ON MODAL TECHNIQUES FOR MODEL REDUCTION

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Abstract. The general applicability of the modal approach for model reduction is restricted by the lack of guaranteed bounds for approximation errors and of a satisfactory modal dominance analysis procedure. Functional and computational enhancements of this approach are proposed. Functional enhancements arise by combining the modal techniques with other methods and by using improved dominance analysis techniques. The computational enhancements are the results of employing numerically reliable algorithms for both dominance analysis as well as for model reduction.

1. INTRODUCTION

The modal approach to model reduction proposed initially by Davison [1] was later extended with new variants by several authors: Marschall [2], Chidambara [3], Fossard [4], Litz [5] and others. The importance of the modal approach as a useful model reduction technique resides in its applicability to reduce high order systems as those arising for example from modelling of large mechanical structures or of large power systems. The method can handle models with lightly damped modes and even unstable systems. In case of very large order systems, the modal technique is one of the very few applicable methods.

Several limitations of the modal approach raise problems for a general use of this approach. In the first place, the lack of a generally applicable modal dominance analysis method prevents the use of this method in many cases as for example when the original system has multiple poles. The existing methods fail sometimes even to detect exact structural nonminimality, that is, poles which are uncontrollable or unobservable. Another weekness of this approach is the lack of a guaranteed bound for the approximation error which has as consequence the frequent need to experiment on a trial and error basis with different approximations.

In this paper we shortly survey the main existent modal reduction approaches and some of available techniques for dominance analysis. Then we discuss possible enhancements of the modal reduction approach. These enhancements consist in: 1) combining the modal techniques with other approaches; 2) using a new, more powerful method for modal dominance analysis; and 3) using numerical techniques with guaranteed numerical reliability. The proposed new approach is well suited for robust software implementation.

2. MODAL REDUCTION TECHNIQUES

Consider the n-th order original state-space model $G := (A, B, C, D)$ with the $p \times m$ *transfer-function matrix* (TFM) $G(\lambda) = C(\lambda I - A)^{-1}B + D$, and let $G_r := (A_r, B_r, C_r, D_r)$ be an r-th order approximation of the original model ($r < n$), with the TFM $G_r = C_r(\lambda I - A_r)^{-1}B_r + D_r$. The modal approach to model reduction can be interpreted as performing a similarity transformation Z yielding

$$\left[\begin{array}{c|c} Z^{-1}AZ & Z^{-1}B \\ \hline CZ & D \end{array}\right] := \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D \end{array}\right], \quad (1)$$

where A_1 and A_2 contains the r *dominant* and respectively, the $n - r$ *non-dominant* eigenvalues (modes) of A, and then defining the reduced model on the basis of this partitioned representation. The above partition of system matrices is equivalent with the additive decomposition $G = G_1 + G_2$, where $G_1 := (A_1, B_1, C_1, D)$ and $G_2 := (A_2, B_2, C_2, 0)$ are the dominant and non-dominant subsystems, respectively.

For our discussion of different modal approaches for model reduction we assume that the original system is already additively decomposed. Furthermore we assume that the system is asymptotically stable. This assumption is only a technical one, because for an unstable system the modal approach can be performed on its stable projection.

We consider three basic approaches:

Method 1. Define $G_r := (A_1, B_1, C_1, D)$. This is

basically the modal approximation proposed in [1]. The approximation error $\Delta = G - G_r$ tends to zero at high frequencies. However, the DC-gains mismatch of the original and reduced models could be large.

Method 2. Define $G_r := (A_1, B_1, C_1, D + G_2(\gamma))$, where $\gamma = 0$ for a continuous-time system and $\gamma = 1$ for a discrete-time system. Note that $G(\gamma) = G_r(\gamma)$, and thus this approximation preserves the DC-gain of the original system, but the approximation error at high frequencies could be large. The methods of [2, 3, 4], which compensate the steady-state errors can be viewed as particular cases of Method 2.

