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Procedia Engineering 199 (2017) 78–85

Procedia Engineering

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X International Conference on Structural Dynamics, EURODYN 2017

# Understanding the dynamics of multi-degree-of-freedom nonlinear systems using backbone curves

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**Abstract**

In this paper we will describe how backbone curves can be used to explain complex dynamic phenomena that can occur in coupled multi-degree-of-freedom physical systems. Three examples will be used to demonstrate some key points. We will describe cases when backbone curves can be decoupled. In the case of nonlinear resonance (or modal interaction) we explain how to distinguish how many modes are interacting, their unison and relative phase characteristics. Bifurcation of higher order interaction curves from the lower order curves will also be discussed. Finally we will consider an example based on the transverse vibration of a thin plate with pinned boundary conditions. Both finite element simulations and a low order differential equation model are developed for this system. The results show the importance of the nonlinear coupling terms in replicating the frequency shift phenomena which is known to occur in structures of this type. Despite its much smaller size, the low order model is able to show qualitative agreement with the finite element model. Knowledge of the backbone curve behaviour for this system, is used to explain the forced damped behaviour.


Peer-review under responsibility of the organizing committee of EURODYN 2017.

*Keywords:* backbone, vibration, nonlinear

## 1. Introduction

Traditionally modelling the dynamics of structures has been carried out using linear analysis, typically following the analytical approach first published by Lord Rayleigh [1]. When the behaviour of an engineering system is linear, computer simulations can be used to make accurate predictions of its dynamic behaviour, although we note that in some situations, particularly with complex geometries, even linear dynamic modelling can be difficult. The usefulness of linear theory is due, in large part, to the remarkable property of linear superposition, whereby a dynamic response for a structure can be obtained by adding together individual responses associated with sub-components of the response. Broadly speaking this approach has become known as “modal analysis” [2], where each mode of vibration is related to the physical configuration of the system and a corresponding resonant (or natural) frequency. Considerable effort has been made in recent years to extend this modal concept to the nonlinear domain, with the result that we now have a theory of nonlinear normal modes—see [3,4] and references therein.

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1877-7058 
Peer-review under responsibility of the organizing committee of EURODYN 2017.
10.1016/j.proeng.2017.09.157

As we will discuss in this paper, one geometrical interpretation of a nonlinear normal mode is a so called backbone curve (also sometimes previously called a skeleton curve, although this terminology seems to have fallen out of use). This is a curve that relates the amplitude of the displacement to frequency. We will discuss how this concept can be used to model dynamic behaviour of nonlinear multi-degree-of-freedom (or coupled) oscillators. Furthermore, we will give examples of how backbone curves can be used to understand nonlinear resonance effects, where one part of a nonlinear system interacts with another part.

## 2. Backbone curves

The concept of a backbone curve is shown in Figure 1. Here, the backbone curve is shown as a long dash line starting at a Frequency value of 1. The main idea is that, for most structures, resonances lead to the largest vibration

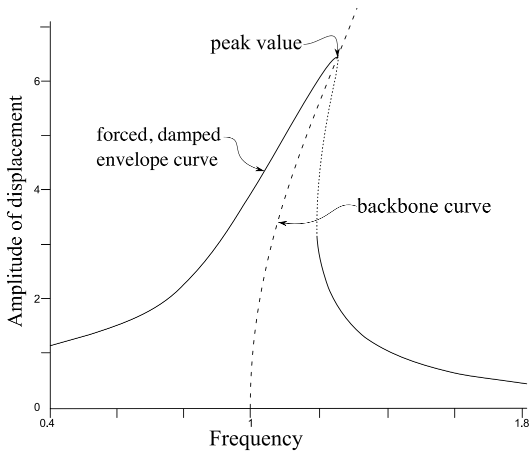


Fig. 1. The concept of a backbone curve, shown as the long dash line, for a single resonance peak. The solid line is stable forced, damped response, and the short dash line is the unstable response.

amplitudes, and so we want to understand the resonant response of the system. This is relatively straightforward for a linear system, because the resonances occur independently of each other, and respond only to external sources of excitation. However, for nonlinear systems, there is the possibility that resonances (often called "modes" in this context) can interact, due to nonlinear coupling between different parts of the structure. For example, membrane stresses (or axial loads) can provide this type of coupling in structures, as we will discuss in the final example of this paper.

