How does a multi-representational mathematical ICT tool mediate teachers' mathematical and pedagogical knowledge concerning variance and invariance?

Alison Clark-Wilson

Institute of Education, University of London

2010

ABSTRACT

This study aims to examine how teachers' mathematical and pedagogical knowledge develop as they learn to use a multi-representational technological tool, the TI-Nspire handheld device and computer software. It is conducted as an enquiry into the learning trajectories of a group of secondary mathematics teachers as they begin to use the device with a focus on their interpretations of mathematical variance and invariance. The research is situated within an English secondary school setting and it seeks to reveal how teachers' ideas shape, and are shaped by, their use of the technology through a scrutiny of the lesson artefacts, semi-structured interviews and lesson observations. Analysis of the data reveals the importance of the idea of the 'hiccup'; that is the perturbation experienced by teachers during lessons stimulated by their use of the technology, which illuminates discontinuities within teachers' knowledge. The study concludes that the use of such a multi-representational tool can substantially change the way in which both the teachers and their students perceive the notions of variance and invariance within dynamic mathematical environments. Furthermore, the study classifies the types of perturbations that underpin this conclusion. The study also contributes to the discourse on the design of mathematical problems and their associated instrumentation schemes in which linked multiple representations offer a new environment for developing mathematical meanings. This thesis makes an original contribution to understanding what and how teachers learn about the concept of mathematical variance and invariance within a technological environment.

KEYWORDS

Hiccup, instrumental genesis, mathematics education, mathematical generalisation, multiple representational technology, teacher development, variance and invariance.

Acknowledgements

I would like to acknowledge the contribution made to this research and thesis by:

Professor Richard Noss who, as my supervisor posed the challenging questions and exhibited great patience in supporting the evolution of this study.

Professor Afzal Ahmed and Honor Williams, whose influences on my personal research and scholarship have undoubtedly shaped the person that I am - and whose friendship and ongoing support I appreciate greatly.

Professor Adrian Oldknow, whose unwavering enthusiasm for new technologies has paved the way for many of my experiences and from whom I continue to learn.

The teachers involved in both phases of the research, and in particular the two teachers involved in the Phase Two, who committed much personal time in familiarising themselves with the technology, planning and evaluating their lessons and making themselves available for interviews and discussions, and without whom the research would not have been possible.

In addition, the data collection carried out during Phase One of the study (and some of the data collection in Phase Two) was funded by Texas Instruments as part of two evaluation research projects, subsequently reported in Clark-Wilson (2009, 2008). My thanks go to Andrea Forbes, Raffaella Fiz and Rob Foshay for their support and for their confidence in me as a 'new researcher'.

Finally, to my husband Alan, whose love and support, both intellectual and practical, exemplifies that of a true critical friend and who undoubtedly enabled me to manage the work-life imbalance necessary to complete this academic achievement!

Author's Declaration

I hereby declare that, except where explicit attribution is made, the work presented in this thesis is entirely my own.

Word count (exclusive of appendices, list of references and bibliography):

77,345 words

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Glossary

BECTa	British Educational and Communications Technology Agency
CAS	Computer Algebra System
DCSF	Department for Children, Schools and Families
DGS	Dynamic Geometry System
GCSE	General Certificate in School Education
ICT	Information and Communications Technology
KS3	Key stage 3 (11-14 years)
KS4	Key stage 4 (14-16 years)
NC	National Curriculum
NCET	National Council for Educational Technology
NOF	New Opportunities Fund
OFSTED	Office for Standards in Education
PGCE	Post Graduate Certificate in Education
QCA	Qualifications and Curriculum Agency
SAT	Standard Assessment Test
SSAT	Specialist Schools and Academies Trust
TDA	Training and Development Agency
.tns	TI-Nspire file extension
TTA	Teacher Training Agency

FOREWORD

In my own initial training to teach mathematics to students aged 11-18 years I was undoubtedly influenced by the strong presence of technological tools. A graphical calculator was loaned to me from the outset of my postgraduate certificate in education (PGCE) course and many taught sessions and tutorials modelled the use of mathematical software displayed to the group through TV screens. I also bought my first personal computer during this period. Two of my lecturers, Adrian Oldknow and Warwick Evans were particularly influential to me at this early stage. They both clearly enjoyed the way in which this recently emerging technology allowed them to experiment with new pedagogies and, perhaps more importantly, how it facilitated a re-examination of our engagement with the mathematics. It was also very apparent that the boundaries between the tutors, students, technology and mathematics were overlapping with learning mathematics at the point of mutual intersection.

Technology was used as a tool to enable problems to be explored quickly and effectively, with plenty of scope for mathematical creativity, in terms of both the depth and analyses of mathematical models. Learning mathematics was presented as a progression of mathematical modelling activities in which tutors introduced new mathematical content as demanded by students' own questions. The domains of mathematics were blurred as we moved between numerical, geometrical, algebraic and statistical interpretations according to the context. I distinctly remember the silhouette of a wine glass being projected onto a screen during a numerical methods module as a starting point to develop mathematical methods to estimate the capacity of the glass.

In 1993, during my first year as a secondary mathematics teacher, I was invited to be a part of the National Council for Educational Technology's (NCET) 'Pilot Evaluation of Portable Computer Technology' project to evaluate the emerging role of handheld technology in the secondary school mathematics classroom. This project was subsequently reported in Stradling et al. (1994) and an example of my personal contribution to the project is recorded in Oldknow and Taylor (2003). My school was loaned a class set of Texas TI-81 graphing calculators, I received ten (funded) days of school-release time and I had the opportunity to report my contribution to the project and submit it as part of a Certificate in Advanced Educational Studies, 'Developing Mathematics with the Microcomputer'.

There are some key factors in my personal response to this initiative:

- The technology was always available to myself and my students;
- The timescale of the project gave me opportunities to explore, experiment, reflect and report;
- The project was linked to my own continuing professional development and I wrote a journal of my classroom activities and reflections which formed the basis for academic accreditation;
- I was working with a group of teachers from similar schools;
- The school had the equipment on 'long-term loan', which meant that both during and beyond the project I was able to work collaboratively with my colleagues to disseminate outcomes, activities and approaches and integrate aspects within the schemes of work.

A couple of years later, a specific activity involving the use of technology also had a profound influence on me. A module within my MA in Mathematics Education course, 'Geometry in a Contemporary Setting', began with the following practical activity.

Take a circle of paper, about 15cm in diameter.

Mark a dot on the paper – anywhere but at the centre.

Fold the edge of the circle so that the circumference just touches your marked point. Make a firm crease.

Unfold the paper.

Fold the paper from a different place, again so that the circumference just touches your mark.

Repeat this to produce an 'envelope' of folds on the circle of paper.

What shape is beginning to emerge?

We then turned to the computers and opened a new file in Cabri-Géomètre (Baulac et al., 1988), which was a new genre of computer software called dynamic geometry software that had been developed at the University of Grenoble in France.

We learned how to construct a circle.

... and a point inside it.

We then created an electronic version of the practical paper-folding activity with which we had just engaged. A discussion then ensued about whether the fold lines were each a perpendicular bisector of a line joining any point on the circumference to the marked point.

A few minutes later we had the following picture.

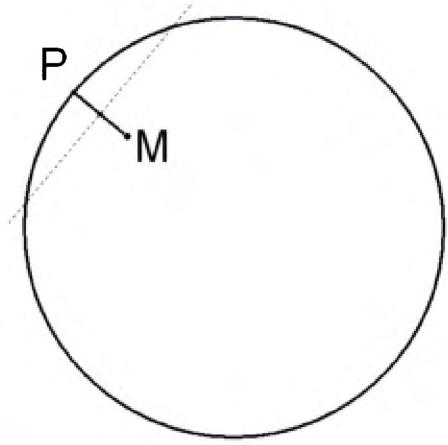


Figure F.0-1 The geometrical model of the paper folding activity where the point M represents the initial point, point P is a chosen point on the circumference and the dotted line indicates the line of fold.

We explored what happened if we moved the point P around the circumference of the circle and noticed the path that the 'fold' line took.

We then used the software to show a series of fold lines[1], by constructing the locus of the 'fold' line as the point P moved around the circumference.

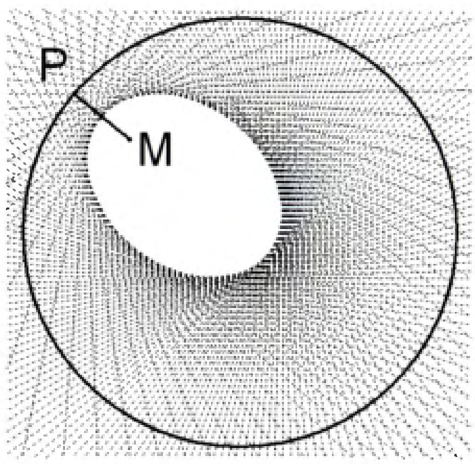


Figure F.0-2 The geometrical model showing the loci of all of the fold lines as the point P was dragged around the circumference of the circle.

This was fascinating enough...

But then we explored what happened if we moved the point M – something that we could not do with the paper circle without starting again...

[1] The number of lines displayed is a function of the software settings within Cabri-Géomètre.

What if it was near the centre? or at the centre? or near the circumference?

or outside of the circle altogether...?

At that critical moment I knew, from my personal experience that technology was going to offer an opportunity for me to rethink how I taught (and learned) mathematics and that it would form an integral part of the experiences that I would plan for my students in their mathematics lessons.

In my current role as Principal Lecturer in Mathematics Education and Head of Research, Development and Consultancy at the Mathematics Centre, University of Chichester, much of my research and scholarship has involved the development of technology based mathematics teaching resources and related professional development activities for teachers within school mathematics. In February 2007 The Mathematics Centre was approached by Texas Instruments with a request to design and lead the pilot evaluation project for *TI-Nspire*, their new handheld and software product for mathematics, which extended the functionality of the graphical calculator by enabling mathematical representations to be implicitly and explicitly linked (Texas Instruments, 2007). This setting presented itself as an ideal opportunity to further my research and scholarship and, as such, the choice of technology was dictated by this project. The relationship between the funded research project and the study reported in this thesis has centred upon the re-examination of the Phase One data (and some of the Phase Two data) after the original project outcomes were reported (Clark-Wilson, 2008, Clark-Wilson, 2009).

My previous experiences as a teacher involved in the early evaluation of handheld technologies had left me feeling positive about its potential in supporting students' to learn mathematics more autonomously (Stradling et al., 1994). In the intervening years, although computer software for mathematics has developed extensively, concerns have been raised about the apparent limited opportunities for students' to access technology during mathematics lessons (Office for Standards in Education, 2008, British Educational Communication and Technology Agency, 2007). Consequently, although my initial impression of the TI-Nspire handheld (operating system v.1.4) was that there were many ways in which both the software and the keypad interface could have been improved, my commitment to this mode of access to technology supported my decision adopt this technology within my doctoral study.

1 INTRODUCTION

1.1 Introduction

In this chapter I have outlined the broad theme and purpose of the study alongside a description of the context that led me to become interested in this area. The chapter also includes definitions of some of the terminology central to the study and it concludes by giving the broad aim for the research which informed the basis of the review of research contained in Chapter 2.

1.2 A description of the study

This is a situated exploratory study in which the unit of attention is *secondary mathematics teachers + educational technology + activity design* and the research sought to expand the discourse on teachers' appropriation of a new technological tool for secondary mathematics classrooms, or more explicitly, how they adapt and mould the tool for their own use. The research lens is particularly trained on the trajectory of their interpretation of the concepts of mathematical variance and invariance. The selected technological tool for this study is TI-Nspire (Texas Instruments, 2007) and an outline of its main features is provided in Section 2.4. This tool offers the functionality to support a range of approaches for the design of mathematical activities that demands a number of mathematical and pedagogical decisions to be made by the subjects of the study, namely the teachers concerned with the design and adaptation of pedagogical materials for secondary mathematics. This setting offers a significant opportunity to progress an understanding of the trajectory of the teachers' learning and classroom practices.

1.3 Why this study is important

The discourse on the appropriation of technological tools for mathematics such as LOGO, Computer Algebra Systems and dynamic geometry systems (DGS) is well-developed in the research literature, and this will be expanded upon in Section 2.4 and the Literature Review in 3. The technological environment offered by the TI-Nspire software and handheld device combines the functionality of a traditional graphing calculator (with statistical functionality) with that of a dynamic geometric environment in which variables defined from numerical, algebraic, geometric and statistical domains can be linked. This leads to two important questions:

Why, when and how would we want to do this?

and

Which problems lend themselves to such approaches within the context of secondary school mathematics?

The range of new possibilities that this introduced to the mathematics teaching setting is of interest to the mathematics education community. Many researchers have called for further studies into the process of integrating technology into classroom practices (Laborde, 2001, Hoyles et al., 2004, Ruthven et al., 2004, Kieran and Yerushalmy, 2004) to support the emerging theories and empirical analysis.

This study has built on the existing discourse to propose an 'ontological innovation' (diSessa and Cobb, 2004) relating to the process of teachers' appropriation in this new environment. The notion of an ontological innovation concerns 'hypothesizing and developing explanatory constructs, new categories of things in the world that help explain how it works' (diSessa and Cobb, 2004, p.77). This is being interpreted within this study in the same sense as the authors conceptualised 'socio-mathematical norms', that is to say an ontological innovation that revealed an element of the hidden curriculum in mathematics classrooms. Within the physical sciences an ontological innovation can be exemplified by concepts such as friction or luminosity, which have been readily adopted and validated by the scientific community. However, within human sciences we do not have a set of concepts to describe the process of teacher learning at a classroom level – something that this thesis sought to develop.

1.4 Context of the study

In February 2007 The Mathematics Centre, University of Chichester was approached by Texas Instruments with a request to design and lead the English pilot evaluation research project for TI-Nspire. As Director of this project, a piece of externally funded Evaluation Research, I worked in collaboration with Texas Instruments to agree a methodology that satisfied their internal requirements, whilst adopting a research-informed approach that did not compromise the need for a substantial professional development experience for the teachers involved and a robust set of research instruments that enabled a rich data set to be analysed. My role was as the sole designer of the research evaluation framework and the associated methodological tools. I was also solely responsible for the subsequent data analysis and reporting I worked with members of the project team (including the teachers) to triangulate my emergent interpretations and findings which ultimately led to the research conclusions. The resulting exploratory study took place in two phases between July 2007 and December 2009 and it involved the case study of fifteen mathematics teachers, all of whom developed and evaluated mathematical activities that involved their students in secondary classroom settings.

1.5 Premises about teaching and learning mathematics

It is not possible to undertake research involving mathematics teachers without confronting the difficulties associated with the body of knowledge we call mathematics. The dual nature it exhibits as both a *self-contained formal system* or *a way of conceptualising the world* (Noss and Hoyles, 1996) creates a dilemma for mathematics teachers in striking a balance between teaching about the mathematical objects and the relationships between them, and providing their students with opportunities to encounter these objects and relationships and construct their own meanings.

As a participant observer in this research, my own theories about mathematics as a body of knowledge, and about its associated pedagogies, were highly relevant as I entered into a research dialogue with teachers. As I move between my role as a teacher of mathematics and that of a researcher, Alba Thompson's seminal work on teachers conceptions of mathematics came to mind and, in particular, her comment,

> *Although the complexity of the relationship between conceptions and practice defies the simplicity of cause and effect, much of the contrast in the teachers' instructional emphases may be explained by differences in their prevailing views of mathematics. (Thompson, 1984)*

In considering Thompson's phrase a 'teacher's view of mathematics', I questioned whether this was this their view of:

- What mathematics as a body of knowledge is (or might be) about?
- The overall cognitive goals and objectives of mathematics teaching and learning?
- The content of the school mathematics curriculum?
- How mathematics should be taught and learned?
- What mathematics is learnable?
- What 'appropriate' mathematics looked like for different groups of students?
- What evidenced mathematical understanding?
- How students should express mathematics?
- How tools and resources might be used in the learning of mathematics?
- How to respond to students' mathematical misunderstandings?

In my many and frequent discussions with mathematics teachers during which I have sought to help them to articulate aspects of their views of mathematics, they have unanimously found this a challenging activity. It is my belief that this is because they have never questioned themselves or others on this subject. Consequently, I cannot agree with simplistic classifications of teachers' beliefs about mathematics (Lerman, 1983, Ernest, 1991, Ernest, 1996) and prefer to adopt Thompson's view that it is conceivable that an individual teacher's conceptions of mathematics are much more fine-grained than Lerman and Ernest suggest and might include aspects that conflict (Thompson, 1992a). By confronting the contradictions, teachers begin to widen their perspective of mathematics as a dynamic body of knowledge. If a dynamic mathematical tool is then introduced to the teaching of a dynamic body of knowledge, how do we retain the sense of the mathematics and what is the teacher's new role in supporting their students' sense-making processes?

For me, mathematics is a language and a set of tools that enable humans to make sense of and describe phenomena that exist in the world that we inhabit. These can be mathematical models of tangible physical situations, such as Newtonian motion or interpretations of natural shape and form as in a classic text such as D'Arcy Thompson's *On Growth and Form* (Thompson, 1917). With respect to the goals of mathematics teaching and learning, as a teacher of mathematics I had the dual role of: supporting my students to appreciate the world through a mathematical lens whilst sharing with them some of the existing body of human mathematical knowledge; enabling them to engage in and contribute to the society in which they found themselves on leaving the formal school system. Having an awareness of the students' cultural setting was a key factor that enabled me to devise a relevant and motivating school curriculum whilst still staying loyal to my beliefs.

As a practising teacher, I appreciated that there was a need for my own students to both participate in and succeed in the formal examination system and that I could not disadvantage them by not providing them with adequate preparation. In order for my students to feel successful, irrespective of their level of attainment, mathematical understanding had to be the central tenet for all that we did in and outside of the classroom. I view mathematics as an essential human activity that is best learned in a social setting and through genuine and interesting problem solving activities, which stimulate us to ask and seek answers to our own mathematical enquiries. My role as a teacher was to select the ideas to provide the initial stimulation for my students, offer a supportive steer for them as they grappled with new mathematical knowledge, and to facilitate their access to appropriate models and tools. I tended not to see curriculum boundaries, only interesting problems, but I was aware that, as students moved into the examination phase, I had the responsibility to ensure that they were prepared in those areas in which they would be formally assessed.

These personal beliefs underpin my approach when working with teachers, as I plan and support teacher development sessions with a firm mathematical basis. My premise is that, in order to discuss issues concerning mathematics pedagogy, it is important for teachers to be actively working on mathematical ideas for themselves. However, from time to time I encourage teachers to adopt a meta-position as they look down on themselves and consider the implications of their own experiences on their classroom role as a teacher. This widens the context to include discussions of pedagogical approaches for mathematics and issues relating to the leadership and management of the mathematics curriculum in the individual classroom and the wider school setting.

1.6 Outline aim for the research

The outline aim for this research was to explore the nature of teachers' cognitive and pedagogical learning as they developed their use of a multi-representational technology (MRT) with a focus on their conception of mathematical variance and invariance. For the purposes of this study, the multi-representational system was being used in a manner consistent with Noss and Hoyles (1996), as *a window on knowledge, on the conceptions, beliefs and attitudes of learners, teachers and others involved in the meaning-making process (p. 5).* This broad aim will be further expanded as an outcome of the Literature review in Chapter 3.

1.7 Overview of the study

The remainder of the thesis is arranged as follows:

Chapter 2 introduces and defines the main terminology used within the thesis and provides an overview of the chosen technology and its functionality.

Chapter 3 contains the literature review, which examines the processes involved in the introduction of new technologies, mathematical variance and invariance and interpretations of mathematics teachers' professional learning. It begins by outlining these themes, followed by a detailed review of selected studies and concludes by stating the more detailed articulation of the aim for the research.

Chapter 4 outlines the two phases of the research and the rationale for its methodology. It continues to detail the methodological actions used in the first phase of the research and the associated data analysis processes.

Chapter 5 presents a summary of the outcomes of the first phase the study, which includes the emergent instrument utilisation schemes and the teachers' conceptions of variance and invariance. It details the rationale for the design of the second phase to incorporate the selection of the case study teachers, the adapted methodology and the process of data analysis.

Chapter 6 introduces the two case study teachers who were studied during the second phase, describes their starting points and includes a detailed analysis of one mathematics lesson from each teacher to provide a deeper sense of the process of data analysis from which the research conclusions evolved.

Chapter 7 details the outcomes of the research, presented as the two case studies of the selected teachers. For each teacher a summary of the trajectory of their learning and the associated classroom events experienced during the study, are articulated.

Chapter 8 offers the discussion of the results in the form of an ontological theory about the nature of teacher learning within the domain of mathematics education.

Chapter 9 concludes the study by summarising its findings and relating these to the existing and emerging research on teacher learning.

2 THE NOTION OF A MULTI-REPRESENTATIONAL TECHNOLOGICAL TOOL FOR MATHEMATICS

> *Information technology will have its greatest impact in transforming the meaning of what it means to learn and use mathematics by providing access to new forms of representation as well as providing simultaneous access to multiple, linked representations. (Kaput, 1986)*

2.1 Introduction

This chapter will define and articulate the important terminology used within the study: technological tools for mathematical learning; the concepts of mathematical variance and invariance and the definition of a multi-representational mathematical environment. A justification for the chosen multi-representational environment will be put forward. The chapter concludes by establishing the basis for the choice of themes included in the Literature Review, which is elaborated within Chapter 3.

2.2 Technological tools for mathematical learning

Mathematical learning tools encompass the whole range of physical, visual and mental resources that have the common purpose to support learners to construct mathematical understandings. In each case the mathematical meaning 'for and with' each tool is constructed by the user, shaping the way in which it is used. Starting from the Russian psychological perspective, which interpreted the notion of activity as 'a unit of analysis that included both the individual and his/her culturally defined environment' (Wertsch, 1981), the work of Lev Vygotsky (which pre-dated technological tools) provided a foundation for the interpretation of human-tool interaction. Vygotsky defined tools or instruments as 'artificial devices for mastering mental processes' and offered a list of examples, which included language, counting systems, algebraic symbol systems and diagrams. Vygotsky suggested that human instrumental processes enabled many more paths to the same epistemological outcome to be facilitated, stating,

> *The most important fact about the instrumental act - a fact that undergirds the instrumental method - is the simultaneous presence of stimulus of both types (i.e. the object and the tool). Each stimulus plays a qualitatively and functionally distinct role. Thus, in the instrumental act, a new intermediate link - the psychological tool, which becomes the structural centre (i.e. the feature that functionally determines all the processes that form the instrumental act) - is inserted between the*

> *object and the psychological operation towards which it is directed. Any behavioural act then becomes an intellectual operation.*
>
> *(Vygotsky, 1981, p.139)*

With the advent of technological tools for learning, a number of researchers revisited Vygotsky's socio-constructivist theories with a view to reconceptualising them within the emerging research field of technology within mathematics education. (Ridgeway, 1997, Verillon and Rabardel, 1995)

The French mathematics education discourse on this theme drew its theoretical framework from the work of Verillon and Rabardel (1995), who elaborated on the instrumental approach and offered a new perspective on the effect that technological tools have on learning processes. Verillon and Rabardel proposed the triad characteristic of Instrumented Activity Situations. Locating such a theoretical framework within the context of this study led to the refined interpretation of the Subject, Instrument and Object that has been adopted as the overarching theoretical framework for the study. The justification of this decision, which includes a discussion concerning the evolution of the refined triad, is considered more deeply within Chapter 3.

An extensive research base exists concerning the development and use of individual technologies for mathematics education (See, for example, Fey, 1989, Kaput, 1992, Hoyles et al., 2004, Hoyles and Lagrange, 2009). Genres of technological tools have been developed, which can be broadly categorised as graphing software, dynamic geometry environments, programming languages, spreadsheets and statistical packages. Most recently there has been a move to develop multi-representational environments that allow for simultaneous movement between the numeric, algebraic, graphical, geometric and statistical domains. The access to these tools has been facilitated through traditional desk top personal computers and more portable technology, such as laptop computers and networkable handheld devices.

Over time, as the implications for teacher development for the integration of technology into 'normal' classrooms has became more understood, researchers have concentrated on particular cases in an attempt to conceptualise the processes involved in assimilating technology into classroom practices. Research has repeatedly highlighted teachers' own dispositions towards mathematics and its pedagogies as a major factor in determining the way in which teachers made sense of the technological tool and created legitimised uses for technology in their classrooms (See, for example Noss et al., 1991, Artigue, 1998, Ruthven et al., 2004, Mousley et al., 2003).

My own observations and experiences have led me to believe that exposing students to a technological resource alone is not necessarily enough to guarantee that any individual will construct mathematical meanings that resonate with those that are legitimised within the school mathematics curriculum. In the classroom setting, the teachers have a role in supporting the process of instrumental genesis and it is the teachers' actions that are of primary interest to this study. By focusing teachers' attentions on the meanings that they (and their pupils) construct, it is anticipated that teachers will begin to consider the nature of useful settings within which their pupils will begin to use a multi-representational system as a tool for the purposeful learning of mathematics.

If the purpose for the use of the technology is to maximise opportunities for teachers to enable pupils to arrive at their own mathematical meanings in relation to the body of knowledge we call mathematics, teachers are immediately faced with a problem. What does a teacher do when the meaning that the pupil has constructed is, in the teacher's view, an 'incorrect' one? Can a meaning be wrong? And if a meaning is 'correct', has it then become an understanding? Is an understanding only valid if it is judged to be correct? These questions epitomise the dilemma that teachers face in the classroom on a day-to-day basis. However, starting from a premise that the use of technology may make these meanings more visible, that is to say providing the 'window' as described by Hoyles and Noss, how do teachers respond to these occurrences, and what do teachers learn from their experiences? This study is designed to explore such scenarios.

For mathematics teachers, the questions above highlight the pedagogical complexities that they face within their role. Moreover, does the technology offer an opportunity for teachers to support their pupils in the process of meaning-making, or does it have the adverse effect, creating more problems than solutions in the eyes of the teachers? Early research by Artigue into the adoption of handheld technology for mathematics concluded that the teacher development process was crucial to its integration (Artigue, 1998). She raised questions about the discrepancies and conflicts between the process of instrumentation of the technology and the institution's underlying philosophy for the way that students engage with mathematics.

The Literature Review will expand on existing research concerning the processes of technology integration and, as a result, articulate the theoretical framework for this study.

2.3 The mathematical focus for the study

On commencing the research, the more tangible mathematical focus appeared to be that of the mathematical variable. In Seymour Papert's words, a mathematical variable is defined as 'The idea of using a symbol to represent an unknown entity', describing 'the concept of symbolic naming through a variable – one of the most powerful mathematical ideas ever invented' (Papert, 1980, p.69). Other definitions of variable differ from an 'algebraic object that can be replaced by a number' (Bardini et al., 2005) to a 'plurality of conceptions: generalised number, unknown, totally arbitrary sign, register of memory' (Malisani and Spagnolo, 2009).

Alphabetic symbol naming results in a number of conflicting interpretations, with the symbol representing an unknown, a variable or a parameter. Within school mathematics the subtle use of particular letters is often intended to impart to the students the various roles of alphabetic labelling. For example, presented with the syntax 4***a*** + 3 = 12, the inference is that the letter ***a*** probably represents an unknown, the value for which is to be found. Whereas, presented with ***y*** = m***x*** + c, the implication is that ***y*** and ***x*** represent the variables and the letters m and c represent parameters (which can also vary), with the complete form representing the family of linear functions within a Cartesian coordinate system. The relationship between the concept of variable and the process of algebraic reasoning has been extensively researched by a number of researchers and theorists (Bednarz et al., 1996, Kaput, 1998, Kieran and Wagner, 1989). What is evident from this research is that most interpretations of variable are firmly situated in the algebraic domain.

An earlier consideration of the role that geometric contexts might have in developing understanding of the concept of a mathematical variable was given by Radford (1996), who adopted a historical perspective to the epistemology of mathematical variables in an attempt to locate possible opportunities for school mathematics. Emanating from this discourse is the notion of mathematical variable as a fundamental building block to support reasoning in mathematics, which is not limited to the domain of number and algebra. Emerging technologies have enabled the creation of dynamic variability and the exploration of relationships in a way that was previously a mental exercise or concept. The process of converting these observations and relationships into a communicable language is something Kaput described as 'algebrafying, a complex composite of five interrelated forms of reasoning' (Kaput, 1998, p.3). Kaput offers a broad definition of algebra as:

> *generalizing and formalizing patterns and regularities, in particular, algebra as generalised arithmetic; syntactically guided manipulations of*

> *symbols; the study of structure and systems abstracted from computations and relations; the study of functions, relations and joint variations; modelling. (Kaput, 1998, p.26)*

Recent curriculum reviews and reports have highlighted that, in English classrooms, school algebra has been focused upon what Kieran (1996) described as generational activities in which variables are used to form expressions and equations, describe quantitative situations and express generality from numeric or geometric contexts. Such generational activities are emphasised over transformational and global meta-level activities, which focus more on the equivalence and relationships between algebraic representations. Again, most of these activities have taken place within the traditional topic of algebra, although more recent curriculum guidance documents in England have sought to promote a broader interpretation, extending to geometry (Qualifications and Curriculum Authority, 2001, Brown et al., 2003, Ruthven, 2003, Qualifications and Curriculum Authority, 2004).

My early work suggested that variable was an inadequate basis on which to focus as variable had historically and ontologically been connected with the realm of algebra within mathematics and the teachers involved in this study would be developing activities that spanned the full curriculum. So, I have taken the decision to train the mathematical lens on the more dynamic conception, 'variance and invariance' for the reasons for which are explained below.

Firstly, the study of variance and invariance, namely responding to the question, 'what is the same and what is different about?' is a fundamental thought process that humans apply in making sense of their surroundings. The process of aligning, reflecting and realigning our thinking is a continuous one, which seems elemental to the way we learn. Within the learning of mathematics, exploring variance and invariance is central to the process of generalisation within both the socio-constructivist and constructionist paradigms and the advent of technological tools has facilitated this process by allowed multiple views and situations to be quickly explored. In a technological setting, the exploration of variance and invariance could imply that dynamic objects will be created and manipulated, allowing generalisations to be hypothesised with a view to formalising mathematical knowledge.

Secondly, the English and Welsh statutory curriculum for secondary mathematics[1]

[1] The statutory secondary curriculum is for all students in English state funded schools aged 11-16 years.

has explicitly stated that all students should be given opportunities to use mathematical reasoning within the context of analysing a mathematical situation in order to 'explore the effects of varying values and look for invariance and covariance' (Department for Children Schools and Families, 2007). This process skill transcends the mathematics topic boundaries and offers a relevant focus for teacher development activities as they begin to adopt the new curriculum. This offers an element of legitimacy for teachers in developing approaches which resonate with the statutory curriculum.

A number of other researchers have observed that many of the uses of technology featured in and reported by the mathematics education literature have commonly offered students the opportunity to 'explore regularity and variation' (Stacey, 2008, Hennessy, 2000, Wilson et al., 2005). This could be explained in two ways. Firstly, that teachers are identifying these sorts of activities as being important within the mathematics curriculum they offer and secondly that the researchers or experts involved in these studies might be promoting such use, thereby distorting the picture. Classroom based case studies reported by Ruthven et al would seem to substantiate the second interpretation. Their observations and interpretations of the natural use of dynamic geometry software by three secondary mathematics teachers revealed that 'more occasionally, and less explicitly, it was a matter of dragging to examine dynamic variation' (Ruthven et al., 2004). This highlighted an area for my study to probe during both Phase One and Phase Two.

2.4 Multi-representational environments for mathematics

In the introduction of his seminal paper of 1986, James Kaput began by offering his own definition of mathematics to embody both a body of knowledge and a language within which exists 'a network of representational systems which interlock, not only with each other, but interact differently with different kinds of mathematical knowledge as well as with non-mathematical representation systems such as natural language and pictures' (Kaput, 1986). In this paper, Kaput presented a vision for the way in which technology might support higher-level engagement with mathematics and he emphasised the role that multi-representational environments might have in this process.

The subsequent discourse on the nature and role of representational environments is well-established, with several key texts devoted solely to this theme (Confrey, 1990, Janvier, 1987, Kaput, 1989). A number of researchers and theorists conceived and developed technological environments for mathematics in which several representations were offered simultaneously alongside the functionality to vary values, objects and graphical or figural images in various ways. In early technologies, the changes were made through a keyboard input. However the advent of the on-screen pointer and mouse facilitated a new way of interacting, through dragging.

