lemma will be used later to guarantee the existence of (uniformly) definable translaters. Recall that if $B \subseteq R^n$ is ball, $\ulcorner B \urcorner$ stands for the code of B in $\mathrm{RV}^{\mathrm{eq}}$ (page 9). Following our previous notion of *respecting* (page 5) and under Convention 1.2.1, we say that a function $\alpha : X \longrightarrow Y$ *respects* the tuple of sets and maps $((A_i)_{i\leq l}, (\rho_j)_{j\leq m})$ if $\chi_{A_i} \circ \alpha = \chi_{A_i}$ on $A_i \cap X$ and $\rho_j \circ \alpha = \rho_j$ on the intersection of X and the domain of ρ_j, for all $i \in \{0, \ldots, l\}$ and $j \in \{0, \ldots, m\}$.

Lemma 4.3.2. *Suppose that $A \subseteq R \cup \mathrm{RV}^{\mathrm{eq}}$ and let $(S_i)_{i\leq n}$ be an A-definable t-stratification of R^n reflecting the A-definable map $\rho : R^n \longrightarrow \mathrm{RV}^{\mathrm{eq}}$. Fix $0 < d \leq n$ and let $B \subseteq R^n$ be a ball disjoint from $S_{\leq d-1}$. If $W := tsp_B((S_i)_{i\leq n}, \rho)$ is exhibited by the projection $\pi : R^n \longrightarrow R^d$, then there is a $(A\cup\{\ulcorner B \urcorner\})$-definable translater $(\alpha_x)_{x\in\pi(B-B)}$ witnessing the W-translatability of $((S_i)_{i\leq n}, \rho)$ on B.*

Proof. Take $Q := \pi(R^n)$ and let $\chi : Q \times R^{n-d} \longrightarrow \mathrm{RV}^{\mathrm{eq}}$ be the map $(q, x) \mapsto \rho(q\hat{\ }x)$, where $q\hat{\ }x$ denotes the concatenation of the tuples q and x. Also set $S_{i,q} := S_i \cap \pi^{-1}(q)$

and $\chi_q := \chi|_{\pi^{-1}(q)}$ for $q \in Q$. Since $S_{d-1} \cap B = \varnothing$, $((S_i)_{i\leq n}, \chi)$ is d-translatable on B and this implies that in [26, Proposition 3.19 (1)-(3)] $Q' = Q$ and χ' is a constant map. Therefore, from [26, Proposition 3.19 (3)] we obtain a compatible $(A \cup \ulcorner B \urcorner)$-definable family of risometries $(\alpha_{q,q'} : \pi^{-1}(q) \longrightarrow \pi^{-1}(q'))_{q,q'\in Q}$ such that for every $q, q' \in Q$, the risometry $\alpha_{q,q'}$ takes $(S_{i,q})_{i\leq n}$ onto $(S_{i,q'})_{i\leq n}$ and respects $((S_i)_{i\leq n}, \chi)$. Since π is an exhibition of W, the hypotheses of [26, Proposition 3.19 (3')] hold too; so, $\mathrm{dir}(\alpha_{q,q'}(z) - z) \in W$, for all $q \neq q'$ in Q and $z \in \pi^{-1}(q)$.

For each $x \in \pi(B - B)$ and each $z \in B$, set $q := \pi(z)$ and then set $\alpha_x(z) := \alpha_{q,q+x}(z)$. This defines a family of maps $(\alpha_x : B \longrightarrow B)_{x\in\pi(B-B)}$, which we claim to be a translater witnessing W-translatability of $((S_i)_{i\leq n}, \rho)$ on B. The proof of this claim is straightforward and technical.

We first prove (i)-(iv) in Definition 1.2.10 and afterwards we prove that each α_x is a risometry. Let $x, x' \in \pi(B - B)$, $z \in B$ and set $q := \pi(z)$. The map α_x respects $((S_i)_{i\leq n}, \rho)$ because $\alpha_{q,q+x}$ respects $((S_i)_{i\leq n}, \chi)$, proving (i). The compatibility of $(\alpha_{q,q'})_{q,q'\in Q}$ means that $\alpha_{q',q''} \circ \alpha_{q,q'} = \alpha_{q,q''}$ for all $q, q', q'' \in Q$. Hence, the following equations prove (ii),

$$\alpha_{x'} \circ \alpha_x(z) = \alpha_{q+x,q+x+x'} \circ \alpha_{q,q+x}(z) = \alpha_{q,q+x+x'}(z) = \alpha_{x+x'}(z).$$

For (iii), $\alpha_{q,q+x}$ takes the set $\{q\} \times R^{n-d}$ to $\{q + x\} \times R^{n-d}$, so $\pi(\alpha_{q,q+x}(z)) = q + x$. It follows that $\pi(\alpha_x(z) - z) = \pi(\alpha_{q,q+x}(z) - z) = x$. Item (iv) follows easily because, if $x \neq 0$, $\mathrm{dir}(\alpha_x(z) - z) = \mathrm{dir}(\alpha_{q,q+x}(z) - z) \in W$, from properties of the maps $\alpha_{q,q'}$.

Now we check that each α_x is a risometry. Fix $x \in \pi(B - B)$. Clearly, α_x is bijective. Let $z, z' \in B$ and set $q := \pi(z)$ and $q' := \pi(z')$. We want to verify that

$$\mathrm{rv}(\alpha_x(z) - \alpha_x(z')) = \mathrm{rv}(z - z'). \tag{4.1}$$

If $q = q'$, (4.1) is clear, for in that case $\alpha_x(z) = \alpha_{q,q+x}(z)$ and $\alpha_x(z') = \alpha_{q,q+x}(z')$, and

$\alpha_{q,q+x}$ is already a risometry. From now on we assume $q \neq q'$.

Let $w := \alpha_{q,q'}(z)$; notice that $\pi(w) = q'$. We would like to use Lemma 1.2.8 (a) with $z-w$ and $w-z'$, and thus we first need to show that $\hat{\mathrm{v}}(z-z') = \min\{\hat{\mathrm{v}}(z-w), \hat{\mathrm{v}}(w-z')\}$. Since $\mathrm{dir}(z-w) \in W$ and π exhibits W, $\hat{\mathrm{v}}(z-w) = \hat{\mathrm{v}}(\pi(z-w)) = \hat{\mathrm{v}}(q-q')$. If the needed equation fails, we would have $\hat{\mathrm{v}}(z-w) < \hat{\mathrm{v}}(z-z') \leq \hat{\mathrm{v}}(q-q') = \hat{\mathrm{v}}(z-w)$, a contradiction. Thus, $\hat{\mathrm{v}}(z-z') = \min\{\hat{\mathrm{v}}(z-w), \hat{\mathrm{v}}(w-z')\}$ holds. To show (4.1) it then suffices to prove the equations,

$$\mathrm{rv}(\alpha_{q,q+x}(z) - \alpha_{q',q'+x}(w)) = \mathrm{rv}(z-w) \tag{4.2}$$

and

$$\mathrm{rv}(\alpha_{q',q'+x}(w) - \alpha_{q',q'+x}(z')) = \mathrm{rv}(w-z'). \tag{4.3}$$

Equation (4.3) is clear because $\alpha_{q',q'+x}$ is a risometry. It remains to prove (4.2). By Lemma 1.2.8 (c), it is sufficient to show that $\pi(\alpha_{q,q+x}(z) - \alpha_{q',q'+x}(w)) = \pi(z-w)$ and $\mathrm{dir}(\alpha_{q,q+x}(z) - \alpha_{q',q'+x}(w)) = \mathrm{dir}(z-w)$. Indeed,

$$\pi(\alpha_{q,q+x}(z) - \alpha_{q',q'+x}(w)) = (q+x) - (q'+x) = q - q' = \pi(z-w),$$

which proves the first equation. For the second equation, since π is an exhibition of W, we have that

$$\begin{aligned}\tilde{\pi}(\mathrm{dir}(\alpha_{q,q+x}(z) - \alpha_{q',q'+x}(w)) &= \mathrm{dir}(\pi(\alpha_{q,q+x}(z) - \alpha_{q',q'+x}(w))) \\ &= \mathrm{dir}(\pi(z-w)) = \tilde{\pi}(\mathrm{dir}(z-w)),\end{aligned}$$

and therefore $\mathrm{dir}(\alpha_{q,q+x}(z) - \alpha_{q',q'+x}(w)) = \mathrm{dir}(z-w)$ because $\tilde{\pi}|_W$ is an isomorphism. This finishes the proof. ❐

Before proving our main theorem, we prove the following about the dimension of tangent cones.

Lemma 4.3.3. *For $X \subseteq R^n$ definable and $p \in R^n$,* $\dim(\mathcal{C}_p(X)) \leq \dim(X)$.

Proof. As usual we assume that $p = 0$. As in the proof of Theorem 4.1.4, we let $D(X) := \{(x, r) \in R^n \times R_{>0} \mid rx \in X\}$. Then $\mathrm{cl}(D(X)) \cap (R^n \times \{0\}) = \mathcal{C}_0(X) \times \{0\}$ and by Proposition 1.1.12 (g),

$$\dim(\mathcal{C}_0(X)) = \dim(\mathrm{cl}(D(X)) \setminus D(X)) < \dim(D(X)) \leq \dim(X) + 1.$$

The result follows. ❐

For their use in the proof of the upcoming theorem, we introduce the *Grassmanians* of subspaces of $\overline{R}^n$. Fix $d \leq n$. We let $\mathbb{G}_d(\overline{R}^n)$ be *the Grassmanian* of d-dimensional subspaces of $\overline{R}^n$, which is simply the set of all d-dimensional subspaces of $\overline{R}^n$. We can identify the elements of $\mathbb{G}_d(\overline{R}^n)$ with matrices with coefficients in $\overline{R}$: every $V \in \mathbb{G}_d(\overline{R}^n)$ corresponds (by considering the orthogonal projection along V) to a symmetric matrix $M \in \mathrm{Mat}_n(\overline{R})$ such that $M^2 = M$ and whose trace equals d. By this identification, it is clear that $\mathbb{G}_d(\overline{R}^n)$ can be seen as an L-definable (in fact, semi-algebraic) subset of $\overline{R}^{n^2}$. There are other approaches to this view of the Grassmanian, for example, that in [4, Subsection 3.4.2] which turns a Grassmanian into a *real algebraic variety*. The specific way in which the Grassmanian is realised as an L-definable set in the residue field is inconsequential.

We now prove the main theorem of the chapter; this is Theorem D in the introduction.

Theorem 4.3.4. *Let X be a definable subset of R^n and let $p \in R^n$. Suppose that $(S_i)_{i \leq n}$ is a t-stratification of X. Then $(\mathcal{C}_{p,i})_{i \leq n}$ is a t-stratification of $\mathcal{C}_p(X)$.*

Proof. For fixed $d \leq n$, we introduce the notation $\mathcal{C}_{p,\leq d}$ for the set $\mathcal{C}_{p,0} \cup \cdots \cup \mathcal{C}_{p,d}$. We first prove Definition 1.3.1 (1), i.e. that for all $d \leq n$, $\dim(\mathcal{C}_{p,\leq d}) \leq d$. Indeed, by the previous lemma, $\dim(\mathcal{C}_p(S_{\leq i})) \leq \dim(S_{\leq i}) \leq i$ for all $i \leq n$, and from this and properties of dimension,

$$\dim(\mathcal{C}_{p,\leq d}) = \dim\Big(\bigcup_{0<i\leq d} \mathcal{C}_p(S_{\leq i}) \setminus \mathcal{C}_p(S_{\leq i-1})\Big) \leq \max\{\dim(S_{\leq i}) \mid i \leq d\} \leq d.$$

Now we prove that Definition 1.3.1 (2') holds for $(\mathcal{C}_{p,i})_{i\leq n}$ and $\mathcal{C}_p(X)$. From now on, as usual, we assume that $p = 0$. We also assume that 0 is a non-isolated point of X; a simpler argument than the one below works otherwise. We need to prove that for each $d \leq n$ and ball $B \subseteq R^n$ disjoint from $\mathcal{C}_{0,\leq d-1}$, the tuple $((\mathcal{C}_{0,i})_{i\leq n}), \mathcal{C}_0(X))$ is d-translatable on B.