So, when there are multiple degrees-of-freedom, backbone curves can be used to help understand the complexities of the resonant behaviour. This is because the resonant response of the forced system is closely linked to the unforced response, and a backbone curve represents the unforced, undamped response. The reasons to use them are that (i) they are generic in the sense that they have an influence on all forced-damped cases, and (ii) they are considerably easier to compute, than the forced, damped response for the system. One other point is worth noting, and that is that backbone curves are consistent with both linear and nonlinear multi-degree-of-freedom dynamics, unlike linear modes based on eigenvalues. So as systems get more complex, we can get physical insight into the resonant behaviour, by studying the backbone curves.

This idea is based on some key assumptions:

1. The damping is light, such that near resonance the forced, damped system is close to (i.e a small perturbation away from) the undamped system
2. The forced-damped envelope curve crosses the backbone curve at, or very close to, a bifurcation point i.e. in Figure 1 the crossing happens very close to the saddle-node bifurcation indicated at the peak value
3. The analysis is restricted to periodic orbits

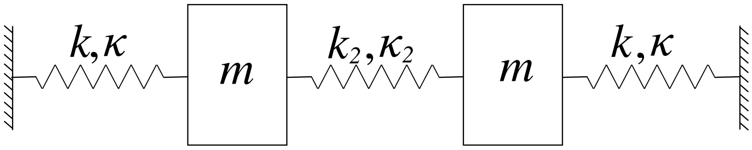


Fig. 2. The two degree-of-freedom example.

In addition to these points, although not strictly a requirement, the nonlinearity we will consider is entirely based on the restoring force (i.e. stiffness). Furthermore, the typical approach is to assume viscous damping and sine wave forcing. That said, we will show a plate example with random inputs at the end of this paper.

### 2.1. Theory

The backbone curves can be obtained using a variety of analytical or numerical methods, and here we use the direct normal form transformation described by [5]. The process is to start with the undamped, unforced version of the system

$$\ddot{\mathbf{x}} + M^{-1}K\mathbf{x} + \epsilon\mathbf{N}_x = 0, \tag{1}$$

where $\mathbf{x}$ is the displacement vector, $M$ is the mass matrix, $K$ is the stiffness matrix, the nonlinear terms are contained in a vector $\mathbf{\Gamma}$, such that $\mathbf{N}_x = M^{-1}\mathbf{\Gamma}$, and the $\epsilon$ parameter is introduced to indicate that the nonlinear terms are assumed to be small. Then we apply a linear modal transform $\mathbf{x} = \mathbf{\Phi}\mathbf{q}$, where $\mathbf{\Phi}$ is a matrix consisting of the linear eigenmodes of (1) with $\mathbf{N}_x = 0$ to give

$$\ddot{\mathbf{q}} + \Lambda\mathbf{q} + \epsilon\mathbf{N}_q = 0, \tag{2}$$

where $\Lambda$ is now a diagonal matrix containing the squared linear natural frequencies of the system. These linear natural frequencies, $\omega_{ni}$ for $i = 1, 2, 3...N$, give us information about the linear (undamped) resonances. Next we carry out a near identity transform $\mathbf{q} = \mathbf{u} + \epsilon\mathbf{h}$ to give

$$\ddot{\mathbf{u}} + \Lambda\mathbf{u} + \epsilon\mathbf{N}_u = 0 \tag{3}$$

where the vector of nonlinear terms $\mathbf{N}_u$ is now in it simplest (or normal) form. This step tells us the nonlinear resonances via the homological equation — see [5] Finally we carry out a harmonic balance (which is exact at this stage) using an assumed solution

$$u_i = U_i \cos(\omega_{ri}t - \phi_i) \tag{4}$$

where $\omega_{ri}$ is the response frequency, and $\phi_i$ is the phase. This gives relationships for the backbone curves. Note also that a particular type of frequency detuning is used in this process, and further details can be found in [5].