It is not surprising that, as the early opportunities to develop multi-representational systems for mathematics were often influenced by the pragmatic need to improve the school experience for learners, much of the research was polarised into the multi-representational 'families' of:

- graphical, tabular, function, coordinates;
- geometric figure, calculated and/or measured values;
- statistical data, graphs, calculated and or measured values.

The Literature Review provided in Chapter 3 will expand on the published research with a view to identifying the features of the existing knowledge and theories that have relevance for this study. Of interest will be studies that have considered the nature of teachers' uses of multi-representational software within the design of classroom activities and the way in which this has, or has not, provided opportunities for substantial teacher learning about the broader concept of mathematical variance and invariance.

As previously stated, the choice of the handheld technology for this study was in part opportunistic. However, it offered an instantiation of a multi-representational environment for mathematics with the additional possibility of the development of wireless connectivity within the research timeline.

2.4.1 The selected multi-representational technology and its functionality

TI-Nspire incorporates a set of 'applications' that has been developed from a suite of existing educational technologies for mathematics: Cabri-Géomètre (Baulac et al., 1988); Fathom (Finzer, 2007); Derive (Soft Warehouse, 1988); and a spreadsheet. It utilises a directory file management structure within which collections of pages (in the form of the applications described in Sections 2.4.2 to 2.4.6 below) are saved as software files. The facility for selecting and dragging on-screen objects is interfaced by a mouse within the computer software and a navigation pad (often referred to as the NavPad or Donut[2]) on the handheld device

[2] Throughout the study the teachers also refer to the NavPad as the Navigational Pad and the Donut.

(v.1.6 and v.1.7).

THIS IMAGE HAS BEEN REDACTED DUE TO THIRD PARTY RIGHTS OR OTHER LEGAL ISSUES

Figure 2-1 The chosen technology for the study – the TI-Nspire handheld v1.6

A brief description follows of the functionality within each of the environments alongside an exemplification of its use within the context of a multi-representational media for mathematics. A more detailed technical description of TI-Nspire can be found in the product manual (Texas Instruments, 2007a, 2007b).

2.4.2 Calculator application

Within the Calculator application, it is possible to execute calculations, define and evaluate functions, display and calculate fraction calculations, define variables and evaluate associated expressions (not shown).

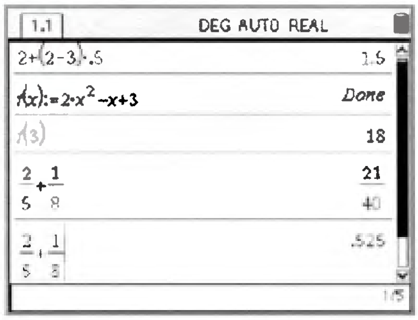


Figure 2-2 A sample calculator screen using the TI-Nspire handheld

Data that has been stored as a variable within another application can be evaluated and manipulated within this application, as can the validity of statements. Figure 2-3 shows the output of a students' response to an activity [GBA5], which also demonstrates how the screen can be split to show more than one application simultaneously.

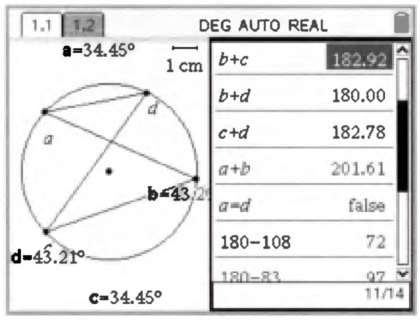


Figure 2-3 A dynamic geometry model alongside the evaluation of related measurements and the testing of mathematical statements

2.4.3 Graphs and Geometry application

The Graphs and Geometry application integrates a function graphing environment with a dynamic geometry application.

In Figure 2-4, a quadratic function has been defined on the entry line and displayed in the graphing pane. Standard linear and quadratic functions can be transformed by dragging as well as by editing the function.

In Figure 2-5, a Graphs and Geometry page is shown (in plane analytic view with the axes and function entry line hidden) on which a dynamic geometric figure has been constructed and some angle measurements made. The measurements have been saved as variables, which enabled calculations to be carried out on this page (or others) in the file.

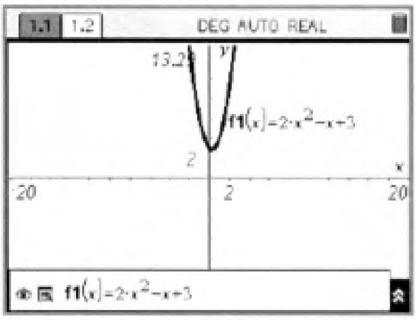


Figure 2-4 A sample graphing screen using the TI-Nspire handheld

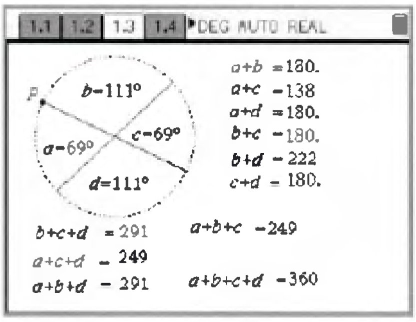


Figure 2-5 A geometric figure and related measurements and calculations.

2.4.4 Lists and spreadsheet application

Figure 2-6 shows an example of a Lists and Spreadsheet page, containing the data for 20 throws of a standard dice. The spreadsheet adheres to conventional layout and functionality. Labelling the columns facilitates the construction of statistical graphs based on the data in the columns and statistical calculations to be made. In addition, the cells in the row marked ♦ can be used to define functions for the data in the column, which includes a data capture facility from dynamic variables that have been defined elsewhere in the document.

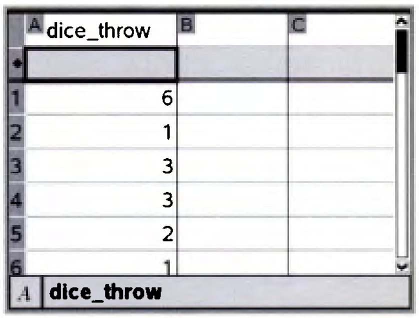


Figure 2-6 A sample spreadsheet screen using the TI-Nspire handheld

2.4.5 Data and statistics application

A full range of statistical plots and associated statistical calculations can be produced from variables defined anywhere in the file. Statistical graphs can be dynamically explored by dragging individual data points within the data set. For example, in Figure 2-7, any of the data points in the plot can be dragged, with the effect on the calculated mean being displayed simultaneously in the bottom pane.

The effect on the calculated mean after a data point has been dragged from a 6 to a 4 is seen in Figure 2-9. The corresponding value in the Spreadsheet would also have been simultaneously changed.

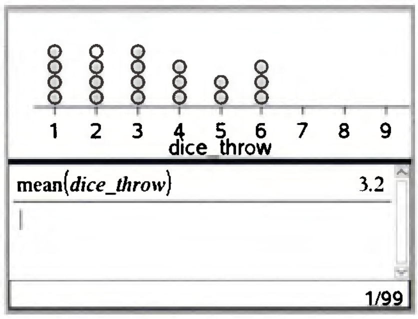


Figure 2-7 A sample spreadsheet screen, statistical plot and corresponding statistical calculation using the TI-Nspire handheld

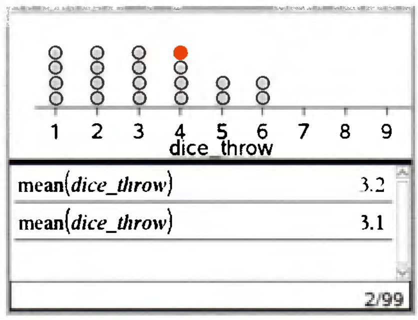


Figure 2-8 A sample spreadsheet screen, statistical plot and corresponding statistical calculation showing the effect of dragging a point on the graph

2.4.6 Notes application

The Notes application supports text input, with some basic mathematical formatting and symbols in addition to the functionality to write mathematical expressions and draw some standard shapes. It also has a comment facility for the teacher or reviewer.

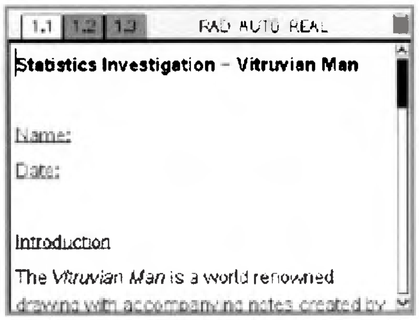


Figure 2-9 A sample notes screen using the TI-Nspire handheld

2.4.7 Defining variables – functionality and pedagogical approaches

The ability to provide simultaneously linked multiple representations within a mathematical model is a significant new feature of mathematical interest to the research study. This multiple representation capability can dynamically link variables from a range of mathematical sources whereby a change in one representation is observable in others, with a draggable interface that can be employed to change geometric figures, statistical plots and certain families of functions. Within the chosen MRT there are a number of ways in which the users can achieve this, all of which have implications for the teachers' pedagogic uses with learners in classrooms.

Some of the links between representations are 'built in' explicitly to the technology. For example, the generation of a graph and table of values from a defined mathematical function, or the generation of a statistical plot from a set of named data within the spreadsheet. In addition, a numerical output from any of the applications can be stored as a variable and then manipulated in any other application. So a measured value in the geometry application could be used to test a conjecture, define a graph or be captured as tabulated 'real' data. (See examples in Figure 2-10 and Figure 2-11).

Measured values of angles are saved as variables **a** and **b** to enable conjectures about their relationship to be tested.

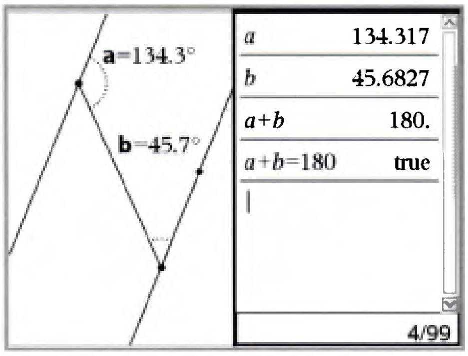


Figure 2-10 A sample dynamic geometric model with measurements and the associated calculations used to test a conjecture

There is a subtlety in the construction of this environment that is relevant to the usefulness of this functionality. In Figure 2.12 the starting point for the construction was the definition of the parallel lines, followed by the construction of the transversal. Had the environment been constructed with two 'free' lines and a transversal, the purpose of which was to enable the students to drag the 'free' lines to discover the equality, the accuracy of the angles measurements would be insufficient to test the truth of any conjecture concerning the angle sum.

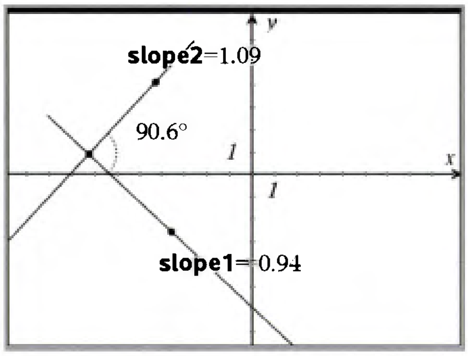
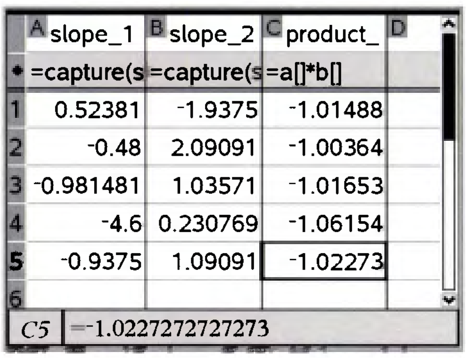


Figure 2-11 A sample graphing screen illustrating how measurements can be taken from two lines that have been constructed geometrically.

Measured values that have been saved as data variables enable manual or automated data capture within the spreadsheet application.



	A slope_1	B slope_2	C product_	D
◆	=capture(s	=capture(s	=a[]*b[]	
1	0.52381	-1.9375	-1.01488	
2	-0.48	2.09091	-1.00364	
3	-0.981481	1.03571	-1.01653	
4	-4.6	0.230769	-1.06154	
5	-0.9375	1.09091	-1.02273	
6				

C5 =-1.0227272727273

Figure 2-12 A sample spreadsheet page showing how captured data can be used as the basis for a calculation of the product of the gradients of the lines in the previous figure

Variables are most usefully defined within subsets of the file, called problems, to prevent global definitions from affecting subsequent activities within the file and, as indicated previously, there are a range of approaches for their definition.

2.5 Summary

This chapter provided a more detailed description of the context for the study, described the main functionality of the technology being used within the study and defined the terminology most relevant to what follows. It has introduced the themes for the Literature Review, which follows in Chapter 3.

3 LITERATURE REVIEW

The pedagogy of mathematics in a virtual culture should logically shift towards fluency in representing problem situations in a variety of systems, and towards students' ability to coordinate among representations as well as create and interpret novel ones.

(Shaffer and Kaput, 1998)

3.1 Introduction

This chapter provides a review of the research relevant to the study, as signalled in Chapter 2, to include the following themes:

- Coming to know new technologies and the role of technology in developing teachers' subject and pedagogic knowledge;
- The concept of mathematical variance and invariance in a multi-representational technological setting;
- Making sense of the process of teacher learning.

More broadly, the literature review aims to: bring together empirical and theoretical research findings from the wider body of published resources and identify areas or issues as relevant to the topic of the study requiring further research; identify key research ideas which could support the development of an interpretive framework for the research data.

The chapter concludes by revisiting the outline aim for the research described in Chapter 1 and expanding this into a refined set of aims in the light of the review of research.

3.2 Coming to know new technologies and the role of technology in developing teachers' subject and pedagogic knowledge

3.2.1 The process of instrumental genesis

During the 1990s, the advent of computational technology prompted a research discourse in education, which has focused on the nature of human appropriation of such technologies within educational settings. A particular strand of this discourse, emanating from various French research teams, has been greatly influenced by the theory of 'Instrumented Activity Situations' first proposed by Verillon and Rabardel (1995), which offered a model to describe how the human-instrument process operated (Figure 3-1).

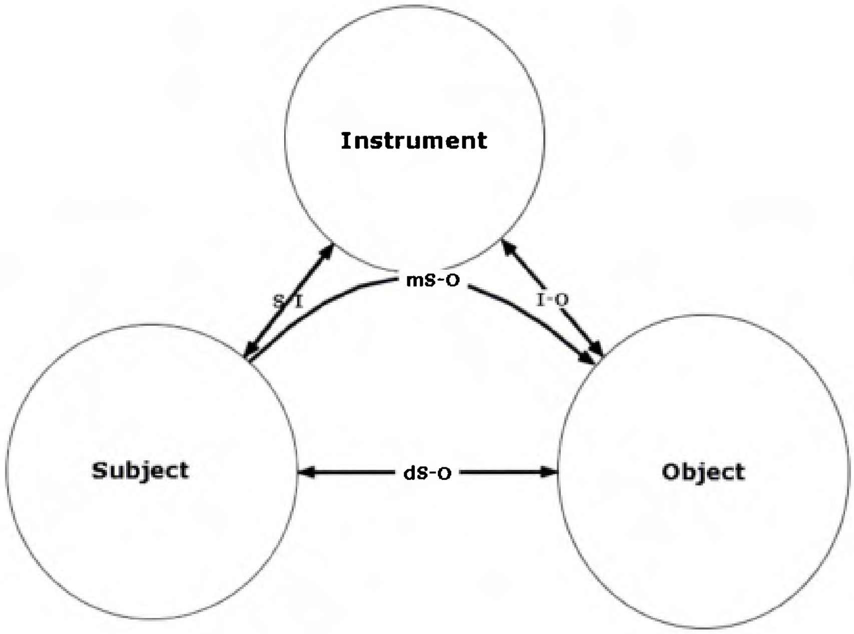


Figure 3-1 The triad characteristic of Instrumented Activity Situations (Verillon and Rabardel 1995 p.85) in which the *Subject* refers to the user, the *Instrument* represents the chosen technology and the *Object* is the activity or purpose for which the technology is being used. The arrows indicate the interactions between Subject and Instrument (S-I), Instrument and Object (I-O) and, in the case of the Subject and the Object, the direct interaction (dS-O) and the mediated interaction (mI-O).

Adopting a Vygotskian perspective for an artefact (i.e. the tool) as both a physical and psychological construct, Verillon and Rabardel consider an artefact as something that has been designed and realised for the purpose of accomplishing a particular activity. The user engages with the artefact and, in doing so, constructs a scheme for its use, with different users constructing different schemes. Over successive elaboration by different users, the instrumental genesis of the artefact takes place and, within this process, the user constructs meanings from the activity that they have carried out. Subsequent Franco-European research in this area has focused further on how this process of 'instrumentation' applies within the domain of technological tools and mathematics (Artigue, 2001, Guin and Trouche, 1999, Trouche, 2004, Drijvers and Trouche, 2008). In the majority of these studies the subjects of the research have been school-age learners, the instrument has been CAS technology and the object has been the accomplishment of mathematical activities. In addition, the majority of these studies have been placed within the French cultural setting in which handheld technology has been promoted by the Ministry of Education since the mid 1990s.

The question is raised as to whether Verillon and Rabardel's model for instrumented activity is sufficient to accommodate situations where the subject is the teacher and the object of the activity is the interlinked process of both 'designing activities for' and 'teaching with' technology. Consequently, an understanding of the process of instrumental genesis as the teacher appropriates a multi-representational technological tool for classroom use is of significant interest to the mathematics education research community. So, in considering Verillon and Rabardel's model as an initial theoretical framework, this study began by proposing an adapted model (Figure 3-2) which considered the position of the teacher as subject.

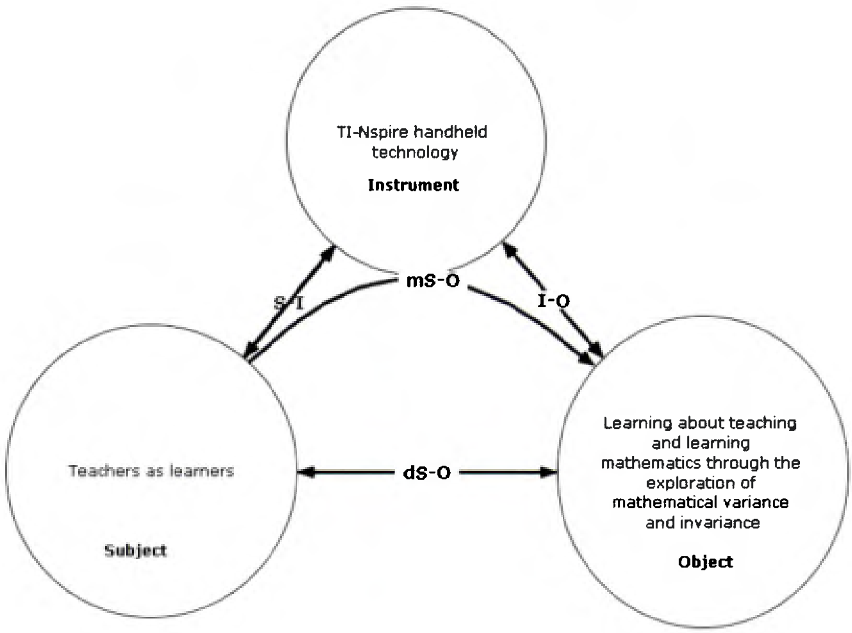


Figure 3-2 This is an adaptation of the previous figure in which the instrument is the TI-Nspire handheld, the subject is 'teachers as learners' and the object is teachers' learning about the teaching and learning of mathematics through the exploration of mathematical variance and invariance.

This research intended to probe how the nature of the design of activities and the experience and evaluation of the subsequent classroom teaching related to the trajectory of the teacher's journey, that is to develop a greater understanding about the nature of the 'object' within the triad when the subject is secondary mathematics teachers.

However, an added complexity concerns the definition of the object within this particular triad. If the context for the research is concerned with the process of teacher learning through the design, implementation and evaluation of technology based learning opportunities, then it reasonable to suggest that an aspect of this learning occurs in the classroom with students.

Hence it is also possible to conceive a second triad characteristic of Instrumented Activity Situations, from the perspective of the students, which exists alongside that of the teacher (Figure 3-3).

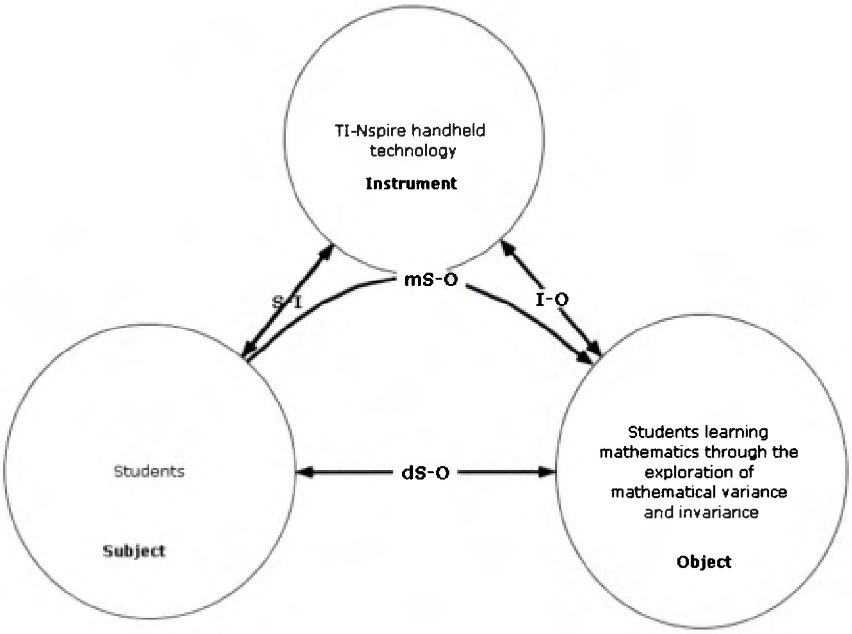


Figure 3-3 A second adaptation of Figure 3-1 in which the instrument remains as the TI-Nspire handheld, the subject is now the students and the object is the students' learning of mathematics through the exploration of mathematical variance and invariance.

In this respect the technology could be conceived as having two roles: mediating teachers' learning in the classroom; and mediating the transformation of knowledge through the way teachers reflect upon students' learning and develop their own theoretical pedagogic ideas. This perspective offers an opportunity to try to conceptualise the notion of pedagogy as an internalisation of these processes and to try to articulate how these two triads might relate to each other.

Verillon and Rabardel define several concepts to support the analysis of instrumented activity, suggesting that artefacts offer both new possibilities and new conditions for organising the action. In relocating these concepts to the situation of this study, for the teachers this involves the process of learning about the affordances of the new technology and then devising the teaching activities and approaches that utilise them. This cycle is described by Verillon and Rabardel as an instrument utilisation scheme (IUS). It was anticipated that, for the teachers involved in the study, the observation of the evolution of the individual instrument utilisation schemes would provide a platform from which to observe and evidence teacher learning.

Furthermore, an important aspect of instrumental utilisation schemes is the 'social utilization scheme', within communities of practitioners, through which a stable base for the tool use emerges (Verillon and Rabardel, 1995, p. 86-7). Developing a theoretical understanding of these social utilisation schemes has been a predominant feature of the research by Ruthven and colleagues at the University of Cambridge, which has sought to articulate the emergent practices of secondary teachers' uses of technology (Ruthven and Hennessy, 2002, Ruthven et al., 2004, Ruthven, 2008, Ruthven et al., 2008, Ruthven et al., 2009). Whilst not a primary consideration for this research, the study may elicit evidence of emergent social utilisation schemes amongst the participating teachers.

Reflecting on the French research studies, Artigue comments on the emergence of new instrumented techniques within school settings and the need for teachers both to be aware of, and to act to stabilise them, within the institutional setting (Artigue, 1998). Whilst my study is not seeking to illuminate the process of the wider integration of the MRT into the school setting, the development of teachers' awareness of evolving utilisation schemes could be a pre-cursor to the stabilisation of practice to which Artigue refers.

Guin and Trouche (1999) contributed to the growing discourse on the process of instrumental genesis within the school mathematics setting, albeit by learners of mathematics, with a set of behaviour profiles that related to the students' use of CAS. They arrived at these profiles through a description of two phases in the students' instrumental genesis, an initial phase of discovery in relation to the affordances of the tool and an organisational phase in which the students sought mathematical consistency between different sources of information. The student behaviour profiles were described as 'work methods' and classified as: random; mechanical; resourceful; rational and theoretical. Although Guin and Trouche do not write explicitly on the role of the teacher in mediating the students' instrumentalisation process, they imply that the teacher needs to:

- make explicit to students the connections between their new knowledge and the National Curriculum;
- highlight to students the important mathematical connections they are making within the broader web of mathematical and technological knowledge;
- have an awareness of existing knowledge concerning mathematics-with-technology to enable the teacher to pre-empt common classroom issues;

- consciously mediate whole class discourse with a view to arriving at consensus with respect to new mathematical knowledge.

These initial observations by Guin and Trouche were the outcomes of a study that had focused on the introduction of handheld technology to a class of students. The unit of analysis was the students and the study did not focus upon the teacher's trajectory with the technology. By contrast, my own study seeks to substantially develop the discourse concerning teachers' utilisation schemes with handheld technology and build upon initial observations, such as those of Guin and Trouche, to try to understand the process through which these utilisation schemes evolve.

In concluding this section, whilst Verillon and Rabardel's triad has been selected and refined as an overarching theoretical framework for the study, it is not detailed enough to provide a useful analytical tool to support the interpretation of the research data. A contrasting perspective involves looking more holistically at what teachers do with technology in the classroom with the intention to identify a framework that would act as a more analytic tool and enable a deeper analysis and interpretation of what happens in classrooms.

3.2.2 Teachers' pedagogical maps

A substantial theoretical framework to describe mathematics teachers' teaching with technology has been developed by Stacey, Pierce and colleagues at the University of Melbourne, the result of their extensive work in this area over the last twenty years (see summary in Pierce and Stacey, 2008). This framework, which the authors describe as a 'pedagogical map', is shown in Figure 3-4.

THIS IMAGE HAS BEEN REDACTED DUE TO THIRD PARTY RIGHTS OR OTHER LEGAL ISSUES

Figure 3-4 The pedagogical map for describing teaching with technology, which considers the functional and pedagogical opportunities for technology within the teaching and learning of mathematics (Stacey, 2008)

The framework emanated from the broad evidence base for the development of the use of technology in mathematics in Australia, the initial focus being on the use of Computer Algebra Software (CAS). In a presentation given in 2008, Stacey argued that, with the facility to use CAS software within a more comprehensive package such as TI-Nspire, rather than as a standalone tool (i.e. Derive or Maple), the boundaries between the various software packages have become less distinct. Also, the Melbourne research team have begun to interpret CAS within the secondary school setting as being 'mathematically able' software (Stacey, 2008). This development has two significant implications for my own study. The first is that the version of TI-Nspire software and handhelds that is available to the teachers in my study is the non-CAS version. In adopting Stacey and Pierce's pedagogical map as an analytical tool for this research, I would need to establish the validity of the map in my own research setting. Secondly, much of the research base for Stacey and her colleagues has been predominantly within the upper secondary and university settings, whereas my own study is within the compulsory secondary phase. Therefore, in order to ensure an element of validity, the first phase of the study would need to establish prior to commencing the main study that there are enough similarities between the research subjects and their settings.

The pedagogical map (Figure 3-4) is divided into four layers and is interpreted from the bottom upwards. The base of the map concerns the affordances of the mathematically able software, which Stacey defines as 'functional opportunities', i.e. those which are dependent on its intrinsic characteristics. She argues that these have the potential to influence both curriculum and assessment change, although she does not address these issues within the paper (Stacey, 2008). In my own study, I will be examining these potential opportunities through the eyes of the teachers, based on their classroom experiences and professional insights. For example, the teachers' emergent instrument utilisation schemes would provide a tangible source of evidence for the perceived functional opportunities.

Working in an upwards direction, the second layer of the map concerns the nature of the mathematical activities in which the students engage, and Kaye and Pierce describe the pedagogical opportunities at the activity level as primarily being about offering improved speed, access and accuracy. This layer is sub-divided into five activity types: scaffold by-hand skills; use real data; explore regularity and variation; simulate real situations and link representations. Stacey comments that the most common pedagogical use for technology observed in Australian classrooms involved activities that explored regularity and variation (Stacey, 2008). This relates strongly to, and provides external validity for, my own mathematical focus on variance and invariance.

The third level of the pedagogical map relates to the 'classroom layer' and the way in which the introduction of mathematically able software can be a catalyst for change in the classroom dynamic through improved shared displays and the students' personal access to an authority other than the teacher. Stacey and Pierce sub-divide this theme by separately considering the changed classroom dynamic and the changed didactic contract, and provide the evidence for each as outlined in Table 3-1.

Changed classroom dynamic	Changed didactic contract
• the students' work becoming the focus of discussion; • the teacher seeding questions and pointing out features;	• the empowering of students to become source of information to share about mathematics and technology; • the CAS acting as another classroom authority - frustratingly with a mind of its own;
• the use of shared technology encouraging the use of group work; • the technology acting as another authority in the classroom - providing issues to explore and puzzles to resolve; • the technology becoming another member of the class or group – what does the technology think?	• an explosion of methods, which gives students more to contribute.

Table 3-1 The evidence for the pedagogical opportunities afforded by the classroom layer within the pedagogical map (Stacey, 2008)

The top (and final) layer of the pedagogical map concerns the reassessment of the goals and methods for the mathematics teaching and this is sub-divided into three themes:

- the exploitation of the contrast between the ideal and machine mathematics, which in practice relates to the use of anomalies and limitations within activities and approaches;
- a re-balancing of the emphasis on learning skills, concepts and applications, which could include the order of teaching these elements or the earlier introduction of real-world problems that are scaffolded by the technology;
- building metacognition and overview by offering students a bird's eye view of a new topic before looking at the details or a macro view of mathematics by encapsulating a multi-step process as a single command within CAS.

Stacey and Pierce argue that, as teachers develop their use of technology, most initially adopt an approach that they describe as 'functional', which focuses on the technology being used to accelerate routine activities and to compensate for by-hand skills. In both of these usage cases the teachers do not fundamentally change what they teach or how they teach it. Stacey and Pierce provide vignettes for four further types of teachers, namely 'progressive, radical, conservative and concerned', and pose the questions, 'are there common paths through the pedagogical maps?' and 'what are the major considerations when exploiting each of these pedagogical opportunities?' (Stacey, 2008). The context for my own study offers an opportunity to consider the second question in detail, particularly in relation to the opportunity for 'exploring regularity and variation'.

Furthermore, Stacey makes the following comments about the map, which highlight its particular relevance to this research study:

- The context for the development of this framework has been predominantly the introduction of CAS into mathematics classrooms, but more recently considers the MRT selected for my own study (TI-Nspire);
- In categorising the types of mathematical activities developed with technology, 'Explore regularity and variation' and 'Link representations' resonate strongly with my own research topic.

The pedagogical map has been adopted as an organisational framework to support the data analysis process during the second phase of the study, which is described more fully in Chapter 5.

3.2.3 Teachers' emergent practices with technology

The third area of existing research of direct relevance to this study concerns the emergent practices of secondary mathematics teachers with technology. A number of research studies have focused on the practices of secondary mathematics with technology with the explicit aim to offer theories concerning legitimised uses within school settings (Monaghan, 2001b, Ruthven et al., 2008, Ruthven et al., 2004, Ruthven and Hennessy, 2002, Rodd and Monaghan, 2002, Monaghan, 2001a, Monaghan, 2004, Godwin and Sutherland, 2004, Noss et al., 1991). The earlier research tended to focus on specific technologies, for example LOGO, spreadsheets, dynamic geometry software or graphing software (and calculators), resulting in theories of use with mathematics specificity. More recent research is seeking to validate these emergent theories across a broader range of mathematics education technologies (Ruthven, 2008, Ruthven et al., 2009). The research approaches have predominantly fallen between two methodological paradigms, which concern *what teachers say* about their classroom uses of technology alongside *what teachers do* in their classrooms with technology.

Ruthven and colleagues at the University of Cambridge have focused on eliciting the nature of teachers' 'craft knowledge' through a number of studies. They have developed what they describe as a 'practitioner model' that has evolved from the analysis of teachers' own descriptions of their technologically enhanced classroom practices, supported by selected classroom observations. Participants in their studies were identified for their 'successful practice' and the resulting categories, which constitute the practitioner model, include: effecting working processes and improving production; supporting processes of checking, trialling and refinement; overcoming pupil difficulties and building assurance; focusing on overarching issues and accentuating important features; enhancing the variety and appeal of classroom activity and fostering pupil independence and peer exchange (Ruthven et al., 2004, Ruthven et al., 2009).