Fix d in $\{1, \ldots, n\}$. Let $x_0 \in \mathcal{C}_0(S_d)$ and let $\mu \in \Gamma$ be such that $B := B(x_0, > \mu)$ is a ball disjoint from $\mathcal{C}_{0,\leq d-1}$. Appealing to Theorem 4.1.4, let $\gamma : (0, 1) \longrightarrow S_d$ be a definable curve for which $\lim_{t\to 0^+} \gamma(t) = 0$ and $\lim_{t\to 0^+} t^{-1}\gamma(t) = x_0$. For each $t \in (0, 1)$, set $B_t := B(\gamma(t), > \mu + \mathrm{v}(t))$. We prove the following claim.

Claim 1. The collection $(S_i)_{i\leq n}$ is at least d-translatable on B_t for all sufficiently small $t \in (0, 1)$.

Proof of Claim 1. For the sake of a contradiction, we assume that there always exist arbitrarily small t in $(0, 1)$ for which $(S_i)_{i\leq n}$ is not d-translatable on B_t. From the proof of [26, Proposition 3.19 (1)] we can deduce that d-translatability of a t-stratification is always a definable condition (the last six lines of the formula written there show this explicitly). So the set $T := \{t \in (0, 1) \mid (S_i)_{i\leq n}$ is not d-translatable on $B_t\}$ is a definable subset of R. It follows that T is a finite union of definable convex sets, so there exists $\varepsilon \in (0, 1)$ such that $(S_i)_{i\leq n}$ is not d-translatable on B_t for all $t \in (0, \varepsilon)$. For such t, we have that $B_t \cap S_{\leq d-1} \neq \varnothing$, otherwise d-translatability on B_t would be forced by $(S_i)_{i\leq n}$ being a t-stratification. The definability of $B_t \cap S_{\leq d-1}$ (taking t as a parameter) and the existence of definable Skolem functions for $T_{\text{convex},c}$ (see Remark 2.2.4), where $c \in R$ is an arbitrary positive element of positive valuation, imply the existence of a definable function η with domain $(0, \varepsilon)$ and such that $\eta(t) \in B_t \cap S_{d-1}$ for each

$t \in (0, \varepsilon)$. By Proposition 2.2.1, we may actually assume that η is a differentiable curve and $\lim_{t \to 0^+} \eta(t) = 0$. The function $t^{-1}\eta(t)$ is then bounded and Lemma 2.2.2 states that $\lim_{t \to 0^+} t^{-1}\eta(t)$ exists; we let y denote this limit. By construction, $y \in \mathcal{C}_0(S_{\leq d-1})$, and there is no loss of generality in assuming that $y \notin \mathcal{C}_0(S_{\leq d-2})$; hence, $y \in \mathcal{C}_{0,\leq d-1}$. Notice that for all sufficiently small t,

$$\hat{\mathrm{v}}(y - x_0) = \hat{\mathrm{v}}\left(t^{-1}\eta(t) - t^{-1}\gamma(t)\right) > \mu,$$

so, in fact, $y \in B \cap \mathcal{C}_{0,\leq d-1}$, which is a contradiction. This proves our claim.

Claim 1 implies that for all sufficiently small t, there is a vector subspace W_t of $\overline{R}^n$ of dimension at least d such that $(S_i)_{i \leq n}$ is W_t-translatable on B_t. Our next claim is that we can find uniformity on the direction of translatability.

Claim 2. There exists a subspace W_0 of $\overline{R}^n$ with $\dim(W_0) \geq d$ and such that $(S_i)_{i \leq n}$ is W_0-transltable on B_t for all sufficiently small t.

Proof of Claim 2. In order to guarantee the definability of some objects in our argument, we point to a specific way of picking the spaces W_t, that is, we let $W_t = \mathrm{tsp}_{B_t}((S_i)_{i \leq n})$ (see definition on page 24). By d-translatability of $(S_i)_{i \leq n}$ on B_t, each W_t is indeed at least d-dimensional. The main advantage of this choice is that by [26, Lemma 3.14] the map F sending t to W_t is a definable map from $\Gamma_{>0}$ into $\mathbb{G}_{\leq d}(\overline{R}^n) := \mathbb{G}_1(\overline{R}^n) \cup \cdots \cup \mathbb{G}_d(\overline{R}^n)$. By the discussion just before the statement of the theorem, F can be regarded as a definable map into a cartesian power of $\overline{R}$. It follows from Proposition 2.1.11 (d) that F has finite image. Recall too that since T is power-bounded, the full structure on Γ is o-minimal by Proposition 2.1.11 (c). It follows that there exists $\lambda_0 \in \Gamma_{>0}$ such that F is constant on $(\lambda_0, \infty) \subseteq \Gamma$. The constant value of F on (λ_0, ∞) serves as the required subspace W_0. Finally, if $\varepsilon_0 \in R_{>0}$ is any element with $\mathrm{v}(\varepsilon_0) > \lambda_0$, then $W_t = W_0$ holds for all $t \in I := (0, \varepsilon_0)$. Claim 2 is proved.

In the remainder of the proof we show that $((\mathcal{C}_{0,i})_{i \leq n}, \mathcal{C}_0(X))$ is W_0-translatable on B.

We show this by explicitly constructing a translater for $((\mathcal{C}_{0,i})_{i\leq n}, \mathcal{C}_0(X))$ on B from given translaters of $(S_i)_{i\leq n}$ on the balls B_t, $t \in I$. Let π be an exhibition of W_0. For each $t \in I$, by W_0-translatability of $(S_i)_{i\leq n}$ on B_t, Lemma 4.3.2 provides us with a $\ulcorner B_t \urcorner$-definable translater $(\alpha_{t,u})_{u\in\pi(B_t - B_t)}$ witnessing the W_0-translatability of $(S_i)_{i\leq n}$ on B_t (with respect to π). Being a translater of a t-stratification is expressible in a first-order way since the conditions (i)-(iv) in Definition 1.2.10 are all first-order expressible. Thus, a compactness argument ensures that there are finitely many formulas defining all the translaters $((\alpha_{t,u})_{u\in\pi(B_t-B_t)})_{t\in I}$; ultimately, there is a single formula that takes (at least) t as parameter and defines all these translaters.

We now start our construction of a translater $(\alpha_x)_{x\in\pi(B-B)}$ witnessing W_0-translatability of $((\mathcal{C}_{0,i})_{i\leq n}, \mathcal{C}_0(X))$ on B. Let $z \in B$ and $x \in \pi(B - B)$. Let $\gamma_z : I \longrightarrow \bigcup_{t\in I} B_t$ be the map given by $\gamma_z(t) := \gamma(t) + t(z - x_0)$ for each $t \in I$. Clearly, $\hat{\mathrm{v}}(\gamma(t) - \gamma_z(t)) > \mu + \hat{\mathrm{v}}(t)$, and so $\gamma_z(t) \in B_t$ for each $t \in I$. Furthermore, $\hat{\mathrm{v}}(t^{-1}\gamma_z(t) - z) = \hat{\mathrm{v}}(t^{-1}\gamma(t) - x_0)$ is true for all $t \in I$, and this implies that $\lim_{t\to 0^+} t^{-1}\gamma_z(t)$ exists and, necessarily, equals z. Notice that $\alpha_{t,tx}(\gamma_z(t)) \in B_t$ for each $t \in I$ and $t \mapsto t^{-1}\alpha_{t,tx}(\gamma_z(t))$ is a definable map thanks to the fact that all the translaters $((\alpha_{t,u})_{u\in\pi(B_t-B_t)})_{t\in I}$ are defined by a single formula. Appealing to Lemma 2.2.2, we set

$$\alpha_x(z) := \lim_{t\to 0^+} t^{-1}\alpha_{t,tx}(\gamma_z(t)).$$

This defines $\alpha_x(z)$, and a simple calculation shows it belongs to B. Even though we used the specific curve γ_z to define $\alpha_x(z)$ we can show that $\alpha_x(z)$ does not depend intrinsically on such curve. To this end, suppose that $\eta : I \longrightarrow \bigcup_{t\in I} B_t$ is another definable curve for which $\eta(t) \in B_t$ for each $t \in I$ and $\lim_{t\to 0^+} t^{-1}\eta(t) = z$. Since each risometry $\alpha_{t,u}$ is in particular an isometry, the following equation holds for every $t \in I$,

$$\hat{\mathrm{v}}\left(t^{-1}\alpha_{t,tx}(\gamma_z(t)) - t^{-1}\alpha_{t,tx}(\eta(t))\right) = \hat{\mathrm{v}}\left(t^{-1}\gamma_z(t) - t^{-1}\eta(t)\right).$$

Since the right hand side goes to ∞ when $t \to 0^+$, it is clear that $t^{-1}\alpha_{t,tx}(\gamma_z(t))$ and

$t^{-1}\alpha_{t,tx}(\eta(t))$ have the same limit when $t \to 0^+$.

We have thus defined the maps $(\alpha_x : B \longrightarrow B)_{x\in\pi(B-B)}$. We now prove that such a family of definable maps is a translater of $((\mathcal{C}_0, i)_{i\leq n}, \mathcal{C}_0(X))$ witnessing W_0-translatability on B. The proof is straight-forward but technical. First we prove the bijectivity of each α_x. By replicating the construction of α_x with the translaters $(\alpha_{t,u}^{-1})_{u\in\pi(B_t-B_t)}$, we obtain a map $\beta_x : B \longrightarrow B$. For $z \in B$ we have

$$\beta_x \circ \alpha_x(z) = \beta_x(\lim_{t\to 0} t^{-1}\alpha_{t,tx}(\gamma_z(t))) = \lim_{t\to 0} t^{-1}\alpha_{t,tx}^{-1}(\alpha_{t,tx}(\gamma_z(t))) = z;$$

so $\beta_x \circ \alpha_x$ is the identity on B. Symmetrically, we can show that $\alpha_x \circ \beta_x$ is the identity on B too. Hence each α_x is bijective. Concurrently, for any $z \neq z'$ in B and a sufficiently small t,

$$\begin{aligned}\hat{\mathrm{rv}}(\alpha_x(z) - \alpha_x(z')) &= \hat{\mathrm{rv}}(t^{-1}\alpha_{t,tx}(\gamma_z(t)) - t^{-1}\alpha_{t,tx}(\gamma_{z'}(t)))\\ &= \hat{\mathrm{rv}}(t^{-1}\gamma_z(t) - t^{-1}\gamma_{z'}(t)) = \hat{\mathrm{rv}}(z - z');\end{aligned}$$

thus, each α_x is a risometry.

Lastly, we show that (i)-(iv) in Definition 1.2.10 hold for $(\alpha_x)_{x\in\pi(B-B)}$. Let $z \in B$, $x, x' \in \pi(B - B)$. We let $\eta : I \longrightarrow \bigcup_{t\in I} B_t$ be any definable curve such that $\lim_{t\to 0^+} t^{-1}\eta(t) = z$ (for concreteness, we could put $\eta = \gamma_z$ below, where γ_z is the curve defined earlier for z).

To prove (i), that α_x respects $((\mathcal{C}_{0,i})_{i\leq n}, \mathcal{C}_0(X))$, we need to prove that for each $i \leq n$, $z \in \mathcal{C}_{0,i}$ if and only if $\alpha_x(z) \in \mathcal{C}_{0,i}$ and $z \in \mathcal{C}_0(X)$ if and only if $\alpha_x(z) \in \mathcal{C}_0(X)$. Fix $i \leq n$. If $z \in \mathcal{C}_{0,i}$, we may, without loss of generality, assume that $\eta(t) \in S_i$ for all sufficiently small t. So $\alpha_{t,tx}(\eta(t)) \in S_i$ for all t, because the family $(\alpha_{t,u})_{u\in\pi(B_t-B_t)}$ respects S_i. Then clearly $\alpha_x(z) = \lim_{t\to 0^+} t^{-1}\alpha_{t,tx}(\eta(t)) \in \mathcal{C}_{0,i}$. In the other direction, if $\alpha_x(z) \in \mathcal{C}_{0,i}$, a similar argument applied to α_x^{-1} tells us that $z = \alpha_x^{-1} \circ \alpha_x(z) \in \mathcal{C}_{0,i}$. That $z \in \mathcal{C}_0(X)$ if and only if $\alpha_x(z) \in \mathcal{C}_0(X)$ is proved similarly.