### 2.2. Two degree-of-freedom example

The first example is a well known two degree-of-freedom system studied by many previous authors — see [3–5] and references therein. Consider the example symmetric system shown in Figure 2 and governed by the following equation of motion

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{bmatrix} k+k_2 & -k_2 \\ -k_2 & k+k_2 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \kappa x_1^3 + \kappa_2(x_1 - x_2)^3 \\ \kappa x_2^3 + \kappa_2(x_2 - x_1)^3 \end{pmatrix} = 0 \tag{5}$$

Applying the direct normal form method gives

$$\left[\omega_{n1}^2 - \omega_{ri}^2 + \tfrac{3\kappa}{4m}\left\{\ U_1^2 + U_2^2\left(2 + \mathrm{e}^{+\mathrm{i}2(\phi_1-\phi_2)}\right)\right\}\right]U_1 = 0\,, \tag{6}$$

$$\left[\omega_{n2}^2 - \omega_{ri}^2 + \tfrac{3\kappa}{4m}\left\{\gamma U_2^2 + U_1^2\left(2 + \mathrm{e}^{-\mathrm{i}2(\phi_1-\phi_2)}\right)\right\}\right]U_2 = 0\,, \tag{7}$$

where $\gamma = 1 + (8\kappa_2/\kappa)$. There are two independent backbone curve solutions

$$S1\text{: when} \qquad U_2 = 0,\ \Omega^2 = \omega_{n1}^2 + \frac{3\kappa}{4m}U_1^2\,, \tag{8}$$

$$S2\text{: when} \qquad U_1 = 0,\ \Omega^2 = \omega_{n2}^2 + \frac{3\kappa\gamma}{4m}U_2^2\,. \tag{9}$$

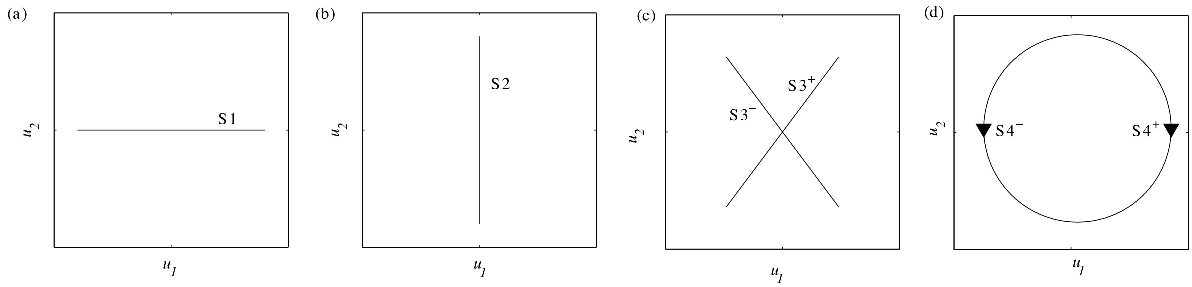


Fig. 3. Spatial configuration of possible modes for the two degree-of-freedom system example. Each plot shows a schematic representation the relative modal motion for (a) $S1$ backbone curve, (b) $S2$ backbone curve (c) $D12_i^{\pm}$ backbone curves with the phase between $u_1$ and $u_2$ being 0 or $\pi$ for $D12_i^+$ and $D12_o^-$ respectively and (d) $D12_o^{\pm}$ backbone curves in which the phase between $q_1$ and $q_2$ is $\pi/2$ or $-\pi/2$ for $D12_o^+$ and $D12_o^-$ respectively.

