Falcón-Cortés, A., Boyer, D., Giuggioli, L., & Majumdar, S. N. (2017). Localization Transition Induced by Learning in Random Searches. *Physical Review Letters*, *119*(14), [40603]. https://doi.org/10.1103/PhysRevLett.119.140603

Supplemental Material

I. NO-RETURN PROBABILITY FOR LÉVY FLIGHTS: RECURRENT VS. TRANSIENT BEHAVIOR

Consider a d-dimensional Euclidean lattice. A random walker moves on the sites of this lattice with random jumps at each time step. The jump lengths are independent and identically distributed (i.i.d) random variables drawn from a normalized distribution $p(\boldsymbol{\ell})$. The walker starts at some arbitrary site ($\boldsymbol{x}_0$ at time $t = 0$). Then the probability of no return to the initial site is given by the well known formula

$$P_{no-return} = \frac{1}{\int_{\mathcal{B}} \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{1}{1-\tilde{p}(\mathbf{k})}} \tag{1}$$

where $\tilde{p}(\mathbf{k})$ is the Fourier transform of the jump distribution

$$\tilde{p}(\mathbf{k}) = \sum_{\ell} p(\boldsymbol{\ell}) e^{-i\mathbf{k}\cdot\boldsymbol{\ell}}\,. \tag{2}$$

Thus $P_{no-return}$ in Eq. (1) is nonzero or zero depending on whether the integral in the denominator is finite or divergent. The divergence of this integral depends on the small $|\mathbf{k}|$ behavior of $\tilde{p}(\mathbf{k})$. In general, for Lévy flights, the small k behavior is given by

$$\tilde{p}(\mathbf{k}) \simeq 1 - K_\mu |\mathbf{k}|^\mu \tag{3}$$

where the Lévy index $0 < \mu \leq 2$. For $\mu < 2$, the second moment of the jump distribution is divergent, while for $\mu = 2$, the second moment is finite. Hence, standard Euclidean random walks with nearest neighbour jumps correspond to $\mu = 2$, with $\tilde{p}(\mathbf{k}) = 1 - D_0|\mathbf{k}|^2$. From now on, we will consider the general $0 < \mu \leq 2$ case, and it will include the $\mu = 2$ case corresponding to standard nearest neighbour random walks. Substituting the small k behavior in the integral in the denominator of Eq. (1), it is evident that this integral diverges if $d < \mu$ and is finite if $d > \mu$. Thus, for $d < \mu$, $P_{no-return} = 0$, while it is non zero for $d > \mu$. Thus, for Lévy flights with index μ ($0 < \mu \leq 2$), the critical dimension is $d_c = \mu$ that separates the recurrent ($d < \mu$) behavior from the transient ($d > \mu$) behavior. For ordinary random walks ($\mu = 2$), $d_c = 2$.

II. CRITICAL BEHAVIOR OF THE ORDER PARAMETER P_0

We first consider the critical value q_c (for fixed γ) that separates the delocalised phase with $P_0 = 0$ for $q < q_c$ and the localised phase with $P_0 > 0$ for $q > q_c$. In the main text, we have shown that the value of q_c is given by the formula

$$q_c = \frac{(1-\gamma)P_{no-return}}{\gamma + (1-\gamma)P_{no-return}} \tag{4}$$

where $P_{no-return}$ is given in Eq. (1). So, clearly, for Lévy flights with index $0 < \mu \leq 2$ (including standard random walks corresponding to $\mu = 2$), using results on $P_{no-retun}$ from the previous Section I, we have

$$q_c = \frac{(1-\gamma)P_{no-return}}{\gamma + (1-\gamma)P_{no-return}} > 0 \quad \text{for} \quad d > \mu \tag{5}$$

$$= 0 \qquad\qquad \text{for} \quad d < \mu\,. \tag{6}$$

We now consider how P_0 increases from its value 0 as q increases above q_c. We want to show here that in general, as $q \to q_c^+$,

$$P_0 \sim (q - q_c)^\beta \tag{7}$$

where the exponent β depends continuously on μ and d in the $\mu - d$ plane. We will show below that

$$\beta = \begin{cases} 1 & \text{for} \quad d > 2\mu \\ \frac{\mu}{d-\mu} & \text{for} \quad \mu < d < 2\mu \\ \frac{d}{\mu-d} & \text{for} \quad d < \mu \end{cases} \tag{8}$$

where, we recall, that in the last case ($d < \mu$), $q_c = 0$.