For (ii)-(iv) we use that for each $t \in I$, $(\alpha_{t,u})_{u \in \pi(B_t - B_t)}$ is already a translater of $(S_i)_{i \le n}$ witnessing W_0-translatability on B_t. For (ii), using that $\alpha_{t,u} \circ \alpha_{t,u'} = \alpha_{t,u+u'}$ for all $t \in I$ and $u, u' \in \pi(B_t - B_t)$, we have that

$$\alpha_x \circ \alpha_{x'}(z) = \lim_{t \to 0} t^{-1} \alpha_{t,tx} \circ \alpha_{t,tx'}(\eta(t)) = \lim_{t \to 0} t^{-1} \alpha_{t,t(x+x')}(\eta(t)) = \alpha_{x+x'}(z).$$

So $\alpha_x \circ \alpha_{x'} = \alpha_{x+x'}$. We turn to (iii). For each $t \in I, u \in \pi(B_t - B_t)$ and $y \in B_t$, $\pi(\alpha_{t,u}(y) - y) = u$. So,

$$\pi(\alpha_x(z) - z) = \lim_{t \to 0} t^{-1} \pi(\alpha_{t,tx}(\eta(t)) - \eta(t)) = \lim_{t \to 0} t^{-1}(tx) = x.$$

Lastly we prove (iv). Suppose that $x \neq 0$. For all $t \in I, 0 \neq u \in \pi(B_t - B_t)$ and $y \in B_t$, we have that $\mathrm{dir}(\alpha_{t,u}(y) - y) \in W_0$. Therefore, for sufficiently small t,

$$\mathrm{dir}(\alpha_x(z) - z) = \mathrm{dir}(t^{-1}\alpha_{t,tx}(\eta(t)) - t^{-1}\eta(t)) = \mathrm{dir}(\alpha_{t,tx}(\eta(t)) - \eta(t)) \in W_0.$$

The proof is now complete. ❐

In Section 5.3 we present a consequence of this theorem in an archimedean setting.

Chapter 5

Archimedean t-stratifications

Throughout this chapter we keep T as a power-bounded o-minimal theory in a language $L \supseteq L_{\text{or}}$. To make sense of most definitions, we also assume that the real field $\mathbb{R}$ can be made into a model of T (this restricts what T can be; in a brief note at the end of the chapter we discuss removing this assumption). We present applications of the results in previous chapters to the o-minimal setting of $\mathbb{R}$. An ultrapower of $\mathbb{R}$ is naturally made into a T-convex field, which allows us to consider t-stratifications and tangent cones. By taking the standard part of these objects, we obtain results on stratifications and tangent cones of L-definable sets in $\mathbb{R}$.

5.1 From archimedean to non-archimedean fields, and vice versa

From now on we regard the real field $\mathbb{R}$ as an L-structure satisfying T. We let ${}^*\mathbb{R}$ denote a *non-standard model* of $\mathbb{R}$; by this we mean that ${}^*\mathbb{R} = \mathbb{R}^{\mathbb{N}}/\mathcal{U}$, where $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{N}$. Then ${}^*\mathbb{R}$ is a model of T and, since $\mathcal{U}$ is non-principal, ${}^*\mathbb{R}$ is a non-archimedean field[1]. We conventionally use s, t and u for elements of $\mathbb{R}^n$ while we use

[1] i.e. the set of natural numbers $\mathbb{N}$ is bounded in ${}^*\mathbb{R}$.

x, y and z for elements of ${}^*\mathbb{R}^n$.

Notation 5.1.1. We follow the usual notation from non-standard analysis; see [25, Section 3.7 and Chapter 4] for precise details.

- The canonical injection from $\mathbb{R}$ into ${}^*\mathbb{R}$ is denoted by the * operation, i.e. for $s \in \mathbb{R}$, ${}^*s$ is the class in ${}^*\mathbb{R}$ of the constant sequence $(s, s, \dots)$. Nevertheless, we frequently identify ${}^*s$ with s for $s \in \mathbb{R}$; so, $\mathbb{R} \subseteq {}^*\mathbb{R}$.

- We employ the *-transform of sets $A \subseteq \mathbb{R}^n$, functions and relations on $\mathbb{R}$. We treat, respectively, ${}^*X$, ${}^*f$ and ${}^*E$ as extensions of the set $X \subseteq \mathbb{R}^n$, the function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and the relation $E \subseteq \mathbb{R}^n$, i.e. by identifying ${}^*s$ with s whenever $s \in X$, $X \subseteq {}^*X$, ${}^*f|_X = f$ and $E \subseteq {}^*E$.

- The *-transform ${}^*\varphi(x, {}^*a)$ of the L-formula $\varphi(x, a)$ is defined as usual by induction on the structure of φ, see [25, Section 4.4].

The following is the main result we need to transfer statements from ${}^*\mathbb{R}$ to $\mathbb{R}$ and vice versa; for an exposition of the result see, e.g., [25].

Fact 5.1.2 (The transfer principle)**.** $\mathbb{R}$ is an elementary L-substructure of ${}^*\mathbb{R}$.

By Example 2.1.2 (3) the convex hull $O_{{}^*\mathbb{R}}$ of $\mathbb{R}$ in ${}^*\mathbb{R}$ is a T-convex subring. Thus, from now on we work with the T-convex field $({}^*\mathbb{R}, O_{{}^*\mathbb{R}})$. Notice that $O_{{}^*\mathbb{R}}$ equals $\{x \in {}^*\mathbb{R} \mid \exists N \in \mathbb{N}(-N \leq x \leq N)\}$, the set of all *finite numbers* of ${}^*\mathbb{R}$. The residue field of $({}^*\mathbb{R}, O_{{}^*\mathbb{R}})$ is $\mathbb{R}$ and the residue map $\mathrm{res} : O_{{}^*\mathbb{R}} \longrightarrow \mathbb{R}$ is traditionally known as the *standard part map*. Recall from Chapter 2 that $\|\cdot\|$ denotes the usual Euclidean norm on $\mathbb{R}^n$. We let $\|\cdot\|_*$ denote the norm on ${}^*\mathbb{R}$ given by setting $\|(x_1, \dots x_n)\|_* := \sqrt{\sum_{i=1}^n x_i^2}$, for each $(x_1, \dots, x_n) \in {}^*\mathbb{R}^n$. By L-definability, $\|\cdot\|_*$ coincides with the *-transform of $\|\cdot\|$.

Let (s_m) be a sequence of elements of $\mathbb{R}^n$. The class of (s_m) in the ultrapower ${}^*\mathbb{R}^n$ will be denoted by $[s_m]$. Let $s \in \mathbb{R}$ and suppose that (s_m) is a sequence in $\mathbb{R}^n$ converging to s. Since $\mathcal{U}$ must contain every cofinite subset of $\mathbb{N}$, we have that for all $n \in \mathbb{Z}^+$, $\{m \in \mathbb{Z}^+ \mid \mathbb{R} \models \|s_m - s\| < 1/n\} \in \mathcal{U}$. It follows that for all $n \in \mathbb{Z}^+$, ${}^*\mathbb{R} \models (\|[s_m] - {}^*s\|_* < 1/n)$, and hence that $\hat{\mathrm{v}}([s_m] - {}^*s) > 0$. We have thus showed that for every sequence (s_m) converging to s, $[s_m] \in \mathrm{res}^{-1}(s)$. Conversely, if $[t_m] \in \mathrm{res}^{-1}(s)$, then for each $n \in \mathbb{Z}^+$ the set $\{m \in \mathbb{Z}^+ \mid \|t_m - s\| < 1/n\}$ is in $\mathcal{U}$ and is thus not empty; let t_{m_n} be in this set. Clearly, (t_{m_n}) is a sequence converging to s and $[t_{m_n}] = [t_m]$. In short, $\mathrm{res}^{-1}(s)$ consists of all $[s_m]$ with (s_m) a sequence converging to s. This turns convergence in $\mathbb{R}^n$ into closeness in ${}^*\mathbb{R}^n$—the paradigma of non-standard analysis.

The fact in the previous paragraph will be used in the form of the following lemma. Recall that $\mathrm{cl}(X)$ denotes the topological closure of X.

Lemma 5.1.3 ([26, Lemma 7.9])**.** *Let $X \subseteq R^n$ be L-definable and let $s \in \mathbb{R}^n$. Then, $s \in \mathrm{cl}(X)$ if and only if ${}^*s \in \mathrm{cl}({}^*X)$ if and only if $\mathrm{res}^{-1}(s) \cap {}^*X \neq \varnothing$.*

Proof. For the first equivalence, $s \in \mathrm{cl}(X) \Leftrightarrow \mathbb{R} \models \forall \varepsilon \in \mathbb{R}_{>0} \exists t \in X(\|s - t\| < \varepsilon) \Leftrightarrow {}^*\mathbb{R} \models \forall \varepsilon \in {}^*\mathbb{R}_{>0} \exists y \in {}^*X(\|{}^*s - y\|_* < \varepsilon) \Leftrightarrow {}^*s \in \mathrm{cl}({}^*X)$. For the remaining equivalence, $s \in \mathrm{cl}(X) \Leftrightarrow$ there exists a sequence (s_m) of elements of X converging to $s \Leftrightarrow$ there exists a sequence (s_m) of elements of X such that $[s_m] \in \mathrm{res}^{-1}(s) \cap {}^*X \Leftrightarrow \mathrm{res}^{-1}(s) \cap {}^*X \neq \varnothing$. ❐

5.2 Archimedean t-stratifications

We keep the notation used in previous chapters to work in the T-convex field $({}^*\mathbb{R}, O_{{}^*\mathbb{R}})$. In particular, we make this valued field into an $L_{\mathrm{RV}^{\mathrm{eq}}}$-structure $({}^*\mathbb{R}, \mathrm{RV}^{\mathrm{eq}})$. As in previous chapters, 'definable' will mean '$L_{\mathrm{RV}^{\mathrm{eq}}}$-definable'.

5.2.1 T-stratifications in $({}^*\mathbb{R}, \mathrm{RV}^{\mathrm{eq}})$

We now turn to study t-stratifications in $({}^*\mathbb{R}, \mathrm{RV}^{\mathrm{eq}})$.

Fact 5.2.1. Since T is power-bounded, for any $A \subseteq {}^*\mathbb{R} \cup \mathrm{RV}^{\mathrm{eq}}$, $B_0 \subseteq {}^*\mathbb{R}^n$ and A-definable map $\chi : B_0 \longrightarrow \mathrm{RV}^{\mathrm{eq}}$, there exists an A-definable t-stratification reflecting χ.

Proof. By Theorems 3.1.1 and 3.2.8, Theorem 1.4.6 holds for $T_{\mathrm{RV}^{\mathrm{eq}}}$. The fact is simply the conclusion of Theorem 1.4.6 specialised to the model $({}^*\mathbb{R}, \mathrm{RV}^{\mathrm{eq}})$. ❐

The following result will be needed in the next subsection and is considered as an explicit 'regularity' condition observed in t-stratifications.