Method 3. Define $G_r := (A_1, B_1, C_1 + C_2E, D)$, where E is to be determined such that $G(\gamma) = G_r(\gamma)$. This approximation automatically ensures small errors at high frequencies. From the equality of DC-gains follows that E should satisfy

$$C_2(\gamma I - A_2)^{-1}B_2 = C_2E(\gamma I - A_1)^{-1}B_1.$$

This is a system of pm linear equations with $(n - r)r$ unknowns and a solution (with possibly minimal norm) generically exists provided $pm \leq (n - r)r$, a condition fulfilled in most applications. The method of [5] results if we impose the stronger condition

$$(\gamma I - A_2)^{-1}B_2 = E(\gamma I - A_1)^{-1}B_1,$$

which usually leads to an E with higher norm. The generical solvability condition in this case is $m \leq r$, which in most applications is also fulfilled. The additional freedom arising from the non-unicity of E can be used to optimally tune the free parameters of E to minimize for instance the output error norm.

One difficulty in using the modal approach is the lack of and *a priori* computable bound for the resulting approximation error $\Delta = G - G_r$. The actual error can be computed only after that a choice has been made, and thus the model reduction can be done only on a *trial and error* basis. In contrast, methods based on balancing, as for example the *balance & truncate* (B&T) method [6, 7], provide *a priori* information (the Hankel-singular values) which can be used to select the appropriate order for an acceptable approximation error.

It is possible to combine the modal approach with other techniques. For example, if the system is already decomposed as in (1), then the reduction can be performed separately on G_1 and G_2. Let $G_r = G_{1r} + G_{2r}$ be the resulting reduced model, where G_{1r} and G_{2r} are the resulting reduced subsystems computed say with the B&T method. If for the separate reduction of terms we have that $\|G_i - G_{ir}\| \leq \varepsilon_i$ for $i = 1, 2$, then $\|G - G_r\| \leq \varepsilon_1 + \varepsilon_2$. Thus, by reducing individually the terms, we can also control the resulting global error by choosing appropriate orders for the reduced subsystems. The technique can be readily extended to additive decompositions with more than two terms (see the next section) and many variations of it are possible by employing alternative model reduction methods.

The real advantage of such combinations is more evident when we have to reduce very large order models, as those which typically result from finite-element analysis of large mechanical structures. Because the large orders of such models, the modal approach is frequently the only method which can be used for order reduction. This reduction is often only a preliminary reduction which makes tractable further reductions with the help of more powerful methods.

3. MODAL DOMINANCE ANALYSIS

The main limitation of the modal approach to model reduction is the lack of a reliable, general purpose method for modal dominance analysis. The existence of such a method is highly questionable because for any of existing methods counterexamples can be easily constructed showing their failures in producing useful dominance information. An counterexample to the method of Litz [5] is given in [8], where a 12-th order system with distinct and equally dominant poles is presented for which a good 4-th order approximation can be computed. We can see this as a basic limitation of the modal approach which can permanently occur, because often the identified dominant parts have still too large orders and thus further reductions should have recourse to alternative techniques.

In this section we discuss the limitations of existing dominance analysis techniques and we propose an alternative approach to overcome them. The new technique allows an easy handling of systems with multiple poles or of systems which are exactly or nearby non-minimal.

Consider the system $G = (A, B, C, D)$ with the state matrix A in a *block-diagonal form* (BDF)

$$A = \mathrm{diag}(A_1, \ldots, A_k) \tag{2}$$

and the matrices B and C partitioned accordingly

$$B = [\, B_1^T, \ldots, B_k^T \,]^T, \qquad C = [\, C_1, \ldots, C_k \,]. \tag{3}$$

This partition of system matrices is equivalent with the additive decomposition $G = D + \sum_{i=1}^{k} G_i$, where $G_i(\lambda) = C_i(\lambda I - A_i)^{-1}B_i$, for $i = 1, \ldots, k$. We use this decomposition to present an unifying treatment of modal dominance analysis methods.

The earlier modal reduction methods [1, 2, 3, 4] concerns exclusively with continuous-time systems and always assume that A is *diagonalizable*, and thus all blocks in (2) are 1×1. An eigenvalue λ_i is called *dominant* (or *slow*) if it is situated not too far from

the imaginary axis and *non-dominant* (or *fast*) otherwise. The fast modes lying far from the imaginary axis are always neglected, even if they have a substantial contribution to the system dynamics.