However, there is a tension in the design of studies that seek to research teachers' practices in a more naturalistic way, which relates to the difficulty in locating embedded practices that use the more mathematically interesting technology in English secondary schools. Small surveys into secondary mathematics teachers' use and perceptions provide localised snapshots, however, fail to uncover the finer nuances of classroom practices (Hyde, 2004, Knights, 2009). The implications of this for the design of my own study are addressed more substantially in Chapters 4 and 5, which detail the research methodologies in each phase.

3.2.4 Implications for my study

In considering the three research perspectives concerning the evolution of educational technology in mathematics classrooms that have just been described I observed some distinct similarities in some of the ideas that have emerged. For example, the processes of instrumentation and instrumentalisation are an inherent element of most studies. In addition, Verillon and Rabardel's 'social utilisation schemes' resonate strongly with the notion of 'archetypical practice' uncovered by Ruthven et al's lesson case studies (Ruthven et al., 2009, 2004).

However, none of the studies cited have explored the nature of the teachers' individual learning that occurs over time as they begin to use technology in their classrooms. Although Stacey et al use their pedagogical map to classify teachers' uses of technology and suggest aspects of their map in which teachers' practices develop over time; their research has not specifically probed the process by which this occurs. My own study seeks to address this gap in the research by adopting a longitudinal case study approach over a three-year period.

3.3 Mathematical variance and invariance

The justification for the choice of the mathematical focus for the research, the concept of variance and invariance in a multi-representational technological setting, has already been presented in Sections 2.3 and 2.4. This Section aims to summarise the existing research on this topic to highlight interpretive theories, insightful methodologies and emergent classroom teaching approaches in terms of their relevance to this study.

3.3.1 The development of representation systems for mathematics

Early multiple representation technologies for mathematics focused on traditional mathematics topic clusters. For example, the representational cluster of functions-graphs-tables-coordinates that tends to reside within the algebra curriculum results in a number of graphing software environments, many of which had common functionality that allowed the user to:

- enter a function and observe the resulting graph being plotted on the screen;
- observe a table of x and y values, and in some cases select the initial values displayed and change the increment between them;
- observe particular values for the defined function on the graph itself, often through a trace facility;
- enlarge or reduce a graph's appearance by either rescaling the axes or using a 'zoom' facility.

Within this type of representational system, the starting point was predominantly the definition of the function (often requiring a particular syntax) and, in order to explore the variance and invariance of a situation, the teacher would need to devise activities in which students were required to focus on particular objects and/or values within the system. However, researchers such as Kaput have been careful to point out that,

> *Our initial attention to symbol systems should not be misread as an assertion that mathematics is, and hence the curriculum should be, about symbols and syntax. On the contrary, our ultimate aim is to account for the building and expressing of mathematical meaning through the use of notational forms and structures. (Kaput, 1989, pp 167-8)*

The broader ideal that representational systems might form part of the environment in which learners can experience and express mathematical generalities is one that has underpinned the work of researchers in the field (Moreno-Armella et al., 2008, Sutherland and Mason, 1995). Mason writes about the dual processes of 'seeing a generality through the particular and seeing the particular through the general' as being integral to the learning of mathematics (Mason, 1996, p.65).

A number of 'typical' pedagogical approaches have become evident in English secondary classroom practices, which attempt to adopt such an approach, for example using a suitable technology to vary the values of the parameters m and c within the generalised linear function $\boldsymbol{y} = m\boldsymbol{x} + c$ (Ruthven et al., 2009). However, the research evidence also suggests that this approach falls short of the ideal as students will often make generalisations that, although correct, are not those which the teacher intended.

Mason writes extensively about the subtle decisions and actions that teachers should make in order to support the students to move fluently from the example that forms the particular to consider the generality and then, from a consideration of the generality, find examples of the particular. He argues that the teacher's choice of mathematical examples that initiate this process in the classroom is crucial, with students often assuming generalities from the examples teachers provide, which are not those that the teacher had intended (Mason, 1996). In my own study I am interpreting variance and invariance in the sense that, through the examples that the teachers select and develop, I am able to gain an insight into their perceptions of the exploration of generality as the central act within their mathematics classroom. Mason poses the question 'Does software that provides particular instances necessarily lead pupils to awareness of generality?' (Mason, 1996, p.70). Within the context of this study I would rephrase this question to read, how do teachers develop activities and approaches that privilege students' awareness of generality within such software environments?

In the mid 1980s, what we know now as dynamic geometry systems began to emerge and these were initially concerned with the construction and manipulation of geometrically defined figures in a pure Euclidian sense (Baulac et al., 1988, Key Curriculum Press, 2003, Schwartz and Yerushalmy, 1985). Such software adopted a notional concept of scale and included measurement functionality. The original intention was that 'measurement can play only a suggestive function, not a conclusive one' (Kaput, 1986), which in some cases led the designers to limit the accuracy of measurement to just two decimal places (Schwartz and Yerushalmy, 1985). Later developments led to the overlaying of a Cartesian graphing plane to the geometric plane, enabling mathematical activities to be designed that allow the measurements from dynamic figures to be output to tables and graphs. This has led to what Lakatos described as a 'quasi-empirical' approach to be developed within mathematical activity design and an expansion and redefinition of the epistemological context (Lakatos, 1976). The collection of 'experimental data', generated from dragging objects or varying values with a view to arriving at generalities using mathematical modelling approaches, has become a recognised genre of technology use in mathematics classrooms (Oldknow, 2009, Arzarello and Robutti, 2008).

3.3.2 Key research studies

James Kaput's seminal paper of 1986 speculated about how the software that was emerging at that time offered new insights into the nature of mathematics as a body of knowledge (i.e., its content and processes) and the possibilities for the resulting human engagement (Kaput, 1986). The primitive mathematical definitions that are designed into mathematical software have a fundamental influence on the resulting mathematical interactions and Kaput was intuitive in his description of how technology might 'open new representational windows' (ibid. p187). Kaput gave teacher training support the highest priority with respect to whether or not new representational forms for mathematics within technological environments would shape the direction of mathematics teaching and learning, saying, 'Limited imagination, lack of confidence, or a totalitarian or almost negative attitude towards students' intellect can close the shades on any representational window' (ibid. p.199). Whilst this might seem negative and judgemental, the intervening twenty-five years had not revealed any groundbreaking developments in the role and use of technology within the English mathematics classroom setting to suggest that Kaput's prediction was inaccurate.

A recent systematic review of the research into the use of technology to support the teaching and learning of algebra by Goulding and Kyriacou (2008) on behalf of the Evidence for Policy and Practice Information and Coordinating Centre (EPPI) concluded that further research was needed on:

- the links between tables of values, symbolic representation and graphical representation, which could provide the bridge between functions and the solution of equations;
- critical use of graph-plotters, including how changing the scale can alter the appearance of the graph, how to use the zoom function, how to change windows, and how to interpret pixel displays.

The first recommendation suggests the need for investigations into particular pedagogical approaches for the use of the technology, for example the design and support for activities that privilege the links between the different representations. However, the second recommendation resonates with the processes of instrumentation and instrumentalisation in that the teacher and students are required to be aware of the various functional techniques suggested, before then being able to utilise this functionality within activities to achieve the criticality that is demanded. This implies the need for design experiments in which researchers and teachers co-construct classroom activities that are valid within and relevant to the school curriculum.

3.3.3 Implications for my study

This element of the literature review, concerning the mathematical focus for the study, has led me to clearly define how mathematical learning is predominantly concerned with the privileging of students' opportunities to generalise and specialise as a means to constructing their own mathematical meanings. Within the context of this study, the teacher's role is to design and orchestrate classroom activities and approaches, which use the various functionality of the MRT to achieve this. However, as teachers' individual belief systems about mathematical learning (and the role of technology within this) will undoubtedly influence their decisions and actions, the trajectory of teacher development to which I refer will reveal evidence of these preconceptions. The section of the literature that follows will examine the process and content of mathematics teacher learning in an attempt to develop a research methodology that will enable teachers' learning to be fore-grounded.

3.4 Interpreting mathematics teachers' professional learning – what and how?

As this study concerns mathematics teachers' professional learning, a consideration of the literature needs to include the following two aspects:

- Definitions and interpretations concerning the nature of mathematics teachers' content knowledge to encompass personal subject knowledge, subject knowledge for teaching and pedagogic knowledge.
- Constructs concerning the process of teacher learning.

The outcomes of this element of the literature review will significantly inform the development of the methodology for the research, particularly concerning tools and structures developed to evidence teachers' learning as appropriate to the two phases of the study.

3.4.1 Teacher knowledge and interpretations of pedagogy

Much has been written about the tacit, implicit or craft knowledge of teaching mathematics, particularly within initial teacher education and the first few years of teaching. This discourse has been greatly influenced by Lee Shulman's presidential address to the American Educational Research Association in 1985 (Shulman, 1986) in which he suggested three categories of content knowledge for teaching: subject matter content knowledge, pedagogical content knowledge and curricular knowledge.

Shulman's definition of subject matter content knowledge extends beyond facts and concepts to encompass the following aspects:

- Substantive structures - the organisation of concepts and principles to incorporate facts.
- Syntactic structures - 'the set of ways in which truths, falsehoods, validity and invalidity are established... ...like a grammar' (Shulman, 1986, p.9).
- Incorporation of an understanding of the importance of the substantive and syntactic structures within the discipline.
- Why particular topics are central to the discipline.

Shulman argued that all of these areas of knowledge are essential for teachers in contrast to the lay person who is probably only concerned with the subject matter alone. He considered pedagogical content knowledge as a second form of content knowledge, which 'embodies the aspects of its content most germane to its teachability' (Shulman, 1986, p.9). Such knowledge is defined by:

- The most useful forms of representations for a given topic idea and the explanatory structures that support others to understand the topic through these representations. This requires teachers to stay abreast with new insights and developments, a particularly relevant aspect when considering the rapid developments within educational technology.
- Aspects of the topic that make it easy or difficult for students at different levels of progress (to include those commonly accepted or misunderstood) and teaching strategies for responding to these.

Shulman calls directly for teachers to have an awareness of research-based knowledge in this area, describing it as being 'at the heart of our definition of needed pedagogical knowledge' (ibid. p.10). This seems highly relevant in my own study, which calls for teacher to build an awareness of how research-informed approaches with technology might support the teaching of particular topics in mathematics.

Shulman's final category of subject content knowledge, namely curricular knowledge, is concerned with a teacher's knowledge of the landscape from which teaching resources and approaches are selected and the resulting student pathways created (or inferred) by these actions. Such resources would typically include the teaching materials, practical resources, assessment approaches and in more recent years, ICT resources to support both teaching and learning. This landscape takes account of the students' own perspectives in that Shulman expects teachers to be aware of the students' wider curriculum, not just the teacher's own subject area.

Shulman furthers his concept of teacher knowledge by conceptualising the forms of representation of the content knowledge that have previously been described. He labels these as 'propositional knowledge, case knowledge and strategic knowledge' and, as all three elements of content knowledge can be described in these forms, this suggests the existence of a three by three matrix. To aid the interpretation of Shulman's theory within the context of this study, and as a means to establishing its usefulness as an overarching interpretation of teacher knowledge, I have attempted to locate possible elements of a secondary teacher's mathematics knowledge concerning the topic 'learning about linear functions using technology' to each of the nine cells (see Table 3-2).

	subject matter content knowledge	pedagogical content knowledge	curricular knowledge
propositional knowledge (principles, maxims and norms)	*Knowledge of commonly used aide memoire such as 'change in y over change in x' for calculating the gradient of a linear function. (maxim)*	*Knowledge of a systematic variation approach to the values of 'm' and 'c' within graph plotting software. (principle)*	*Knowledge of the real data collection approach for the purpose of linear modelling for data analysis. (norm)*
case knowledge[1] (prototypes, precedents and parables)	*Knowledge of existing classroom approaches for teaching linear function with technology. (precedent)*	*Knowledge of empirical research studies, which have reported on the teaching of linear function with technology.(parable)*	*Knowledge of cross-curricular approaches for the teaching of linear function through interpretations of speed-distance-time.*
strategic knowledge	*Knowledge of machine constraints when working on linear functions with technology.*	*Knowledge of the way in which students make sense of their interactions with technology concerning linear functions.*	*Knowledge of the way in which students relate the modelling of a 'line of best fit' from real data to the mathematical definition of function.*

Table 3-2 My exemplification of Shulman's theory of teacher content knowledge for the mathematical topic of linear function (Shulman, 1986)

Shulman's definition of 'strategic knowledge', namely the knowledge the teacher draws upon when confronted by 'particular situations or problems, whether theoretical, practical, or moral, where principles collide and no simple solution is possible' (ibid. p.13), offered the closest interpretation of the type of knowledge developed through situated experiential learning within the context of my own study. This would rely on my interpretations of the evidence from teachers that

[1] Shulman argues that case knowledge refers to 'specific, well documented and richly described events' (ibid. p. 11).

would be located within each of the cells on the bottom row of Table 3-2. In completing this non-trivial mapping activity, I very quickly concluded that, illuminating as Shulman's theory was, it offered far too complex an interpretation of teacher knowledge to be useful for my own research purposes, particularly as Shulman's further sub-divisions of propositional knowledge and case knowledge suggested even more cells within the matrix. Besides, I was looking to develop my own ontological innovation for an interpretation of teacher knowledge and learning based on the empirical evidence from my own study.

Consequently, I looked to the research of Rowland, Husktep and Thwaites which, although located within their work with pre-service primary teachers, developed an alternative interpretation of teacher knowledge through a grounded methodological approach (Rowland et al., 2005). By analysing lesson observation data and video tapes relating to twenty four lessons (from twelve different trainees), they identified a 'knowledge quartet' to describe the four dimensions through which the teacher's mathematics-related knowledge could be observed, namely, 'foundation, transformation, connection and contingency' (ibid. p.259). Whilst it could be argued that primary trainees are a very different group of research subjects to the more experienced secondary teachers involved in my own study, it was the researchers' classroom-situated, grounded approach that sought to develop a framework that was 'manageable and not overburdened with structural complexity' that attracted me to it. Rowland et al's aim was to find out what their subjects did know, rather than articulating what they ought to know, an element of their research in contrast with Shulman's work. Rowland et al focussed their data analysis on the elements they identified as concerning subject matter knowledge and pedagogic content knowledge[2] and, through a robust process arrived at eighteen codes which were then grouped into the four previously mentioned 'units' (Rowland et al., 2005, p.258). On reviewing closely the descriptions of each unit, the first of these, 'foundation knowledge', concerned the teacher's underlying knowledge and beliefs about mathematics and its pedagogies. The descriptions of the units of 'transformation' and 'connection' linked with 'knowledge-in-action' were related to 'deliberation and choice in planning and teaching'. However, I was particularly interested in the notion of contingent knowledge, which Rowland et al define as:

> *Knowledge-in-interaction as revealed by the ability of the teacher to 'think on her feet' and respond appropriately to the contributions made*

[2] Rowland et al offered no explanation as to why they omitted to code their data for curriculum knowledge, a surprising decision as primary teachers could possibly be expected to have a wider perspective on this compared to their secondary counterparts.

> *by her students during a teaching episode. On occasion this can be seen in the teacher's willingness to deviate from her own agenda to develop a student's unanticipated contribution:*
> - *might be of special benefit to that pupil, or*
> - *might suggest a particularly fruitful avenue of enquiry for others.*
>
> (Rowland et al., 2005, pp. 265-6)

It was the inclusion of the phrase 'knowledge-in-interaction' which resonated strongly with my own insightful observations of teachers using technology in mathematics classrooms. The interactions, as described by Verillon and Rabardel's adapted triad offered a broader range than that proposed by Rowland et al, who were observing teaching that did not involve the use of technology. This provided a worthwhile line of enquiry for the study as I sought to expand upon this notion of contingency in the secondary setting with the added complexity of a multi-representational tool. However, researching the occurrence and nature of the contingent moments would require systematic, focussed lesson observations, which would form part of the methodology developed for the second phase of the study.

Other researchers have alluded to the existence of such 'moments' in lessons and commented on their contribution to teachers' epistemological development. For example, Zodik and Zaslavszky researched teachers' uses of mathematical examples in lessons and concluded that the teachers' reflections on the outcomes of their use of spontaneous[3] examples had the potential to provide learning opportunities for the teachers (Zodik and Zaslavsky, 2008). However, this looked to be an under-researched area within the domain of technology use in secondary mathematics classrooms.

In concluding this section, a consideration of the nature of 'tacit knowledge' seemed to be a central idea when investigating teachers' professional development (Polanyi, 1962, Polanyi, 1966). Polanyi justified the existence of tacit knowledge by saying 'we know more than we can tell'. With respect to the craft of teaching secondary mathematics, it is the teacher's insightful knowledge that underpins their thoughts and actions. In my study, researching the acquisition of this tacit knowledge required a subsidiary awareness to the subtleties of teachers' thoughts and actions within their practice. One of the challenges for the design of my research methodology, particularly during the second phase would be to develop a robust set of research tools and strategies that would reveal teachers' tacit knowledge in

[3] Zodik and Zaslavsky define spontaneous examples as those 'for which there is evidence that choosing it involved to some extent in-the-moment decision making' (ibid p.171)

relation to the research topic.

3.4.2 Making sense of the process of teacher learning

The process of teacher learning is a relatively under-defined area within mathematics education and a central construct that has emerged from the literature is the notion of reflective practice. This was first proposed by Schön (1984) who examined the practices of different professionals in situ. In defining the starting point for his research, Schön seems to build upon Polanyi's earlier concept of tacit knowledge when he stated,

> *I begin with the assumption that competent practitioners usually know more than they can say. They exhibit a kind of knowing-in-practice, most of which is tacit. (Schön, 1984, p.viii)*

Following Schön's early research the concept of 'teachers as reflective practitioners' was taken up by a number of researchers within the context of mathematics education (Thompson, 1992b, Mason, 2002, Jaworski, 1994, Ahmed and Williams, 1997). Mason (2002) critiqued Schön's classifications of 'reflection-on-action' and 'reflection-in-action' stating that neither takes account of the quality or depth of the reflection that may be taking place. Mason added the classification 'reflection-through-action,' which he described as 'becoming aware of one's practice through the act of engaging in that practice' (ibid, p.15). Mason also introduced the idea of the 'professional incident' as the starting point for the reflection, claiming that these starting points can lie within,

> *psychological-historical-structural aspects of a specific topic or practice, on processes of learning-teaching, on institutional forces in which I[4] am embedded, or on socio-cultural-political forces which constitute the encompassing system. (Mason, 2002, p15)*

Mason argues that the reflective accounts of such incidents would form the basis for the enhanced communication of insights and analyses as appropriate to the community within which the incident is located. I have a strong personal resonance with this notion as I can recall many 'incidents', which have been triggered by all of the starting points identified by Mason and which led me to take actions in my own professional practice.

Several researchers have developed strategies for stimulating and encouraging reflective practice by mathematics teachers, which have served as part of the

[4] Mason refers to this in the first person – I interpret this as 'institutional forces in which the teacher is embedded'.

research methodology (Britt et al., 2001, Cooney and Krainer, 1996) or have been developed as outcomes of the research process (Jaworski, 1994). These strategies have focussed on:

- Structures to support the reflective process such as the Jaworksi's 'teaching triad' - management of learning; sensitivity to students and mathematical challenge' (Jaworski, 1994).
- The existence and nature of 'professional conversations' as being central to the reflective process - discussions among those who share a complex activity or profession in order to improve their understanding of, and efficacy in what they do. (Britt et al., 2001, p.31).
- The development of 'teachers' logs' as an integral part of the reflection process to include lesson evaluation structures and prompts. (Ahmed, 1987, Ahmed and Williams, 1997)

Cooney and Krainer (1996) reiterate the importance of recording and analysing experiences through the written word as part of the reflective process. They contrast the differences between the different types of writing, for example writing for oneself as opposed to writing something for the public domain, and argue that the latter places an expectation above 'normal' teacher's practises as they have to:

> *gather data, analyze and reflect on the data and take action; write down their findings (and not just communicate them orally), and formulate these results for other people (and not just practise something in their own classrooms).(Cooney and Krainer, 1996, p.1174)*

Within the context of teachers' professional learning, their expectations as to the nature of the programme of activity are of some significance. Schielack and Chancellor (1994) described the frustrations they felt when teachers participating in their professional development courses seemed to be interested only in accumulating activities to take back to the classroom. They responded by changing the balance of the sessions to give a much longer proportion of time for reflection, acknowledging that 'stopping time for reflection' was likely to be viewed critically by the teachers who might not have seen this as a valid form of professional activity. Schielack and Chancellor noted that they 'would need to offer a supportive environment until they began, as individuals, to reap the benefits of a reflective attitude towards their teaching' (ibid, p.305).

Several studies that have focused on teacher learning involving technology have included the role of teacher reflection within discussions and writings following classroom teaching as an implicit element of the research methodologies (Noss et al., 1991, Laborde, 2001, Norton et al., 2000). However, none of these particular studies included a discussion of the role of active reflection as an integral element of the research design. In most cases the teachers' comments and notes were part of the data collection process and not seen as fundamental to their teachers' subsequent learning. The opportunity for teachers' reflections to be heard through their writings in addition to their oral contributions is of significant value within any research process, which focuses on aspects of teacher learning and, as this was the main aim for this study my methodology would need to take account of these issues.

3.4.3 Implications for my study

The review of literature referring to the content, nature and process of teacher learning led me adopt a broad interpretation of knowledge as proposed by Shulman's 'knowledge for teaching'. It also highlighted the complexities of the process of teacher learning and the methodologies that would be needed to capture this in line with my desire to describe teachers' trajectories of development within the domain of the study.

3.5 Summary and refined aims for the study.

Chapter One concluded with the description of the broad aim for the study, which was to explore the nature of teachers' cognitive and pedagogical learning as they developed their use of a MRT with a focus on their conception of mathematical variance and invariance. The literature review has expanded this question to incorporate the following subsidiary aims:

- What are the trajectories in the teachers' learning and how can these be described in terms of the activity, classroom and subject layers?
- What do the emergent practices say about the teachers' conceptions of mathematical variance and invariance?
- What form of instrument utilisation schemes emerge?

4 THE RESEARCH METHODOLOGY

4.1 Introduction

In concluding their research study into teachers' perceptions of the use of technology, which involved secondary mathematics departments in seven English schools, Ruthven and Hennessy (2002), highlighted the need for,

> *naturalistic studies which provide analyses of the perspectives and practices of groups of teachers across different settings within the context of using technology to support teaching and learning of mathematics. (p.50)*

Although this study cannot claim to be wholly naturalistic, the reasons for which are explained later in this chapter, it did aim to gain a deep insight into the perspectives and practices of secondary mathematics teachers as they developed their use of the MRT in the classroom. In particular, it sought to explore the process through which the teachers' perceptions, mathematical and pedagogical knowledge concerning variance and invariance evolved over the period of the study. The study adopted a broad interpretation of a mathematical variable as a mathematical object that can be varied within a static and dynamic setting through the process of changing an input or dragging an object (in a dynamic context). This extended the concept from the more traditional numerical and algebraic domains to include the geometric domain, and also resonated with recent changes to the English curriculum (Department for Children Schools and Families, 2007), offering an element of legitimacy for the teachers involved.

The research was carried out in two phases, the first between July 2007 – November 2008 and the second between April 2009 and December 2009. In each of these phases, a group of teachers was selected and a series of methodological tools developed to capture rich evidence of their use of the technology in classrooms to enable the aims of the study to be realised. The methodological approaches for each phase were distinct and these are described in detail in Chapter 4 and Chapter 5, accompanied by a justification of the decisions that were made.

4.2 Theoretical stance for the research process

My research adopts the perspective that reality is socially constructed and that the 'activity of the researcher is to understand the multiple social constructions of meaning and knowledge' (Robson, 2001, p. 27). Consequently I have sought to privilege the voices, actions and meanings of the teachers as the main data sources and ensure the reliability of the study through a robust process of data analysis leading to a valid set of conclusions.

One of the difficulties encountered in designing research studies that focus on secondary mathematics teachers' uses of technology is that, despite the wide range of technologies available, inspection reports and large scale surveys have concluded that there is limited use in classrooms of the type that fully exploits the potential of the available technology (see for example, British Educational Communication and Technology Agency, 2007, Office for Standards in Education, 2008, Office for Standards in Education, 2006). It is very difficult to identify a group of teachers who are consistently using any given technology and further problems arise in designing and funding longitudinal research into their trajectories of practice. By their nature they would be a disparate group and it would be time-consuming, managerially difficult and geographically challenging to research their personal learning concerning technology over a realistic timescale.

Consequently, there were two naturalistic features within both phases of my study. Firstly the teachers were developing and evaluating activities and approaches that used technology in their own classrooms with learners, as opposed to in a research environment. Although the selected technology was new to each classroom setting, it had to be assumed that the 'non technology' practices and classroom ethos were already established in each classroom. Therefore both the teacher's and students' responses would be naturalistic within the context of the introduction of a new technological environment.

Secondly, the responsibility to determine the curriculum content of the activities and approaches that were developed lay with the teachers themselves. The teachers were not all provided with the same set of pre-designed activities and approaches for them to use and evaluate, although the professional conversations that would ensue during the study would be expected to lead to opportunities for the teachers to learn about (and share) activities and approaches that used similar technologies in addition to developing new ideas.

My research approach was similar to that of 'design experiments in educational research' as described by Cobb et al (2003), which is consistent with the desire to arrive at an ontological innovation (diSessa and Cobb, 2004) as outlined in Section 1.3.

Cobb et al describe design experiments thus,

> *Prototypically, design experiments entail both engineering particular forms of learning and systematically studying those forms of learning within the context defined by the means of supporting them. (Cobb et al., 2003, p.9)*

My experiment was primarily concerned with the 'design and ecology' of teacher learning and my methodology is of the type and scope that Cobb et al describe as pragmatic, 'In-service teacher development studies in which researchers collaborate with teachers to support the development of the professional community' (ibid p.9).

The purpose of the research was to develop a class of theories about the process of teacher learning. This required me to adopt an interventionist approach by engineering situations in which I was both supporting the teachers to develop their uses of the technology, whilst also researching the underlying processes and resulting outcomes. The research was also prospective in that its design was implemented with a hypothesised learning process. I had my own research-informed idea of the nature of the activities that might use the MRT to support learners to make sense of mathematical situations concerning variance and invariance. The research was also reflective in that it fostered the emergence of other potential pathways for learning and development by capitalising on the research outcomes that arose as the research was underway. Examples of this occurred during the sharing of mathematical activities and pedagogical approaches between the participating teachers involved in the first phase of the research.

However, in my study there was not going to be a clear single 'product' in development, a commonly interpreted pre-requisite for a piece of design research in the form of a final design of an artefact or an exemplar set of materials. However, the teachers and I were all working together to further our understanding of the technology and its role and use in mathematics education. There would be some examples of iterative design as activities and approaches underwent 'cycles of invention and revision' (Cobb et al., 2003, p.8) through an initial design process, classroom implementation, review and sharing process with other teachers and, in some cases, re-design and re-implementation in another classroom setting. This could not be a planned aspect of the methodology as it would be dependent on how the initial ideas resonated with the individual teachers and whether they had an appropriate class of students with which they might be able to use that idea. In addition, the imposition on teachers of a set of ideas and approaches using the technology would have jeopardised the naturalistic aspect of the study. This could indicate a tension between the interpretation of a design experiment methodology as 'highly interventionalist' when working within the in-service education of a professional learning community where the participants need to have some autonomy within the decision making processes. To some extent, the responsibility for elements of the design became a shared activity between the researcher and the participants. This demanded a methodology in which the participants' views and decisions were made as explicit as possible to enable them to be central to the research data.

Finally, in accordance with the definition of a design experiment methodology in education described by Cobb et al,

> *Any theories developed from the research are humble not merely in the sense that they are concerned with a domain specific learning processes but also that they are accountable to the activity of the design. (Cobb et al., 2003, p.8)*

The research was phenomenological in its approach as it sought to explore one particular phenomenon, namely the trajectory of teachers' knowledge and practice concerning mathematical variance and invariance as they learned to use a multi-representational technological tool (Vandenberg, 1997). The study was situated within the English secondary school mathematics classroom setting, alongside teachers who were beginning to use a multi-representational technological tool.

4.3 Phase one: Introducing the MRT to the teachers

The first phase of the research included the introduction of the chosen MRT to the teachers involved in the study, which took place from July 2007 to November 2008. During this period, the teachers engaged in seven days of face-to-face support when they learned to use the different functionality of the technology and devised activities and approaches to use in their classrooms with learners. The teachers were introduced to the methodological tools that had been designed to support them to capture and describe their classroom outcomes, with an emphasis on how they were using the functionality of the MRT to provide a substantial learning environment in which their students might learn mathematics. Opportunities were provided for the teachers to discuss and present their classroom outcomes and, over the course of this phase of the research, share the emerging practices, which involved the considered use of the multi-representational environment.

Through the analysis of the Phase One data I anticipated that I would arrive at:

- a clearer understanding of the ways that the teachers used the multi-representational environment to emphasise different conceptions of variance and invariance in ways that demonstrated educational legitimacy for teachers within the secondary school mathematics context;

- an insight into what the process of teacher learning might involve, elicited from the teachers' perceptions of the lesson (as evidence by their lesson evaluation data);
- a decision concerning the validity of the pedagogical map (Stacey, 2008) within the setting;
- the identification of the teachers who would become the subjects of the research in its second phase.

In common with other teacher development projects, with and without ICT, there was a clear rationale underpinning the approach taken to establish the ethos and ways of working within the project (Ahmed, 1987, Watson et al., 2003). There was a clear sense of 'researching-with', which resulted from privileging the teachers' classroom stories and by supporting teachers to substantiate their own claims of changed mathematical experiences and outcomes on the part of their students.

4.3.1 Selection of the teachers for the study

Pairs of teachers were selected from seven English state schools that were part of the Specialist Schools and Academies Trust (SSAT) network. The review of literature had revealed that the teachers would have a better support network away from the face-to-face elements of the project if they had a colleague in their school who was also actively involved (Office for Standards in Education, 2008, Ahmed and Williams, 1997). However existing literature had also indicated that it would be important that at least one of the pair of teachers in each school had a level of confidence and competence with the use of technology in the mathematics classroom. Consequently, the initial selection of schools was based on information provided by colleagues in the SSAT and Teachers Teaching with Technology (T^3) networks.

Obviously, in agreeing to participate in a pilot project seeking to implement a new technology, the lead project teachers in each school demonstrated a sense of 'pioneer militancy' (Artigue, 1998), immediately placing this project in the category of similar studies, which sought to research ordinary teachers but, by their nature, only involved extraordinary teachers. In an attempt to mitigate this position, each lead project teacher was asked to involve a second teacher in their school and, in some cases, the second teacher was asked to participate because they said they lacked confidence with technology. This approach resonates with Ofsted evidence in relation to the development of ICT expertise in secondary mathematics department.

> *Where a department has at least one member who is a confident and enthusiastic user of ICT, this enables others to share their expertise and take the first steps; before long this becomes self-perpetuating and the department as a whole develops a momentum which benefits all pupils. In contrast, schools with limited facilities or resident expertise need external support, often of a sustained nature, in order for teachers to become effective users of a range of appropriate ICT mathematics applications. (Office for Standards in Education, 2008)*

The head teachers from each of the selected schools signed an agreement, which outlined the expectations of both the school and the teacher involved (Appendix 1). At this stage, I did not make it explicit to the teachers that my specific research focus concerned how they made sense of the concepts of mathematical variance and invariance, as I did not want to jeopardise their engagement in the project. There was a danger that, if the teachers felt that they were being treated as research subjects, they would not perceive a personal relevance for them in their involvement and this might have caused them to be de-motivated or less engaged. I felt confident that I would be able to embed my own research within a broader project concerning the mathematics curriculum holistically in the spirit of a more naturalistic enquiry. Whilst this could be interpreted as a lack of focus or weakness in the research design, I would argue that it strengthened the approach as the teachers engaged fully with the study. From their perspective it related strongly to the National Curriculum for secondary mathematics, which they were all expected to be implementing from September 2008. In making this decision, it also enabled me to explore how a multi-representational technology, which facilitated the linking of algebra with geometry, would add to the existing research on MRT that had mainly concerned the algebraic and numeric domain, often focussing on the concept of function.