Proposition 5.2.2 ([26, Corollary 7.6])**.** *Let $A \subseteq {}^*R \cup \mathrm{RV}^{\mathrm{eq}}$. Suppose that $(S_i)_{i \leq n}$ is an A-definable t-stratification of an A-definable ball $B_0 \subseteq {}^*\mathbb{R}^n$. Let $B \subseteq B_0$ be a ball and let d be maximal such that $B \subseteq S_{\geq d}$. Then there exists a $(A \cup \{\ulcorner B \urcorner\})$-definable set $M \subseteq \Gamma$ such that: for $j > d$, $x \in S_d \cap B$ and $y \in S_j \cap B$, if $\hat{\mathrm{v}}(x - y) \notin M$, then for any ball $B' \subseteq S_{\geq j}$ containing y, $\mathrm{dir}(x - y) \in \mathrm{tsp}_{B'}((S_i)_{i \leq n})$.*

Proof. This in fact holds in arbitrary T-convex fields (with T power-bounded). In Chapter 3 we proved that $T_{\mathrm{RV}^{\mathrm{eq}}}$ satisfies Hypotheses 1.1.9 and has the Jacobian property; additionally, Proposition 2.1.11 (d) states that the value group Γ and the residue field $\overline{R}$ are orthogonal. The result is then a particular case of [26, Corollary 7.6]. ❐

Later we will apply this proposition when $A \subseteq \mathbb{R}$ and B is definable with parameters only from $\mathbb{R}$. In that case, the set M above is definable with parameters only from $\mathbb{R}$, so the orthogonality of Γ and the residue field $\mathbb{R}$ implies that M must be either $\varnothing$ or $\{0\}$. Indeed, by Remark 2.1.12, there exist 0-definable subsets $M_1, \ldots, M_k$ of Γ such that $M = \bigcup_{i \leq k} M_i$. By the definition of L_Γ on page 45, it follows that each M_i is 0-L_Γ-definable. Furthermore, by Proposition 2.1.11 (c), Γ as an L_Γ-structure is simply an expansion by definitions of a non-trivial ordered vector space over the field of exponents E of T in the language $\{+, 0, <, \{r\}_{r \in E}\}$ (note that E must be a subfield of the real field $\mathbb{R}$ since T has an archimedean model). The map $x \mapsto 2x$ is an automorphism of $(\Gamma, +, 0, <, \{r\}_{r \in E})$ that fixes 0 and nothing else, so the definable closure of $\varnothing$ equals $\{0\}$. It follows that each M_i is either empty or equals $\{0\}$, and thus the same holds for

M.

5.2.2 Archimedean t-stratifications and Whitney stratifications

Definition 5.2.3. Let $X \subseteq \mathbb{R}^n$ and $(S_i)_{i\leq n}$ be an L-definable partition of $\mathbb{R}^n$. We say that $(S_i)_{i\leq n}$ is an *archimedean t-stratification* of X if $({}^*S_i)_{i\leq n}$ is a t-stratification of ${}^*X$.

The following is the first main result in this chapter; it is a consequence of Fact 5.2.1, thus ultimately a corollary of the results in Chapter 3, and an application of Theorem 3.3.8.

Theorem 5.2.4. *Let $A \subseteq \mathbb{R}$. Every A-L-definable set $X \subseteq \mathbb{R}^n$ admits an A-L-definable archimedean t-stratification.*

Proof. Since ${}^*X$ is in particular $L_{\mathrm{RV}^{\mathrm{eq}}}$-definable, by Fact 5.2.1 it admits an A-$L_{\mathrm{RV}^{\mathrm{eq}}}$-definable t-stratification. By Theorem 3.3.8 it follows that ${}^*X$ admits an A-L-definable t-stratification $(S_i)_{i\leq n}$. Then $(\mathrm{res}({}^*S_i \cap O_{{}^*\mathbb{R}}))_{i\leq n}$ is an A-L-definable archimedean t-stratification of X. ❒

In the rest of the section we aim to prove that archimedean t-stratifications are C^1-Whitney stratifications. The proof of I. Halupczok's in the case when $L = L_{\mathrm{or}}$ and $T = \mathrm{RCF}$ works verbatim in our setting and we present it below. The proof consists of no more than exploiting Proposition 5.2.2 via non-standard analysis. We first explain how this works intuitively.

The set of all non-trivial vector subspaces of $\mathbb{R}^n$, $\mathbb{G}(\geq 1, \mathbb{R}^n)$, admits an Euclidean topology given by the metric

$$d(V, W) := \sup_{v\in V, \|v\|=1} \inf \{\|v - w\| \mid w \in W\};$$

where we assume that $\dim(V) \geq \dim(W)$. This is the topology we use to work with limits of sequences of vector spaces below. If X is a submanifold of $\mathbb{R}^n$, by $T_s(X)$ we denote the tangent space of $X \subseteq \mathbb{R}^n$ at $s \in X$, whenever it exists (e.g. when X is a C^k-manifold for some $k \geq 1$).

Definition 5.2.5. Let X and Y be C^1-submanifolds of $\mathbb{R}^n$. We say that the pair (X, Y) satisfies *Whitney's condition* (b) if for every $s \in X$ and every pair of sequences (s_m) and (t_m) of elements of X and Y, respectively, both converging to s, if $\lim\limits_{m\to\infty} \mathbb{R}(s_m - t_m)$ and $\lim\limits_{m\to\infty} T_{t_m}(Y)$ exist, then $\lim\limits_{m\to\infty} \mathbb{R}(s_m - t_m) \subseteq \lim\limits_{m\to\infty} T_{t_m}(Y)$.

Historically, *Whitney's condition* (a) is another condition originally considered but it was soon showed to follow from Whitney's condition (b) (see, e.g., [36, Proposition 2.4]). This is why we only work with Whitney's condition (b).

In a *Whitney stratification*, Whitney's condition (b) appears as a requirement for every pair of strata (S_i, S_j) with $i < j$ (see below). On the other hand, it is not reasonable to required this condition for t-stratifications. There are two obvious ways in which a t-stratification might not satisfy Whitney's condition (b): first, sequences might not be enough to witness convergence in a valued field (so the limits above might not make sense), and second, even if we allow longer sequences for convergence, t-stratifications were never asked to be smooth at any level, rendering the consideration of tangent spaces to the strata meaningless. However, not all is lost as we explain below.

The driving idea is that if $(S_i)_{i\leq n}$ is a t-stratification of ${}^*\mathbb{R}^n$ then for each $x \in S_d$ and suitable ball B containing x, $\operatorname{tsp}_B((S_i)_{i\leq n})$ serves as an approximate tangent space to S_d at x. (The definition makes obvious that this approximate tangent space does not depend too much on x—any other $x' \in S_d \cap B$ has the same approximate tangent space—and this may be seen as evidence of the non-local nature of t-stratifications; see [26, Section 1].) With this notion, Proposition 5.2.2 can be seen as an analogue of Whitney's condition (b) since it ensures that, with a few exceptions, the direction of secants between points close to x belongs to the approximate tangent space $\operatorname{tsp}_B((S_i)_{i\leq n})$. Applying Lemma 5.1.3, the character of 'approximate' above is dropped after transferring to $\mathbb{R}^n$.

Notation 5.2.6. We explain some abuses of notation used below. For any $X \subseteq {}^*\mathbb{R}^n$ we write $\mathrm{res}(X)$ for the set $\mathrm{res}(X \cap O^n_{^*\mathbb{R}})$. Another convention is that the brackets notation can be applied to sequences of subspaces of $\mathbb{R}^n$; for instance, for a convergent sequence (W_m) of vector subspaces of $\mathbb{R}^n$ we get that $[W_m] \in \mathrm{res}^{-1}\big(\lim_{m\to\infty} W_m\big)$, as we did before for sequences of points.

Let $(S_i)_{i\leq n}$ be an archimedean t-stratification of $\mathbb{R}^n$. We assume for now that each S_i is a C^1-submanifold of $\mathbb{R}^n$. Fix $d \leq n$. Recall that $\mathbb{G}_{\geq 1}(\mathbb{R}^n) := \mathbb{G}_1(\mathbb{R}^n) \cup \cdots \cup \mathbb{G}_n(\mathbb{R}^n)$ can be regarded as an L-definable set in a power of $\mathbb{R}$ (see the discussion just before Theorem 4.3.4). We can then see that the map from S_d to $\mathbb{G}_{\geq 1}(\mathbb{R}^n)$ sending each $s \in S_d$ to $T_s(S_d)$ is L-definable. By the transfer principle, we deduce that each $x \in {}^*S_d$ is associated to a vector subspace of ${}^*\mathbb{R}^n$, which we conveniently denote by $T_x({}^*S_d)$. By definition and the transfer principle, if $s \in S_d$, $\mathrm{res}(T_{^*s}({}^*S_d)) = T_s(S_d)$. To be able to use Proposition 5.2.2 we need to associate the latter space to $\mathrm{tsp}_{B'}\,({}^*S_i)_{i\leq n}$ for some B'. We prove something slightly more general.

Lemma 5.2.7. *With the notation and assumptions in the paragraph above, for any $x \in {}^*S_d$ there is a ball $B' \subseteq S_{\geq d}$ containing x such that $\mathrm{res}(T_x({}^*S_d)) = \mathrm{tsp}_{B'}(({}^*S_i)_{i\leq n})$.*

Proof. By the definition of tangent spaces in $\mathbb{R}^n$ and the transfer principle, there is a ball B'' containing x such that for all $x' \in {}^*S_d \cap B''$ with $x' \neq x$, $\mathrm{dir}(x - x') \in \mathrm{res}(T_x({}^*S_d))$. Let $B' \subseteq B'' \cap {}^*S_{\geq d}$ be a ball containing x for which $\dim(\mathrm{tsp}_{B'}(({}^*S_i)_{i\leq n})) = d$. Clearly, ${}^*S_d$ is $\mathrm{tsp}_{B'}(({}^*S_i)_{i\leq n})$-translatable on B'. It follows from Lemma 1.4.2 (a) and the property of $T_x({}^*S_d)$ above that $\mathrm{tsp}_{B'}(({}^*S_i)_{i\leq n}) \subseteq \mathrm{affdir}({}^*S_d \cap B') \subseteq \mathrm{res}(T_x({}^*S_d))$. Moreover, tangent spaces of S_d have dimension $\dim(S_d) = d$, so by definability of dimension and the transfer principle, $\dim(T_x({}^*S_d)) = d$. Hence, the desired equation follows because $\mathrm{tsp}_{B'}(({}^*S_i)_{i\leq n})$ and $\mathrm{res}(T_x({}^*S_d))$ are both d-dimensional. ❐

We are now ready to prove that archimedean t-stratifications are C^1-Whitney stratifications. The latter were introduced by H. Whitney in [53] and a classical definition of them can be found in [4, Subsection 9.7]; we provide a slightly different definition.

Definition 5.2.8. An L-definable partition $(S_i)_{i\leq n} = (S_0, \ldots, S_n)$ of $\mathbb{R}^n$ is said to be a C^1-*Whitney stratification* if the following hold.

(1) For each $d \leq n$, S_d is a C^1-submanifold of $\mathbb{R}^n$;

(2) for each $d \leq n$, either $S_d = \varnothing$ or $\dim(S_d) = d$;

(3) for all $d \leq n$, $S_{\leq d} := S_0 \cup \ldots, \cup S_d$ is topologically closed;

(4) for all $d < j \leq n$, the pair (S_d, S_j) satisfies Whitney's condition (b).

Furthermore, if $X \subseteq \mathbb{R}^n$, we say that $(S_i)_{i\leq n}$ is a Whitney stratification of X if the following extra property holds.

(5) X is a union of some of the connected components of the strata $S_0, \ldots, S_n$.

Theorem 5.2.9. *Let $X \subseteq \mathbb{R}^n$ be L-definable. Then every non-archimedean t-stratification $(S_i)_{i\leq n}$ of X is a C^1-Whitney stratification of X.*

Proof. This is (essentially) Halupczok's proof in the case when $L = L_{\text{or}}$ and $T = \text{RCF}$ in [26, Subsection 7.3]; only more details and adaptations needed to move from L_{or} to L are new.

