CAPILLARY EFFECTS AND SHORT-SCALE INTERACTION IN A WEAKLY VISCOUS SUPERCRITICAL OVERFALL

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Summary We consider a thin liquid film past a semi-infinite horizontal plate under the action of gravity acting vertically, surface tension, and relatively low viscosity. This scenario comprises a manifold of effects at play, given the two disparate length scales involved: distance from jet impingement generating the layer to the trailing edge of the plate, height of the film. The yet not fully understood behaviour of a developed viscous film near the edge and previous studies on bores and hydraulic jumps in weakly viscous horizontal layers stimulate the present investigation. In sharp contrast to these, here the flow remains supercritical, and isolated regimes of strong viscous–inviscid interaction are dictated by the short length scale rather than the common shallow-water approximation. Specifically, we show how viscosity produces standing waves upstream of localised interaction and how weak capillarity modifies drastically the potential-flow singularity close to the edge, which in turn affects crucially its viscous regularisation.

MOTIVATION AND SCOPE

Higuera [1] was the first who elucidated in detail the asymptotic structure of a steady developed viscous liquid film as this passes the plate edge in the configuration given in the summary under the assumption of shallow-water theory and supercritical initial conditions. He showed how the film undergoes a hydraulic jump, for a developed film meaning transition from super- to subcritical film at a streamwise scale that measures the distance from its initial stages to the edge, before it finally becomes critical in the sense of the Burns–Lighthill criterion for long waves. However (and as he conceded), the type of singularity the solution of the thin-film equations terminates in at the edge is not unambiguous. An alternative to the one proposed in [1] is provided by the expansive singularity in freely interacting hypersonic boundary layer (BL) flow as already pointed out in [2]. In addition, it is not fully understood how the flow becomes strongly interactive and how Higuera's singularity accommodates to the situation of a very supercritcal film and the associated shortening of the streamwise scale at play around the edge.

This contribution is devoted to the clarification of this issue, where we address the problem from the viewpoint of the high-Reynolds-number (Re) limit. Then a BL forms in an otherwise uniform flow, and the associated separation of scales gives rise to a myriad of intriguing phenomena not encountered in developed flow. Specifically, the short scale becoming essential and finally provoking localised strong viscous-inviscid interactive is given by the shallow-water parameter, ϵ, rather than the largeness of the Froude and/or Weber numbers, as in the investigation [3] of strong hydraulic jumps in a developed film. Also, the situation is unlike that in the preceding study [4] where those were considered in the high-Re and long-wave limit. Hence, our study is a paradigm for genuine short-scale interaction and the associated plate-normal momentum transfer in such flows as its generation is imprinted by the potential flow falling off the edge. The flow is then governed by the reciprocal forms G and T of the squared Froude and the Weber number, respectively, apart from the asymptotic parameters $\epsilon \ll 1$ and $Re \gg 1$, the latter formed with the speed $\tilde{U}$ and heigth $\tilde{H}$ of the unperturbed film and reduced by ϵ. The flow is critical/sub-/supercritical in the sense adopted in shallow-water theory for $G = / > / < 1$. Let us tacitly refer to figure 1 (a) subsequently: the flow velocity and Cartesian coordinates x, y are non-dimensional with $\tilde{U}$ and $\tilde{H}$; the long scale is thus represented by $1/\epsilon$.

We first justify strict supercriticality of the flow, then envisage the short-scale characteristics of the weakly/strongly interactive BL flow by advanced asymptotic techniques. As an exciting finding, a triple deck (TD) structure having an extent of $O(1)$ emerges at a distance $-x = O(\ln \epsilon)$. This new TD allows for the destabilising effect of viscosity and the generation of stationary waves upstream but their suppression downstream of it. Also, the solutions of the BL equations form a singularity at and thus a further interactive stage around the edge. We finally use this surprisingly rich flow picture to approach the situation of a developed flow addressed in [1] by letting ϵ (the plate) to become so small (so long) that strict supercriticality is questioned and local critical conditions do or do not point to the terminal structure of the BL close to the edge.

POTENTIAL-FLOW LIMIT: THERE IS NO STEADY SUBCRITICAL WATERFALL

We commence the investigation by analysing the potential flow falling off the edge, parametrised by G where first $T \ll 1$ is assumed as in [1]. Interestingly, though posing a fundamental member of closely related (more complex) ideal-fluid flow problems of great importance in hydraulics, cf. [5], converged numerical solutions to it are rare and not reliable at all for $G > 1$. This is not so surprising as one commonly associates subcritical flows with standing waves far upstream, which here

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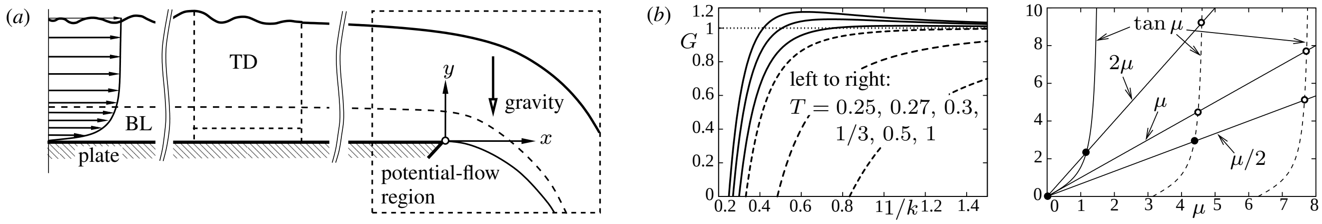