4.3.2 Process of teacher development

The teachers involved in the first phase of the project engaged in thirty hours of face to face professional development over seven days, the objectives of which were to:

- introduce them to the technical features of the MRT and develop their personal confidence and competence to use it in their classrooms;
- consider how these features might be useful to them in supporting students'

learning experiences in their mathematics lessons;

- enable the teachers to design activities and develop the necessary support materials, for example students' files and activity sheets;
- provide opportunities for the teachers to reflect upon the outcomes of their own and their students' classroom experiences with the MRT and use a range of research tools to support them to evaluate this.

The teachers were encouraged to behave as action researchers by adopting a cyclical approach to evaluating their classroom experiences of using the MRT with their learners. They were each asked to submit a minimum of five activities, which they felt exemplified their 'best' use of the MRT during first phase of the study.

4.3.3 Methods of data collection during Phase One

The qualitative nature of the study required a range of data collection instruments and strategies to be developed. In some cases these evolved in response to unplanned incidents or observations within the research as a direct consequence of an iterative design methodology. This section describes the data collection strategies used and links them to the research aims.

4.3.3.1 Teacher questionnaires

During Phase One four separate questionnaires were designed and administered at different times during the study.

At the beginning of the study, the teachers were asked to complete the questionnaire shown in Table 3-1 which was designed to obtain information about: their professional backgrounds and related qualifications and experience; their previous uses of technology for mathematics and the ICT resources for mathematics to which they currently had access, and how they used them.

Q1-1	Name
Q1-2	Date of Birth
Q1-3	What is the highest qualification you hold in mathematics? i.e. O level/GCSE, A level, degree
Q1-4	What teaching qualification do you hold? i.e. Cert Ed, PGCE, GTP
Q1.5	What subject and age range did you originally train to teach?
Q1.6	What year did you originally qualify to teach?
Q1.7	How many years teaching experience do you consider you have?
Q1.8	Are you currently studying or do you hold a Master's Degree in a mathematics education related area?

Q1.9	If so, please describe your study.
Q1-10	Are you are Member of any professional education bodies? i.e. the ATM or MA
Q1-11	If so, please state which one(s) and describe your level of activity. i.e. read journals, attend conferences, go to branch meetings etc.
Q1-12	If you are (or have been) involved in any ICT related projects, pilots or network groups, please describe them and your level of activity.
Q1-13	How would you describe your previous experience with handheld technology for learning mathematics?
Q1-14	Please be as specific as possible - include dates, CPD or projects you have been involved in, resources developed or used etc.
Q1-15	How would you describe your level of confidence to use handheld technology in mathematics lessons?
Q1-16	Please add any other background details about yourself that you think might be relevant to the project. i.e. My previous career was a software designer!

Table 4-1 Questionnaire 1: Identifying the teachers' backgrounds and experiences as relevant to the study.

The information that the teachers provided in their responses to Questionnaire 1 helped me to contextualise each teacher's actions and activity within the project in relation to their background, experience and starting points. Also, as the processes of designing activities using the MRT began, it would enable me to challenge the individual teachers to develop their mathematical and pedagogical approaches. For example, a teacher who had prior experience with the use of graphing calculators would be encouraged to develop activities, which built on this experience and consider how the explicit linking of representations or the use of different representations might complement previously developed approaches.

A second questionnaire was designed as a common structure for the teachers' to provide detailed descriptive accounts of their classroom uses of the MRT throughout Phase One of the study (see Table 4-2). Only the MRT activities which included a completed Questionnaire 2 were included in research data.

Name:	Date:
School:	Class (and year):
Approx NC level	Number of students:
Q2-1 **During this mathematics lessons my students used:** *(please circle)* Only handheld device Only software on a PC Handheld and Software Neither	Q2-2 **For the follow up mathematics homework my students used:** *(please circle)* Only handheld device Only software on a PC Handheld and Software Neither

Q2-3	What did you want the students to learn?
Q2-4	What activity did you choose (or develop)?
Q2-5	And what mathematical learning took place?
Q2-6	How did you introduce the activity?
Q2-7	What were students' initial reactions/questions?
Q2-8	Approximately how many of the students could develop strategies to fully pursue the activity with little or no guidance from you?
Q2-9	What, if any, guidance did you have to give to the other students? Please indicate how many students approximately needed additional guidance?
Q2-10	Give examples of the sort of interventions you made
Q2-11	How did the activity enable students to take more responsibility for their mathematical learning?
Q2-12	Give a brief summary of the students' work/conclusions
Q2-13	How did the idea influence further work?
Q2-14	What aspect(s) of the idea would you use again?
Q2-15	What changes would you make?
Q2-16	Any other observations...? pupils' comments...? other teachers' comments...?
Q2-17	In your view, did the use of the technology enhance the mathematical understanding of the students? If yes, what evidence would you use to support this?
Q2-18	Feedback from pupils *Did you enjoy the lesson?* *Did you have any problems using TI-Nspire?* *Did you have any mathematical surprises in the lesson?* *How was the lesson different for you as a pupil?*
Q2-19	

	Key process	**Example**
Representing	identify the mathematical aspects of the situation or problem	
	choose between representations	
	simplify the situation or problem in order to represent it mathematically using appropriate variables, symbols, diagrams and models	
	select mathematical information, methods and tools for use	

Analysing	make connections within mathematics	
	use knowledge of related problems	
	visualise and work with dynamic images	
	look for and examine patterns and classify	
	make and begin to justify conjectures and generalisations, considering special cases and counter examples	
	explore the effects of varying values and look for invariance	
	take account of feedback and learn from mistakes	
	work logically towards results and solutions, recognising the impact of constraints and assumptions	
	appreciate that there are a number of different techniques that can be used to analyse a situation	
	reason inductively and deduce	
Use appropriate mathematical procedures	make accurate mathematical diagrams, graphs and constructions on paper and on screen	
	calculate accurately, using a calculator when appropriate	
	manipulate numbers, algebraic expressions and equations and apply routine algorithms	
	use accurate notation, including correct syntax when using ICT	
	record methods, solutions and conclusions	
	estimate, approximate and check working	
Interpreting and evaluating	form convincing arguments based on findings and make general statements	
	consider the assumptions made and the appropriateness and accuracy of results and conclusions	
	be aware of strength of empirical evidence and appreciate the difference between evidence and proof	
	look at data to find patterns and exceptions	
	relate findings to the original context, identifying whether they support or refute conjectures	
	engage with someone else's mathematical reasoning in the context of a problem or particular situation	
	consider whether alternative strategies may have helped or been better	
Communicating and reflecting	communicate findings in a range of forms	
	engage in mathematical discussion of results	
	consider the elegance and efficiency of alternative solutions	
	look for equivalence in relation to both the different approaches to the problem and different problems with similar structures	
	make connections between the current situation and outcomes, and ones they have met before	
Q2-20		

Personal qualities	*Developing good work habits*	Being imaginative, creative or flexible	
		Being systematic	
		Being independent in thought and action	
		Being cooperative	
		Being persistent	
	Developing a positive attitude to mathematics	Developing a fascination for mathematics	
		Showing interest and motivation	
		Showing pleasure and enjoyment from mathematical activities	

		Showing an appreciation of the purpose, power and relevance of mathematics	
		Showing satisfaction derived from a sense of achievement	
		Showing confidence in an ability to do mathematics at an appropriate level	

Table 4-2 Questionnaire 2: Lesson evaluation form designed for Phase One of the study.

In designing Questionnaire 2, the response section Q2-19 was adapted from the draft National Curriculum for mathematics (Qualifications and Curriculum Authority, 2007), the justification for which is provided later in this chapter. The questionnaire response section Q2-20 was adapted from an early guidance document for mathematics which pre-dated the introduction of the first National Curriculum but offered a useful set of probes that acknowledged the students' personal qualities when engaging in mathematical activity (Her Majesty's Inspectorate, 1985).

Questionnaire 3 was designed and administered during Phase One to capture some of the more subtle information concerning the design of each of the teacher's TI-Nspire activities (Table 4-3).

Name:	
Lesson title and code:	
About your planning and preparation....	
Q3-1	If the activity was pre-constructed for this lesson, outline the story of its development *i.e. did you work with colleagues and/or mentors?*
Q3-1	If the TI-Nspire file was constructed as an outcome of the lesson, outline how you did this, or supported this to happen.
Q3-3	On reflection, was the planning and preparation for this lesson sufficient? If not, how could you have improved on this?
About the TI-Nspire activity itself	
Q3-4	Did the TI-Nspire activity include linked multiple representations? If so, what specifically did you link? (be specific to the activity please).
Q3-5	Did your TI-Nspire activity include curriculum content from more than one NC Attainment area? Which ones?

Table 4-3 Questionnaire 3: Probing the process of each teacher's activity design during Phase One of the study.

The final questionnaire designed during Phase One, Questionnaire 4 (Table 4-4), was administered as a task during one of the face-to-face meetings in which the teachers were asked to consider one classroom activity where they felt that the use of the MRT had enriched their students' mathematical experiences.

Representing mathematics	Analysing	Use appropriate mathematical procedures
Interpreting and evaluating	Personal Qualities	Communicating and reflecting

Table 4-4 Questionnaire 4: Task cards designed to support the teachers to understand the terminology of Q2-19 and Q2-20 when completing Questionnaire 2.

The six categories were accompanied by a set of statements (aligned with each category using colour coded card) and each teacher worked with a colleague involved in the project to identify those that they felt had been privileged within their chosen MRT activity.

For example, the category *Representing mathematics* was accompanied with cards stating:

- identify the mathematical aspects of the situation or problem;
- choose between representations;
- simplify the situation or problem in order to represent it mathematically using appropriate variables, symbols, diagrams and models;
- select mathematical information, methods and tools for use.

These categories and the supporting statements included identical text to that within sections Q2-19 and Q2-20 of Questionnaire 2 (Table 4-2). The teachers then recorded the evidence from the activity that they felt substantiated a claim.

Questionnaire 4 was designed in response to my realisation that, although the teachers were conscientiously completing Questionnaire 2 (Table 4-2), the depth of their responses to sections Q2-19 and Q2-20 was not detailed enough for me to begin to analyse them in the way I had originally anticipated in my methodology. This supported Polanyi's claim that 'we know more than we can tell' (Polanyi, 1966). My individual discussions with a few of the teachers had intimated that they were not completely confident with their interpretations of the terminology within this part of the questionnaire and so were reluctant to record anything. The subsequent design of Questionnaire 4 allowed them to identify a lesson and discuss in depth each of the Key Process statements in relation to that lesson. A group discussion at the end of the task elicited the particular terminology that had challenged the group and between us we were able to arrive at a better group understanding, including exemplification from familiar activities using the MRT.

Questionnaires 2, 3 and 4 were designed to gather as much information as possible from the teachers about the activities they designed using the MRT throughout the study. As the lessons during Phase One were not observed by myself, it was important that the responses to Questionnaire 2 included the finer nuances of the lesson and that the teachers were encouraged to submit supportive evidence in the form of their students' work, comments and feedback, in addition to the teachers' own post-lesson reflections. In this respect the data represented the teachers' perceptions of what had happened in their classrooms and, consequently, provided an insight into each teacher's underlying beliefs, classroom actions and reflections.

For example, behind each of the reported activities, the teacher would have made decisions about what mathematics they wanted the students to learn through the use of the MRT. In probing each teacher's thinking about the design of the activity and how they had introduced and developed it through the course of the lessons, it would be possible to gain a perspective on how the teacher perceived the mathematical content alongside their pedagogical approaches. I was particularly focused on eliciting evidence concerning their interpretations of variance and invariance in addition to evidence that might indicate some form of teacher learning through the activity design, implementation and evaluation processes.

4.3.3.2 Personal research journal

My personal research journal formed a significant evidence base within this study as it documented my decisions and reflections within the iterative design and described the incidental and unplanned interactions, which had implications on subsequent activities and interpretations. These notes recorded my own reflections relating to my progress through the research and, in doing so, highlighted the struggles within my own learning as I sought to overcome frustrations, identify issues and note successes. These research notes fully described my own personal journey. In addition I kept notes of my informal discussions with the teachers and the content of email exchanges between us. Extracts from my research journal provided additional data that enabled me to triangulate other data sources and give external validity to the research findings.

4.3.3.3 Teachers' lesson plans, pupil resources and software files

For each of the lessons designed by the teachers, there was a significant amount of supplementary materials, which comprised the data for that lesson. These materials might include some or all of the following:

- an activity plan in the form of a school lesson planning proforma or a hand-written set of personal notes;
- a lesson structure for use in the classroom (for example a Smart NoteBook or PowerPoint file);
- A software file developed by the teacher for use by the teacher (to introduce the activity or demonstrate an aspect of the activity);
- a software file developed by the teacher for use by the students, which would normally need to be transferred to the students' handhelds in advance or at the beginning of the lesson;
- a activity or instruction sheet developed by the teacher for students' use;
- students' written work resulting from the activity;
- students' software files captured during and/or at the end of the activity.

A summary of the actual reported data for each of the lessons submitted by the teachers during Phase One is included in (Appendix 3).

The teachers were encouraged to seek ways to capture the students work during and at the end of the lesson. However, during Phase One there was no easy method for them to do this, as the prototype teacher software only facilitated a maximum of four handhelds to be connected to the teacher's computer at any one time. If the teachers chose to use the software, they were more able to manage file distribution and collection. The teachers were also encouraged to obtain their students' feedback and views about the activities. This data was collected through questionnaires and interviews designed and implemented by the teachers, which they then reported to the study.

4.3.3.4 Teachers' presentations

There were three occasions during the first phase when the teachers were asked to prepare presentations in which they provided feedback on their classroom activities using the MRT. These sessions were audio recorded and provided additional data through which to triangulate other data sources. In particular, the presentations made during the final project meeting held at the end of the first phase provided an opportunity for the teachers to reflect more publicly on their professional journeys during this period.

4.3.3.5 Labelling and managing the data

A systematic approach to the data labelling was adopted throughout the study with each school given a letter code and each teacher identified by the two initials of their first and last name. The lessons were numbered in the order in which they were submitted by each teacher, which may not necessarily have been chronological.

For example data provided by Eleanor L of Coastway School would be preceded by CEL. The complete set of data codes is provided in Table 4-5.

School (Code)	Teacher	Code
Bishop Brown School (B)	Annie K	BAK
	Jeff J	BJJ
Coastway School (C)	Eleanor L	CEL
	Helena S	CHS
Greenmount School (G)	Amie S	GAS
	Becky A	GBA
	Robert E	GRE
Hightrees School (H)	Rosalind G	HRG
	Andrea H	HAH
Lime Street School (L)	Malika F	LMF
	Sam M	LSM
Pierview School (P)	Carla T	PCT
	Sophie H	PSH
Stadium School (S)	Tim P	STP
	Judith K	SJK

Table 4-5 Data codes for each school and teacher adopted throughout the study.

In addition, the range of different types of data that was contributed to the study necessitated a second level of labelling to be developed. The data labelling scheme adopted throughout the study is shown in Table 4-6.

Data source	**Code**	**Example in thesis**
My personal PhD Journal, which included: meeting notes; lesson observation notes; personal correspondence; ongoing thoughts and reflections and research action plan.	Journ	[Journ]
Student's lesson evaluation	Eval-S	CEL1(Eval-S)
Interviews with teacher	Int	CEL1(Int)
Teacher's plan for the activity	Plan	CEL1(Plan)
Teacher's response to Questionnaire *N* *i.e.* Teacher's lesson evaluation	Quest*N*-T	CEL(Quest*N*-T)
Screen capture of students' responses (by activity)	ScreenCapt-Activity*n*	CEL(ScreenCapt-Activity1)
Teacher's activity structure for use in the classroom (for example a Smart NoteBook or PowerPoint file);	Struct	CEL1(Struct)
Student's work (by activity)	StudWork-Activity*n*	CEL1(StudWork-Activity*n*)
Activity or instruction sheet developed by the teacher for students' use	Activity-S	CEL1(Activity-S)
Software file developed by the teacher for use by the students.	tns-S	CEL1(tns-S1), STP1(tns-S2) etc
Software file developed by the teacher for use by the teacher (to introduce the activity or demonstrate an aspect of the activity);	tns-T	CEL1(tns-T)
Transcript of the lesson audio recording	Trans	CEL1(Trans)
Video clip of activity	Video*n*	CEL1(Video1)
Digital image of the teacher's handwritten whiteboard notes.	Image*n*	CEL1(Image1)

Table 4-6 Data labelling scheme adopted throughout the study.

Throughout the study Nvivo8 software (QSR International, 2008) was used to support both the management of the data and its subsequent analysis.

4.3.4 My role as researcher/in-service teacher educator

As mentioned in Section 4.2 this was an interventionist study in which I had dual roles as both a researcher and an in-service teacher educator. Each role carried its own responsibilities of which I was acutely aware, and I sought throughout to maintain the objectivity of the research process alongside managing the participating teachers' expectations of me in supporting their development.

During the first phase, whilst my knowledge and experience of a range of technologies for mathematics education put me in a position that was ahead of most of the teachers involved in the project, I was wary of imposing my own knowledge on the teachers. I was aware of many of the issues and obstacles that teachers faced when developing their use of a new technological tool but also appreciated how there were opportunities for substantial learning when teachers took risks by trying approaches that were new for them. A simple example of this was an early tendency for teachers to attempt to translate traditional paper and pencil mathematical activities into those involving technology, resulting in activities that do not utilise the functionality in a way that might support learners to understand the underlying mathematics. Indeed exactly this scenario took place during Phase One of the study when Helena and Eleanor devised the activity *Triangle angles* [CEL2], which emulated a 'traditional' exploratory activity whereby students draw a series of different triangles on paper, measure the interior angles and tabulate and sum each triangle's interior angles to arrive at a generalisation for all triangles.

Figures 4.3.1 and 4.3.2 below show the screens of the activity they designed, which required the students to use the technology to measure the interior angles and then input these measurements into a spreadsheet page to observe the resulting angle sums for each of the four given triangles. The teachers were seeking to overcome the difficulties students often have with accurately measuring angles in the paper and pencil environment. However they did not anticipate the potential of rounding errors as their students measured the angles to a degree of accuracy they had fixed to one decimal place.

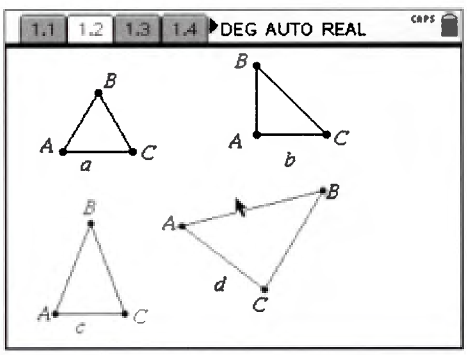


Figure 4-1 A set of triangles constructed within the dynamic geometry application [CEL2(tns-T)]

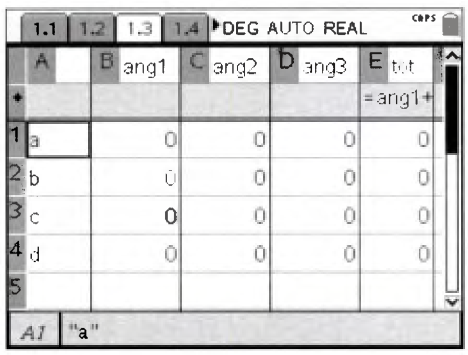


Figure 4-2 A blank spreadsheet page in which the angle measurements would be entered [CEL2(tns-T)page3]

So the transition to the technology had automated the angle measuring process but it did not take into account how, by drawing a single triangle and measuring its angles, students could have explored many different cases by dragging its vertices and observing the data, captured and displayed in the spreadsheet as shown in Figure 4-3 and Figure 4-4.

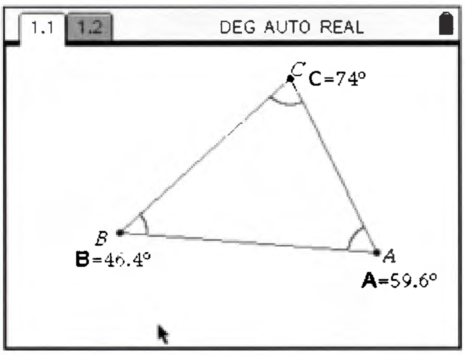
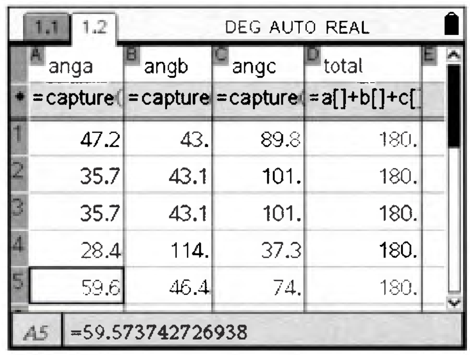


Figure 4-3 A dynamic triangle constructed within the Geometry application



	A anga	B angb	C angc	D total
•	=capture(	=capture	=capture	=a[]+b[]+c[
1	47.2	43.	89.8	180.
2	35.7	43.1	101.	180.
3	35.7	43.1	101.	180.
4	28.4	114.	37.3	180.
5	59.6	46.4	74.	180.

A5 =59.573742726938

Figure 4-4 The spreadsheet application showing the captured angle measurements and associated angle sums when the vertices of the triangle in the previous figure were dragged

However, I judged that, through the process of evaluating and reflecting on the original activity and considering alternative approaches to its design, the two teachers involved would experience significant learning regarding the adaptation of familiar activities within a technological context that they would be able to apply in future contexts.

In my role as the researcher, whilst I cannot claim to have held an equal stance, I worked hard to ensure the sense of 'researching-with' by developing a relationship with the teachers (and wider project group) that valued the teachers' unique contributions and used these to arrive at a non-judgemental view of how teachers' conceptions of variance and invariance had emerged through the project. There were features of the MRT selected for the study that presented new mathematical and pedagogical opportunities, so in some respect I was on a parallel learning journey with the teachers. However, these journeys had not started from the same position, nor were the learning velocities the same. However, by problematising the process of developing, teaching and evaluating the classroom experiences, we were all involved in a genuine professional discourse.

My experiences as a teacher researcher in previous projects (Stradling et al., 1994, National Council for Educational Technology, 1998) had undoubtedly influenced my own behaviours and actions during the study. I wanted to establish a culture within our project community where we were all actively involved in the research process whilst also engendering an atmosphere that valued open discussion, individual views and honesty. Although I clearly had personal goals embedded in the project, I did not want these to override those of the individual teachers. Consequently, I planned carefully each of the professional activities that we undertook, ensuring that I was clear about what we were doing, why we were doing it and how the outcome would support all of us to learn more about the situation. For example, the development of Questionnaire 4 (Table 4-4) in response to the teachers' reluctance to respond to Q2-19 and Q2-20 within Questionnaire 2 (Table 4-2). It would have been easy for me to provide a prompt sheet with more detailed descriptions and exemplifications but I felt it important that I was seen to be part of this struggle rather than having the answers at my fingertips.

Throughout the study, I saw myself as the bridge between the existing knowledge and research in this domain that would influence my own decisions, actions and interpretations whilst respecting the professional learning of the teachers involved.

4.3.5 Other support for the teachers during Phase One

During Phase One, the teachers also had access to technical, mathematical and pedagogic support from the wider project team, which included both staff from Texas Instruments' and consultants employed by them to stimulate teaching ideas and approaches appropriate to the secondary (11–16 years) mathematics curriculum. The project team were in agreement that the onus was on the teachers to seek out this support where needed, although during the face to face sessions, some whole-group activities were planned to introduce the teachers to new pedagogic approaches and technical functionality, for example the instrumentation processes of defining and linking variables within the different applications (see Section 2.4.7) or selecting and dragging objects within the different applications. At the initial project meeting when the whole project team had access to the handheld and software for the first time, one activity was designed that I used with the teachers to overcome some of the early instrumentation issues relating to moving between different pages and recognising the different applications. There is evidence that this activity, 'Four fours', was viewed by the teachers as a valid classroom activity to introduce their learners to the technology, since seven of them decided to use it as an introductory activity with their learners.

4.3.6 Process of data analysis during Phase One

The initial analysis of the Phase One data required the identification of activities in which the teachers had privileged the students' explorations of variance and invariance as an explicit act in the activity design or pedagogical approach. Having identified these lessons, it would then be possible to probe deeper into how this had been interpreted within the activity design and implementation from the teachers' descriptions.

The evidence for this was sought from the multiple data sources provided by the teachers concerning the activities and approaches they had developed. By contrasting the teachers' different mathematical interpretations and the resulting pedagogical implications, it was anticipated that a greater insight would be gained that could suggest a broader definition of the teachers' interpretations of the concepts of variance and invariance within the multi-representational technological environment. It was also anticipated that a number of teachers would be selected to participate in the second phase of the study on the basis of their response to this first phase.

The process involved collating and labelling all of the different data sources submitted by the teachers to the study for each activity. This raw data was imported to Nvivo8 where it was subsequently scrutinised and coded to elicit three elements:

- a broad description of the lesson;
- an inference concerning the teacher's interpretation of variance and invariance within the designed activity;
- the suggested instrument utilisation scheme.

So, for Judith's lesson 'Prime factorisation' [SJK1], the summary shown in Table 4-7 was produced.

Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
SJK1 Prime factorisation Students created a new file and used the *factor()* command within the Calculator application to explore different inputs and outputs to encourage generalisation. Students recorded their work on a worksheet prepared by the teacher.	 Q. What aspect(s) of the idea would you use again? *Definitely the worksheet idea as this enables pupils to work at their own pace. I needed the worksheet for me as well as for them. I was able to refer to the sheet and that helped my confidence. The sheet also allowed pupils to continue with the work whilst I went round to help students with a problem. BUT - the sheet could have been more structured i.e. not jump around haphazardly but be more systematic. Factor (1), Factor (2), Factor (3), etc... I liked Exercise 1 but questions 1 and 2 were too hard for this group. I was nervous to use the device even though I am a very experienced teacher of maths. I needed the worksheet as support. Having done one lesson I would now be confident to try again. The worksheet could have been more interesting. Pupils seemed to enjoy the lesson.* [SJK1(Quest2)]	Teacher had constructed a worksheet (with support from her mentor) that did lead students through a set of suggested input numbers that progressed in their level of complexity. **Variance** = changing the input number (manual text input to calculator application using factor () syntax) **Invariance** = all prime numbers had only two factors.	**IUS1**: *Vary a numeric or syntactic input and use the instrument's functionality to observe the resulting output in numeric, syntactic, tabular or graphical form.* **Private utilisation scheme**: Students worked with a numerical input and output within one application responding to the same set of numbers, as provided by the teacher. No use of multiple representations using the technology. **Social utilisation scheme**: Whilst JK developed this lesson in close collaboration with TP, who taught a similar lesson [STP1]. The lessons had very different utilisation schemes. JK provided a structured worksheet for the students that contained a variety of questions that did not appear to follow any conceptual progression.

Table 4-7 The summary data for Judith's activity 'Prime factorisation' [SJK1].

The complete summaries for all of the activities submitted during Phase One is provided in Appendix 4. A more thorough explanation of the presentation of this data and its implications for the subsequent data analysis process is provided within Chapter 5.

At the outset it was anticipated that this initial data analysis would enable the selection of a subset of activities that were judged to have involved the MRT in some way. However, a first examination of the data resulted in only nine of the sixty three reported lessons utilising the technology in this way, which led me to decide that the complexities of the instrumentation process on the part of the teachers made this an unrealistic expectation in this first phase of the research. For example, one activity was submitted [STP1] that used the calculator application to explore the output of the built-in command 'factor(n)' for different integer inputs. The underlying purpose to this activity was that the students would be encouraged to vary the inputs with the intention that they look for variance and invariance in the outputs, an activity that did not involve other mathematical representations other than the syntactical and numerical. However, it was a valid activity within the domain of the project as a 'first activity' for both teachers and learners within the project. For this reason I decided that it was not essential for activities to involve multiple representations as it would have excluded a number of activities that provided strong evidence of teachers' conceptions of variance and invariance within the multi-representational environment. In addition, the complexity of the selected technology meant that there was a range of applications available to the teachers. Hence it was predictable that the initial activities would focus on single applications whilst both the teachers (and their learners) were engaged in the instrumentation phase with the new technology.

The Nvivo8 data coding process required a close scrutiny of the teachers' written evaluations for the submitted activities. This led to the development of a section of the teachers' activity evaluation form (Questionnaire 2) around the draft curriculum statements concerning the mathematical *Key Processes* (Qualifications and Curriculum Authority, 2007), which are shown in Table 4-8. Each process has been coded to ease its identification purposes.

<table>
<tr><th></th><th>Code</th><th>Key process (KP)</th></tr>
<tr><td rowspan="4">Representing</td><td>KPR1</td><td>identify the mathematical aspects of the situation or problem</td></tr>
<tr><td>KPR2</td><td>choose between representations</td></tr>
<tr><td>KPR3</td><td>simplify the situation or problem in order to represent it mathematically using appropriate variables, symbols, diagrams and models</td></tr>
<tr><td>KPR4</td><td>select mathematical information, methods and tools for use</td></tr>
<tr><td rowspan="8">Analysing</td><td>KPA1</td><td>make connections within mathematics</td></tr>
<tr><td>KPA2</td><td>use knowledge of related problems</td></tr>
<tr><td>KPA3</td><td>visualise and work with dynamic images</td></tr>
<tr><td>KPA4</td><td>look for and examine patterns and classify</td></tr>
<tr><td>KPA5</td><td>make and begin to justify conjectures and generalisations, considering special cases and counter examples</td></tr>
<tr><td>KPA6</td><td>explore the effects of varying values and look for invariance</td></tr>
<tr><td>KPA7</td><td>take account of feedback and learn from mistakes</td></tr>
<tr><td>KPA8</td><td>work logically towards results and solutions, recognising the impact of constraints and assumptions</td></tr>
</table>

	Code	Key process (KP)
	KPA9	appreciate that there are a number of different techniques that can be used to analyse a situation
	A10	reason inductively and deduce
Use appropriate mathematical procedures	KPMP1	make accurate mathematical diagrams, graphs and constructions on paper and on screen
	KPMP2	calculate accurately, using a calculator when appropriate
	KPMP3	manipulate numbers, algebraic expressions and equations and apply routine algorithms
	KPMP4	use accurate notation, including correct syntax when using ICT
	KPMP5	record methods, solutions and conclusions
	KPMP6	estimate, approximate and check working
Interpreting and evaluating	KPIE1	form convincing arguments based on findings and make general statements
	KPIE2	consider the assumptions made and the appropriateness and accuracy of results and conclusions
	KPIE3	be aware of strength of empirical evidence and appreciate the difference between evidence and proof
	KPIE4	look at data to find patterns and exceptions
	KPIE5	relate findings to the original context, identifying whether they support or refute conjectures
	KPIE6	engage with someone else's mathematical reasoning in the context of a problem or particular situation
	KPIE7	consider whether alternative strategies may have helped or been better
Communicating and reflecting	KPCR1	communicate findings in a range of forms
	KPCR2	engage in mathematical discussion of results
	KPCR3	consider the elegance and efficiency of alternative solutions
	KPCR4	look for equivalence in relation to both the different approaches to the problem and different problems with similar structures
	KPCR5	make connections between the current situation and outcomes, and ones they have met before

Table 4-8 Key processes involved in learning mathematics as specified in the English and Welsh National Curriculum. The shaded processes indicate those of predominant interest to the study.

Whilst all of these key processes have a role in mediating the mathematical activities of the teacher and students, some of these processes resonate more strongly with mathematical activity within the context of engagement with a MRT. What follows is the justification for the key processes that have been shaded within Table 4-8 as they are of predominant interest to the study.

Within a multi-representational environment, the key process of representing mathematics is of fundamental importance. For example, within work on arithmetic sequences, the sequence can be represented as a list in a spreadsheet, be plotted as coordinate pairs or generated using the built in command 'Seq'. Depending on this choice of representation, the resulting mathematical opportunities for both the teacher and learners would differ. Within the context of the research study, the teachers would make the initial decisions regarding the design of the activity and, based on this design or intention, the students would have the opportunity to interact with the activity within the multi-representational environment in a number of ways. In order to engage with the mathematical scenario, the student would need to be able to identify the mathematics of the problem, albeit by focussing in on the important or relevant representation. Within a multi-representational environment, this initial engagement is crucial to the student being able to engage with the mathematics in the way that the teacher intended. The extent to which the students would engage in Key Processes KPR2, KPR3 and KPR4 would be highly dependent on the teachers' plan for the progression of the activity and the amount of autonomy they gave to their students. For example, although an activity might use multiple representations in its design, if the teacher had not provided the students with the opportunity to make choices, then it would not be possible to conclude that the students had engaged in such decision making. However, from the research perspective, this would provide some rich topics for discussion with the teachers as they began to engage in the terminology and vocabulary of the draft curriculum document and make sense of it in relation to their own students' mathematical experiences.