Fix $d \leq n$. To prove that S_d is a C^1-submanifold of $\mathbb{R}^n$ we first claim that

$$\text{affdir}({}^*S_d \cap \text{res}^{-1}(s)) = \text{tsp}_{\text{res}^{-1}(s)}({}^*(S_i)_{i\leq n}). \tag{5.1}$$

We denote $\text{res}^{-1}(s)$ and $\text{tsp}_{\text{res}^{-1}(s)}({}^*(S_i)_{i\leq n})$ by B_s and V_s, respectively. We showed earlier (just before Lemma 5.1.3) that B_s consists of all points $y = [s_m]$ with (s_m) a sequence in $\mathbb{R}^n$ converging to s. Lemma 1.3.4 (a) states that ${}^*s$ is in a ball entirely contained in ${}^*S_{\geq d}$; it follows that $B_s \subseteq {}^*S_{\geq d}$, and thus that $({}^*S_i)_{i\leq n}$ is d-translatable on B_s. We prove (5.1) below.

Let $\pi : {}^*\mathbb{R}^n \longrightarrow {}^*\mathbb{R}^d$ be an exhibition of V_s. We claim that each fibre of π intersects ${}^*S_d \cap B_s$ in a single point. By V_s-translatability, it is enough to prove the claim for the fibre $F := \pi^{-1}(\pi({}^*s))$. Notice that $(S_d \setminus \{s\}) \cap \text{res}(B_s \cap F) = \varnothing$, so we get that ${}^*s \notin \text{cl}((S_d \setminus \{{}^*s\}) \cap B_s \cap F)$ by Lemma 5.1.3. Thus ${}^*S_d \cap B_s \cap F = \{{}^*s\}$. If $x, x' \in {}^*S_d \cap B_s$ are distinct, a translater $(\alpha_q)_{q\in\pi(B_s - B_s)}$ witnessing V_s-translatability of $(^*S_i)_{i\leq n}$ on B_s must be such that $\alpha_{\pi(x-x')}(x) = x'$, so $\mathrm{dir}(x - x') \in V_s$. We have proved that $\mathrm{affdir}(^*S_d \cap B_s) \subseteq V_s$. The equality must hold because both spaces are of dimension d. Indeed, the maximality of B_s as a ball such that $\mathrm{res}(^*S_d \cap B_s) = \{s\}$ and translatability imply that $\dim(V_s) = d$; on the other hand, $^*S_d$ has local dimension d (Lemma 1.3.4) , so $d = \dim(^*S_d \cap B_s) \leq \mathrm{affdir}(^*S_d \cap B_s)$.

We have proved that (5.1) holds. We are now able to prove the conditions (1)-(5) in Definition 5.2.8.

(1) We first need to construct an atlas for S_d. By translatability of S_d in balls contained in $S_{\geq d}$ and the fact that the valuative and the norm topologies coincide on $^*\mathbb{R}$, the following holds for any $s \in S_d$: there are an open neighbourhood B of $^*s$ and a coordinate projection $\pi : {}^*\mathbb{R}^n \longrightarrow {}^*\mathbb{R}^d$ such that π is a bijection from $B \cap {}^*S_d$ to an open neighbourhood in $^*\mathbb{R}^d$. Using the transfer principle it follows that for any $s \in S_d$ there are an open neighbourhood $D \subseteq \mathbb{R}^n$ of s and a coordinate projection $\bar{\pi} : \mathbb{R}^n \longrightarrow \mathbb{R}^d$ such that $\bar{\pi}$ is a bijection from $S_d \cap D$ to an open set in $\mathbb{R}^d$. This defines a coordinate chart $(S_d \cap D, \bar{\pi})$ and therefore we have an atlas for S_d. To prove that S_d is a C^1-manifold we actually show that the tangent space of S_d at $s \in S_d$ is V_s. We prove that for any two sequences (s_m) and (s'_m) of elements of S_d converging to s, with $s_m \neq s'_m$ for all m, if the sequence of secant lines $\mathbb{R}(s_m - s'_m)$ converges, then the limit is a subspace of V_s. By (5.1), $\mathrm{dir}\left([s_m] - [s'_m]\right) \in V_s$, so the span $\mathrm{res}(^*\mathbb{R}\left([s_m] - [s'_m]\right))$ of this vector is a subspace of V_s. Hence,

$$\lim_{m\to\infty} \mathbb{R}(s_m - s'_m) = \mathrm{res}\left([\mathbb{R}(s_m - s'_m)]\right) = \mathrm{res}(^*\mathbb{R}\left([s_m] - [s'_m]\right)) \subseteq V_s,$$

as required.

(2) Lemma 1.3.4 (b) states that either $^*S_d = \varnothing$ or $\dim(^*S_d) = d$, so, using the transfer principle and that in Definition 1.1.11 we can swap valuative open balls with open Euclidean discs, we have that either $S_d = \varnothing$ or $\dim(S_d) = d$.

(3) By Lemma 1.3.4 (a) $^*S_{\leq d}$ is closed, hence, using Lemma 5.1.3, for any $s \in \mathbb{R}^n$,

$s \in \mathrm{cl}(S_{\leq d}) \Leftrightarrow {}^*s \in \mathrm{cl}({}^*S_{\leq d}) = {}^*S_{\leq d} \Leftrightarrow s \in S_{\leq d}$, proving that $S_{\leq d}$ is closed.

(4) Fix $d < j \leq n$. Let $s \in S_d$ and let (s_m) and (t_m) be sequences of elements of S_d and S_j, respectively, both converging to s and such that $s_m \neq t_m$ for all m. As earlier, let $B_s := \mathrm{res}^{-1}(s)$. Then clearly, d is maximal such that $B_s \subseteq {}^*S_{\geq d}$. By Proposition 5.2.2, there is a set $M \subseteq \Gamma$ definable over $\ulcorner s \urcorner$ plus the parameters in $\mathbb{R}$ used to define $(S_i)_{i \leq n}$, and such that for any $x \in {}^*S_d \cap B_s$ and $y \in {}^*S_j \cap B_s$, if $\hat{\mathrm{v}}(x - y) \notin M$ and $B' \subseteq {}^*S_{\geq j}$ is any ball containing y, then $\mathrm{dir}(x - y) \in \mathrm{tsp}_{B'}(({}^*S_i)_{i \leq n})$. Moreover, it follows by the discussion following Proposition 5.2.2 that M must be either $\varnothing$ or $\{0\}$, for it is definable with parameters solely from the residue field. In either case, since $\mathrm{res}([s_m]) = \mathrm{res}([t_m]) = s$, we have that $\hat{\mathrm{v}}([s_m]-[t_m]) \notin M$, so $\mathrm{dir}([s_m]-[t_m]) \in \mathrm{tsp}_{B'}(({}^*S_i)_{i \leq n})$ for any ball B' containing $[t_m]$. Thus, $\mathrm{res}({}^*\mathbb{R}([s_m] - [t_m]))$ is a subspace of $\mathrm{tsp}_{B'}(({}^*S_i)_{i \leq n})$. This in particular holds for the ball B' in Lemma 5.2.7, so

$$\begin{aligned}
\lim_{m\to\infty} \mathbb{R}(s_m - t_m) = \mathrm{res}\left([\mathbb{R}(s_m - t_m)]\right) &= \mathrm{res}\left([{}^*\mathbb{R}([s_m] - [t_m])]\right) \\
&\subseteq \mathrm{tsp}_{B'}(({}^*S_i)_{i \leq n}) = \mathrm{res}(T_{[t_m]}({}^*S_j)) \\
&= \mathrm{res}([T_{t_m}(S_j)]) = \lim_{m\to\infty}(T_{t_m}(S_j)).
\end{aligned}$$

This proves Whitney's condition (b).

(5) By general topology, it is enough to show that for each $d \leq n$ both $S_d \cap X$ and $S_d \setminus X$ are open in S_d. Fix $d \leq n$. In general, if $D \subseteq \mathbb{R}^n$ is L-definable, D is open if and only if

$$\mathbb{R} \vDash \forall s \in D \exists \varepsilon \in \mathbb{R}_{>0} \forall t \in \mathbb{R}^n (\|s - t\| < \varepsilon \longrightarrow t \in D),$$

and by the transfer principle this is equivalent to

$${}^*\mathbb{R} \vDash \forall x \in {}^*D \exists \varepsilon \in {}^*\mathbb{R}_{>0} \forall y \in {}^*\mathbb{R}^n (\|x - y\|_* < \varepsilon \longrightarrow y \in {}^*D).$$

So it suffices to show that both ${}^*S_d \cap {}^*X$ and ${}^*S_d \setminus {}^*X$ are open in ${}^*S_d$. We prove this simultaneously by showing that for each $x \in {}^*S_d$ there exists a ball $B \subseteq {}^*S_{\geq d}$

containing x for which either ${}^*S_d \cap B \subseteq {}^*S_d \cap {}^*X$ or ${}^*X \cap B = \varnothing$. Indeed, it is enough to take B small enough such that the fibres of an exhibition $\pi : {}^*\mathbb{R}^n \longrightarrow {}^*\mathbb{R}^d$ of $V := \mathrm{tsp}_B(({}^*S_i)_{i \leq n})$ intersect ${}^*S_d \cap B$ in a single point (this can be done by translatability, e.g., take B such that $\mathrm{affdir}({}^*S_d \cap B) = V$ and then repeat the argument for Lemma 3.2.2 (c) replacing C with ${}^*S_d \cap B$ to show that the required intersection is a singleton). Then suppose that ${}^*X \cap B \neq \varnothing$. By the property of π, ${}^*S_d \cap B$ must be straightened to the same single coset of V to which ${}^*X \cap B$ is straightened. Since any straightener is a bijection, it follows that ${}^*S_d \cap B \subseteq {}^*S_d \cap {}^*X$, as required. ❐

Through Theorems 5.2.4 and 5.2.9 we obtain a new proof of the existence of Whitney stratifications for L-definable set in $\mathbb{R}^n$. This is not a new result; and in fact is weaker, for T has been assumed to be power-bounded: T. L. Loi proved in [34] that Whitney stratifications exist for all definable sets in *any* o-minimal expansion of the real field.

Corollary 5.2.10. *Let $A \subseteq \mathbb{R}$ and let $X \subseteq \mathbb{R}^n$ be A-L-definable. Then X admits an A-L-definable C^1-Whitney stratification.*

5.3 Classical tangent cones and archimedean t-stratifications

Our goal in this section is to obtain an analogue of Theorem 4.3.4 in the archimedean setting.

Recall from page 89 that the tangent cone $C_p(X)$ of $X \subseteq \mathbb{R}^n$ at $p \in \mathbb{R}^n$ is the set $\{t \in \mathbb{R}^n \mid \forall \varepsilon \in \mathbb{R}_{>0} \exists s \in X, r \in \mathbb{R}_{>0} (\|s - p\| < \varepsilon \wedge \|r(s-p) - t\| < \varepsilon)\}$. Notice that if $X \subseteq \mathbb{R}^n$ is L-definable and $p \in \mathbb{R}^n$, then ${}^*C_p(X) = \mathcal{C}_{{}^*p}({}^*X)$ since $C_p(X)$ is L-definable as well.

The following can be seen as a definition of $C_p(X)$ in the style of non-standard analysis.

Lemma 5.3.1. *Let $X \subseteq \mathbb{R}^n$ be L-definable and fix $p \in \mathbb{R}^n$. For any $t \in \mathbb{R}^n$, $t \in C_p(X)$ if and only if*

$$({}^*\mathbb{R}, \mathrm{RV^{eq}}) \models \exists x \in {}^*X, r \in {}^*\mathbb{R}_{>0} \left(\hat{\mathrm{v}}(x - {}^*p) > 0 \wedge \hat{\mathrm{v}}(r(x - {}^*p) - {}^*t) > 0\right). \quad (5.2)$$

Proof. We assume as usual that $p = 0$. Since the valuative topology on ${}^*\mathbb{R}^n$ coincides with the topology induced by $\|\cdot\|_*$, (5.2) is equivalent to,

$$\text{for all } n \in \mathbb{Z}^+, {}^*\mathbb{R} \models \exists x \in {}^*X, r \in {}^*\mathbb{R}_{>0} \left(\|x\|_* < 1/n \wedge \|rx - {}^*t\|_* < 1/n\right),$$

which by the transfer principle (Remark 5.1.2) is in turn equivalent to,

$$\text{for all } n \in \mathbb{Z}^+, \mathbb{R} \models \exists s \in X, r \in \mathbb{R}_{>0} \left(\|x\| < 1/n \wedge \|rs - t\| < 1/n\right).$$

Clearly this last statement holds if and only if $t \in C_0(X)$. ❒

Using this lemma we now prove a result of the nature of Proposition 4.2.1.