A more satisfactory approach was proposed by Litz [9]. As dominance index for an eigenvalue λ_i he used the quantity

$$R_i = \|D_1 G_i(0) D_2\|, \tag{4}$$

where D_1 and D_2 are diagonal output and input scaling matrices, respectively, and $\|F\| := \sum_{i,j} |f_{ij}|$ or $\|F\| := \max_{i,j} |f_{ij}|$. Those eigenvalues having the largest dominance indices are called *dominant* and are retained in the reduced model. In order to evidence week dynamic interactions, Litz also introduced a somewhat heuristically defined frequency-weighted dominance index. The choice of matrices D_1 and D_2 should reflect the relative importance of different output and input variables. A possible choice for the diagonal elements of these matrices is to take them as the reciprocal of the absolute maximum values of the the corresponding output and input variables. Note that dominance indices equivalent with (4) can be defined by using any norm for TFMs as for instance the 2-, ∞- or Hankel-norm. Each TFM $G_i(\lambda)$ being of the form $C_i B_i/(\lambda - \lambda_i)$, the evaluation of these norms can be done by using easily computable explicit formulas: $\|G_i\|_\infty = \Gamma_i$, $\|G_i\|_2 = \sqrt{|\lambda_i|/2}\Gamma_i$, $\|G_i\|_H = \Gamma_i/2$, where $\Gamma_i = \|G_i(0)\|_2$.

The main limitation of using such dominance indices is the requirement for A to be diagonalizable. Even if A is diagonalizable, all discussed dominance indices are not appropriate for detecting exact or nearby structural non-minimality, as evidenced by the following simple example $A = \mathrm{diag}(-1, -1, -10)$, $B = [\,1\ 1\ 1\,]^T$, $C = [\,1\ -1\ 1\,]$. Apparently the *slow* eigenvalues $\lambda_1 = \lambda_2 = -1$ should be kept in the reduced model and the *fast* eigenvalue $\lambda_3 = -10$ should be removed. The dominance indices $R_1 = R_2 = 1$, $R_3 = 0.1$ computed with (4) support this decision. However, it is easy to observe that an *exact* minimal realization of this system is $A = -10$, $B = 1$, $C = 1$.

The possible enhancements of the modal dominance analysis are directed towards handling the cases of multiple eigenvalues, or of exact or nearby non-minimality. We assume that in the BDF (2), any of two diagonal blocks have no common eigenvalues. Let n_i be the order of the i-th block and let $\sigma_j^{(i)}$, $j = 1, \ldots n_i$ the decreasingly ordered *Hankel singular values* (HSV) of the subsystem $G_i = (A_i, B_i, C_i)$ (the square-roots of the eigenvalues of the product of the corresponding gramians). The eigenvalues of a diagonal block A_i for which $\sigma_{n_i}^{(i)} > \varepsilon$, are called *dominant*, where ε is a given tolerance on the HSV. If $\sigma_1^{(i)} \leq \varepsilon$ then the eigenvalues of A_i are called *non-dominant*. If $\sigma_j^{(i)} > \varepsilon$ for $j = 1, \ldots r_i$, then r_i of the eigenvalues are dominant and $n_i - r_i$ are non-dominant. To uncontrollable and/or unobservable eigenvalues correspond null singular values. Thus, by setting $\varepsilon = 0$, the dominant eigenvalues are those which are both controllable and observable. The non-dominant part of a subsystem $G_i = (A_i, B_i, C_i)$ can be removed by applying one of several powerful model reduction methods, as for instance the *balancing-free square-root* variant of B&T method [10].