The process of initially engaging in a mathematical activity through KPR1 is closely linked with that of analysing the mathematical context or situation provided by the teacher within the multi-representational environment. Depending on the nature of the activity and the accompanying guidance or instructions provided by the teacher, the students would then respond to the activity in a way that could involve all of the processes of analysis as listed in Table 4-8.

The teachers' individual activity designs evidenced their expectations with respect to their students' interactions with the MRT, with respect to the extent to which students would engage in the key processes. For example, if the teacher had designed or selected an activity, which required the students to manipulate and investigate a dynamic image with a view to identifying patterns, both KPA3 (visualise and work with dynamic images) and KPA4 (look for and examine patterns and classify) would be inherent in the students' response to the activity. The wider research literature resoundingly concludes that technology has a fundamental role in facilitating KPA7 (take account of feedback and learn from mistakes) and this study was particularly concerned with how the MRT mediates this from the teachers' perspective, both in designing activities and making sense of their students' responses to activities. On a personal note, I am uneasy with the phrase 'learn from mistakes', which holds a negative connotation with respect to the experience of learning mathematics. I totally accept that, within the process of formal mathematical assessment, mistakes exist as notable occurrences of mis-alignment with the expected response. However, within the process of learning mathematics, students' ideas can be 'in formation', 'half-baked' and not yet aligned with the accepted mathematical conventions and rules. Consequently I would be more comfortable with KPA7 if it were reworded as 'take account of and learn from feedback'.

4.4 Ethical considerations

In addition to the project agreement, which clearly set out the expectations for the involvement of each school and its teachers, the individual teachers also signed an ethical agreement in accordance with the guidelines published by BERA (British Educational Research Association, 2004). A copy of this is provided in (Appendix 2). The research was approved in accordance with the University of Chichester Ethical Review guidelines in June 2007 and, as such, satisfies the Institute of Education's own ethical approval process (Institute of Education, 2006).

4.5 Summary

In this section I have provided the theoretical approach for the methodology adopted by the research and detailed the methodology adopted in the first phase. To reiterate, the first phase of the study was primarily designed to lead to the following outcomes:

- a clearer understanding of the ways that the teachers used the multi-representational environment to emphasise different conceptions of variance and invariance in ways that demonstrated educational legitimacy for teachers within the secondary school mathematics context;
- an insight into what the process of teacher learning might involve, elicited from the teachers' perceptions of the lesson (as evidence by their lesson evaluation data);
- a decision concerning the validity of the pedagogical map (Stacey, 2008) within the setting;
- the identification of the teachers who would form the subjects of the research in its second phase.

In addition, it was fully anticipated that there would need to be a redesign of the methodology adopted in this first phase to take account of the need to closely observe a sub-set of the teachers in Phase Two. This would necessarily involve classroom observations and individual interviews. The way in which the Phase Two methodology was developed and the justifications for the decisions taken are included in Chapter 5 which follows.

5 Outcomes of Phase One and the design and methodology of Phase Two.

5.1 Introduction

This chapter provides a brief analysis of the first phase of the research, its main outcomes and how these informed: a deeper understanding of teachers' conceptions of variance and invariance; a greater insight into process of associated teacher learning; the establishment of the validity of the pedagogical map (Stacey, 2008) within the setting; and the identification of the teachers who would form the subjects of the research in its second phase. The chapter concludes by describing how it was envisaged that the data analysis process during the second phase would enable the research questions to be more fully addressed.

5.2 Description of Phase One of the research

During the first phase of the study the teachers developed and evaluated their classroom uses of the MRT with learners aged 11-16 years. This was carried out with support from the wider project community and the teachers were encouraged to exchange emerging activities and approaches. As described previously it was anticipated that the data provided by the teachers would give a clearer understanding of the activities that had educational legitimacy within the classroom setting and contain examples in which the teachers privileged explorations of variance and invariance within the multi-representational environment. The insight gained from the analysis of the selected lessons would also support the development of a broader understanding of the teachers' interpretations of variance and invariance. The data analysis process provided an important opportunity to describe some of the emergent instrument utilisation schemes, which would be probed further during the second phase.

In addition, I trialled the use of the pedagogical map as a means to illuminate aspects of the teachers' uses of the MRT in the classroom context. I was very aware that in order to be able to achieve the aims of the research and articulate a deeper understanding of the nature of the object within the revised triad of instrumented activity (Figure 3-2) I would be required to develop some intermediary tools and techniques.

As I began the data analysis process, whilst the evidence began to emerge concerning the teachers' perceptions of mathematical variance and invariance, another obvious trend in the data also became apparent. Within their lesson evaluations, reflective writings and conversations the teachers made noticeable comments about their own learning. They were describing much broader observations, incidents and reflections that had come about through their uses of the technology in the classroom. In my role as a teacher educator, I had a moment of realisation as I hypothesised that it was most likely that these instances would influence the conceptions that the teachers were evolving concerning the role of the technology within their secondary mathematics classrooms. Consequently, I began to consider how it might be possible to research this element of teachers' learning more closely during the second phase of the study.

5.3 Analysis of the Phase One data

During the first phase of the research the fifteen teachers reported sixty-six activities, all of which had been used in the classroom to support students' mathematical learning, mostly within single one hour lessons. A very small number of the reported activities spanned more than one lesson. Also, as many of the teachers planned the lessons together, the same lesson activities sometimes occurred more than once. Some teachers worked as a 'school pair' to evaluate the classroom activities and, particularly in the early stages in the project, the wider project team supported this process. They did this through a deeper questioning of teachers' initial responses to Questionnaire 2 (Table 4-2) and by asking the teachers to substantiate their claims by providing tangible evidence, particularly in relation to how the students were engaging with the technology during the activities. The data for each lesson was imported into Nvivo8 software and initially, the National Curriculum mathematical processes identified to be most relevant to the study (Table 4-8) were used to highlight the evidence concerning each of these key processes, an example of which is shown in Figure 5-1.

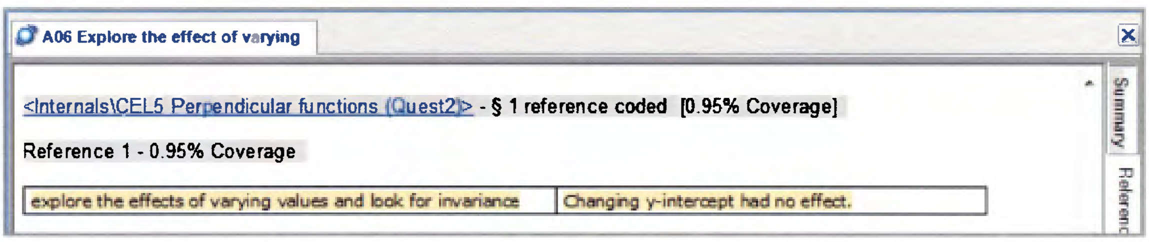


Figure 5-1 An example of an evidence source for Key Process A06, 'Explore the effects of varying values and look for invariance' taken from Eleanor's activity 'Perpendicular functions' [CEL5]

Taking this particular code as an example, in some cases it was very easy to identify the teacher's intention to focus the activity towards an exploration of variance and invariance as, in devising the activity, the teacher had explicitly stated that this was one of the lesson objectives. For example, Judith wrote about a lesson that had focused on 'straight line graphs', stating,

> *I wanted the students to know about the straight line graph, y=mx+c - what effect changing the 'm' has on the straight line , keeping 'c' constant and what effect changing the 'c' has on the straight line, keeping the 'm' constant. Trying to get from them what the 'm' is and what the 'c' is. [SJK4(Quest2)]*

In other activities, although the teachers described the mathematical content of the lesson, they were less explicit about how the learners would use the technology to engage with the mathematics. Consequently, the teacher's interpretation of variance and invariance was gleaned from the implied instrument utilisation schemes as evidenced by the lesson data they had submitted to the research project.

For example, in her lesson 'circle theorems' [GBA5], Amie stated her lesson objectives thus:

> *I wanted them to discover two circle theorems – "opposite angles in a cyclic quadrilateral make 180" and "angles in the same segment are equal". [GBA5(Quest2)]*

Whereas in her description of her students' main activity she wrote,

> *I got the students to construct a cyclic quadrilateral in a geometry page. I used a spilt page layout to enable them to have a calculator in the other half of the page. I got them to measure each of the angles within the cyclic quadrilateral and store them as variables. This enabled them to move the quadrilateral and/or the circle around and see what happened to the angles. I asked them to calculate various sums each time they had adjusted the diagram and then note down what they noticed. [GBA5(Quest2)]*

Another complexity within this aspect of the data analysis was that, although Questionnaires 2 and 4 (Table 4-2 and Table 4-4) had been constructed to support the teachers to report on whether their designed activities provided their students with opportunities to explore variance and invariance, there were only a few teachers who responded consistently to this question. This did not mean that such opportunities had not existed within the reported activities, but it merely signified that the teachers had not decided to comment on that aspect specifically within their written evaluations. Consequently, the evidence was extracted more often from the instrument utilisation scheme for the activity rather than the teacher's explicit comments within their written or oral reflections.

It was appreciated that, within the context of any single activity or approach there would be a number of instrument utilisation schemes in use. These might include:

- the teacher's intended instrument utilisation scheme for the students' participating in the activity;
- the teacher's actual instrument utilisation scheme as evident from the introductory or demonstration phase of the activity;
- the students' many individual instrument utilisation schemes adopted in response to the activity.

As the latter two instrument utilisation schemes could only be elicited through systematic classroom observation, possibly supported by the automated recording of real-time actions, it was not possible to comment on these from the data collected during Phase One. Consequently, the instrument utilisation schemes that were elicited are implied from the teachers' descriptions of the mathematical activities and approaches as evidenced by the data they submitted to the study. They represented my objective interpretations of the teachers' intentions. As part of the data validation process, and in keeping with the ethical agreement, the teachers were invited to amend these descriptions and, where the teachers responded, their suggested changes have been incorporated.

For each of the activities and approaches submitted by the teachers during the first phase, a summary text was written which included:

- the lesson code and title and a brief description;
- relevant screenshots from the MRT;
- extracts from the data that evidenced aspects of the teacher's learning;
- a commentary on the inherent interpretation of variance and invariance;
- the implied instrument utilisation scheme.

The complete data set for the Phase One activities and approaches is provided in Appendix 4 and examples of the summary text relating to two activities submitted by Sam and Amie are shown in Table 5-1 and Table 5-2 which follow.

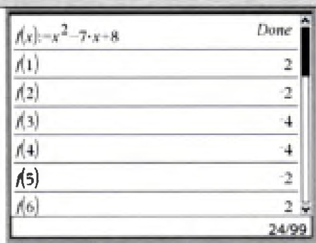
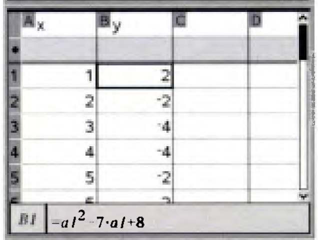
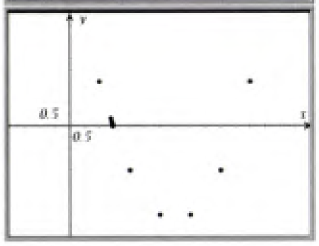
Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
LSM4 Trial and Improvement Students created a new file and used the Calculator application to define a given quadratic function using function notation. They then evaluated the function by substituting different values of x to try to find the solutions of the equation. Students then used a Spreadsheet page to automate the process. They then added a Graph and geometry page to construct a scatter plot of the values of x and f(x). Students recorded their observations in their exercise books.	*When students started to look for values to fill into their tables, it would have been better to tell them to use f(x) notation. In fact it was left open to students to find a method, but the f(x) idea was explained to a few and spread, which was inefficient. It could be that not suggesting anything would have led to interesting discussion about the efficiency of different methods, but in this case it would be important for the teacher to avoid giving any method.* *It was difficult to explain how to set up the spreadsheet in the middle of the lesson.* *Student data collection is vital – the table in the book ensured students could reflect on their values.* *Very clear activity instructions are vital for the smooth running of the lesson.* [LSM4((Quest2)]	**Variance** = the change in the evaluated y-values for different inputted x-values for a defined function. This was also shown in a tabulated form and graphical form. **Invariance** = that the x-value for the particular case where the y-value equalled zero – is called the 'solution'. This was observable as the point of intersection between the function and the x-axis within the graphing representation.	**IUS5:** *Vary a numeric or syntactic input and use the instrument's functionality to observe the resulting output in numeric, syntactic, tabular or graphical form. Use another representational form to add insight to or justify/prove any invariant properties.*

Table 5-1 The data analysis summary for Sam's activity 'Trial and improvement' [LSM4].

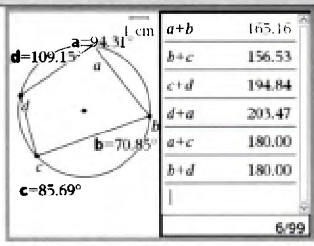
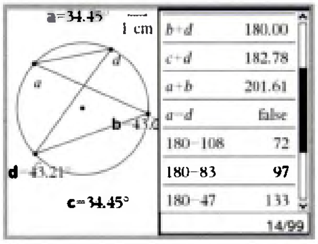
Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
GBA5 Circles theorems Students created a new .tns file and inserted a split page with a Graphs and Geometry (analytic window) alongside a Calculator page. They constructed a circle and an inscribed quadrilateral, measuring the four interior angles and saving the measurements explicitly as variables. They used the calculator page to check angle sums using the variable notation and made generalisations relating to their explorations. (In the subsequent lesson they explored the angle properties of a quadrilateral with three vertices on the circle's circumference and the fourth vertex positioned at the centre of the circle.)	 [GBA5 (tns-S2)]  [GBA5 (tns-S3)] *Getting the pupils to construct the situation and play about with it themselves is definitely a valuable activity that I would use again.* [GBA5((Quest2)] *I feel that my teaching of this topic has been enhanced by this activity. Although it could have been done on a geometry program in a computer room, the use of the TI-Nspire was less disruptive and as this class have used them before it was easier for them to access.* *I feel that the pupils will remember these circle theorems as they have had a practical lesson to discover them rather than being shown or told them and then simply applying them.* [GBA5(Quest2)] *Having discovered one circle theorem in the previous lesson they now seemed more confident in making generalisations about the angles that they had measured and any connections they were noticing. They were again able to move things around and see what happened to the angles.* [GBA5(Quest2)]	**Variance** = the size of the circle, the position of the vertices of the quadrilateral or 'bow' shape. **Invariance** = opposite angles of a cyclic quadrilateral sum to 180°; angles subtended on the same arc are equal.	**IUS9**: Construct a graphical or geometric scenario and then vary the position of a geometric object by dragging to observe the resulting changes. Save measurements as variables and test conjectures using a syntactic form.

Table 5-2 The data analysis summary for Amie's activity 'Trial and improvement' [GBA5]

Whilst coding the data for the selected key processes (Table 4-8), I also coded the data using a second set of thick codes: 'activity aims' and 'instrument utilisation scheme'. This was because I had begun to notice the more implicit evidence in these two areas that appeared to substantiate some evolution of teachers' practices coinciding with the project timeline. For example, over time the lesson aims appeared to be becoming more thoughtful, particularly around supporting the students' instrumentation phase, as did aspects of the teachers' implied instrument utilisation schemes with respect to exploring variance and invariance. A full analysis of the data concerning teachers' interpretations of variance and invariance (as evidenced by the activity design and implied instrument utilisation schemes) and the implications of this for the design of Phase Two of the study are presented later in the chapter in Section 5.3.1.

Finally, as I continued to scrutinise the data, a third thick coding category emerged, which I called 'teacher learning'. The data analysis process then took on a more grounded approach. A fuller description of this data category; its subsequent sub-analysis; and implications for the Phase Two methodology are provided in Section 5.3.2.

5.3.1 Designing activities that focus on variance and invariance – teachers' emergent practices

The initial lesson coding revealed that, of the sixty-six activities reported to the study, eight activities provided insufficient data to ascertain precisely how the students were expected to engage with the mathematics using the technology during the activity, or were judged not to have given students sufficient opportunity to explore variance and invariance during the lessons. For those with insufficient data, often the teachers had described the lesson aims and some of the outcomes but they did not actually report in enough detail the progression in mathematical activity in which they expected their students to engage, nor the way in which the technology was to be used (by themselves or their students). Consequently, the following activities were rejected from the Phase One data analysis process: [LMF6], [HAH2], [HAH5], [HRG1], [HRG3], [BAK2], [BJJ7] and [BAK10/BJJ10]

The detailed descriptions of the interpretation of variance and invariance from each of the selected activities (as exemplified by Table 5-1), and the subsequent analysis of these descriptions led to the nine implied instrument utilisation schemes that follow. The schemes are described with respect to the mathematical representation of the 'input', which were classified as being 'numeric', 'syntactic' or 'geometric'.

A numeric input might involve entering numeric values into a spreadsheet or changing an input for a numeric variable, for example the value of the fraction in [STP2] (see Appendix 6).

A syntactic input is considered to encompass both the syntactic forms of conventional mathematical notation in addition to the syntax required when using specific functionalities of the MRT. For example, the need to use function notation when generating a graph or to use the specific syntax of the built-in 'Factor' command. In this respect the word syntactic is not being interpreted in a wholly linguistic sense but it does embrace Shulman's sense of 'syntactic structures' (Shulman, 1986).

The classification of geometric inputs initially encompassed the 'positional' aspects of objects defined within both two-dimensional geometric representations and those with an underlying representational construct such as a Cartesian coordinate system. This was because these were both accessed from the same TI-Nspire application 'Graphs and geometry'. As I began to classify the nature of the 'outputs' representational system I initially used the same three categories. However, it quickly became apparent that the analysis became more informative if some sub-divisions of the initial three categories were made. Hence the 'numeric' category was sub-divided into 'measured', 'calculated' and 'tabulated' and the 'geometric' category was subdivided into 'graphical (data points)', graphical (function graphs) and geometric (positional).

The resulting scheme map in Figure 5-2 is used throughout the study to illuminate the flow of the representational input(s) and outputs of the mathematical objects under scrutiny in a diagrammatic form.

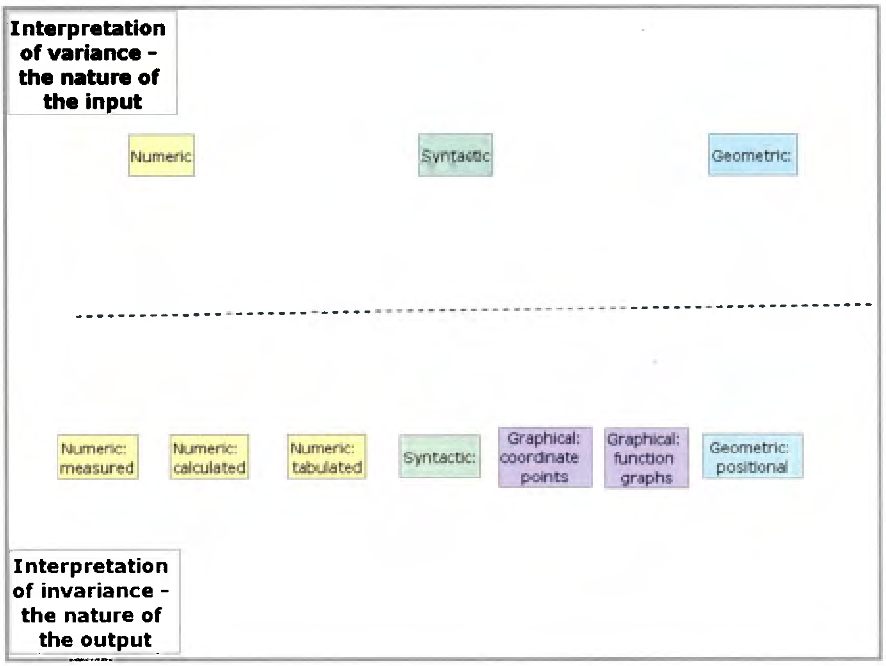


Figure 5-2 The IUS map developed to support the data analysis process, which enables the flow of representations through each activity to be illustrated

For example, in Eleanor's Phase One activity 'Triangle angles' [CEL2], her opening representation was a geometric one, from which the students were expected to make angle measurements (a numeric measured output), which they were then expected to enter into a pre-prepared spreadsheet (a numeric tabulated output). The corresponding Instrument Utilisation Scheme map is shown in Figure 5-3.

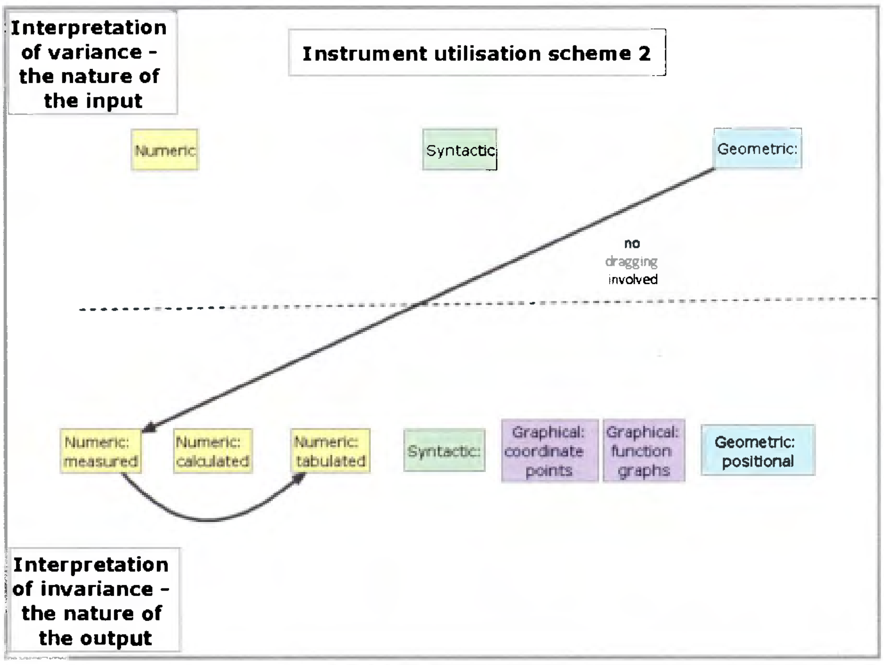


Figure 5-3 The Instrument Utilisation Scheme for Eleanor's activity 'Triangle angles' showing the flow of representational forms as indicated by the arrows [CEL1].

For each scheme that follows, an example from the Phase One research data is provided along with the activity codes for the other activities classified as being of the same type.

IUS1: *Vary a numeric or syntactic input and use the instrument's functionality to observe the resulting output in numeric, syntactic, tabular or graphical form.*

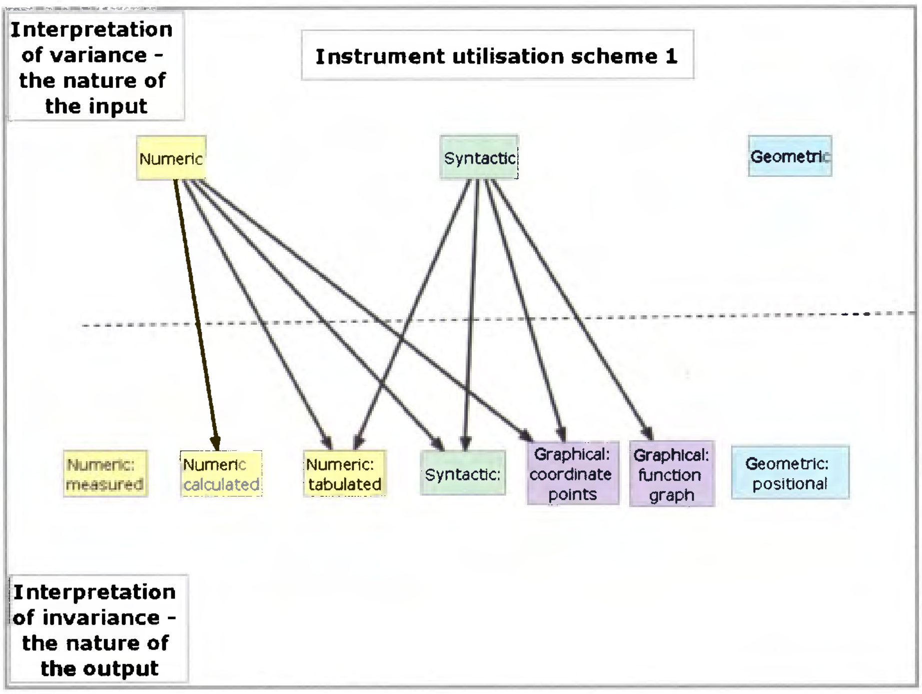


Figure 5-4 Instrument Utilisation Scheme 1 shown diagrammatically using the IUS map showing the flow of representational forms as indicated by the arrows.

Example: SJK4 – Students manually entered a function into the entry line of the Graphing application in the form 'm***x*** +c' with the aim of observing the invariant properties.

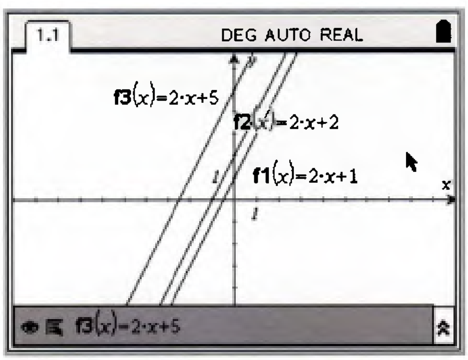


Figure 5-5 Exemplification of IUS1 from the activity [SJK4]

Other similar activities were: BAK1/BJJ1 BAK3/BJJ3 BAK5/BJJ5 CHS1 CHS3 GBA1 GBA2 GRE1 GRE2 HAH1 HAH3 HAH4 HRG2 LSM1 LSM2/LMF1 LSM3 LMF2 LMF3 LMF4 PCT1 PSH1 STP1 SJK1 SJK3 SJK5

IUS2: *From a given set of static geometric objects, make measurements and tabulate data to explore variance and invariance within the measured data in numeric and tabular forms.*

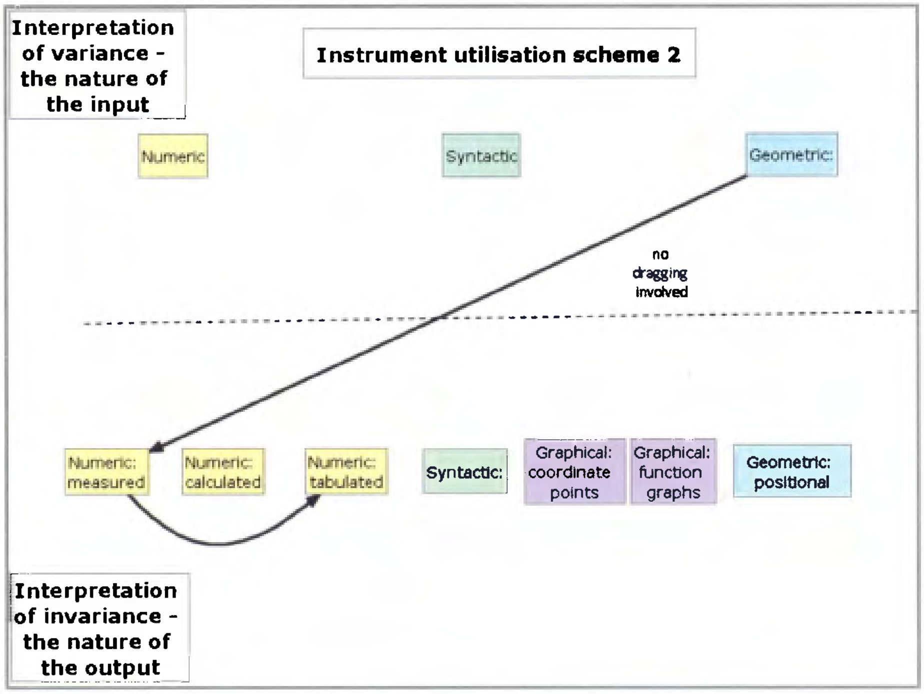


Figure 5-6 Instrument Utilisation Scheme 2 shown diagrammatically using the IUS map showing the flow of representational forms as indicated by the arrows.

Example: CHS2 - Students measured the interior angles of different triangles and entered the values into a spreadsheet, which calculated the angle sum.

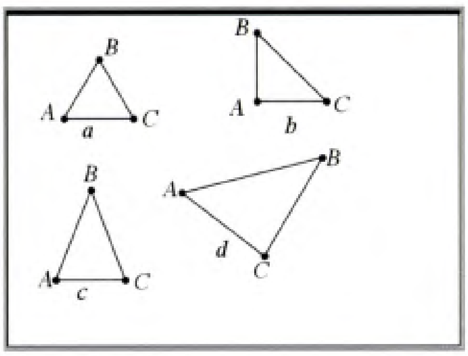
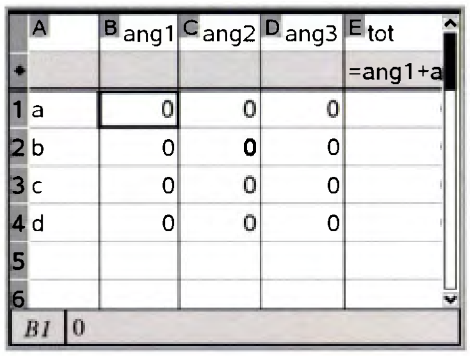


Figure 5-7 Exemplification of IUS2 [CHS2(tns-T)page1]



	A	B ang1	C ang2	D ang3	E tot
◆					=ang1+a
1	a	0	0	0	
2	b	0	0	0	
3	c	0	0	0	
4	d	0	0	0	
5					
6					

B1 0

Figure 5-8 Exemplification of IUS2 [CHS2(tns-T)page2]

Other similar activities were: CEL1 CEL2 PSH5

IUS3: *Vary the position of an object (by dragging) that has been constructed in accordance with a conventional mathematical constraint and observe the resulting changes. (Use another representational form to add insight to or justify/prove any invariant properties).*

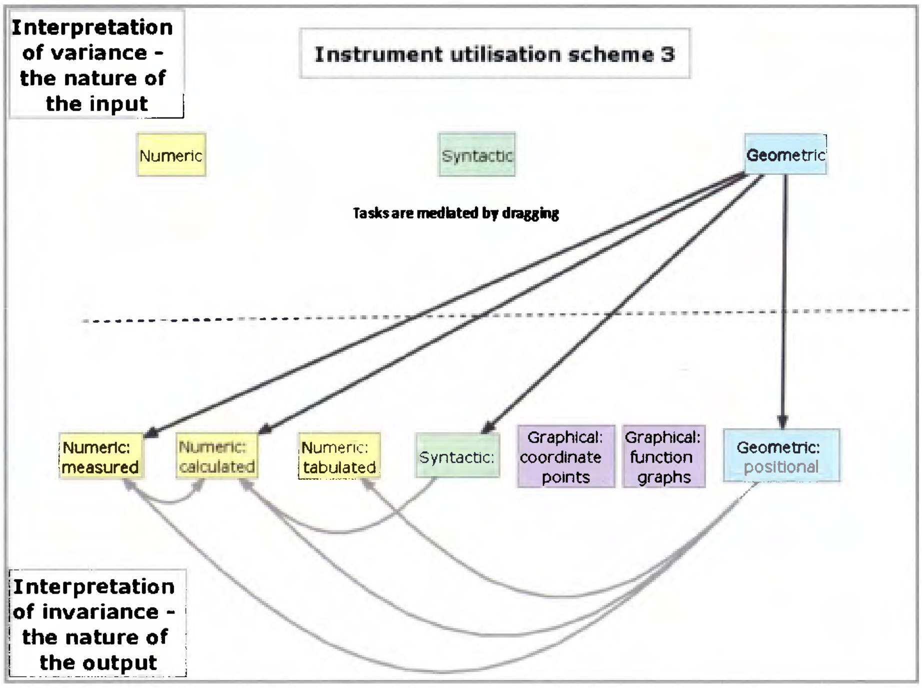


Figure 5-9 Instrument Utilisation Scheme 3 shown diagrammatically using the IUS map showing the flow of representational forms as indicated by the arrows. As not all activities involved a second output representation, the arrows on the diagram are shown in grey.

Example: PCT3 – Students dragged a specified point fixed onto the circumference of a given circle (of variable size) with the aim of observing invariant angle properties: equality of vertically opposite angles; sum of angles around the centre point.