Proposition 5.3.2. *Let $X, Y \subseteq \mathbb{R}^n$ be L-definable and let p be a non-isolated point of X and Y. If there is a definable risometry φ on $B({}^*p, > 0)$ taking ${}^*X \cap B({}^*p, > 0)$ onto ${}^*Y \cap B({}^*p, > 0)$ and fixing ${}^*p$, then $C_p(X) = C_p(Y)$.*

Proof. We assume that $p = 0$ and denote the ball $B(0, > 0)$ by B_0. If $t \in C_0(X)$, by Proposition 5.3.1, there are $x \in {}^*X$ and $r \in {}^*\mathbb{R}_{>0}$ such that $\hat{\mathrm{v}}(x) > 0$ and $\hat{\mathrm{v}}(rx - {}^*t) > 0$. In particular, $\hat{\mathrm{v}}(rx) \geq 0$, because $\hat{\mathrm{v}}({}^*t) \geq 0$. Then $\varphi(x) \in {}^*Y \cap B_0$ and, since $\hat{\mathrm{rv}}(\varphi(x)) = \hat{\mathrm{rv}}(x)$,

$$\hat{\mathrm{v}}(r\varphi(x) - {}^*t) \geq \min\{\hat{\mathrm{v}}(r\varphi(x) - rx), \hat{\mathrm{v}}(rx - {}^*t)\} > \min\{\hat{\mathrm{v}}(rx), 0\} = 0.$$

By Proposition 5.3.1, $t \in C_0(Y)$. We have proved that $C_0(X) \subseteq C_0(Y)$. The containment $C_0(Y) \subseteq C_0(X)$ is proved similarly. ❒

Under the hypotheses of the last proposition, Proposition 4.2.1 implies that there is a risometry between $\mathcal{C}_p({}^*X) \cap B_0$ and $\mathcal{C}_p({}^*Y) \cap B_0$. This can be used to give another proof of Proposition 5.3.2.

We now turn to study archimedean t-stratifications of $C_p(X)$. Imitating the definition of $(\mathcal{C}_{p,i})_{i \leq n}$ for a t-stratification (Definition 4.3.1) we define the partition $(C_{p,i})_{i \leq n}$ as follows.

Definition 5.3.3. For an archimedean t-stratification $(S_i)_{i \leq n}$ of $\mathbb{R}^n$ and fixed $p \in \mathbb{R}^n$, the partition $(C_{p,i})_{i \leq n}$ is defined as $C_{p,0} := C_p(S_0)$ and $C_{p,i} := C_p(S_{\leq i}) \setminus C_p(S_{\leq i-1})$ for $0 < i \leq n$.

Naturally, by the L-definability of each S_i, we have that ${}^*C_{p,i} = \mathcal{C}_{{}^*p,i}$ for all $i \leq n$, where the latter comes from the t-stratification $({}^*S_i)_{i \leq n}$. If $(S_i)_{i \leq n}$ is an archimedean t-stratification of $X \subseteq \mathbb{R}^n$ and $p \in \mathbb{R}^n$, we say that $(S_i)_{i \leq n}$ *induces an archimedean t-stratification* on $C_p(X)$ if $(C_{p,i})_{i \leq n}$ is an archimedean t-stratification of $C_p(X)$. Our next result is that this is always the case whenever X is L-definable; this theorem is an immediate but important corollary to Theorem 4.3.4. The second part follows from Theorem 5.2.9.

Theorem 5.3.4. *Suppose that $X \subseteq \mathbb{R}^n$ is L-definable and fix $p \in \mathbb{R}^n$. If $(S_i)_{i \leq n}$ is an archimedean t-stratification of X, then $(C_{p,i})_{i \leq n}$ is an archimedean t-stratification of $C_p(X)$. Particularly, $(C_{p,i})_{i \leq n}$ is a C^1-Whitney stratification of $C_p(X)$.*

The second part of the statement reveals a property of archimedean t-stratifications that Whitney—and other classical kinds of—stratifications do not posses: Whitney and, e.g., *Verdier stratifications*, do not induce Whitney stratifications on tangent cones in general. A *Verdier stratification* is a Whitney stratification whose strata satisfy (the stronger) condition (w) in [52, Subsection 2.1], or, in a more model-theoretic context, satisfy the 'Verdier condition' in [34, Section 1].

Example 5.3.5. Consider the set $X := \{(x, y, z) \in \mathbb{R}^3 \mid x^3 - y^2 - z^2 = 0\}$. The sets $S'_0 := \{0\}$, $S'_1 := \emptyset$, $S'_2 := X \setminus \{0\}$ and $S'_3 := \mathbb{R}^3 \setminus X$ constitute a Verdier stratification (hence a Whitney stratification) of X. The tangent cone $C_0(X)$ is the set $\mathbb{R}_{\geq 0} \times \{0\} \times \{0\}$. Following Definition 5.3.3, we obtain the sets $C'_{0,0} = \{0\}$, $C'_{0,1} = \emptyset$, $C'_{0,2} = \mathbb{R}_{>0} \times \{0\} \times \{0\}$ and $C'_{0,3} = \mathbb{R}^3 \setminus (\mathbb{R}_{\geq 0} \times \{0\} \times \{0\})$. The dimension of $C'_{0,2}$ is 1 and this makes it impossible for $(C'_{0,i})_{i \leq 3}$ to be a Whitney stratification of $C_0(X)$. On the other hand, as an example of an archimedean t-stratification, the sets $S_0 := \{0\}$, $S_1 := \mathbb{R}_{>0} \times \{0\} \times \{0\}$, $S_2 := X \setminus \{0\}$ and $S_3 := \mathbb{R}^3 \setminus (X \cup \mathbb{R}_{>0} \times \{0\} \times \{0\})$ form an archimedean t-stratification of X. In this case we obtain the sets $C_{0,0} = \{0\}$, $C_{0,1} = \mathbb{R}_{>0} \times \{0\} \times \{0\}$, $C_{0,2} = \emptyset$ and $C_{0,3} = \mathbb{R}^3 \setminus (\mathbb{R}_{\geq 0} \times \{0\} \times \{0\})$, and they do constitute a Whitney stratification of $C_0(X)$ as expected.

A generalisation of this chapter's results

The condition that T has archimedean model was mainly needed to make sense of the objects in Subsection 5.2.2, namely, manifolds, Whitney stratifications, etcetera. After removing said condition on T, there is no obstacle to obtain the straight analogues of Theorem 5.2.4, Proposition 5.3.2 and the first part of Theorem 5.3.4 after exchanging $\mathbb{R}$ and ${}^*\mathbb{R}$ with R and ${}^*R$, respectively, where R is an arbitrary model of T and ${}^*R$ is a (non-principal) ultrapower of R. While this allows us to remove the condition that T possesses an archimedean model, we consider that remaining in the archimedean setting of $\mathbb{R}$ is more suitable for applications.

Afterword

We discuss a few open problems orbiting the topics in this thesis. First of all, we direct the reader's attention to [26, Section 9] where several open problems on t-stratifications in general valued fields are introduced and discussed.

Problem 1. [26, Problem 9.3] was partially addressed in Chapter 5 of this thesis, where we showed that, if T is power-bounded, the existence of t-stratifications for definable sets in T-convex fields implies the existence of C^1-Whitney stratifications for definable sets in $\mathbb{R} \models T$. It remains open to find the suitable context (the appropriate valued fields) from which the existence of Whitney stratification follows for more general definable sets in $\mathbb{C}$ and $\mathbb{R}$. For instance, we suspect that the existence of C^∞-Whitney stratifications for analytic subsets of $\mathbb{C}^n$ follows from considering t-stratifications in valued fields with analytic structure [7]; the difficulty here is obtaining the analogue of Theorem 3.3.8.

Problem 2. As mentioned in page 25, it is still open how to define stratifications analogous to t-stratifications in valued fields of mixed characteristic.

Problem 3. Determine how the *valuative Lipschitz stratifications* introduced in [28] relate to t-stratifications; this question is of relevance now that both have been found to exist for all closed L-definable sets in T-convex fields if T is power-bounded. As mentioned in the introduction, there are good reasons to believe that every valuative Lipschitz stratification is a t-stratification. If proved true, this could also lead to proving that in the setting of $\mathbb{R}$, Lipschitz stratifications are archimedean t-stratifications.

The next two problems were a constant concern during the work on Chapters 4 and 5.

Problem 4. Develop applications to local geometry from the results on tangent cones and t-stratifications in Chapter 4. The pioneering paper Whitney [53] already deals with some interactions between such kinds of objects.

Problem 5. Determine the right notion of equi-singularity that archimedean t-stratifications entail. As mentioned earlier, Whitney stratifications entail *normal equi-singularity* of points in the same stratum. We expect that a stronger notion of equi-singularity holds for points in the same stratum of an archimedean t-stratification. Perhaps understanding the equi-singularity implicit in Lipschitz stratifications could direct these investigations. The paper [48] could start off this project.

Appendix A: O-minimality and weakly o-minimality

A.1. O-minimality

For a thorough account of o-minimality see [15]. Let L be a first order language containing a symbol $<$ invariably interpreted as a linear order. Open intervals are sets of the form $(a, b) = \{x \in M \mid a < x < b\}$, with $a, b \in M \cup \{\pm\infty\}$.

Definition A.1.6. We say that the L-structure M is *o-minimal* if it is densely ordered and every L-definable set $X \subseteq M$ is a finite union of points and open intervals. An L-theory T containing the axioms for dense linear orders is said to be o-minimal if each of its models is an o-minimal structure.

O-minimality is preserved under elementary equivalence [32], so a complete theory is o-minimal if and only if one of its models is. We only deal with complete theories from now on. Recall that $L_{\text{or}} := \{+, -, \cdot, 0, 1, <\}$.

Example A.1.7. (1) The real field $\mathbb{R}$ is an o-minimal L_{or}-structure. The L_{or}-definable sets in this structure are also known as *semi-algebraic sets*.

(2) The ordered group of rational numbers $\mathbb{Q}$ is an o-minimal structure in the language $L_{\text{og}} := (+, -, 0, <)$. On the other hand, the field structure of $\mathbb{Q}$ in the language L_{or} is not. The L_{or}-definable set $\{x \in \mathbb{Q} \mid x^2 < 2\}$ can not be written as a finite union of points

and intervals.

Definition A.1.6 says that all the L-definable subsets of M are definable using only the order $<$ on M. While nothing is asked explicitly about definable sets in more variables, plenty of properties can be proved for such sets. See [15, Chapter 3 §2] for the prominent *Cell-decomposition Theorem*, briefly mentioned in Section 3.3. The foundation of these results is the *Monotonicity Theorem*. We fix an o-minimal L-structure M.

Theorem A.1.8 (Monotonicity Theorem)**.** *Let $A \subseteq M$ and let $f : M \longrightarrow M$ be an A-L-definable function. Then there are A-definable points $a_1 < \cdots < a_n$ in M such that, after putting $a_0 := -\infty$ and $a_{n+1} := \infty$, for each $i \in \{0, \ldots, n\}$, the function $f|_{(a_i, a_{i+1})}$ is continuous and either constant or strictly monotone.*

Continuity is relative to the topology determined by the order. If $L \supseteq L_{\text{or}}$ and M expands an ordered field, $f|_{(a_i, a_{i+1})}$ can be made differentiable.

The dimension of a definable set is expressed in Definition 1.1.11, with the obvious difference that in M^n an *open ball* is simply a product of open intervals. This definition coincides with the dimension defined through cells, and also with the algebraic dimension—exploiting that the exchange principle holds for M.