Here it is assumed that there is a common response frequency $\Omega = \omega_1 = \omega_2$, which relates to the case when the system is forced at a single frequency. For the nonlinear resonance (i.e. "modal" interaction) case $U_1 \neq 0$ and $U_2 \neq 0$, and so the bracket in Eqs. (7) must be zero. In addition, the phase difference must be real, so we have

$$p = \mathrm{e}^{\,\mathbf{i}2|\phi_1-\phi_2|} = \pm 1\,, \tag{10}$$

$p = +1$ corresponds to $|\phi_1 - \phi_2| = 0, \pi$, which corresponds to the in-unison case, with either in-phase (0) or out-of-phase ($\pi$) cases,
$p = -1$ corresponds to $|\phi_1 - \phi_2| = \pm\pi/2$, which is the out-of-unison case, [6], with $\pm 90°$ out-of-phase options.

Setting $p = +1$ yields two backbone curves, labelled $D12_i^+$ and $D12_i^-$, with the phase differences

$$D12_i^+ : \quad |\phi_1 - \phi_2| = 0\,, \qquad\qquad D12_i^- : \quad |\phi_1 - \phi_2| = \pi\,. \tag{11}$$

Note that the notation is $D$ for double (i.e. two mode) interaction, followed by the modes that interact (1 and 2 in this case), the subscript $i$ means in-unison, and plus (in-phase) or minus (out-of-phase). For $p = +1$ the amplitude and response frequency relationships are

$$D12_i^{\pm} : \qquad U_1^2 = \left(1 - 4\frac{\kappa_2}{\kappa}\right) U_2^2 - \frac{2m}{3\kappa}\left(\omega_{n2}^2 - \omega_{n1}^2\right)\,. \tag{12a}$$

$$D12_i^{\pm} : \qquad \Omega^2 = \frac{3\omega_{n1}^2 - \omega_{n2}^2}{2} + \frac{3(\kappa - \kappa_2)}{m} U_2^2\,. \tag{12b}$$

For $p = -1$ there are two further backbone curves, denoted $D12_o^+$ and $D12_o^-$, where the subscript $o$ denotes out-of-unison. These are characterised by the phase differences

$$D12_o^+ : \quad \phi_1 - \phi_2 = +\pi/2\,, \qquad\qquad D12_o^- : \quad \phi_1 - \phi_2 = -\pi/2\,. \tag{13}$$

For $p = -1$ the amplitude and response frequency relationships

$$D12_o^{\pm} : \qquad U_2^2 = \frac{m}{6\kappa_2}\left(\omega_{n1}^2 - \omega_{n2}^2\right)\,, \tag{14a}$$

$$D12_o^{\pm} : \qquad \Omega^2 = \frac{\kappa}{8\kappa_2}\left(\gamma\omega_{n1}^2 - \omega_{n2}^2\right) + \frac{3\kappa}{4m} U_1^2\,. \tag{14b}$$

As this example only has two degrees-of-freedom, the spatial configuration for each of the backbone curves can be plotted in the $u_1$ vs $u_2$ space, as shown in Figure 3. From Figure 3 it can be seen that the $S1$ and $S2$ backbone curves