Figure 1: (*a*) flow configuration; (*b*) $\delta_0 = 0$: $G > 1$ can be double-valued for $T < 1/3$ ($\mu = 0$), μ jumps as G exceeds 1 ($T = k = 0$).

conflict with the required approach towards uniform flow. Moreover, in the aforementioned related situations solutions have been observed to exist if G does not exceed some critical value $G_c < 1$. Proving existence of such a threshold with sufficient rigour and determining its value analytically has not been successful so far. However, we demonstrate that $G_c = 1$ holds for the classical downfall problem; this value might even be lower if a sill or dent modifies locally the horizontal surface.

If q denotes the flow speed, the proof essentially exploits the extremal properties of $\ln q$ in consideration of the isotaches and their possible branching points. We first show that the detached streamline has negative curvature, i.e. bends towards the direction of gravity, throughout. Evaluating the global momentum balance in horizontal direction then yields the remarkable result $G < 4$. In particular, the flow past a flat plate is shown to be strictly accelerating, which in combination with its far-upstream variation finally yields $G \leq 1$. For $T \geq 0$, the latter is algebraic for $G = 1$ and purely exponential otherwise, and our restriction to the as appealing as geometrically simple overfall scenario includes the delicate near-critical case $G \to 1_-$.

The first part of that proof implies forced flow detachment at the edge for $T = 0$. With q equal to $q_0 := \sqrt{1+2G}$ there in this limit, we find the position $z := x + \mathrm{i}y$ very close to the edge as a function of the complex flow potential, w, in the form

$$q_0\bar{z} \sim w - \mathrm{i}c_1 w^{3/2} - \frac{9c_1^2 w^2}{16} - \left[\frac{2c_1 G}{5q_0^3}\Big(\frac{\mathrm{i}\ln w}{\pi}+1\Big) + \mathrm{i}c_2\right] w^{5/2} - T\left[\frac{3c_1}{2q_0}\Big(\frac{\mathrm{i}\ln w}{\pi}+1\Big)+\mathrm{i}d\right] w^{1/2} + O\big(w^3\ln w, T^2 w^{-1/2}\big) \quad (1)$$

for $w \to 0$, $T \to 0$. Here we have $\bar{z} := z - x_d$, actual detachment at some $z = x_d(T) = O(T\ln T)$, and undetermined coefficients $c_1(G) > 0$, $c_2(G)$, $d(G)$. Resolving the non-uniformity of (1) proves a delicate matter; see [5] and the relevant references therein. At the critical stage $z = O(T)$, the procedure involves the (non-unique) solution of the celebrated *dock problem* by the Wiener–Hopf method. One finally finds the bottom streamline detaching at a stagnation point $z = x_d$ and an angle $\sim -\alpha c_1 T^{1/2}\ln T$ with the plate. This and the solution to the separation problem given in [5] differ by the number $\alpha > 0$ and the logarithmic term, which crucially affecting its viscous modification by the aforementioned second interaction process.

UPSTREAM BOUNDARY LAYER AND SHORT-SCALE INTERACTION

The exponentially varying short-scale disturbances imposed far upstream of the trailing-edge region $|z| = O(1)$ interact strongly with the BL if the rescaled reference wavelength $\Lambda := \epsilon Re^{3/2}$ is kept fixed. They satisfy the dispersion relation

$$\Lambda^{1/3}\,\delta_0 \sim \chi\,\delta_1, \quad [\delta_0, \delta_1] := [\lambda - (G - T\lambda^2)\tan\lambda,\ \lambda^{4/3}(G - T\lambda^2 + \lambda\tan\lambda)], \quad \chi := \Gamma(\tfrac{1}{3})(2/3)^{2/3}\beta^{-5/3} \doteq 7.2059. \quad (2)$$

for steady weakly viscous linear gravity–capillary waves with wavenumbers $k := \Im\lambda$ and damping rates $\mu := \Re\lambda > 0$; $\beta \doteq 0.4696$ is the wall shear rate of the unperturbed Blasius BL. The imposed, purely non-oscillatory perturbations are recovered for $\Lambda \gg 1$; for $G > 1$ they could also be strictly oscillatory: see figure 1 (*b*). Even slightly subcritical flows might be possible given the weak damping of these inviscid waves. The localised nonlinear lower-deck pocket associated with the new TD absorbs the waves upstream of it and described by (2) and achieve consistency with the overall flow structure by shedding the non-wavy potential-flow perturbations downstream. Accordingly, the generalised Fourier transforms $-\mathcal{A}$, $\mathcal{P}$ of the displacement function and the induced pressure perturbations, respectively, satisfy the interaction law $\mathrm{i}k\mathcal{A}(k) = \mathcal{P}(k)\,(\delta_0/\delta_1)(\mathrm{i}k)$.

The current efforts are directed towards resolving the trailing-edge singularity for $\Lambda \ll 1$ and the new TD. Specifically, we expect the limit $G \to 1$ to reveal how the long-wave TD structure in [6] governing transcritical flow resides in our setting.

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