To derive this result for β, we start from the equation in the main text that determines P_0 for any given q, namely

$$\frac{1}{(2\pi)^d} \int_{\mathcal{B}} \frac{d^d\mathbf{k}}{(1-q)[1-\tilde{p}(\mathbf{k})] + q\gamma P_0} = \frac{1-\gamma}{q\gamma(1-\gamma P_0)} . \tag{9}$$

Of course, at $q = q_c$, $P_0 = 0$ and this gives us

$$\frac{1}{(2\pi)^d} \int_{\mathcal{B}} \frac{d^d\mathbf{k}}{(1-q_c)[1-\tilde{p}(\mathbf{k})]} = \frac{1-\gamma}{q_c\,\gamma} , \tag{10}$$

which indeed leads to the expression for q_c in Eq. (4).

We are now ready to see how P_0 increases from 0 as q increases above q_c. For this we consider two cases separtaely.

Case I: $\mathbf{q}_c > 0$. As we have seen before, this corresponds to the transient regime where $P_{no-return} > 0$. For Lévy flights, this means $d > d_c = \mu$. To proceed, we first subtract Eq. (9) from Eq. (10) which gives

$$\int_{\mathcal{B}} \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{[q\gamma\delta - (q-q_c)(1-\tilde{p}(\mathbf{k}))]}{(1-\tilde{p}(\mathbf{k}))[(1-q)(1-\tilde{p}(\mathbf{k})) + q\gamma P_0]} = \frac{(1-q_c)(1-\gamma)(q-q_c-q\gamma P_0)}{qq_c\gamma(1-\gamma P_0)} . \tag{11}$$

We then set $q = q_c + \epsilon$ with $\epsilon \to 0$ and $P_0 = \delta$ with $\delta \to 0$. Our goal is to find how δ scales with ϵ to leading order in small ϵ. In this limit, the leading contribution to the integral on the left hand side (lhs) of Eq. (11) comes from the small k region, where we can replace $\tilde{p}(\mathbf{k})$ by Eq. (3). Keeping only the leading order terms and simplifying, we obtain

$$\delta I(\delta) + O(\delta) = A\epsilon \tag{12}$$

where $A = (1-\gamma)(1-q_c)K_\mu^2/(\gamma^2 q_c^3)$ is just a constant and $I(\delta)$ is the integral

$$I(\delta) = \int_{\mathcal{B}} \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{1}{|\mathbf{k}|^\mu[|\mathbf{k}|^\mu + b\delta]} \tag{13}$$

where $b = q_c\gamma/(K_\mu(1-q_c))$ is a constant. We now need to analyse the integral $I(\delta)$ as $\delta \to 0$. There are again two cases: (1) $d > 2\mu$ and (2) $\mu < d < 2\mu$. We consider them separately.

1. $d > 2\mu$: In this case, if we put $\delta = 0$ in $I(\delta)$ in Eq. (13), the integral converges as $k \to 0$, making $I(0)$ finite. Hence, from Eq. (12), we get

$$\delta \sim \epsilon \quad \text{implying} \quad \beta = 1 \quad \text{for} \quad d > 2\mu . \tag{14}$$

2. $\mu < d < 2\mu$: In this case, the integral $I(0)$ in Eq. (13) is divergent. Hence, to extract the leading singularity, we rescale $k \to \delta^{1/\mu} y$ in Eq. (13).

$$I(\delta) \sim \delta^{\frac{d}{\mu}-2} \int_0^\infty \frac{dy\, y^{d-1-\mu}}{y^\mu + b} . \tag{15}$$

Note that the integral in Eq. (15) is convergent in both limits $y \to 0$ and $y \to \infty$, as long as $\mu < d < 2\mu$. Hence, substituting Eq. (15) in Eq. (12) we get, to leading order

$$\delta \sim \epsilon^{\frac{\mu}{d-\mu}} \quad \text{implying} \quad \beta = \frac{\mu}{d-\mu} \quad \text{for} \quad \mu < d < 2\mu . \tag{16}$$

Case II: $\mathbf{q}_c = 0$. This case corresponds to the recurrent case when $P_{no-return} = 0$, making $q_c = 0$. As discussed before, for Lévy flights with index $0 < \mu \le 2$, this happens when $d < d_c = \mu$. In this case we analyse directly Eq. (9) by substituting $q = \epsilon$ and $P_0 = \delta$. Again, keeping only the small $\mathbf{k}$ contribution to the integral, we get to leading order

$$\int_{\mathcal{B}} \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{1}{|\mathbf{k}|^\mu + \epsilon\delta} \sim \frac{1}{\epsilon} \tag{17}$$