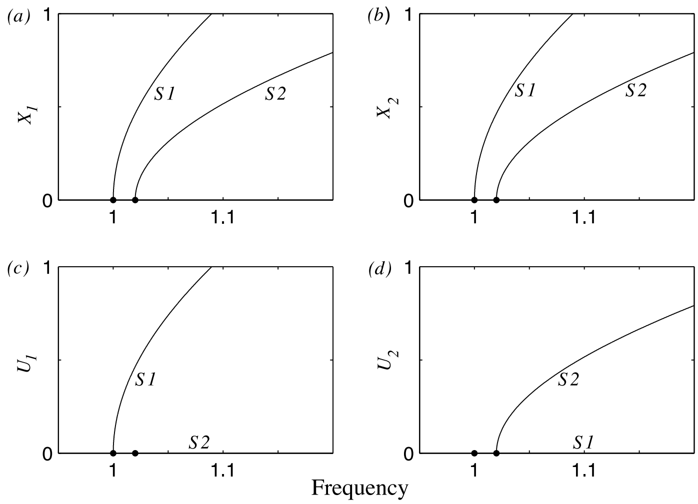


Fig. 4. Backbone curves for the two degree-of-freedom example showing the backbone curves in both physical space (a) & (b) and in the normal from coordinates (c) & (d). $X_i$ and $U_i$ are amplitudes in physical and normal form coordinates respectively. The key parameter value is that $\kappa < 4\kappa_2$.

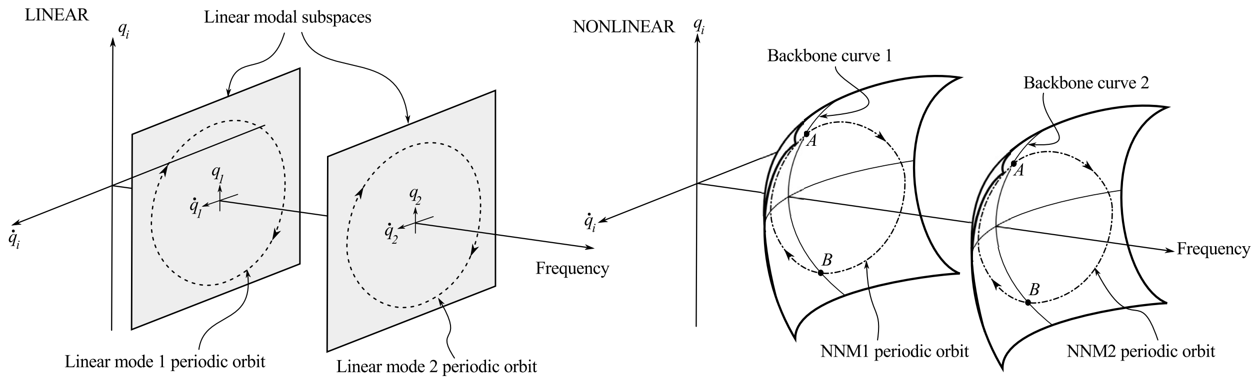


Fig. 5. Schematic drawing of two linear and nonlinear modal subspaces and manifold nonlinear normal mode (NNM) concept.

have very simple straight line relationships, and are effectively 'decoupled in the $u_i$ coordinates. This is in contrast to the $D12_i^{\pm}$ and $D12_o^{\pm}$ backbone curves which are functions of both $u_1$ and $u_2$, i.e. they are not decoupled. It turns out that $S1$ and $S2$ exist for all parameter values, while $D12_i^{\pm}$ and $D12_o^{\pm}$ are dependent on certain parameters, which in this case is $\kappa \geq 4\kappa_2$. The backbone curves for the simpler case are shown in Figure 4.

Notice that in the physical space (Figure 4 (a) & (b)) both $S1$ and $S2$ backbone curves appear in the response amplitude plots for physical amplitudes $X_1$ and $X_2$. However, in the transformed, normal form, coordinates (Figure 4 (c) & (d)) only $S1$ or $S2$ appears in the response amplitude plots for normal form amplitudes $U_1$ and $U_2$ respectively. In fact, in both cases the other backbone curve is there, but on the zero axis, as indicated by the black dots, which correspond to the start of each of the backbone curves. So the normal form transformation has decoupled the response into two separate parts in this case. In the case where $\kappa \geq 4\kappa_2$ it is possible for the $D12_i^{\pm}$ and $D12_o^{\pm}$ backbone curves to be present, and these appear as bifurcations from the $S1$ and $S2$ curves. Further details can be found in [5].