If M expands an ordered group (and accordingly $L \supseteq \{+, -, 0, <\}$), it is not difficult to check that M has *definable Skolem functions*, that is, for any n-tuple x, m-tuple y and every formula $\phi(x, y)$ there exists a definable function $f_\phi : M^n \longrightarrow M^m$ such that

$$M \models \forall x(\exists y \phi(x, y) \to \phi(x, f_\phi(x))).$$

The following result is useful when working with convergence in M. Recall that with $\mathrm{cl}(X)$ we denote the topological closure of X.

Proposition A.1.9 (O-minimal curve selection lemma)**.** *Assume M expands an ordered group. Let $X \subseteq M^n$ be L-definable and let $p \in \mathrm{cl}(X) \setminus X$. Then there is an L-definable injective continuous function $\gamma : (0, 1) \longrightarrow X$ such that $\lim_{t \to 0^+} \gamma(t) = p$.*

If M expands a field, then the function γ above can be assumed to be differentiable.

The following are further and important examples of o-minimal structures.

Examples A.1.10. (1) The L_{or}-structure of $\mathbb{R}$ enriched with restricted analytic functions (see [12] for the original model-theoretic approach) is o-minimal. This structure is denoted by $\mathbb{R}_{\text{an}}$, its language by L_{an}, and its theory by RCF_{an}.

(2) The L_{or}-structure of $\mathbb{R}$ with the exponential function $\exp : \mathbb{R} \longrightarrow \mathbb{R}$ added is o-minimal (a result combining work of A. Wilkie and of A. Khovanskii, see [54]). This expansion is denoted by $\mathbb{R}_{\exp}$, its language by $L_{\exp} := L_{or} \cup \{\exp\}$, and its theory by $\text{RCF}_{\exp}$.

The examples above illustrate two different kinds of behaviours in o-minimal structures. In $\mathbb{R}_{\text{an}}$ every L-definable function from $\mathbb{R}$ to itself is eventually bounded by a polynomial (see [12, Page 192]). In $\mathbb{R}_{\exp}$, that is obviously not true. C. Miller proved in [37] that in fact in every o-minimal structure on $\mathbb{R}$ either all definable functions are eventually bounded by a polynomial—in which case the structure is said to be *polynomially bounded*—or the exponential function is definable in the structure. Below we discuss the generalisation of this fact to arbitrary o-minimal fields, also due to C. Miller.

Let $L \supseteq L_{\text{or}}$ and let R be an o-minimal L-structure expanding an ordered field. It is not difficult to see that in particular, R is a real closed field.

Definition A.1.11. A *power function* in R is an L-definable group morphism from $(R_{>0}, \cdot, 1)$ to itself.

A power function g is differentiable at 1 and, moreover, $g'(1)$ determines g entirely (if h is another power function with $h'(1) = g'(1)$, then $h = g$). Additionally, once $g'(1)$ is known, it can be easily checked that g is differentiable everywhere and that the equation $xg'(x) = g'(1)g(x)$ holds. The name of these functions comes from the case when $g'(1) = m$ is an integer, because in that case g equals the map $x \mapsto x^m$ on R.

The set $E := \{g'(1) \mid g : R_{>0} \longrightarrow R_{>0}$ is a power function in $R\}$ forms a subfield of R

called the *field of exponents of* R. For example, the field of exponents of $\mathbb{R}$ as an L_{or}-structure is (isomorphic to) the field of rational numbers. The field of exponents of an o-minimal theory T is defined as the field of exponents of the prime model of T (which always exists, after expanding T by definitions if necessary, see [16, (2.3)]). Whenever T has an archimedean model, then its field of exponents is (isomorphic to) a subfield of $\mathbb{R}$.

Definition A.1.12. We say that R is *power-bounded* if for every L-definable function $f : R \longrightarrow R$ there exist a power function g and $x_0 \in R_{>0}$ such that $|f(x)| \leq g(x)$ for all $x \geq x_0$. Naturally, we say that the complete o-minimal theory T is power-bounded if all of its models are power-bounded (by [14, 3.2], this is equivalent to the existence of *one* power-bounded model of T).

Polynomial-boundedness, is defined analogously by requiring g above to be a polynomial x^m, for some $m \in \mathbb{Z}$. Clearly then, whenever the field of exponents is archimedean, power-boundedness and polynomial-boundedness are equivalent.

The following theorem is the dichotomy discovered by C. Miller.

Theorem A.1.13 ([38]). *If $L \supseteq L_{or}$ and R is an o-minimal L-structure expanding an ordered field, then either R is power-bounded or there exists an L-definable isomorphism* $e : (R, +, 0) \longrightarrow (R_{>0}, \cdot, 1)$.

In the second case of the statement we call e an *exponential map on* R and we say that 'R defines an exponential'. It follows that a complete o-minimal theory $T \supseteq$ RCF is either power-bounded or one of its models define an exponential (eq. all models of T define an exponential).

The main results in this thesis are exclusive to power-bounded o-minimal fields; the presence of an exponential map does complicate several results and makes others fail (see e.g. Section 3.4).

A.2. Weak o-minimality

Weakly o-minimal structures were introduced by A. Dickmann in [11] and their theory was extensively developed by D. Macpherson, D. Marker and C. Steinhorn in [35]. We again let L be a language containing a symbol $<$ for a linear order.

Definition A.2.14. We say that the L-structure M is *weakly o-minimal* if every L-definable $X \subseteq M$ is a finite union of convex sets. We say that an L-theory containing the axioms for dense linear orders is weakly o-minimal if each of its models is a weakly o-minimal structure.

Examples A.2.15. (1) Since points and intervals are convex sets, any o-minimal structure is weakly o-minimal. Accordingly, any o-minimal theory is weakly o-minimal.

(2) Let $L = \{<, D\}$, where D is unary predicate. In $\mathbb{Q}$, D is interpreted as the set $\{x \in \mathbb{Q} \mid x^2 < 2\}$. Then $\mathbb{Q}$ is a weakly o-minimal L-structure. Note that this structure is not o-minimal. In general, it is true that an o-minimal structure expanded by predicates for convex sets is weakly o-minimal (see [1, §4]).

Contrary to the situation in o-minimality, weak o-minimality is not preserved under elementary equivalence, see [35, Example 2.5].

Appendix B: Proof of Lemma 2.4.1

In this appendix we present the proof of Lemma 2.4.1 due to Y. Yin [57]. This appendix does not contain original material—we only present Yin's proof with more details.

We keep the notation of Section 2.3. We let T be a power-bounded complete o-minimal theory expanding RCF in a language $L \supseteq L_{\text{or}}$, and we fix a T-convex field (R, O_R). We let (R, RV) be the corresponding L_{RV}-structure (R, RV) (Definition 2.3.1).

We fix a large saturated model (R^*, O_{R^*}) of T_{convex}, which we take to be an elementary extension of (R, O_R). By transforming (R^*, O_{R^*}) into an L_{RV}-structure we obtain a big saturated model (R^*, RV^*) of T_{RV}, which is of course an extension of (R, RV). By 'substructure' we will always mean 'L_{RV}-substructure of (R^*, RV^*)', and such substructures will be denoted as simply M, N, etcetera. For a substructure M, we make use of the (self-explanatory) notation $\text{VF}(M) := M \cap R^*$ and $\text{RV}(M) := M \cap \text{RV}^*$. If M is a substructure, then $\text{rv}(\text{VF}(M)) := \{\text{rv}(x) \mid x \in \text{VF}(M)\} \subseteq \text{RV}(M)$, and the strict containment could occur. When the equality holds, i.e. when $\text{rv}(\text{VF}(M)) = \text{RV}(M)$, we say that M is *field-generated*. For $A \subseteq R^* \cup \text{RV}^*$, $\langle A \rangle$ denotes the substructure generated by A. Recall that if $X \subseteq R^*$, $\langle X \rangle_L$ denotes the L-substructure of R^* generated by X, and that in fact $\langle X \rangle_L = \text{VF}(\langle X \rangle)$.

Remark B.1.16. (a) If $A \subseteq R^*$, then $\langle A \rangle$ is field-generated.

(b) The substructure M is an elementary substructure of (R^*, RV^*) (i.e. a model of T_{RV}) if and only if it is field-generated and $\text{v}(M) \neq \{0\}$.

Proof. (a) Note that $\text{VF}(\langle A \rangle) = \langle A \rangle_L$, so it is enough to show that $(\langle A \rangle_L, \text{rv}(\langle A \rangle_L))$

is the minimal substructure containing A; this is true because $\mathrm{rv}(\langle A\rangle_L)$ will always be contained in the RV-sort of any other L_{RV}-structure with $\langle A\rangle_L$ as field-sort .
(b) This is easily justified by looking at the axiomatisation of T_{RV} given in [57, Definition 1.3] (which we decided not to present here). ❒

As we have done before, every model of T_{convex} can be easily (and canonically) transformed into a model of T_{RV}. It can also be seen that each model of T_{RV} can be canonically turned into a model of T_{convex}: for this it is enough to recall that the valuation ring can be defined from the language L_{RV} (see the paragraph following Definition 2.3.1). This is frequently exploited to turn arguments about L_{RV}-morphisms between substructures into arguments about L_{convex}-morphisms between L_{convex}-substructures of (R^*, O_{R^*}), and vice versa. By 'morphism' between substructures we will always mean 'L_{RV}-morphism', and similarly for embeddings and isomorphisms.

Definition B.1.17 ([57])**.** Let M be a substructure. An embedding $\sigma : M \longrightarrow (R^*, \mathrm{RV}^*)$ is said to be *immediate* if $\sigma(\xi) = \xi$ for all $\xi \in \mathrm{RV}(M)$. The notion of an immediate isomorphism between two substructures M and N is defined in accordance.

Notice that if there is an immediate isomorphism between M and N, then necessarily $\mathrm{RV}(M) = \mathrm{RV}(N)$.

If M is a substructure and $x \in R^* \cup \mathrm{RV}^*$, Mx denotes the set $M \cup \{x\}$. Also, for a substructure M we denote the set $M \cap O_{R^*}$ by $O(M)$. When M is field-generated, $\Gamma(M)$ denotes is its value group and $\overline{M}$ its residue field; recall that $\overline{M}$ is regarded as a model of T by Proposition 2.1.6 (b).

Lemma B.1.18 ([57, Lemma 1.13])**.** *Let M and N be substructures and let $\sigma : M \longrightarrow N$ be an immediate isomorphism. Suppose that $a \in R^* \setminus M$ and $a' \in R^* \setminus N$ are such that $\mathrm{rv}(a - c) = \mathrm{rv}(a' - \sigma(c))$ for all $c \in M$. Then σ can be extended to an immediate isomorphism $\sigma' : \langle Ma\rangle \longrightarrow \langle Na'\rangle$ such that $\sigma'(a) = a'$.*

Sketch of proof. We omit some details whose justification would need far more material from [14, 16] than the presented so far. We aim at illustrating the important steps in the proof employing some results from the said papers as blackboxes. For now we assume that M and N are field-generated, in the last paragraph of the proof we comment on why this is enough. The advantage of this assumption is that all the substructures considered below correspond immediately to L_{convex}-substructures.

According to σ, a and a' make the same kind of cut on R^*, so there is an L-isomorphism $\alpha : Ma \longrightarrow Na'$ that extends σ on M and takes a to a'. The rest of the proof is to ensure that we can extend α to an immediate L_{RV}-isomorphism. We first focus on just extending α to an L_{RV}-isomorphism, immediateness is dealt with later. We need to ensure that the elements of $\text{RV}(\langle Ma\rangle)\setminus\text{RV}(M)$ can be sent suitably to elements of $\text{RV}(\langle Na'\rangle)\setminus\text{RV}(N)$ (recall that $\text{RV}(M) = \text{RV}(N)$). We do this by analysing how the new elements of $\text{VF}(\langle Ma\rangle)$ lie among the elements of $\text{VF}(M)$. We particularly ask whether an element in $\text{VF}(\langle Ma\rangle)$ lies between $O(M)$ and the rest of M; regardless of the case, we show that α maps $O(\langle Ma\rangle)$ bijectively to $O(\langle Na'\rangle)$. Having done this, α is then naturally an L_{convex}-isomorphism from $(\text{VF}(\langle Ma\rangle), O(\langle Ma\rangle))$ to $(\text{VF}(\langle Na'\rangle), O(\langle Na'\rangle))$, and by previous comments, α then induces an L_{RV}-isomorphism from $\langle Ma\rangle$ to $\langle Na'\rangle$.