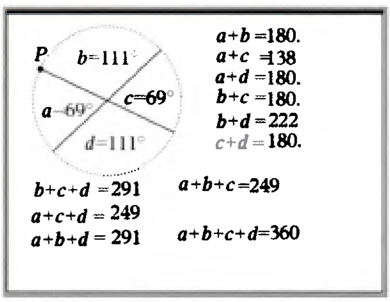


Figure 5-10 Exemplification of IUS3 [PCT3]

Other similar activities were: BAK4/BJJ4 BAK9/BJJ9 CEL3 CEL4 CHS4 PCT2 PCT3 PSH2 STP3 SJK2

IUS4: *Vary a numeric input and drag an object within a related mathematical environment and observe the resulting visual output.*

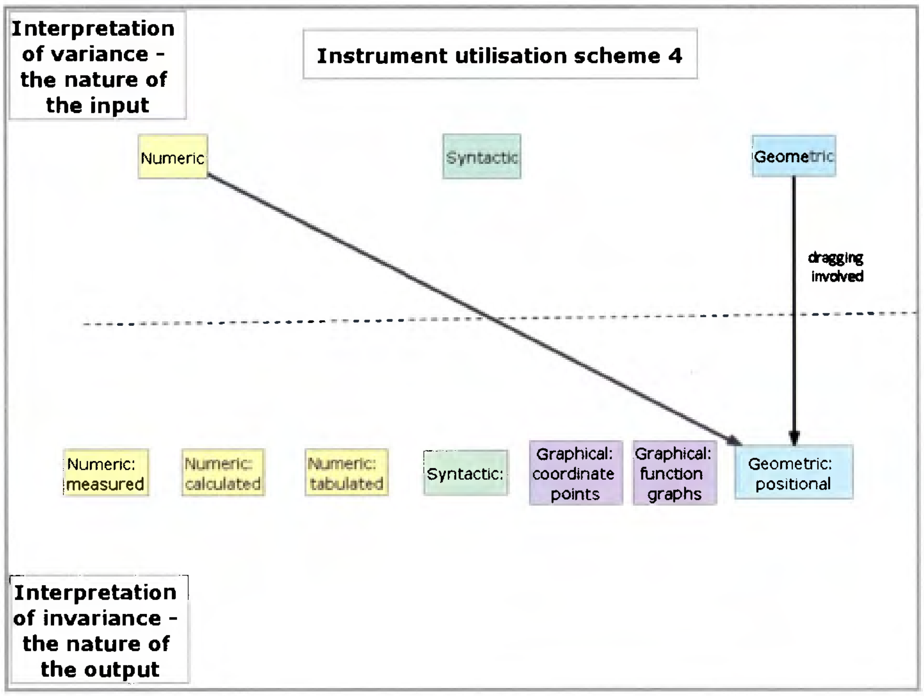


Figure 5-11 Instrument Utilisation Scheme 4 shown diagrammatically using the IUS map showing the flow of representational forms as indicated by the arrows.

Example: STP2 – Students input the numeric values of the numerators and denominators for two different fractions and the value of a common denominator and observed the resulting fraction pictures. Students then dragged a point on a line segment (from left to right) to simulate adding one fraction to the other and observed the resulting fraction picture.

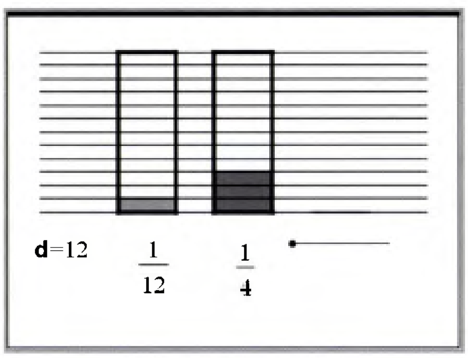


Figure 5-12 Exemplification of IUS4 [STP2]

Other similar activities were: STP5

IUS5: *Vary a numeric or syntactic input and use the instrument's functionality to observe the resulting output in numeric, syntactic, tabular or graphical form. Use another representational form to add insight/justify/prove any invariant properties.*

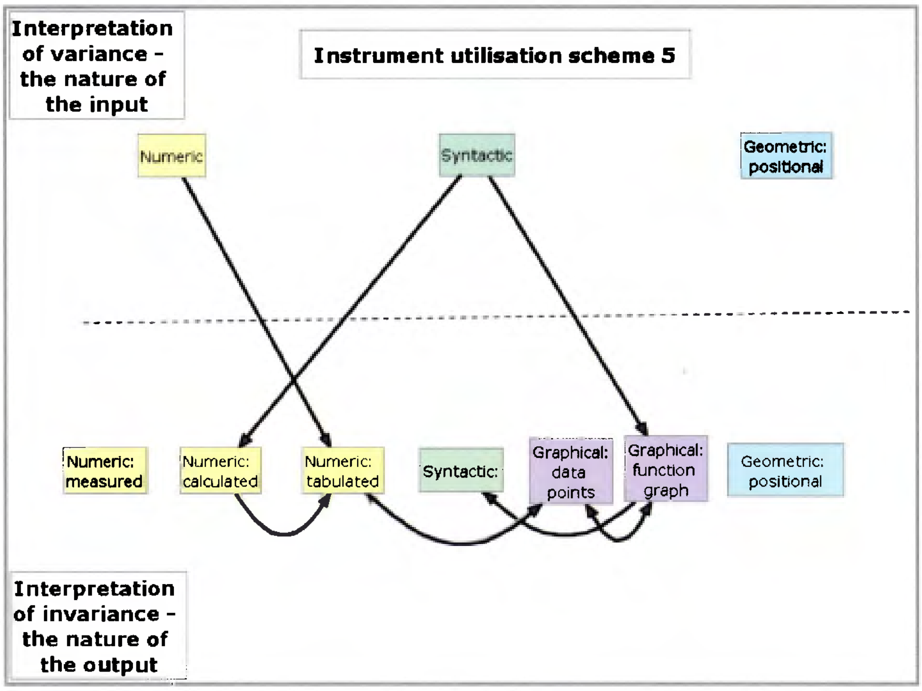


Figure 5-13 Instrument Utilisation Scheme 5 shown diagrammatically using the IUS map showing the flow of representational forms as indicated by the arrows.

PCT4 – Students defined a given linear and a given quadratic function and used the instrument's functionality to graph these and find the point(s) of intersection. Students then substituted the solutions using function notation within the Calculator application to ascertain their significance.

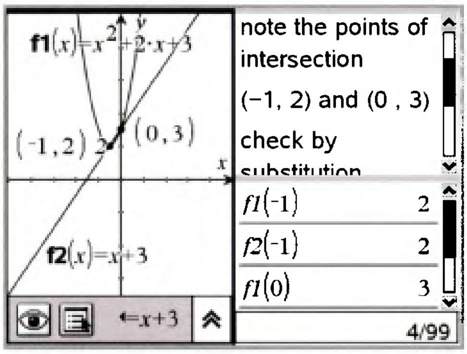


Figure 5-14 Exemplification of IUS5 [PCT4]

Other similar activities were: BAK6/BJJ6 GBA3/GAS1 GRE3 LSM4 PSH4

IUS6: *Vary the position of an object that has previously been defined syntactically (by dragging) to satisfy a specified mathematical condition.*

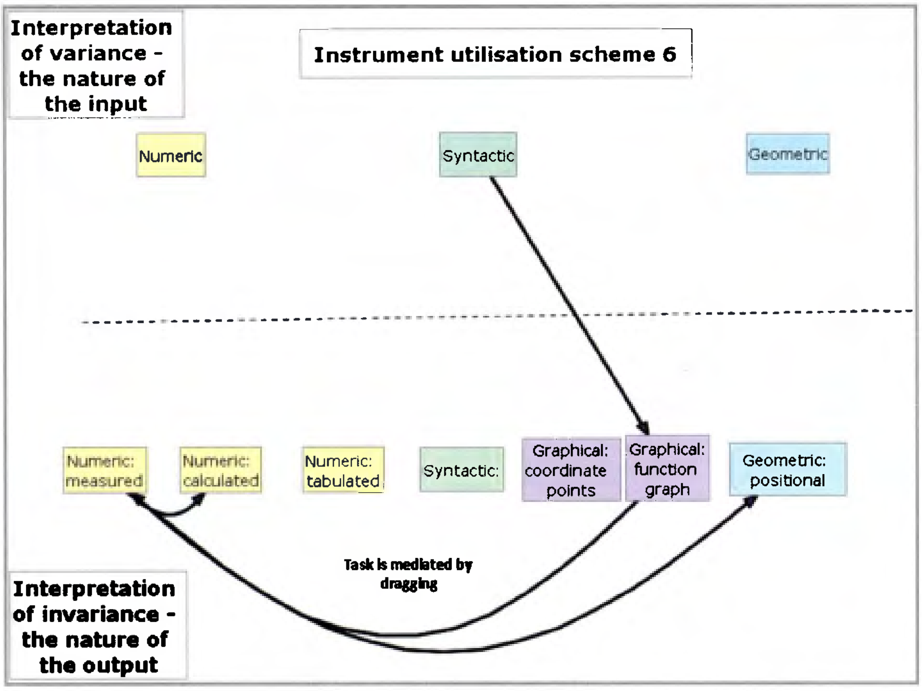


Figure 5-15 Instrument Utilisation Scheme 6 shown diagrammatically using the IUS map showing the flow of representational forms as indicated by the arrows.

PSH3 – Students entered two linear functions and measured the angle between them. They then dragged the second function such that it was perpendicular to the first and observed the resulting gradient properties.

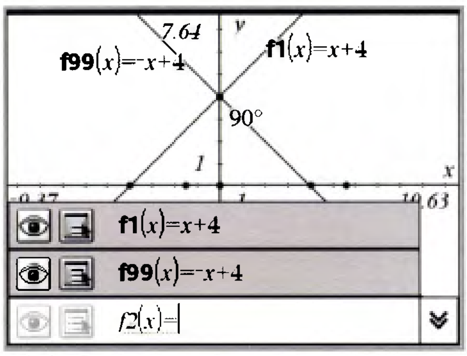


Figure 5-16 Exemplification of IUS6 [PSH3]

No other activities used this IUS.

IUS7: *Construct a graphical and geometric scenario and then vary the position of geometric objects by dragging to satisfy a specified mathematical condition. Input syntactically to observe invariant properties.*

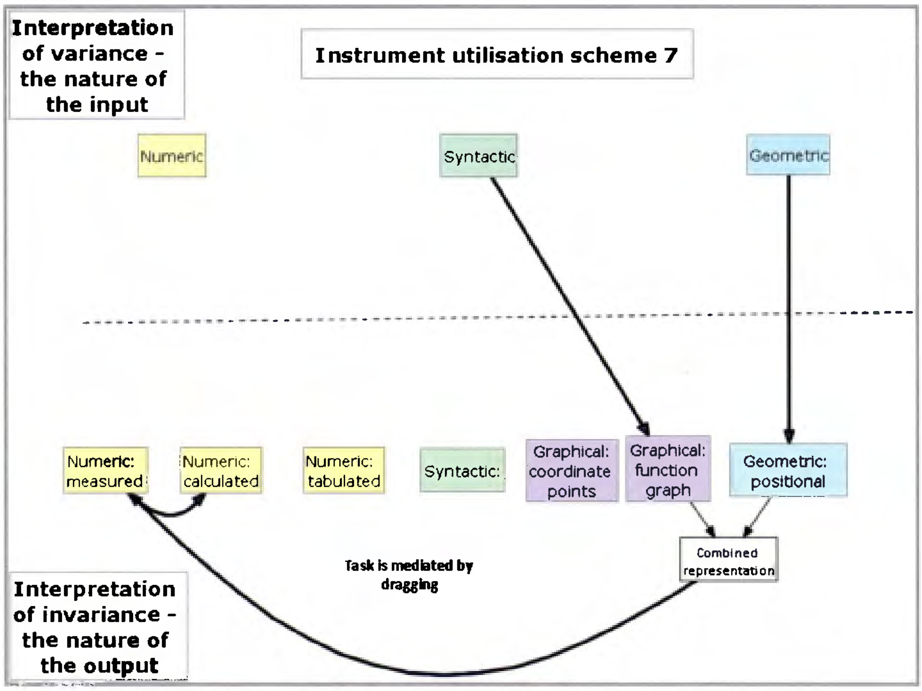


Figure 5-17 Instrument Utilisation Scheme 7 shown diagrammatically using the IUS map showing the flow of representational forms as indicated by the arrows.

CEL5 – Students defined a linear function in the Graphs application using f(x) syntax. They then constructed a geometric line that appeared to be perpendicular, measured the angle between the two lines and dragged the geometric line such that the measurement was close to 90°. Students then defined families of functions that appeared to be parallel to the geometric line and observed patterns in the resulting gradients.

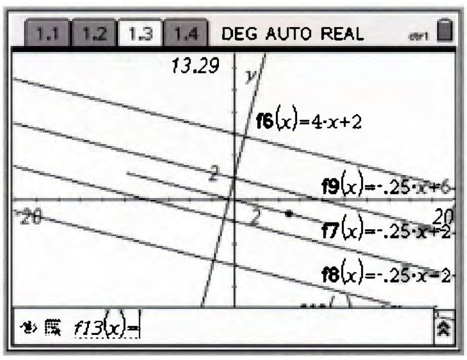


Figure 5-18 Exemplification of IUS7 [CEL5]

No other activities used this IUS.

IUS8: *(Construct a geometric scenario and then) vary the position of objects (by dragging) and automatically capture measured data. Use the numeric, syntactic, graphical and tabular forms to explore, justify (and prove) invariant properties.*

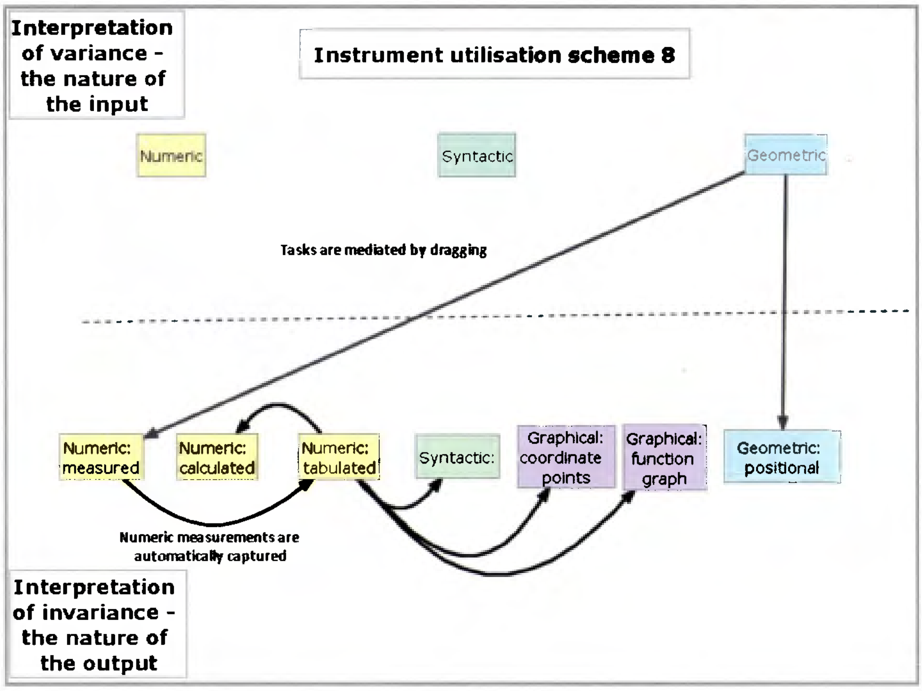


Figure 5-19 Instrument Utilisation Scheme 8 shown diagrammatically using the IUS map showing the flow of representational forms as indicated by the arrows.

LMF5 – Students constructed a circle and its diameter. They measured the lengths of the radius, diameter, circumference and area and used the data capture functionality to calculator application to calculate the ratio for different instantaneous values.

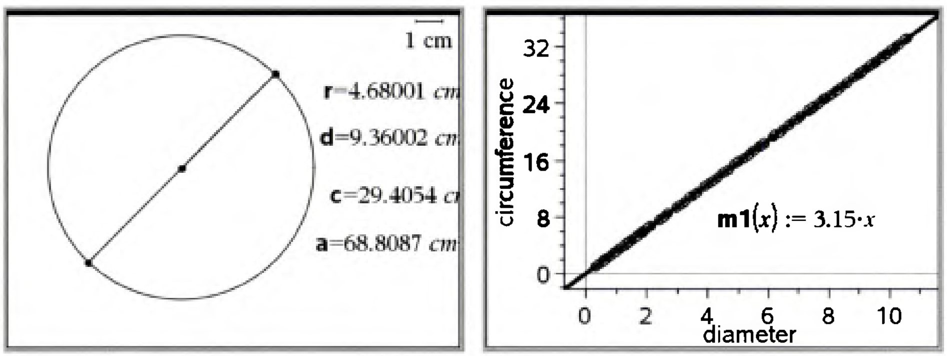


Figure 5-20 Exemplification of IUS8 [LMF5]

Other similar activities were: BAK8

IUS9: (*Construct a graphical or geometric scenario and then) vary the position of a geometric object by dragging to observe the resulting changes. Save measurements as variables and test conjectures using a syntactic form.*

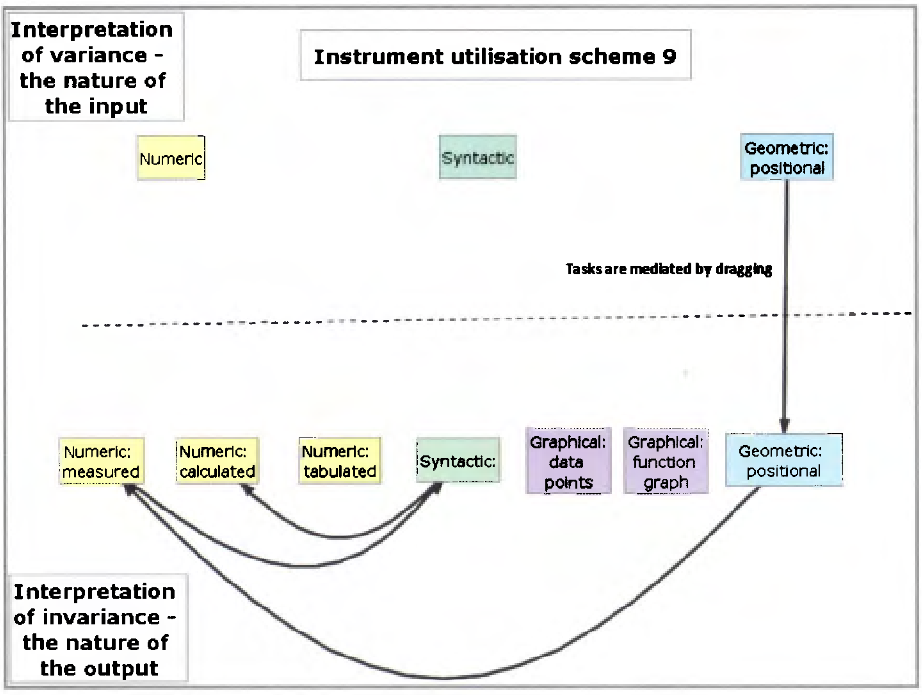


Figure 5-21 Instrument Utilisation Scheme 9 shown diagrammatically using the IUS map showing the flow of representational forms as indicated by the arrows.

Example: GBA5 - Students created a new .tns file and inserted a split page with a Graphs and Geometry page alongside a Calculator page. They constructed a circle and an inscribed quadrilateral, measuring the four interior angles and saved the measurements as variables. They used the calculator page to check angle sums using the syntactic notation and made generalisations relating to their explorations.

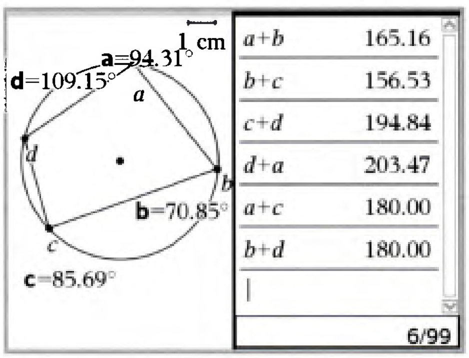


Figure 5-22 Exemplification of IUS9 [GBA5]

Other similar activities were: GBA4/GAS2

This analysis of the activities led to some very clear patterns in the data.

- The propensity for teachers, particularly in the early stage of the project, to use activities of type IUS1 was very high, with just under half of the reported lessons during this first phase adopting this approach. This implied strongly that, in the early stages of adoption of a complex new educational technology, the teachers' instincts were to familiarise themselves with its inherent functionality within one representational domain and then develop uses for identified elements of this functionality within the classroom setting.
- Some teachers did not develop instrument utilisation schemes outside of IUS1 for the duration of the project, although they expanded the range of applications and functionalities used.
- In some activities, following an initial approach of type IUS1, the output data was then used within another application to include another mathematical representation. This was defined as a new instrument utilisation scheme, IUS5.
- There were a few teachers who not only used a diverse range of instrument utilisation schemes during the project but actively sought to design activities and approaches that combined aspects of the tools functionalities in new and interesting ways.

The instrument utilisation schemes are presented in an order that intimates a sense of progression alongside the project timescale, but these cannot be interpreted in a linear way. A metaphor to help make sense of the way in which the instrument utilisation schemes evolved during the first phase of the project is one that relates to seed dispersal and germination. All except two of the teachers were present at all of the face to face meetings when activities and approaches were shared within the project group[1]. In that sense the seeds of the ideas could be considered to have been widely dispersed amongst the group. However, the rate at which these ideas were carried forward by the teachers varied. If the seed was successfully germinated, the idea or approach appeared in a later lesson evaluation from the teacher. The Phase One data analysis seems to support a hypothesis that for most teachers, the propensity of their later instrument utilisation schemes involving a diversification of earlier schemes was a function of time. If this was the case, it would strongly indicate that the timescale of Phase One was insufficient for the majority of the teachers to develop a wider repertoire of instrument utilisation

[1] A change in teaching staff at Greenmount School led to Becky replaced Robert in January 2008.

schemes with such a complex technology. On the other hand, these seeds may lie dormant for some time awaiting a future opportunity to be developed by the teacher.

In completing the Phase One data analysis, there were several observations that I made whilst trying to elicit and articulate an interpretation of variance and invariance within the teachers' activity designs. Firstly, in some activities it appeared to be a very straight forward process. For example in STP1 (see Appendix 6) the activity design was 'mathematically tight' in that the students were constrained to use limited technological functionality, but they were also 'mathematically free' to explore a range of numbers in an order of their own choosing. Other activities were more complex to decode in that, although the teacher's implied instrumentation scheme may have appeared to require the students to focus their attention and activity onto particular aspects of variance and invariance, the environment offered a mathematically wider landscape, which presented other elements of variation. For example, there were several reported activities that required the students to explore the effect of varying the gradient and intercept values for different linear equations ([SJK4] [HRG2] [BAK3] [BJJ3]). It seemed that the teachers generally accepted that, by doing this activity, their students would automatically 'see' the effect on the gradient and intercept as they input different functions whereas some teachers actually reported that their students noticed other variant properties.

In an example of students' work reported to the project, a pair of students (11-12 years) made the comments shown in Figure 5-23 and Figure 5-24 in relation to their exploration of varying the gradient and intercept [BAK3][2].

[2] The students at this school used the TI-Nspire software on their own laptop computers and not the TI-Nspire handheld devices, hence the different appearance of the screenshots.

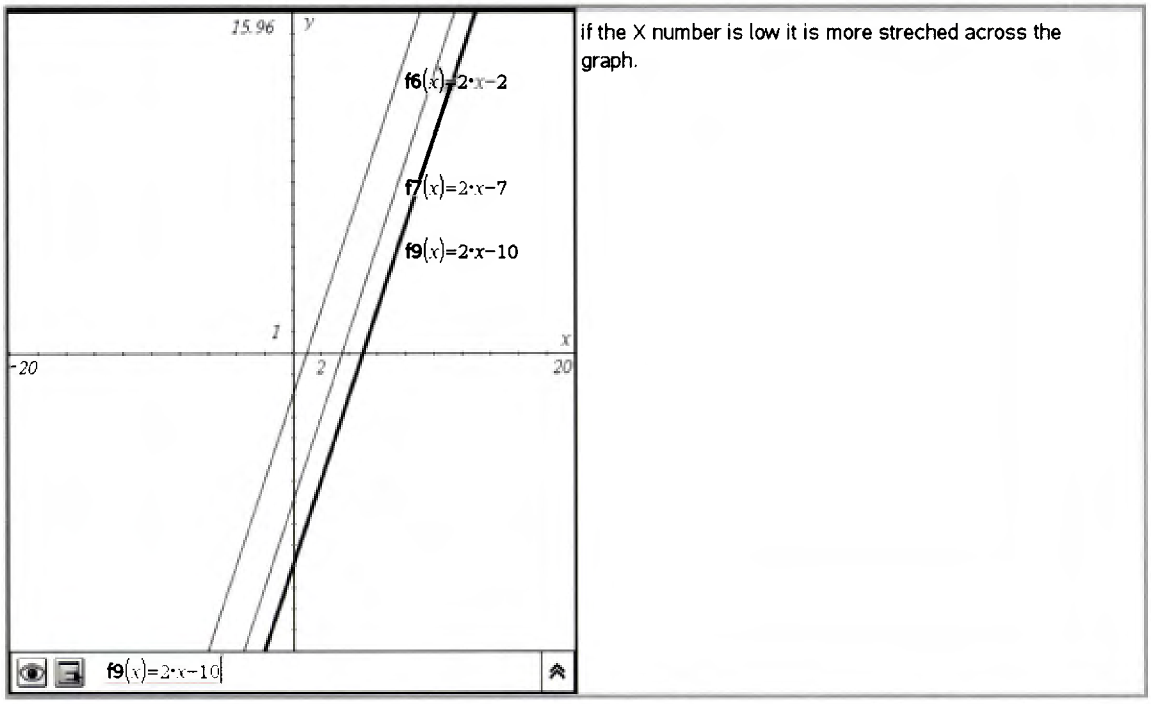


Figure 5-23 Students' comments about the gradient and intercept properties of straight line graphs [BAK3(tns-S)page3]

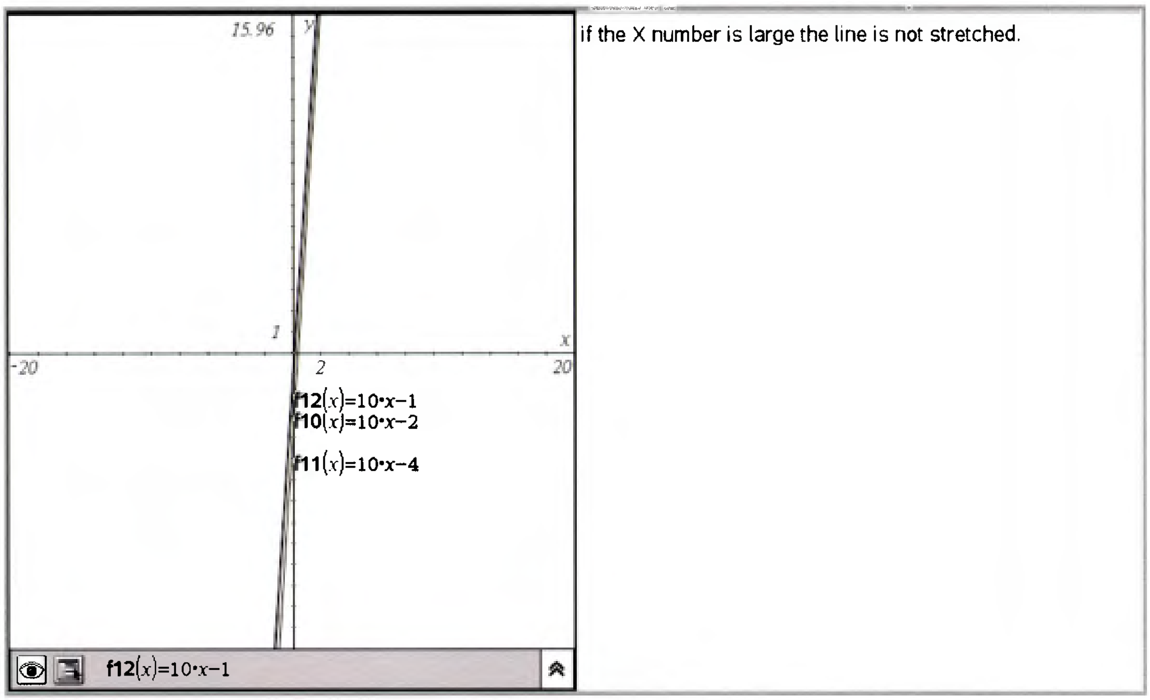


Figure 5-24 Students' comments about the gradient and intercept properties of straight line graphs [BAK3(tns-S)page4]

The students' description of the lines being 'stretched' offers a legitimate line of enquiry concerning their observation of variation. However, without having been present in the classroom to view whether the students' response was noticed by the teacher (or their peers) and if so, the nature of the discourse that ensued, I am unable to comment upon whether the role of technology in this scenario influenced the teacher's professional knowledge in any way. The need to research such situations more closely through systematic classroom observation was becoming increasingly apparent.

Staying within the context of activities that required students to 'vary m and c', another reported response was for students to notice variation in the lengths of the resulting lines. This offered another potential route for mathematical enquiry that could lead to a discussion concerning the meaning of gradient. However this opportunity did not seem to be construed as a 'legitimate' route of enquiry for the teachers concerned. There was a sense that the students' observations needed to comply with the 'intended learning outcomes' that the teachers had devised. The established social utilisation schemes for the use of technology within explorations of linear function are well-documented and the practice exhibited by the teachers in this particular context resonates strongly with the archetypal practices reported by Ruthven and Hennessey (2009).

A second insight that I made in relation to the analysis of the instrument utilisation schemes concerned the small number of teachers who were devising new conditions for the instrumentation of the technology. These offered what appeared to be innovative environments in which students were expected to explore variance and invariance.

A good instance of such a new approach was the lesson devised by Sophie, 'Perpendicular lines' [PSH3] previously included as an exemplification of IUS6 on page 110. In this activity she wanted her students to establish their own definitions of the gradient property of perpendicular lines so that they could achieve her desired lesson outcome, which was 'To recognize and generate the equations of lines parallel and perpendicular to given straight line graphs' [PSH3(Plan)]. In her description of this activity Sophie reported that she had instructed her students to use the Graphs and Geometry application and generate the function y=x. She asked them to investigate finding a perpendicular line to y=x and reported that her students were quick to respond with y=-x. She challenged with the question 'how can you prove to me it's perpendicular?' and, on their response that they wanted to measure the angle between the lines, showed the pupils how to do this. Traditional graph plotting technologies would not have facilitated the ability to switch from an algebraic approach to a geometric one. In this example, Sophie was using the knowledge she had gained about the available functionality of the technology to permit her students' request to make an angle measurement to be pursued. Having established the 90° angle measurement (and justified this according to the symmetry of the situation), Sophie then used a second aspect of the functionality to 'encourage pupils to then change f1(x) and f2(x) to avoid having to re-measure.' Finally, having asked the students to vary f1(x), Sophie showed them how they could vary f2(x) by dragging to get close to 90° and observe the displayed value of function f2(x). A static image of this is shown in Figure 5-25.

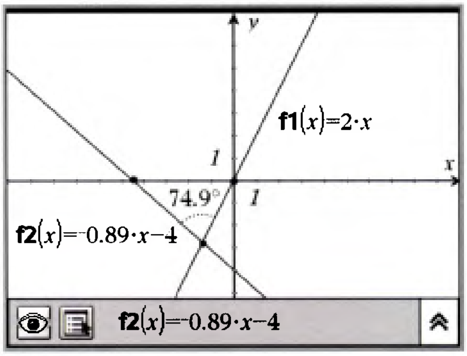


Figure 5-25 Sophie's activity design that required students to drag f1(x) and f2(x) such that they appeared perpendicular as a means to generalising about the gradient property of perpendicular lines.

This brought an empirical aspect to the activity, which in Sophie's evaluation, supported more of her students to be able to conjecture the relationship and express it in their own words. Sophie supported this claim by submitting an example of one of her student's written comments, 'You have to divide 1 by the amount of x's.' [PSH3(tns-S)]. This description of Sophie's activity design alongside her report of its outcomes provides an insight into the complexities of how she used her new knowledge about the technology and its functionality. She combined her desire for her students to construct their own meanings with her sound understanding of the underlying mathematics to create a new pedagogical approach, which had been afforded by the technology.