For the non-resonant case, when just $S1$ and $S2$ exist (i.e. as shown in Figure 4) we can make a comparison with the theories for both linear modes and nonlinear normal modes. This is shown in Figure 5, where in both plots two modal subspaces are shown superimposed onto a frequency axis. The modal subspaces can be thought of as containing a family of periodic orbits, and the coordinates are that of the phase space for that mode (note that the phase space is rotated $90^o$ from its normal position). On the left we show the linear case where the subspaces spaces are planar, and the backbone curves would just be straight lines in the $q_i$ vs Frequency plane. On the right we show the nonlinear

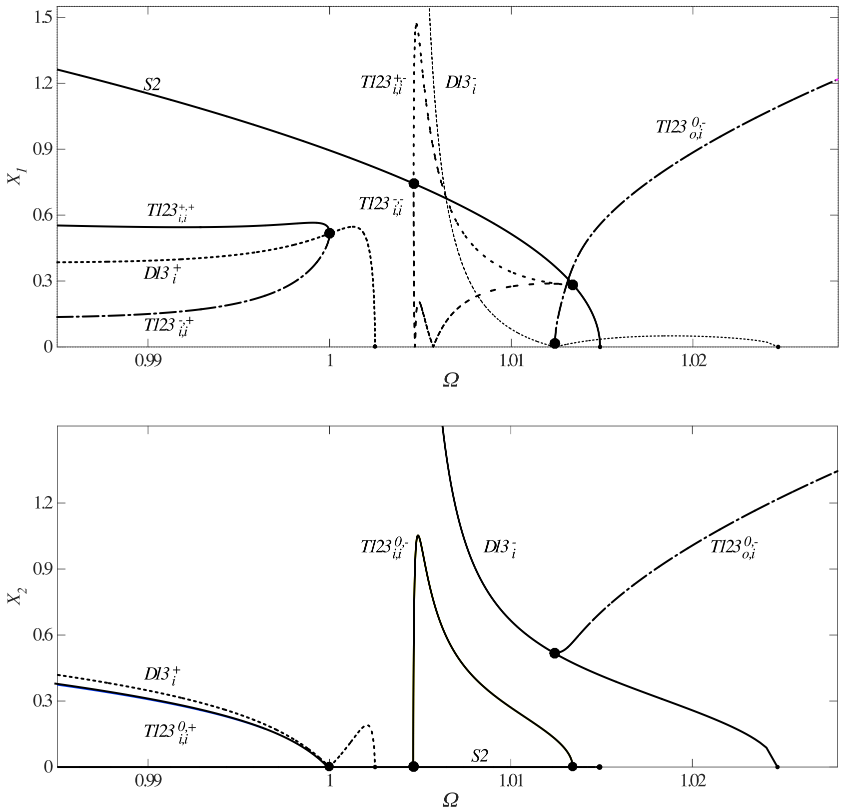


Fig. 6. Backbone curves for a three degree-of-freedom system. $X_3$ is not shown as it is qualitatively the same as $X_1$ except for changes in the phase. The subscripts $i$ and $o$ denote the in-unison and out-of-unison resonant interaction respectively and the plus or minus signs represent the phase, except where a superscript 0 occurs which means that that phase case does not exist. Full details are given in [8]. Small black dots are where the back bone curves start, and large black dots are bifurcation points on the backbone curves.

case, where each family of periodic orbits are contained in a manifold [7]. The intersection of these manifolds with the $q_i$ vs Frequency plane are the backbone curves. Note that the shape of the manifolds in the nonlinear case will change depending on the exact form of the nonlinearity being considered. In Figure 5 the nonlinear manifolds follow a similar trend to those shown in the two degree-of-freedom example. In fact the backbone curves would be similar to plotting Figure 4 (c) & (d) simultaneously.