The following straightforward procedure can be used to compute reduced order models by combining the modal approach with a suitable model reduction method capable to handle non-minimal systems:

1. Reduce the system (A, B, C, D) to the additively decomposed form (2)-(3), where $\lambda(A_i) \cap \lambda(A_j) \neq \phi$ for $i \neq j$.
2. For $i = 1, \ldots, k$ determine r_i, the number of dominant eigenvalues of block A_i.
3. For each n_i-th order subsystem $G_i = (A_i, B_i, C_i)$ compute its r_i-th order dominant part $G_{ir} = (A_{ir}, B_{ir}, C_{ir}, D_{ir})$ by using a suitable model reduction algorithm.
4. Construct $G_r = (A_r, B_r, C_r, D_r)$, where $A_r = \mathrm{diag}(A_{1r}, \ldots, A_{kr})$, $B_r = [\,B_{1r}^T\ \ldots\ B_{kr}^T\,]^T$, $C_r = [\,C_{1r}\ \ldots\ C_{kr}\,]$, $D_r = D + \sum_{i=1}^k D_{ir}$.

This procedure can be easily implemented to determine a reduced system of a *specified* order or a reduced system G_r satisfying $\|G - G_r\| \leq \varepsilon_a$, where ε_a is a given absolute error tolerance. In the latter case, the orders r_i of reduced subsystems G_{ir}, $i = 1, \ldots, k$ can be usually determined automatically. For instance when using the B&T method we can choose r_i such that for a given ε_a we have

$$\|G - G_r\|_\infty \leq 2 \sum_{i=1}^{k} \sum_{j=r_i+1}^{n_i} \sigma_j^{(i)} \leq \varepsilon_a,$$

where we used the expressions of bounds derived in [7] for the B&T method. Note however that the actual error is generally greater (sometimes even much greater) than that resulting from the application of the B&T method directly to the whole system. Various other aims (DC-gain matching, phase preserving) can be accommodated by using alternative techniques (see [11] for a survey of model reduction methods). It is easy to see that when A has distinct eigenvalues, then the above procedure can be so devised to be equivalent with any of mentioned modal methods.

4. NUMERICAL ASPECTS

The model reduction procedure of previous section can be implemented by using exclusively numerically

reliable algorithms. For the computation of the BDF at step 1 the algorithm of [12] can be used followed possibly by the reordering and enlarging of diagonal blocks. Note however that in many cases (finite-element models, non-minimal TFM realizations) A is already block-diagonal. In such cases only the reordering of blocks is necessary in order to include nearby eigenvalues in the same blocks.

For the dominance analysis at step 2 the HSV can be computed very accurately by using the square-root algorithm of [13]. The same algorithm is applicable to both continuous- and discrete-time systems. The only difference consists in solving continuous- or discrete-time Lyapunov equations to compute the corresponding gramians. The term *square-root* designates a class of new model reduction methods with enhanced accuracy in which the computation of reduced models is based exclusively on square-root information as for instance the Cholesky factors of the gramians. The computation of Cholesky factors can be done by solving directly for these factors the corresponding Lyapunov equations by using the algorithms proposed in [14].

The reduction at step 3 can be done by using any of the recently developed model reduction algorithm with enhanced accuracy (the so-called *square-root* or *balancing-free square-root* methods) (see the references in the companion paper [11]). All these methods are appropriate to handle exact or nearby non-minimality and thus can be also used very effectively as minimal realization procedures. At both steps 2 and 3 additional computational efficiency arises by exploiting the particular quasi-upper triangular form of diagonal matrices A_i which results usually from the reduction to BDF.

Because of usually low dimensions of subsystems G_i, the involved computational effort is mainly due to the reduction to the BDF and thus is about $15n^3$ operations. If the procedure is properly implemented, all computations can be done practically with minimum additional storage (at most n^2 locations if the reduction to BDF is necessary).

5. CONCLUSIONS

A model reduction procedure based on an enhanced modal dominance analysis technique has been proposed. The proposed procedure fulfills the basic requirements (generality, numerical reliability, enhanced accuracy) for a satisfactory numerical algorithm and thus can serve as basis for robust software implementation. The new procedure extends the range of applicability of the modal approach to the reduction of arbitrary continuous- or discrete-time systems. In the same time, it can be seen as enlarging also the applicability of many powerful model reduction methods to very large order systems.

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