The analysis of the Phase One data had provided a substantial insight into the teachers' emergent instrument utilisation schemes concerning explorations of variance and invariance and it had also highlighted differences in individual teachers' responses to their involvement in the project. This influenced my selection of the teachers that would be involved in the second phase of the study, which is elaborated upon later in this chapter in Section 5.4.1.

5.3.2 Evidence of teacher learning – developing the research lens

As I became increasingly immersed in the data, I found myself becoming more aware of the teachers' responses in which they had expressed surprise or described unexpected outcomes relating to the activities. There was clear evidence that the evolution in their classroom practices with technology had spanned a wide range of mathematical content and pedagogical approaches.

In all, there were 268 thickly coded examples of 'teacher learning' from 100 data sources and, using a constant comparison data analysis approach, the individual examples were grouped into the 34 categories shown in Table 5-3 .

Category of teacher learning as evidenced in Phase One
TL - Amendment to students' IUS
TL - Amendment to teacher's introductory IUS
TL - Appreciation of 'fundamental' aspect of mathematics
TL - Appreciation of the need for a tighter focus on generalisation
TL - Balance of construction to exploration within IUS
TL - Better initial examples
TL - 'blend' of ICT and paper-pencil approach
TL - Classroom management issues
TL - Fundamental change to teacher pedagogy
TL - Good 'first' activity
TL - Importance of teacher demonstration re IUS
TL - Need for supporting resources for teachers and pupils
TL - Need for teacher to practise own skills - IUS
TL - Need to discuss machine v real mathematics
TL - New ways of organising classes for mathematical purpose
TL - Pupils enjoyed the activity
TL - Pupils working faster
TL - Software supporting classroom discourse
TL - Speed of student uptake
TL - Student reluctance to use technology
TL - Students' instrumentation issues
TL - Students' instrumentation successes
TL - Students' need for a big picture view of learning
TL - Students' need for teacher to renegotiate the didactic contract
TL - Students use of technical language beyond teachers' expectation
TL - Students working at own pace
TL - Surprise at students; ability to use technology
TL - Teacher observation of genuine student surprise and intrigue
TL - Teacher realisation of another way of learning
TL - Teachers' appreciation of improved student accuracy
TL - Trying a new approach
TL - Visual imagery provided scaffolding
TL - When not to use technology
TL - Whole class mediation – IUS

Table 5-3 The broad categories of teacher learning as evidenced by the analysis of

the Phase One data.

These broad categories were then grouped into four domains: Activity design; Expectations of students; Instrument utilisation schemes and Meta-level ideas. These four themes are described in Table 5-4. (The categories from Table 5-3 that were associated with each domain are listed in Appendix 5).

Domain of teacher learning	**Description**
Activity design	• the teacher's initial choice of examples, its appropriateness and potential for non-trivial mathematical exploration and extension; • the features of a good 'first activity' when introducing the technology or new instrumental techniques to the students; • the need to balance the construction and exploration elements within the design of activities; • teachers' reflections on their personal instrument utilisation scheme when introducing the activity to the students and during whole class discussions; • the relationship between the students' learning of relevant mathematical concepts with technology and the traditional 'by hand' or 'paper and pencil' approaches - and reflections surrounding the teacher's role in supporting students to connect these experiences; • an appreciation of the need for a tighter focus on or attention to specific mathematical generalisations within activities.
Expectations of students	• the pace at which the students worked with the technology, which was sometimes faster and sometimes slower than the teacher expected; • teachers' estimations of the students' existing knowledge and abilities and perceptions of how the technology had provided them with a deeper insight of this; • the fact that the students could be highly engaged, show enjoyment, genuine surprise and be intrigued when engaging with the mathematics using the technology; • the ease with which most students were able to use the technology when the teachers had carefully planned all aspects of the activity; • the instrumentation difficulties encountered by some students, which often accompanied students' reluctance to use the technology; • the intuitiveness of the mathematical language that some students used when describing unfamiliar mathematical contexts.

Domain of teacher learning	**Description**
Instrument utilisation schemes	• the importance of the role of teacher demonstration when introducing new instrumental techniques and the need to plan this carefully; • the students' experience, familiarity and confidence to use relevant features of the technology, and in particular, selecting and dragging mathematical objects, becoming familiar with syntax and moving between different representations; • the way in which the students evolved their own instrument utilisation schemes, which were not always what the teacher had expected or had demonstrated; • aspects of the mathematics, which promoted a need for the teachers to discuss 'machine' versus 'real' mathematics with the students; • general classroom management issues concerning the use of the technology.
Meta-level ideas	• the appropriateness of the use of technology within specific topics and activities, that is to say when to, and when not to use the technology; • the need for the teachers to have sufficient personal skills with the technology to enable them to accomplish the actual instrumental techniques as envisaged within their activity designs. (Sometimes they conceived approaches that were not actually possible); • an appreciation of the role that the technology may have had in enabling students to work more accurately; • their students' need for a 'bigger picture' of their mathematical learning beyond the single activity they were working on and which may have involved a renegotiation of the didactic contract; • the way in which the technology supported their learners to work at their own pace and to have more autonomy in their own routes of mathematical enquiry; • an appreciation of the role of generalisation as being a 'fundamental aspect' of learning and doing mathematics; • the way in which the different representations, in particular visual imagery supported students to scaffold both the mathematical ideas and the associated classroom discourse.

Table 5-4 The domains of teacher learning as evidenced by the analysis of the Phase One data.

The broad term 'Meta-level ideas' referred to the evidence of teacher learning that seemed to suggest a more fundamental shift in the teachers' thinking concerning the use of the technology within the domain of the study. For example, Jeff concluded at the end of the project that his experience had prompted him to fundamentally rethink his pedagogy concerning his teaching of geometrical constructions. He said, 'Even if I wasn't using the software, this would change the way I approached teaching this topic' and that he and his students had a greater appreciation of 'why we use our compasses in constructions - understanding the importance of circles in constructions.' [Journ] I was intrigued by his conclusion as he was referring to a self-identified 'successful lesson' that I had rejected from my Phase One data analysis as I did not feel there was sufficient data to conclude that the students had had opportunities to explore variance and invariance. However, this highlighted to me again the need for systematic classroom observations during the second phase. Without being present in the classroom, I was relying on the teacher's individual abilities to describe their lessons with a level of detail that articulated the students' mathematical processes. The evidence submitted by Jeff for this lesson did not indicate how the classroom discourse had enabled him and his students to appreciate the role of the circle within geometric constructions.

The breadth and depth of the emergent categories of teacher learning led me to carefully consider the design of the second phase of the study, to enable me to probe further the triggers for such teacher learning through the systematic classroom observation of a subset of the teachers. In particular, I was interested to observe the instances in which it appeared that the teachers were perturbed (in an epistemological sense) as a result of their uses of the MRT with their students.

5.3.3 Establishing the validity of the pedagogical map

The first phase of the study sought to explore the validity of Stacey and Pierce's pedagogical map within my setting. As previously discussed in Section 3.2.2 this map was an outcome of work with upper secondary students using CAS technology. Since my study was in the domain 'variance and invariance within a multi-representational technological setting', I had chosen not to focus on activities that privileged the scaffolding of by-hand skills, the use of real data or the simulation of real situations. Consequently I chose only to focus upon the aspects of the map that I have indicated by the pink shading in Figure 5-26.

THIS IMAGE HAS BEEN REDACTED DUE TO THIRD PARTY RIGHTS OR OTHER LEGAL ISSUES

Figure 5-26 The pedagogical map showing the pedagogical opportunities provided by mathematical software with the specific aspects on which my the study focussed indicated by the pink shading (Stacey, 2008).

The first phase of the project revealed rich data concerning teachers' perceptions of explorations of regularity and variation, which seemed to substantiate Stacey's claim that, from the Australian research, this was the most common form of mathematical activity with technology (Stacey, 2008). There was less evidence of activities that made use of opportunities to link representations. However, as the developing instrument utilisation schemes showed, this could be a function of both teachers' time and familiarity with the technology. On the other hand, for reasons for that are described later in this chapter, it would not be possible for me to pursue this hypothesis during the second phase of the study as I would no longer be working with the whole cohort of Phase One teachers.

Considering the 'Classroom layer', there were many examples where the teachers had articulated their perceptions of how both the social dynamics and didactic contract in their classrooms were being affected as a result of the use of the technology. For example,

> *The way the sums were given (a+b=180), encouraged much more generalised working. This helped to be able to generate conversations related to proof. [BAK4(Quest2)]*

> *They had not been shown how to construct a perpendicular line before,*

> *so were slightly hesitant when we begun using this. They also wanted to know why I didn't just use the rectangle tool – they didn't really appreciate that I was working on the definition, not just drawing a rectangle. [BAK9(Quest2)]*

As an analytical tool to support the descriptive analysis of future classroom observations I could envisage the two elements of the classroom layer (social dynamics and didactic contract) within the pedagogical map as providing a useful construct within the second phase of the study and I had sufficient data to convince me of its validity in my own setting.

Finally, reflecting on the resonance between my own analysis of the first phase data and the 'subject layer' of the pedagogical map, again there was substantial evidence from the teachers to indicate that they were taking advantage of the pedagogical opportunities that related to this layer. Teachers had begun to appreciate the anomalies and limitations of the machine mathematics, alongside their established views of the 'real mathematics'.

> *Some students needed help with regard to finding the range – getting the min and max values. Would a range function be useful? Possibly – though the conversation this generated was good. [BAK5(Quest2)]*

They had also reflected on their approach to learning certain topics in mathematics, and the order and emphases of skills concepts and applications within this. Fewer of the teachers had made comments relating to a sense of building metacognition and overview. However, I could see elements of this within the design of some activities and their associated instrumentation utilisation schemes. For example, the emergence of IUS3[3] did in some cases enable a broader mathematical landscape to be offered to the students than within IUS1[4].

5.4 Designing the methodology for Phase Two

My interest in the teachers' situated learning as they began to use the MRT in their classrooms with learners had been ignited by the detailed analysis of the data from the first phase. I had undoubtedly gained a greater insight into the nature of their learning and associated trajectories of development. The second phase of the research was designed to enable a much more focussed and systematic case study of the selected teachers through classroom observations and individual interviews

[3] IUS3 *Vary the position of an object (by dragging) that has been constructed in accordance with a conventional mathematical constraint and observe the resulting changes.*
[4] IUS1 *Vary a numeric or syntactic input and use the instrument's functionality to observe the resulting output in numeric, syntactic, tabular or graphical form.*

with a view to understanding more about the process of teacher learning. This enquiry remained centred on the teachers' development, implementation and evaluation of their lesson activities and approaches in their classroom. However some elements of the methodology evolved and new methodological tools developed to respond to the case study approach and my underlying justifications are provided in this section.

A multiple case study approach was used, as broadly defined by Denscombe (2003)

> *Case studies focus on one instance (or a few instances) of a particular phenomenon with a view to providing an in-depth account of events, relationships, experiences or processes occurring in that particular instance. (p.32)*

A limitation of the case study method is perceived to be the lack of reliability, meaning that an alternative researcher might reach different conclusions. To minimise the likelihood of this I chose to adopt a reflective reporting style as defined by Anderson and Arsenault (2002) in which 'the researcher's voice is heard throughout the report and data are woven into the story' (p. 159). I sought to triangulate my interpretations through the use of multiple data sources and the internal validity of the research was supported by incorporating a 'tight and interconnected path of recording evidence so that the reader who was not present to observe the case can follow the analysis and come to the stated conclusion'. (ibid p.159)

Case study methodologies, by their nature provide an in-depth insight into the same processes, namely teachers developing their use of the selected technology with the explicit aim to enable their students to explore mathematical variance and invariance. This militates against the ability to generalise the outcomes outside the particular instance. However, there were many elements in both the situation and outcomes of the research that resonated with existing research findings and it would be important for me to make the necessary connections in drawing together my research findings and communicating the outcomes of the research. Although this would not enable any form of statistical generalisability to be inferred, a resonance with existing research findings would contribute to the validity of my findings.

I envisaged that, by adopting a selection of the teachers as individual case studies, I would be entering into a partnership with them as described by Sutherland, who advocates a position where 'both teachers and researchers bring distinctiveness and complementarity to the knowledge building process' (2007, p.82).

5.4.1 Selection of the teachers for Phase Two of the study

In making the selection of the teachers to be involved in the second phase of the study two considerations were made. The first was that the teachers needed to have demonstrated a level of technical competency that showed they had grasped the skills needed to create activities using a range of applications within the MRT. For example, evidenced through the lessons they had designed during Phase One, they had planned activities that involved more than one representation and had provided opportunities for learners to make mathematical connections between these representations as an outcome of the activity. The systematic interpretation of this was that their activities did not remain within any single representation, but they sought to explore different ways of using the technology based on their personal knowledge growth of both the technology and pedagogical opportunities it afforded. The teachers' technical competency was considered important as the study provided an opportunity to research how teachers' learned to use the wide range of functionalities afforded by the MRT.

The second consideration was that the teachers adopted pedagogical approaches that placed the students' mathematical experiences at the centre of the classroom environment. By allowing the students more mathematical choices within the activities they developed, they adopted a more socio-constructivist philosophy in which their students could form their own mathematical meanings with a collaborative, supportive classroom ethos. My rationale for this choice was related to my desire for the study to generate new knowledge about the way that teachers conceive and learn from their own innovations with complex new technologies.

There were four teachers who most demonstrated these attributes (Tim, Eleanor, Carla and Sophie) and all were approached to participate in the second phase. However, as Sophie was moving to a new school in September 2008, and would no longer have access to the chosen technology, she was unable to participate. During the period of the first phase Carla was both the Head of Mathematics and an Advanced Skills Teacher, with the significant responsibility to work outside of her on school on behalf of her local school authority. Although she was willing to be involved, I judged that she would be unlikely to be able to commit fully to the second phase. Consequently, I selected Eleanor and Tim as my two cases for further study.

My decision to continue the research with two case studies alone was justified by my belief that the research tools I would develop for the second phase would help me to draw out the teachers', and my own insightful observations that would be generalisable. In this respect, a single case study would have provided sufficient opportunity to theorise about processes involved in situated teacher learning within the domain of the study. However, the inclusion of the second case would provide an opportunity to contrast the emerging practices of the two teachers and deepen the validity of my resulting theories.

5.4.2 Methods of data collection in Phase Two

The change of methodological approach in Phase Two required a review of the data collection instruments and strategies in order to fully respond to the research aims. The most obvious new strategy was the inclusion of classroom observations as part of the methodology, supported by pre-lesson and post-lesson correspondences and discussions with the teachers. The context for the classroom observations also allowed the teachers to develop activities and approaches that spanned more than a single one-hour lesson.

5.4.2.1 Classroom observations

As already indicated, classroom observations were an important addition to the methodology during the second phase of the research as this was judged to be the only way that it would be possible to evidence the teachers' situated learning in the classroom. The observations replaced the need for the teachers to submit detailed evaluation questionnaires, although I adopted the same questioning format for the post lesson semi-structured interviews.

When observing in the classroom, I chose to position myself at the front of the room and to one side, so that I could closely observe the whole class from the teacher's perspective during episodes of whole class discourse. If the teacher moved to work amongst the students, I moved around the classroom to enable me to have a visual memory of the teacher's actions during the whole lesson. I avoided intervening with students, although there were some inevitable moments when I responded to requests for immediate help.

On arrival in each classroom I made a sketch of the table layout and recorded where the students were sitting as the teacher registered the class (with the help of the students). This enabled me to connect individual students' responses to any discussions that the teacher's had with them during the lesson. Most of the lessons were audio recorded with the teacher wearing a digital recording device during the lesson and some key episodes were video recorded using a small handheld digital camera.

Finally, I took the opportunity to digitally photograph particular examples of the teacher's writings and annotations on the class whiteboard where it had provided the focus for individual or whole class discourse. I judged these to be of importance to the narrative of the lesson, in addition to the relevant screen capture view of the MRT from both the teacher's and the students' handheld devices or the teacher's class computer.

5.4.2.2 Teachers' lesson plans, pupil resources and software files

As in Phase One, an important element of the data collection concerned the suite of materials that constituted the lesson. These were similar to those submitted during the first phase and may have included:

- an activity plan in the form of a school lesson planning proforma, a hand-written set of personal notes;
- a lesson structure for use in the classroom (for example a Smart NoteBook or PowerPoint file);
- a software file developed by the teacher for use by the teacher (to introduce the activity or demonstrate an aspect of the activity);
- a software file developed by the teacher for use by the students that would normally need to be transferred to the students handhelds on advance of the lesson;
- a task or instruction sheet developed by the teacher for students' use;
- students' written work resulting from the activity;
- students' software files captured during and/or at the end of the activity.

However, my presence in the classroom meant that I was in a position to prompt or instigate the collection of some of this data in addition to that which the teacher had provided. For example, if I was aware of a particular student's work that had become the topic for classroom discourse, I was able to ensure that it was saved at that point of the lesson and included in the data set.

Developments in the technological setup also meant that the Phase Two teachers had access to a more efficient form of file distribution and collection than during the first phase. The introduction of the TI-Nspire Navigator (Texas Instruments, 2009) platform in May 2009 enabled the handhelds to be wirelessly networked and resulted in a more efficient means of managing the students' files, in addition to the ease of displaying their individual and group screens in the classroom. Consequently, the lesson data might be supplemented by the following data in relation to particular classroom episodes:

- periodic file collection from all students' handhelds;
- screenshots from all students' (or an individual student's) handhelds;
- short video sequences from the teacher's (or an individual student's) handheld;
- still digital photographs of the teacher's annotations on the traditional whiteboard;

This additional data provided an unexpected rich resource to support the process of reviewing and discussing the classroom observations with the teachers, as it provided a tangible focal point concerning particular classroom episodes. By synchronising the lesson audio with the screenshots and video sequences, it was possible to reconstruct these important episodes as part of the data analysis process. The inclusion of the classroom network also added another layer of instrumentation for both teachers, the implications for which are considered later in the study.

5.4.2.3 Interviews with teachers

The interviews with the teachers preceded and followed the classroom observations and were of great significance in this phase of the study, as they formed the major opportunity to hold in-depth discussions with the teachers about their intentions, actions and reflections. As previously mentioned, a framework for the pre and post-lesson interviews was provided by Questionnaire 2 (Table 4-2). The pre-lesson interviews focussed on eliciting the teacher's intentions for the lesson by adapting Questions 2-3, 2-4 and 2-6 to the future tense. This inevitably led to more detailed discussion regarding all aspects of the activity design during which I would probe each teacher concerning the pedagogical and mathematical implications, particularly concerning the exploration of variance and invariance. The post-lesson interview would enable the teacher's perspective of the lesson outcomes to be explored in more depth. The naturalistic setting for the second phase of the study resulted in all of the lessons being taught as part of the teacher's normal teaching timetable and the inevitable timetable constraints resulted in some of the pre-lesson and post-lesson communications being held through telephone conversations or email exchanges. The face-to face discussions were audio recorded, transcribed and imported into Nvivo8 software for subsequent coding. This data analysis process is described in greater detail later in this chapter in Section 5.4.4.

5.4.2.4 Personal research journal

As in the first phase of the research, I used my (electronic) research notes to record my own insightful observations and potential lines of enquiry, as well as using them as a holding space for my emerging theories and research action plan. These comments related to my observations of activities in the classroom, my reflections on discussions with the teachers and associated email correspondences. I also recorded my own thoughts that connected my experiences during the research and my wider reading within the domain of study. For example, following a discussion with Tim in which he had suggested a possible approach with the technology to support students to develop some meaning for the notion of the solution of a linear and a quadratic equation, I made the following note in my journal (Figure 5-27).

Notes 10-12-09: [redacted]
Planning

Neat idea – plots the function, put a point on it, hide the function... Drag it so that it moves – can you work out what the function is by observing the crossing points?

Figure 5-27 An extract from my research journal showing an incidental comment relating to a teacher's instrumentation scheme for a task [Journ]

I envisaged that my personal notes would enable me to make explicit my role in the research during the second phase and also provide rich data in itself. For example, the extract in Figure 5-27 is revisited later in the study when I discuss Tim's trajectory of development in Chapter 7.

5.4.3 Summary of the Phase Two data

During the second phase I observed eight of Tim's lessons and six of Eleanor's lessons in the period from May 2009 until December 2009. In some cases these were planned teaching sequences of lessons with a particular class, whereas other observations were of single lessons. In common with the first phase, the onus was on Eleanor and Tim to decide upon the mathematical content and progression for each activity and the role for the MRT within this, although in several cases this prompted a dialogue between us that impacted upon the resulting lessons. For each of the observed lessons, all of the raw data was collated, coded and, when it was in a compatible format, imported to Nvivo8[5]. An example of this is shown in Table 5-5 for Tim's lesson 'Pythagoras theorem' [STP6].

[5] TI-Nspire and Smart Notebook files were an unsupported format. Files in these formats were imported into Nvivo8 as image or printed document format (pdf) files.

Data	Description
[Journ]	My personal research notes.
[STP6(tns-T)]	Tim's TI-Nspire file for the lesson.
[STP6(Quest2)]	Tim's written lesson evaluation.
[STP6(Trans)]	The lesson transcript.
[STP6(Eval-S)]	The students' written evaluations of the activity.
[STP6(ScreenCapt-Activity1)]	Digital screen capture image of the students' handheld screens as a Screen Capture view.
[STP6(Video-Activity1)]	Video sequence of Tim's introduction to activity 1.
[STP6(Video-Activity2)]	Video sequence of Tim's introduction to activity 2.
[STP6(Journ-T)]	Tim's written lesson reflection.
[STP6(tns-S)]	Full set of students' TI-Nspire files, collected at the end of the activity.
[STP6(tns-Tv2)]	Tim's TI-Nspire file (revised after the lesson).

Table 5-5 The summary of raw data for the lesson 'Pythagoras exploration' [STP7]

The complete list of the raw data from Tim's Phase Two activities and the associated codes are provided in Appendix 8. A similar list of raw data and associated codes for Eleanor's lessons are included in Appendix 10.

5.4.3.1 Tim's lessons

A brief description follows of each of the lessons developed by Tim during the second phase of the study. The complete descriptions of these lessons, and their accompanying analyses, are provided in Appendix 9.

5.4.3.1.1 STP6 Pythagoras exploration

This exploration was developed as a series of tasks within a single lesson that took place in July 2009 with a class of sixteen more-able 13-14 year old students who had recently begun the GCSE higher tier syllabus. The students began with an activity that required them to drag visible geometric points to transform three squares that shared some common vertices such that they had equal areas. The main activity in the lesson required the students to drag points within a dynamic geometry construction to satisfy a given condition, taking feedback from the numeric, geometric and syntactic outputs. The activity objective was for the students to generalise that, within the specific geometric environment, when the areas of the two smaller squares summed to equal the area of the larger square, the triangle would be right-angled. The final activity within the lesson presented a more traditional Pythagoras type problem with some measurements given, which the students were asked to try to solve. The students' handheld screens were on display to the class periodically during the lesson.

5.4.3.1.2 STP7 Circles and lines

The activity comprised a series of tasks offered over a sequence of two lessons that took place in December 2009 with a 'more able' group of twenty nine 15-16 year old students working towards the GCSE higher tier examination. The two lessons were conducted on the same day. The introductory activity focussed on a discussion about √25 and √(-25) which did not use the technology. The students were then sent a file with a circle defined geometrically on a Cartesian plane and were asked to construct a line on the page, leading to a discussion about the possible number of points of intersection between the circle and any line. The students' handheld screens were on display to the class throughout this discussion. In the final phase of this activity, students made conjectures about the points of intersection between different horizontal lines of the form y=c and attempted to prove their conjectures both syntactically using paper and pencil, and using the MRT.

5.4.3.1.3 STP8 Quadratic curves

The activity comprised a series of tasks offered during a single lesson that took place in December 2009 with twenty-six middle ability 14-15 year old students working towards the GCSE higher tier examination. The students were sent a file in which it was intended that they would enter two values into a spreadsheet that represented 'a' and 'b' in a quadratic of the form y = (x-a)(x-b) and then drag a related point on a Cartesian plane. The point traced the path of the defined quadratic function, which had been hidden from view. The roots of the quadratic were visible as crosses on the axes. The students were asked to make conjectures about the nature of the point's locus and to predict the factorised form of quadratic function. They entered their predictions with the aim to 'reveal' the locus of the moving point. However, a technological problem occurred when the file was transferred, which resulted in the movable point losing its connection with the underlying quadratic curve onto which it had been constructed. Consequently, Tim had to amend the activity during the early part of the lesson and the focus of the activity became one of matching the factorised form of the quadratic with its expanded form, both syntactically and graphically.

5.4.3.1.4 STP9 Equivalent quadratic equations

The activity comprised a series of tasks offered during a sequence of three lessons that took place in December 2009 with a group of sixteen lower ability 14-15 year old students working towards the GCSE foundation tier examination. In the preceding two lessons Tim reported that the students had learned to plot quadratic graphs using paper and pencil methods. The lesson sequence initially focussed on supporting the students to compare graphs of coincident quadratic functions that had been plotted syntactically in both the factorised and expanded form, with a view to making generalisations about the algebraic forms of each matching pair. The second lesson extended this approach and Tim introduced a paper and pencil grid multiplication method to support his explanation of the students' conjectures. In the final lesson, the students worked from a more conventional worksheet in which they practised the grid multiplication method for expanding quadratic functions and used the MRT as a 'class authority' to check their answers to each question.

5.4.3.1.5 STP10 Linear graphs

This activity was designed for a single lesson that took place in December 2009 with a small group of twelve 16-17 year old students who were being prepared to resit their GCSE examination and its mathematical focus concerned the gradient and intercept properties of linear functions. A TI-Nspire file was created during the initial whole-class discourse within the graphing application onto which two coordinate points had been constructed. The 'display grid' functionality had been enabled, resulting in the points being fixed onto integer values. These points were then joined with a line. A second line was added in a similar way that shared a common point with the first line. This file was then sent to the students' handhelds and they were shown how to reveal the equation of the one the lines and asked to discuss the resulting gradient and intercept properties. Tim used a class poll to gain their responses to a question about the gradient of the second line. In the final part of the lesson, he constructed a third line segment and asked the students to predict its gradient.

5.4.3.2 Eleanor's lessons

A brief description follows of each of the lessons developed by Eleanor during the second phase of the study. The complete descriptions of these lessons, and their accompanying analyses, are provided in Appendix 11.

5.4.3.2.1 CEL6 Transforming graphs

A single lesson taught in May 2009 with a group of twenty-nine 15-16 year old girls working towards the GCSE higher tier examination syllabus. The students were sent a file in which they were expected to enter given sets of functions with a view to generalising about the effect of different syntactic forms on the resulting graphs. The students' handheld screens were on public display in the classroom throughout the lesson. Eleanor led a session of whole class discourse towards the end of the lesson that concerned transformations of the type $y=f(x\pm a)$.

5.4.3.2.2 CEL7 Generating circles

A single lesson taught in May 2009 with a group of twenty-nine 15-16 year old girls working towards the GCSE higher tier examination syllabus. The initial activity required the students to offer two numbers using the class poll which, when individually squared and summed, equalled twenty five. The initial answers were shared and the poll repeated, building a set of correct responses. These numbers were then collated into a spreadsheet and the values plotted as coordinate pairs on a Graphing page that had been linked using the MRT. The students' handheld screens were on display to the class periodically during the lesson. Students made conjectures in relation to the shape of the resulting graph and in the final phase they were supported to fit a function to the data points that plotted the circle function syntactically.

5.4.3.2.3 CEL8 Triangles and squares

A single lesson taught in July 2009 with a year 8 group of 30 students working at National Curriculum levels 6-7 who were soon to commence the higher tier GCSE course. This is Eleanor's adaptation of the lesson developed and taught by Tim [STP6]. The students began with an activity that required them to drag visible geometric points to transform three squares constructed on a common horizontal base, such that they had equal areas. The students were then supported to remember how to calculate the side length of any square, given its area. The second activity required the students to drag points within a dynamic geometry construction to satisfy a given condition, taking feedback from the numeric, geometric and syntactic outputs. Towards the end of the lesson, a poll of the students' conclusions was conducted and the results discussed with the group. In the final minutes of the lesson, they dragged points within a traditional Pythagoras construction to create their own numerical problem, for which they were then asked to try to find as many of the missing measurements on their diagram as they could. The students' handheld screens were on display to the class periodically during the lesson.

5.4.3.2.4 CEL9 Crossing linear graphs

A series of three lessons taught in October 2009, with a year 9 class (set 1) of 29 students working at National Curriculum levels 7-8 who were undertaking the higher tier GCSE course. Students began by conjecturing algebraic rules that could describe the relationship between the x and y values for a given coordinate pair. A class poll was conducted to gather their responses and, following an initial discussion, the poll was repeated to allow the students to amend their answers. The students were sent a file which included a graphing page, on which the given coordinate point had been plotted, and they were asked to generate linear functions that coincided with the point. The students' handheld screens were on display to the class during this time, resulting in episodes of discussion. In the final phase the students were shown how to reveal the function table and they used this to support them to reason that the MRT was indeed correct. In the final lesson, one of the student's responses in the previous lesson was used to reiterate the successful strategies that had been devised to generate linear equations through a given point. This process was then reversed by inviting the students to find their own strategies to calculate the point of intersection for any two given linear functions. They were expected to record their approaches using paper and pencil and they also had access to the TI-Nspire handhelds during the lesson.

5.4.4 Process of data analysis during Phase Two

For each of the classroom observations, I began by transcribing the lessons in full and importing the audio and text into Nvivo8 as a synchronised set of data. This enabled me to move smoothly between the audio and transcript and to efficiently locate, replay and code key episodes of the lessons. I was also able to link other related supporting data that had also been imported into Nvivo8, for example the digital image of a student's handheld screen or the teacher's written work on the class whiteboard.

I used this set of interlinked data to support me to write a detailed, accurate and complete lesson narrative, which typically included:

- descriptions of my classroom observations (triangulated by the audio recording and associated data);
- direct quotes from the teacher and/or students for key episodes within the lesson;
- screen shots of the teacher's and/or students' technological display for key

episodes within the lesson;

- photographs of the teacher's written board work;
- photographs of students' written work.

Examples of the detailed lesson narratives for one of Tim's activities and one of Eleanor's activities are provided later in the study within sections 6.2.2 and 6.3.2 respectively.

Following this, I used Stacey and Pierce's pedagogical map as a tool to support the writing of a broad analysis of the lesson (Stacey, 2008). This was done within Nvivo8 by initially coding the lesson for the three 'layers of pedagogical opportunities', namely the task layer, the classroom layer and the subject layer. For example, Figure 5-28 shows the format that I adopted when developing the descriptive analysis for Tim's lesson 'Pythagoras exploration' [STP6].

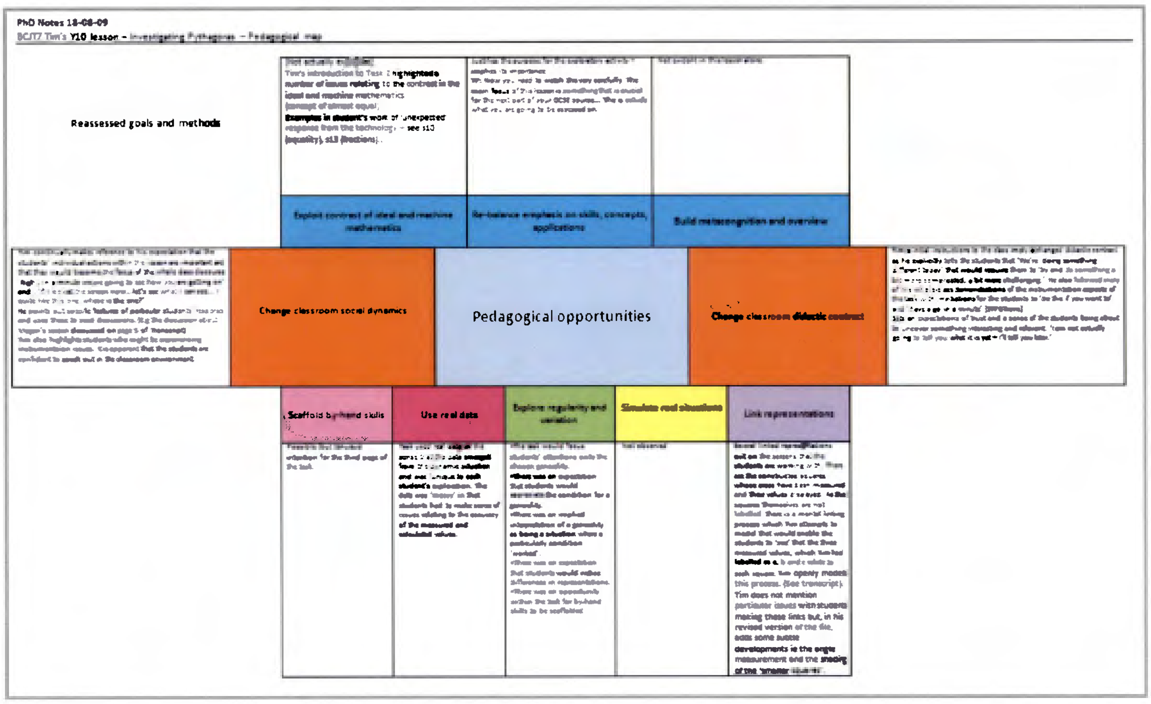


Figure 5-28 An example of how I produced a written analysis of an activity using the themes provided by the pedagogical map for Tim's 'Pythagoras exploration' [STP6].