## 2.3. More than two degrees-of-freedom

When there are $N$ degrees-of-freedom, and $N > 2$, it is possible for interactions between $2, 3, ..., N$ modes, (or $N, N-1, N-2..., 2.$) For example in Figure 6 we show the backbone curves for a three degree-of-freedom system with softening spring stiffness nonlinearity [8]. These backbone curves result in a response where triple interactions are possible (i.e. $T123$) in addition to the single curves (such as $S2$), and double interactions (such as $D13_i^+$). All these backbone curves can be seen in Figure 6, and we note that the higher order interaction backbone curves tend to emanate from the lower order curves via bifurcation points shown as large black dots in Figure 6.

It's not difficult to see how as the number degrees-of-freedom increases, the number of possible interactions will increase significantly as well. In fact, there are also an infinite number of non-phase-locked interactions which also typically exist, [9]. However, when forcing and damping is added to the system, the vast majority of these potential interactions will not persist. In fact, only a few are robust enough to be sustained with damping present, and these tend to be between the lower order resonant modes. The exact process of which modes are significant requires an

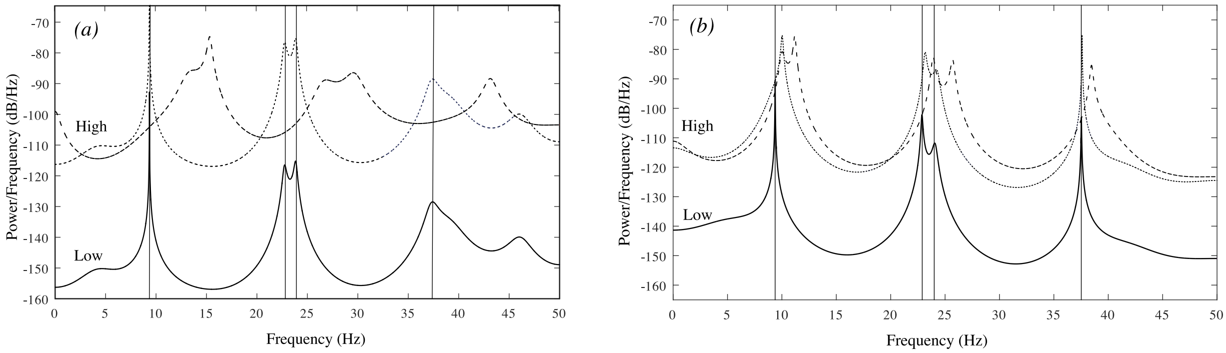


Fig. 7. Autoregressive power spectrum of the displacement response of the plate under random excitation on left-bottom quarter area at two power levels: Low when $A = 1 \times 10^{-2}$ and High when $A = 1$. (a) shows the FE simulation, and (b) the low order model from the Galerkin reduction. The short dash lines represent the integration results excluding the nonlinear coupling effects and the longer dash lines represent the results including the nonlinear coupling. The vertical thin solid lines denote the position of the linear modal frequencies.

understanding of the energy transfer between modes, as described in [10]. We now consider an example where backbone curves can be used to help understand the behaviour of a forced damped system.

### 2.4. A plate example with random forcing

The final example we discuss in this paper is a 500mm×520mm×5mm thin rectangular plate with pinned supports on all edges. In this example the plate was excited using a forcing input based on random data with the sample rate of 10kHz for a period $T = 50$s. The model consisted of 1600 thick shell elements (S8R in Abaqus) which were used to discretize the plate. S8R is used as it includes membrane stretching effects for large displacements, and this is a key part of the physics which we wish to capture. In addition the integrator, Abaqus implicits, in the Abaqus standard solver was used with the value of $\alpha_n = -\frac{1}{6}$ selected to ensure adequate numerical damping. The displacement responses at the centre of top-right quadrant of the plate is used as a metric, which guarantees that the contributions of the first four bending modes are included. The natural frequencies of the four (linear) modes found from the Abaqus model are $\omega_{n1} = 58.707$rads/s, $\omega_{n2} = 143.33$rads/s, $\omega_{n3} = 150.24$rads/s and $\omega_{n4} = 234.83$rads/s. Further details are given in [11].