Case I. No $x \in \text{VF}(\langle Ma\rangle)$ is such that $|O(M)| < x < |\text{VF}(M)\setminus O(M)|$. It also follows, since $\alpha : \text{VF}(\langle Ma\rangle) \longrightarrow \text{VF}(\langle Na'\rangle)$ is an L-isomorphism, that no $x \in \text{VF}(\langle Na'\rangle)$ is such that $|O(N)| < x < |\text{VF}(N) \setminus O(N)|$. We claim that $O(\langle Ma\rangle)$ is simply the convex hull of $O(M)$ in $\text{VF}(\langle Ma\rangle)$. Indeed, if this did not hold, there would be an element $x \in O(\langle Ma\rangle)$ such that $O(M) < x$. By the initial hypothesis, there must be a positive $y \in \text{VF}(M) \setminus O(M)$ such that $y < x$. The convexity of $O(\langle Ma\rangle)$ implies that $y \in O(\langle Ma\rangle)$, so $y \in \text{VF}(M) \cap O(\langle Ma\rangle) = O(M)$, contradicting that $y \notin O(M)$. We can similarly see that $O(\langle Na'\rangle)$ is the convex hull of $O(N)$ in $\text{VF}(\langle Na'\rangle)$. Since α is an L-isomorphism (and in particular it preserves the ordering), it maps $O(\langle Ma\rangle)$ bijectively to $O(\langle Na'\rangle)$. The extension of α to an L_{RV}-isomorphism from $\langle Ma\rangle$ to $\langle Na'\rangle$ is then done as commented earlier.

Case II. There is $b \in \mathrm{VF}(\langle Ma\rangle)$ such that $|O(M)| < b < |\mathrm{VF}(M) \setminus O(M)|$. We aim to show that α maps $O(\langle Ma\rangle)$ bijectively to $O(\langle Na'\rangle)$. If $b \notin O(\langle Ma\rangle)$, [14, Lemma 5.4] implies that $\Gamma(\langle Ma\rangle) = \Gamma(M) \oplus (E \cdot \mathrm{v}(b))$, where E is the field of exponents of T (see page 120). In particular, $\Gamma(\langle Ma\rangle) \supsetneq \Gamma(M)$ and by Proposition 2.1.13 there is $d \in \mathrm{VF}(\langle Ma\rangle)$ such that $\mathrm{v}(a-d) \notin \Gamma(M)$. We then have that $\mathrm{v}(a' - \alpha(d)) \notin \Gamma(N)$. We now use the *Wilkie inequality* as a blackbox. If V is a vector space over E, $\dim_E(V)$ denotes its dimension as vector space. If $S \preceq S' \vDash T$, then $\mathrm{rk}(S' \,|\, S)$ stands for the minimal size of a set of generators of S' over S (equivalently, the maximal size of a dcl_L-independent set $X \subseteq S'$ over S). The Wilkie inequality tells us that $1 = \mathrm{rk}(\mathrm{VF}(\langle Na'\rangle) \,|\, \mathrm{VF}(N)) \geq \dim_E(\Gamma(\langle Na'\rangle)/\Gamma(N)) + \mathrm{rk}(\overline{\langle Na'\rangle} \,|\, \overline{N})$. Since $\mathrm{v}(a' - \alpha(d)) \notin \Gamma(N)$, we get that $\dim_E(\Gamma(\langle Na\rangle))/(\Gamma(N))) \geq 1$, so $\langle Ma\rangle$ and M have the same residue field; this easily implies that $O(\langle Na'\rangle)$ is the convex hull of $O(N)$ in $\mathrm{VF}(\langle Na'\rangle)$. Notice then that if $\alpha(b) \in O(\langle Na'\rangle)$, then, using that α preserves the ordering, $b \in O(\langle Ma\rangle)$, which is absurd. Symmetrically, if we start with $b' \notin O(\langle Na'\rangle)$ we see that $\alpha^{-1}(b') \notin O(\langle Ma\rangle)$, and thus α maps $O(\langle Ma\rangle))$ bijectively to $O(\langle Na'\rangle)$ as required. The extension of α to an L_{RV}-isomorphism follows as previously.

We have thus extended the L-isomorphism $\alpha : \mathrm{VF}(\langle Ma\rangle) \longrightarrow \mathrm{VF}(\langle Na'\rangle)$ to (first an L_{convex}-isomorphism $(\mathrm{VF}(\langle Ma\rangle), O(\langle Ma\rangle)) \longrightarrow (\mathrm{VF}(\langle Na'\rangle, O(\langle Na'\rangle))$ and then naturally to) an L_{RV}-isomorphism $\bar\sigma : \langle Ma\rangle \longrightarrow \langle Na'\rangle$. Clearly also $\bar\sigma$ coincides with σ on M. We now show immediateness, so far hidden in the construction. If the RV-sort of M did not grow when adding a, i.e. if $\mathrm{RV}(\langle Ma\rangle) = \mathrm{RV}(M)$, there is nothing left to check because by immediateness of σ we obtain that $\mathrm{RV}(\langle Ma\rangle) = \mathrm{RV}(M) = \mathrm{RV}(N) = \mathrm{RV}(\langle Na'\rangle)$. So we assume that $\mathrm{RV}(M) \subsetneq \mathrm{RV}(\langle Ma\rangle)$. Similarly as in the valuation property (Proposition 2.1.13) we can find $d \in \mathrm{VF}(M)$ such that $\mathrm{rv}(a-d) \notin \mathrm{RV}(M)$ (we may do cases knowing that $1 = \dim_E(\Gamma(\langle Ma\rangle)/\Gamma(M)) + \mathrm{rk}(\overline{\langle Ma\rangle} \,|\, \overline{M})$ by the Wilkie inequality). Since $\bar\sigma$ is an L_{RV}-isomorphism, $\mathrm{rv}(a-d) = \mathrm{rv}(a' - \bar\sigma(d))$. Set $\xi = \mathrm{rv}(a-d)$. Since $Ma \subseteq \mathrm{rv}^{-1}(\mathrm{RV}(M)\xi)$, clearly $\langle Ma\rangle \subseteq \langle \mathrm{RV}(M)\xi\rangle$. On the other hand, $\xi \in \langle Ma\rangle$ because $a, d \in \langle Ma\rangle$, so in fact $\langle Ma\rangle = \langle \mathrm{RV}(M)\xi\rangle$. It follows that $\bar{\sigma}$ is an L_{RV}-isomorphism from $\langle \mathrm{RV}(M)\xi\rangle$ to $\langle \mathrm{RV}(N)\xi\rangle$, and since σ is already the identity on $\mathrm{RV}(M)$ and $\bar{\sigma}(\xi) = \xi$, we must have that $\bar{\sigma}$ is the identity on $\mathrm{RV}(\langle Ma\rangle) = \mathrm{RV}(\langle \mathrm{RV}(M)\xi\rangle)$.

Lastly, we assume that M and N are not field-generated. Recall that $\mathrm{RV}(M) = \mathrm{RV}(N)$. Let $\xi \in \mathrm{RV}(M) \setminus \mathrm{rv}(M)$ and pick $d \in \mathrm{VF}(M)$ such that $\mathrm{rv}(d) = \xi$. It follows that $\mathrm{rv}(\sigma(d)) = \xi$ and according to σ, d and $\sigma(d)$ make the same kind of cut on R^*. An argument like the one above using further results from [14] allows us to find an immediate isomorphism extending σ from $\langle Md\rangle$ to $\langle N\sigma(d)\rangle$. Iterating this process for all $\xi \in \mathrm{RV}(M) \setminus \mathrm{rv}(M)$, allows us to reach a point at which we have an immediate isomorphism between two field-generated substructures $M' \supseteq M$ and $N' \supseteq N$. This case is thus reduced to the previous one. ❐

The following is an easy and important consequence.

Corollary B.1.19 ([57, Lemma 1.14])**.** *Let M and N be substructures. Then every immediate isomorphism $\sigma : M \longrightarrow N$ can be extended to an immediate automorphism $\bar{\sigma}$ of (R^*, RV^*).*

Proof. Recall that $\mathrm{RV}(M) = \mathrm{RV}(N)$. Since σ is immediate, for all $c \in M$ we have that $\mathrm{rv}(c) = \mathrm{rv}(\sigma(c))$, so $\mathrm{v}(c - \sigma(c)) > \mathrm{v}(c)$. Fix $\xi \in \mathrm{RV}^* \setminus \mathrm{RV}(M)$ and $a \in \mathrm{rv}^{-1}(\xi)$. For all $c \in M$ we have that $\mathrm{rv}(a) = \xi \neq \mathrm{rv}(c)$, so $\mathrm{v}(c) \geq \mathrm{v}(a - c)$. The two inequalities above imply that $\mathrm{rv}(a - c) = \mathrm{rv}(a - \sigma c)$ for all $c \in M$. By Lemma B.1.18, there is an immediate isomorphism from $\langle Ma\rangle$ to $\langle Na\rangle$ extending σ. By iterating this process over all $\xi \in \mathrm{RV}^* \setminus \mathrm{RV}(M)$, we eventually obtain an immediate embedding σ', extending σ, from a substructure M' into (R^*, RV^*), where M' is such that $\mathrm{RV}(M') = \mathrm{RV}^*$. By the quantifier elimination for T_{convex} (Theorem 2.1.8), $\sigma'|_{R^*}$ can be extended to a full L_{convex}-automorphism of R^*. By putting this automorphism on the field-sort and keeping σ' on the RV-sort, we obtain the desired immediate automorphism $\bar{\sigma}$. ❐

As in Chapter 2, dcl and dcl_L denote the definable closure operators with respect to L_{RV}

and L, respectively. We have that VF$(\langle X\rangle) = \mathrm{dcl}(X) \cap R = \mathrm{dcl}_L(X) = \langle X\rangle_L$ for any $X \subseteq R^*$.

Remark B.1.20. Let M be a substructure. For $a, a' \in$ VF(M), $\mathrm{rv}(c-a) = \mathrm{rv}(c-a')$ for all $c \in \mathrm{dcl}_L(\mathrm{VF}(M))$ if and only if $B(a, \geq \mathrm{v}(a-a')) \cap \mathrm{dcl}_L(M) = \varnothing$.

Proof. For $c \in \mathrm{dcl}_L(M)$, $\mathrm{rv}(c-a) \neq \mathrm{rv}(c-a')$ if and only if $\mathrm{v}(c-a) \geq \mathrm{v}(a-a')$. ❒

Lemma 2.4.1 follows easily from the next version of [57, Lemma 2.19].

Lemma B.1.21. *Let M be a substructure. Any M-definable closed ball $B \subseteq R^*$ contains an M-definable point.*

Proof. Suppose otherwise, i.e., that $B \cap \mathrm{dcl}_L(M) = \varnothing$. By the saturation of (R^*, RV^*), there is an open ball $D \subseteq R^*$ such that $D \cap \mathrm{dcl}_L(M) = \varnothing$ and $B \subsetneq D$. Let $a \in B$ and $a' \in D \setminus B$. Since clearly $B(a, \geq \hat{\mathrm{v}}(a-a')) \subseteq D$, it follows from Remark B.1.20 that $\mathrm{rv}(c-a) = \mathrm{rv}(c-a')$, for all $c \in \mathrm{dcl}_L(M)$. By Lemma B.1.18 and Corollary B.1.19, there is an immediate L_{RV}-automorphism σ of (R^*, RV^*) fixing M such that $\sigma(a) = a'$. Hence $\sigma(B) \neq B$, which contradicts the M-definability of B. ❒

Proof of Lemma 2.4.1. Put $M = \mathrm{dcl}(A)$ in Lemma B.1.21. ❒

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