Rescaling $k = (\epsilon\delta)^{1/\mu} y$ gives

$$(\epsilon\delta)^{\frac{d}{\mu}-1} \int_0^\infty \frac{dy\, y^{d-1}}{y^\mu + 1} \sim \frac{1}{\epsilon}\,. \tag{18}$$

Note that the integral in Eq. (18) is convergent in both limit $y \to 0$ and $y \to \infty$ for $0 < d < \mu$. Hence, Eq. (18) then gives

$$\delta \sim \epsilon^{\frac{d}{\mu-d}} \quad \text{implying} \quad \beta = \frac{\mu}{\mu - d} \quad \text{for} \quad 0 < d < \mu\,. \tag{19}$$

This completes the derivation of the result for the exponent β given in Eqs. (8), (8) and (8).

III. LOCALIZATION OF THE $1d$ RANDOM WALK WITH NEAREST NEIGHBORS JUMPS

We derive here an analytical expression for the stationary distribution P_n. We consider the particular case of the random walk with nearest neighbor jumps in one dimension, where the step distribution is given by $p(l) = \frac{1}{2}[\delta_{l,1} + \delta_{l,-1}]$. The Fourier transform of $p(l)$ is $\tilde{p}(k) = \cos k$. In this case, the expression given by Eq. (4) of the main text for the Fourier transform of P_n becomes

$$\tilde{P}(k) = \frac{\gamma P_0[1 - (1-q)\cos k]}{(1-q)(1-\cos k) + q\gamma P_0} = \gamma P_0 + \frac{q\gamma P_0(1-\gamma P_0)}{(1-q)(1-\cos k) + q\gamma P_0}. \tag{20}$$

The form of the steady-state probability can be derived by inverse Fourier transforming. Using the fact that for $a^2 > 1$ [1]:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} dk \frac{\cos(kn)}{1 + a^2 - 2a\cos k} = \frac{1}{(a^2-1)a^{|n|}}, \tag{21}$$

we write the denominator $(1-q)(1-\cos k) + q\gamma P_0$ under the form $b(1 + a^2 - 2a\cos k)$. By identification, we have:

$$2ab = 1 - q \tag{22}$$

$$b(1 + a^2) = 1 - q(1 - \gamma P_0) \tag{23}$$

which yields

$$a = 1 + \frac{\gamma q P_0}{1-q} + \sqrt{\frac{\gamma q P_0}{1-q}\left(2 + \frac{\gamma q P_0}{1-q}\right)}. \tag{24}$$

Using Eq. (21) and (22), the inversion of Eq. (20) gives:

$$P_n = \gamma P_0 \delta_{n,0} + \frac{q\gamma P_0(1-\gamma P_0)}{1-q}\frac{2a}{(a^2-1)a^{|n|}}. \tag{25}$$

By evaluating Eq. (25) at $n = 0$, the above expression can be rewritten in compact form:

$$P_n = \gamma P_0 \delta_{n,0} + (1-\gamma)P_0 a^{-|n|}, \tag{26}$$

which is one of the main result of this section. We are only left with the determination of P_0, the asymptotic probability of occupying the inhomogeneity. For this purpose, we evaluate once more Eq. (25) at $n = 0$, obtaining:

$$2q\gamma(1 - \gamma P_0) = (1-\gamma)(1-q)(a - a^{-1}). \tag{27}$$

Inserting the expression of a given by Eq. (24) into Eq. (27) gives a quadratic equation for P_0 whose only positive root is

$$P_0 = \frac{-(1-q)(1-\gamma)^2 - q\gamma^2}{q\gamma(1-2\gamma)} + \frac{\sqrt{[(1-q)(1-\gamma)^2 + q\gamma^2]^2 + (q\gamma)^2(1-2\gamma)}}{q\gamma(1-2\gamma)}, \tag{28}$$

for $\gamma \neq 1/2$. When $\gamma = 1/2$, the solution is simply $P_0 = q$.

---

[1] Gradshteyn, I. S. and Ryzhik, I. M., *Table of integrals, series, and products*, Eighth ed., (Elsevier/Academic Press, Amsterdam, 2015).