The Abaqus model results are shown in Figure 7 (a) when the random input is applied on to the left-bottom quadrant of the plate. Two different forcing cases were used. An amplitude scaling factor denoted as $A$, was used so that the random force magnitude denoted *Low* in Figure 7 (a) occurs when $A = 1 \times 10^{-2}$. This ensures that the maximum displacement response amplitude of the plate is less than 20% of the thickness. In the second case, denoted *High* in Figure 7 (a), $A = 1$, so that the maximum displacement response amplitude was larger than the thickness of the plate.

For each of the two forcing cases, linear and nonlinear results were computed by switching *Nlgeom* on and off in the Abaqus model. From Figure 7 (a), it can be seen that for the low level excitation situation the linear and nonlinear results are the same (both represented by single solid line) and their resonant frequencies are close to the corresponding linear modal frequencies (vertical lines). When the excitation level is High, the difference between the linear (short dashed line) and nonlinear (longer dash line) results becomes significant. For this case, the resonant frequencies of the linear results are still close to the linear frequencies, but the nonlinear results have all shifted to the right significantly.

To help understand this phenomena, a low order backbone curve model was used. The nonlinear model was taken from a Galerkin decomposition of the governing partial differential equation for a thin plate as described by [5]. This resulted in a model consisting of four coupled second order differential equations that should capture the essential physics of the plate. Six backbone curves were found for this system, four single mode curves, $S1$ to $S4$, and two double mode $D23^{\pm}$ curves were found. The $S1$-$S4$ curves relate to the distortion of the underlying linear response, and the $D23^{\pm}$ curves occur because of the fact that $\omega_{n2}$ and $\omega_{n3}$ are close in value, giving a potential interaction.

As the system has random forcing we would not expect nonlinear resonance to occur, but instead we focus on the feature of the frequency peaks shifting to the right as the forcing amplitude increases. The resonance shifting can be explained using backbone curves when there are significant membrane stresses (axial loads) in the structure, [12]. With random forcing we see the same effect in the finite element model in Figure 7 (a). This can be qualitatively compared with the low order model shown in Figure 7 (b). In this figure, forced damped responses have been computed using analogous conditions as shown in Figure 7 (a). The result show the same type of qualitative behaviour. In particular it shows the role of the nonlinear coupling terms, as they produce the main frequency shifting effect.

The quantitative behaviour is also reasonably close, but it should be remembered that the Abaqus has 1600 element, whereas the low order model is just 4 coupled modal equations. The example demonstrates how a low order model based on backbone curves can help explain a complex forced damped behaviour.

## 3. Conclusions

In this paper we have described how backbone curves can be used to explain the complex dynamic phenomena that can occur in coupled multi-degree-of-freedom physical systems. Starting with the example of a two degree-of-freedom oscillator, we described how backbone curves could be obtained. Furthermore, we pointed out that using the direct normal form transformation, the two backbone curves could be decoupled for a certain parameter range. Following this we discussed a three degree-of-freedom example. In this case, further distinctions of nonlinear resonance (or modal interaction) were described using notation to distinguish how many modes were interacting, their unison and relative phase characteristics. It was also noted that the higher order interaction curves tend to bifurcate from the lower order curves in most cases.

Finally we reviewed an example based on the transverse vibration of a thin plate with pinned boundary conditions. In this case the system was excited with random forcing. Both finite element simulations and a low order differential equation model were developed for this system. The results showed the importance of the nonlinear coupling terms in replicating the frequency shift phenomena which is known to occur in structures with membrane stresses. Despite its much smaller size, the low order model was able to show qualitative agreement with the finite element model. Knowledge of the backbone curve behaviour for this system, was used to explain the forced damped behaviour.

## Acknowledgements

The author would like to acknowledge the contribution of the following collaborators who generated the data for the following figures; Xuanang Liu (Figs. 6. & 7), Tom Hill (Fig. 3) and Simon Neild (Fig. 4). Parts of this work were supported by EPSRC grant EP/K003836/2.

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