For example, in relation to the pedagogical opportunity within this activity to 'explore regularity and variation', the following features were surmised:

- the activity focussed students' attentions onto the chosen generality;
- there was an expectation that students would appreciate the condition for the generality;

- there was an implied interpretation of the generality as being a situation where a particularly geometric condition 'worked';
- there was an expectation that students would notice variation through differences in the representations.

5.4.5 My role during Phase Two

During the second phase of the study my role changed from being predominantly concerned with in-service teacher education to that of an educational researcher. I had invited Eleanor and Tim to be part of the second phase of my study and they were aware that it formed part of my doctoral research. Consequently, I discussed with them my need to observe them in the classroom using the technology with their learners and that I was interested in the sorts of activities they designed that provided opportunities for their learners to explore mathematical variance and invariance.

I offered myself as a critical friend to the two teachers and, although I was still keen that the activities and approaches designed by the teachers were of their own origination, I was aware that technically, mathematically and pedagogically they would want to know my opinion and views on different instrument utilisation schemes. As such, although I was involved in professional conversations with Eleanor and Tim concerning the design and evaluation of lessons, my motive for holding these discussions was more concerned with eliciting rich data for the study. This data was needed to support the triangulation process and, as the study proceeded, provide opportunities for discussions concerning lessons.

From my research perspective, I did not interpret this as a threat to the objectivity of the study as I was including these exchanges as part of the research data through my audio recordings and personal research journal. I would be able to 'declare' my level of involvement at all stages of the activity design and use the associated data to provide further insight into the teachers' trajectories of development. However, as previously stated, my role within the actual lesson observations was that of a non-participant observer as it was crucial that the lessons were as naturalistic as they could be when a researcher was present in the classroom.

5.5 The emergence of the hiccup as an organising principle

As Phase Two commenced and I began to observe the teachers in their classrooms, my attention was increasingly shifting towards the existence and opportunity to analyse what I refer to throughout the thesis as lesson 'hiccups'. These were the perturbations experienced by the teachers during the lesson, triggered by the use of the technology that seemed to illuminate discontinuities in their knowledge and offer opportunities for the teachers' epistemological development within the domain of the study. They were highly observable events as they often caused the teacher to hesitate or pause, before responding in some way. Occasionally the teachers looked across to me in the classroom in surprise and, particularly in the case of hiccups relating to what they considered to be unhelpful technological outputs, they sometimes expressed their dissatisfaction verbally. Consequently, I also started to code each activity for hiccups. An example of such coding for Tim's lesson 'Equivalent quadratic equations' [STP9] is shown in Figure 5-29.

Name	Sources	References
TP9 Hiccup01 - Student's instrumentation issue entering X^2	3	3
TP9 Hiccup02 - Teacher's assumption that students 'notice' the brackets	2	6
TP9 Hiccup03 - Teacher's assumption that students 'see' the invisible multiplication sign	1	1
TP9 Hiccup04 - Students' task sequencing issue.	1	1
TP9 Hiccup05 - Teacher's assumption task sequencing will lead to counter-examples in stu	1	1
TP9 Hiccup06 - Technology fails as teacher goes to use live presenter in final plenary	1	1
TP9 Hiccup07 - Noticing that the worksheet uses letters other than x	1	5
TP9 Hiccup08 - Student doubts authority of MRT 'Why didn't it say false'	1	1
TP9 Hiccup09 - Insufficient specificity about labelling objects under discussion	1	3

Figure 5-29 An example of the hiccups within Tim's lesson 'Equivalent quadratic equations' as coded within the Nvivo8 software [STP9].

The lesson transcripts formed the main data source, supported by my observational notes and, in some cases, triangulated by evidence from the post-lesson interview or the teacher's written lesson reflections. If a particular hiccup was referred to more than once within a single data source this would generally be considered to be a major hiccup. For example, within the Nvivo8 summary shown in Figure 5-29 the hiccup coded as TP9Hiccup07 occurred five times within this particular activity.

As I began to organise the research data to bring the examples of the lesson hiccups to the fore, I became increasingly convinced that they were having a significant role within Tim's and Eleanor's situated learning. Interestingly, although the teachers' immediate reaction to the hiccups was often a negative one, I did not conceive the occurrence in the same way. This might have been because I could see it as an opportunity to focus upon the unanticipated elements of the lessons as a trigger for a substantial professional discussion with each teacher.

The notion of the lesson hiccup is expanded more substantially within Chapter 6 when I identify a single hiccup from each of Eleanor and Tim's lessons and articulate its significance within the thesis.

Finally, whilst coding and re-coding the data I also began to appreciate the subtleties of each teacher's actions both during and after the lessons, which seemed to provide some possible evidence to suggest that the teacher had noticed the hiccup and acted in response to it. Consequently I identified and coded this data as 'Actions'. The coded actions from the lesson [STP9] are shown in Figure 5-30.

Name	Sources	References
TP9 Action01 - Appreciated 'hiccup' over choice of expressions	1	2
TP9 Action02 - Appreciated that the MRT could have accepted any letter as variable	1	1
TP9 Action03 - Made explicit link between Medley's theory and grid multiplication.	1	1
TP9 Action04 - Noticed that students appreciated being sent the same file to work on	1	1
TP9 Action05 - Observed that students were highly motivated by MRT response 'true'	1	1
TP9 Action06 - Privileged 'press control z to undo'	1	1
TP9 Action07 - Responded to students' concerns about needing to type the brackets	2	2
TP9 Action08 - Reversed second section of task 2 to force inverse strategies	1	1
TP9 Action09 - Revised language wrt needing brackets	1	1
TP9 Action10 - Revised the task to account for the L1 hiccup	1	2
TP9 Action11 - Appreciated that student had noticed 'bold' graph	1	1
TP9 Action12 - Appreciated student's connection between 'Medley's rule' and grid multiplica	1	1

Figure 5-30 [STP9] An example of the Tim's actions in response to the hiccups within his lesson 'Equivalent quadratic equations' as coded within the Nvivo8 software [STP9].

This process of coding both the lesson hiccups and list of teacher actions was repeated for each of Tim's and Cindy's observed lessons. The detail of this coding for all of their Phase Two lessons is contained within Appendix 9 and Appendix 11 respectively. The overarching analysis that resulted from this data would ultimately lead to the development of the 'hiccup theory' of situated teacher learning.

5.6 Summary

In this chapter I have detailed the outcomes of the first phase of the research and how these informed an analysis of the teachers' conceptions of variance and invariance within the technological setting. This resulted in nine distinct instrument utilisation schemes. In addition I was able to summarise the elements of the associated teacher learning and to propose the notion of the lesson hiccup as an observable instance of the teacher's situated learning in the classroom. The chapter outlined the methodology for the second phase to include: a justification of the use of the adapted pedagogical map as an analytical tool for the forthcoming lesson observations; the selection of the teachers to be involved; a description of my revised role within the research.

6 INTRODUCING THE TEACHERS AND ANALYSING THEIR MATHEMATICS LESSONS

> *If we could understand more precisely the roles that teachers are called upon to play in computationally based environments, then it would surely throw light on the nature of the mathematical knowledge involved, its relationship with the priorities of the official mathematics curriculum, and help to clarify how the process of integrating technologies productively into classroom practice might be facilitated.*
> *(Hoyles et al., 2004, p. 316)*

6.1 Introduction

This chapter begins by introducing the two teachers selected as cases for the second phase of the study namely, Eleanor and Tim. It provides a concise description of their individual backgrounds and school contexts as relevant to the research. In addition, the chapter includes an overview of their responses to the first phase that served as a point of reference against which to contrast their later practices. This overview refers to the nine Instrument Utilisation Schemes (IUS) concerning activities and approaches that privileged the exploration of variance and invariance that were defined as an outcome of the first phase of the research (Section 5.3.1).

The chapter continues by making explicit to the reader the process through which the data from each of Eleanor and Tim's Phase Two lessons was collected and analysed. By providing the detailed account of a selected lesson from each teacher, followed by its analysis, I aimed to collect a more detailed and measured set of data through which the story of each teacher's trajectory could be elicited. The notion of the lesson 'hiccup' is introduced as an instance of situated teacher learning triggered by the technology, within the domain of study. For each selected lesson, the occurrence and analysis of a chosen hiccup is described with a view to developing a class of theories to describe this new construct that are expanded upon more fully in Chapter 7.

6.2 Introducing Tim

Tim was in his mid-thirties and he held an honours degree in mathematics. When asked about the first time that he had used technology in connection with mathematics he responded,

> *I remember it well... we had tutorials in the first year at* [redacted] *and we were looking at graphs of sin of one over x or something, I can't remember which one it is, and when it is close to zero it goes like this [TP gestured the shape of the graph] ...and having to draw it ...and the tutor got out a [graphical] calculator and said you can zoom in on this, you can see its behaviour and, oh wow. I decided that that I must go out and buy one. [STP(Int-T)]*

Tim commented that this incident, which happened within a first year algebra module, was the one and only time during his degree course that any of his tutors had used technology. The incident prompted him to buy his own programmable graphical calculator (a Texas Instruments TI-85) and, as Tim did not own a personal computer at that time, the portability of the handheld device enabled him to become familiar with its functionalities. He used it throughout the remainder of his degree course.

Tim progressed from his undergraduate degree course straight onto a one-year postgraduate certificate in education course (PGCE) in secondary mathematics at the same university. During this time one of his course tutors introduced Tim and his peers to the dynamic geometry software package, Cabri-Géomètre. Tim reflected on this experience thus:

> *I thought 'why on earth didn't our geometry lecturer in Uni use Cabri?' because I didn't understand my geometry - because we didn't do formal geometry at school, I went through SMP GCSE... I remember sitting in the back of this lecture hall and he had an overhead projector and a felt pen... and the lecturer just couldn't draw - that was it - I didn't get anything. [TP laughed] ...and then we were using Cabri and it's like 'this is making sense to me now' and I wish I'd had that sort of experience - it [the software] was around - this was only a couple of years later, it was around but wasn't available to us and I didn't know about it - and I thought I want it. [STP(Int-T)]*

During his PGCE, Tim became familiar with LOGO and dynamic geometry software and the use of handheld technology was promoted by his PGCE tutors. He borrowed the university's class set of Casio graphical calculators during his final teaching practice to use with his students.

Tim qualified in 1996 and early on in his career he arranged a departmental in-service professional development session at his school for himself and his colleagues that focussed on the use of handheld technology. This led to Tim's active involvement in the Teachers Teaching with Technology (T^3) network, a practitioner network part-funded by Texas Instruments to support the professional development of teachers who had adopted their handheld products. Tim commented upon how this happened thus,

> *I got involved with T^3 because I invited* [redacted] *in to do a presentation at my first school. He looked at a booklet that I had done for my department and he said you should be writing stuff. So that was what started me off with it. [STP(Int-T)]*

Tim was a member of the Association of Teachers of Mathematics (ATM) and he had contributed a number of articles to their journals, all of which have concerned the use of technology. Tim had also contributed to published teaching resources and he regularly led professional development sessions for other teachers concerning the use of technology in secondary mathematics.

Tim would certainly be described as an innovator with respect to his curiosity for the design of new teaching resources and pedagogies for mathematics. At the outset of the study Tim commented that he had 'given some training and shared ideas with the department... but so far this has had little effect' [Journ]. This suggested that he was experiencing some of the difficulties involved with supporting his colleagues to develop and use technology with their learners. Tim also reported that he had registered for a Masters of Philosophy (MPhil) course and that his personal research interests lay in the 'focus of attention in the teaching and learning of linear equations' [STPQuest1)].

In 2007 Tim was appointed to his first post as a head of a mathematics department at Stadium School in the centre of a West Midlands suburb. During the period of the study, Stadium School was a mixed school of around one thousand students aged between eleven and eighteen. The most recent Office for Standards in Education (Ofsted) description of the school described it as serving a diverse area with above average social and economic deprivation. At that time the ethnicity of the students was approximately: one third white British; one quarter Pakistani backgrounds; one eighth Caribbean backgrounds; and smaller numbers from Bangladeshi and African backgrounds. One in ten students were judged to be at an early stage of acquiring English language and one in seven students had designated learning difficulties. The students' mathematical achievement was consistently below the national expectation, although there had been a slowly improving trend in the GCSE results in the period since Tim assumed responsibility for the mathematics department at the school in January 2007.

6.2.1 Tim's participation during Phase One

Within a few weeks of participating in the project Tim had articulated that he saw his students' uses of the MRT as providing them with opportunities to 'try out mathematical conjectures' and 'access the mathematics in a situation more directly'. He provided the following example of this:

> *In an activity exploring gradient of straight line graphs, students were able to generate many graphs in a short space of time and keep their focus on gradient rather than on the physical process of drawing the graphs themselves. [Journ]*

Tim's experience and confidence with handheld technology was demonstrated from the outset of the project as he sought to adapt and reversion familiar activities in addition to designing new activities, which took advantage of the MRT environment and associated functionalities. The detailed description and analysis of Tim's activities using the MRT during the first phase is provided in Appendix 6.

With the exception of activity STP4, all of Tim's activities began with an input in one representational form and led to a single output in another representational form. His activities were classified as using IUS1, IUS3 and IUS4. Figure 6-1 shows diagrammatically an indication of the types of input and output representations that Tim privileged in his early activity designs.

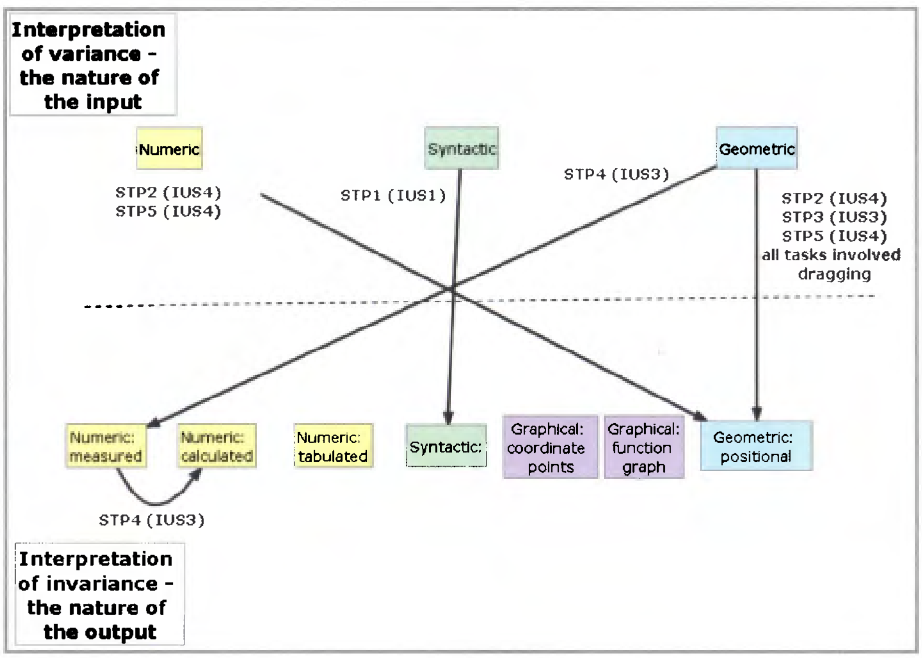


Figure 6-1 The summary of Tim's Instrument Utilisation Schemes produced from the analysis of his Phase One lesson data.

For example, in activity STP2, the students' initial input was numerical and the resulting output gave a geometric image in a particular position on the display screen. This diagram evidences Tim's willingness to explore different representational starting points in his early activity design.

Reflecting on Tim's trajectory of development during the first phase of the study it was possible to draw several conclusions concerning Tim's practices.

- In all of Tim's activities and approaches his actions were consistent with his original aims for the project in that they had all been designed to encourage his students to explore and conjecture with respect to the mathematics under investigation.
- Despite Tim's advanced technical knowledge he was mindful to his students' needs with respect to the process of instrumentation. Although some of his activities were complex in their design, the students only needed to be familiar with basic techniques such as moving between pages, entering numerical data and selecting and dragging objects.
- It was noticeable that Tim chose to display the mathematical representations in the form of the different applications on separate pages within the TI-Nspire files, rather than taking advantage of the functionality to combine applications on a single page. This decision seemed to help his mediation of the activities whereby he was able to support his students to maintain their attentions in a similar direction.
- Tim seemed to have carefully considered what, when and how he wanted his students to make paper and pencil recordings alongside their use of the technology. It was also an established aspect of his practice that his students evaluated on their own learning through an open written comment made towards the end of the activities. This provided Tim with additional insight into their experiences and seemed to be influencing his own learning.

6.2.2 Observing Tim in the classroom - The lesson introducing Pythagoras' theorem

What follows is a description and analysis of the lesson taught by Tim during Phase Two involving his class of fifteen year 10 students aged 14-15 years. The group were all pursuing a compulsory General Certificate of School Education (GCSE) mathematics course with the majority being prepared for the higher tier of entry. Tim asserted that they were currently working at around National Curriculum levels 5-6 (Department for Children Schools and Families, 2007). During the lesson Tim and the students used the TI-Nspire handheld machines. As mentioned in Section

5.4.4 developments in the technological setup had resulted in the further functionality that connected all of the handhelds to the Tim's class computer through a wireless classroom network. It was therefore now possible to display the students' handheld screens publicly through a ceiling mounted data projector onto the classroom interactive whiteboard[1].

The lesson was observed, audio recorded and the computer action for key sequences where Tim led whole-class activities using the live presentation tool were video recorded. The students' files were collected from their handhelds the end of the lesson using the file collection facility. In addition, the students' written work, which included their self-assessment statements about the lesson, was also collected. Where it did not interrupt the classroom discourse, I also made a number of screen captures of the students' handheld displays during the lesson to correspond with key episodes within the lesson when the class view had stimulated whole-class mathematical discourse[2]. The description of the lesson is supported by the relevant software screen shots and extracts from the lesson transcript to provide a context for the subsequent detailed analysis using Pierce and Stacey's framework (2008).

6.2.2.1 A description of the lesson activity

Tim introduced the lesson by informing the students that they were going to be 'doing something different today' [STP6(Trans)], although in his written lesson evaluation, completed after the lesson, he wrote that he had two objectives for the lesson,

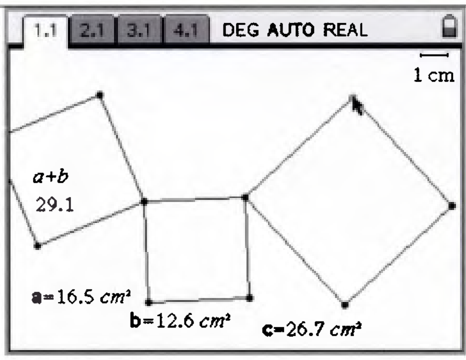
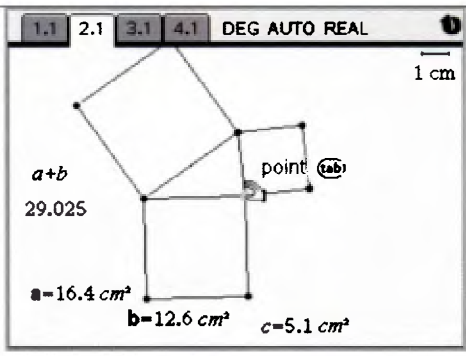
> *Mathematical objectives were for students to appreciate Pythagoras' Theorem, in particular recognising that the sum of the areas of the squares on the two smaller sides will equal the area on the longer side* ***if and only if***[3] *the triangle is right angled*
>
> *...and then, in right angled triangles, find missing sides using the fact that the two smaller areas sum to the larger area [STP6(Quest2]*

In his written evaluation he also added a specific intention for the use of the wireless network, 'Each individual student will explore the triangles on their own handheld – we will use the shared space of screen capture to come to a shared

[1] The class interactive whiteboard can be considered a 'red herring' in this lesson as, at no time did Tim (or his students) use its functionality to drive the computer software.
[2] The only strategy for capturing the screen on class display was accomplished by using the key sequence 'Alt Print Screen' on the class computer and 'pasting' the image into another application which then had to be saved separately.
[3] Tim's emphasis.

agreement about the necessity for the triangle to be right angled' [STP6(Quest2].

The activity was divided into three distinct episodes, hereafter described as tasks, and each task focused around a particular dynamic interactive. Table 6-1 shows a screenshot and description of the construction of each of the activity interactive that had all been designed by Tim without my involvement.

Opening screen	**Description of the construction of the environment**
 Figure 6-2 The opening view of Task 1 in which the students were asked to drag different vertices to produce three squares of equal areas [STP6(tns-T)page1].	**Task 1**: The environment had been constructed in the graphs and geometry application such that three defined, connected squares could be altered in size and orientation by dragging any vertex. The areas of the squares had been measured and assigned to the variables a, b and c. The definition of the screen scale (1 cm) enabled the areas of the squares be given a standard unit. Finally the value of a + b was calculated and displayed.
 Figure 6-3 The opening view of Task 2 in which the students were asked to drag different vertices to make a+b=c [STP6(tns-T)page2]	**Task 2**: In this environment, also in the graphs and geometry application, a triangle had been constructed onto the sides of which three squares had been defined. The areas of the squares had been measured and assigned to the variables a and b. The value of a + b was calculated and displayed.

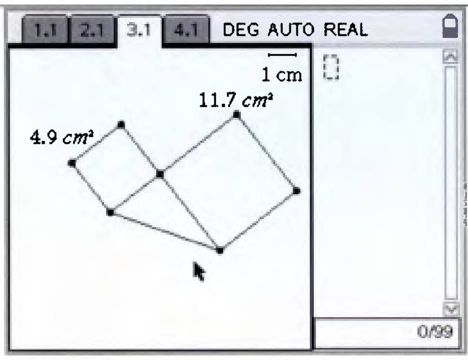
Opening screen	**Description of the construction of the environment**
 Figure 6-4 The opening view of Task 3 in which the students were asked to find the lengths of as many of the lines in the diagram as they were able. [STP6(tns-T)page3]	**Task 3**: The final environment included a split screen with the left hand side displaying the graphs and geometry application and the right hand side the calculator application. A right angled triangle was constructed onto which only two squares had been constructed. Two area measurements were also displayed on the page.

Table 6-1 The design of Tim's activity 'Pythagoras exploration' [STP6]

Tim began Task 1 by using his handheld and the live presentation facility (through the data projector) to demonstrate to the students how to use the handheld's navigation pad to select one of the vertices of the squares and drag it to a different position. During this phase he drew the students' attention to the changing numbers on the screen and told them that the numbers related to the areas of the three squares and which number related to which square, gesturing to their initial positions on the screen. He set the students an initial task within this environment, the purpose for which he told the students was 'to get yourself organised and get your head around this' and gave the following instructions to the class.

> *Try and move the point around so that – it doesn't matter what number you pick – so that that number... [Tim gestured to value of area defined as 'a'] ...and that number... [d.o. 'b'] ...and that number... [d.o. 'c'] are the same - or as close as you can get them to be the same - as each other. Okay so try and make those three areas equal to each other. [STP6(Trans)]*

Tim gave the students three to four minutes to respond to the task and then chose to display to the class a static screen capture of all of their handheld screens (see Figure 6-5).

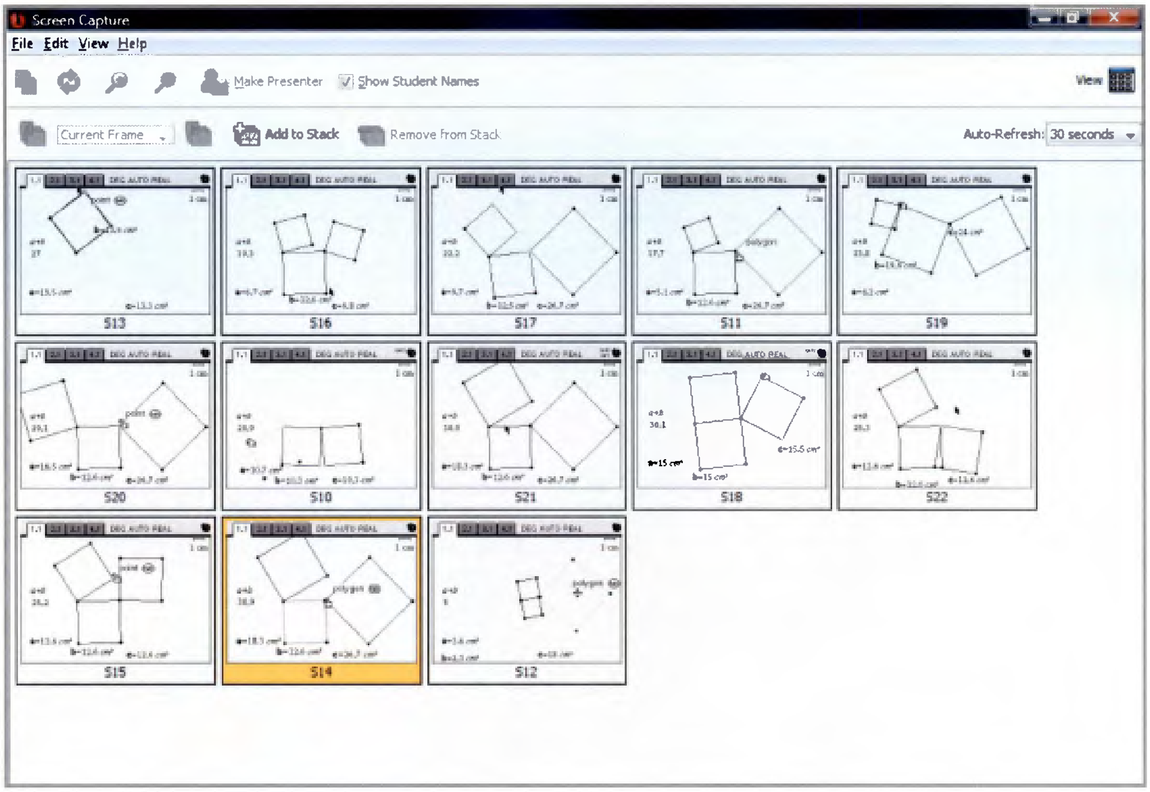


Figure 6-5 An image of the students' handheld screens in response to Task 1[4] captured during the activity [STP6(ScreenCapt)].

Tim then spent a few minutes identifying what he described as 'interesting screens' and, as the students' screens were anonymised[5], identifying which screen belonged to each student. It was noticeable that the students were keen to find their screen within the public view. He picked out two of the displayed screens (S10 and S13) and invited other students to suggest the strategy that was being adopted. At this point Tim also noticed that a number of the students had not managed to make any progress with this first task as their screens appeared unchanged, saying 'We've got some [*students*] that haven't really started – who's S17?' to which a student responded 'I can't do the points thing.' Tim gave the students a second strategy to select and move the vertices of the squares and established that all of the students were then able to accomplish this. The final stage of each student's response to this first task was captured when the files were collected at the end of the lesson. These files provided further evidence of how the students had responded to the various discussions and interventions during the lesson.

Tim then used the live presentation facility to show the students how to move to

[4] Tim had opted for a display setting that highlighted the designated teachers' handheld in yellow (S14).
[5] This was because Tim had not imported his students' names into the software to enable their actual names to be displayed – consequently they appeared as S11, S12 etc.

the second page in the file and gave a very clear rationale for his introduction to the second task:

> *The main focus of this lesson is something that is crucial for the next part of your GCSE course. This is not - I am just doing for the fun of it - this is actually what you are going to be assessed on. Okay? And you need to be focused here. I am not actually going to tell you what it is yet – I'll tell you what it is later. [STP6(Trans)]*

He displayed Task 2 (see page 147) and spent a few minutes showing students how the three squares from the first task had been 'joined together' to 'look more like a triangle in the middle' [STP6(Trans)]. He then moved the triangles around by dragging different vertices, highlighting which area measurement related to which square and setting the aim for Task 2, which was to move the squares around until the area measurements that had been labelled **a** and **b**, when summed, equalled the area measurement that had been labelled **c** saying,

> *So you need to think about which square is which and move them around a bit and I want a and b to add up to make c. Do you kind of get what we have to do? You're trying to change the sides so that a and b adds to make c. [STP6(Trans)]*

At this stage Tim gave the students five minutes to respond to this challenge, during which time he moved around the room supporting students and monitoring their work. In this period the students' handheld screens were on public display to the class, refreshing automatically every thirty seconds. During this period Tim chose to send one student's work [STP6(tns-S22)] to the teacher's computer, which captured the student's response to the task at that point in the lesson. Tim concluded this period of the lesson by alerting the students that they were going to be stopping and reviewing the class display of the individual handheld screens in a few minutes and that they would, 'scroll down and have a little chat about them and see how we're getting on' [STP6(Trans)]. With the students' attention back on the screen capture view of their work, Tim began to pick out screens and check that the numbers displayed satisfied the desired condition by talking out loud. For example, he focused on screen S13, saying,

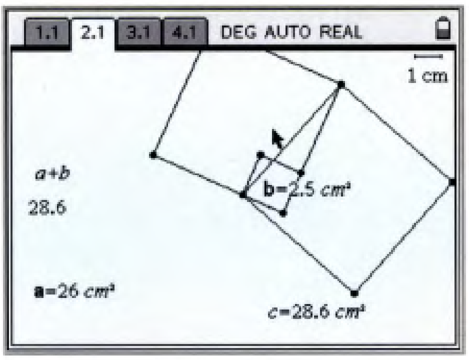


Figure 6-6 Student's handheld screen showing a response to Task 2 [STP6(tns-S13)].

Okay I'll go through these and we'll have a look at them so... a add b is twenty-eight point six-ish, and there's a and b is two point five, add them together that's kind of alright – that's really good. [STP6(Trans)]

He then moved on to screen S16, saying,

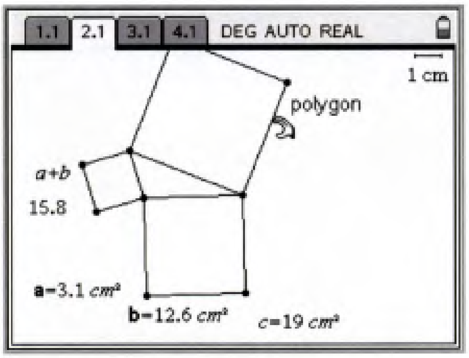


Figure 6-7 Student's handheld screen showing a response to Task 2 [STP6(tns-S16)].

We've got this one here and we've got three and twelve, that's fifteen and that's nineteen so that's close but a little bit off but it's close. [STP6(Trans)]

At this point he reminded the students that '...we're kind of looking at the ones that do work and the ones that don't...' and he invited the students to volunteer their screen number if they thought that their screen 'worked' [STP6(Trans)]. At this point there was a noticeable increase in students' participation and involvement as a number of students were heard to call out 'mine works', '22 works' and 'mine's 12' and Tim tried to locate these screens and move them so that they were visible to the class.

Tim then directed the students by saying,

> *Okay I'd like you to look at the ones that work that we've identified and compare them with the ones that don't work and I want you to look at the shape of the triangle... ...in the middle. This is what I am asking you to look at now. Look at the shape of the triangle. Look at the ones that work, look at the ones that don't work and my question to you and you've thirty seconds to discuss this now, my question to you is, is there anything different about the shape of that triangle in the ones that work*

compared to the ones that don't quite work? You've got 30 seconds to talk about it. [STP6(Trans)]

After a short period of pupil discussion Tim asked if anyone had noticed anything and a student volunteered the response, 'Is it right angled?' [STP6(Trans)].

Tim responded by displaying the following student's handheld screen (Figure 6-8) and making the following comment, directed towards the owner of the screen:

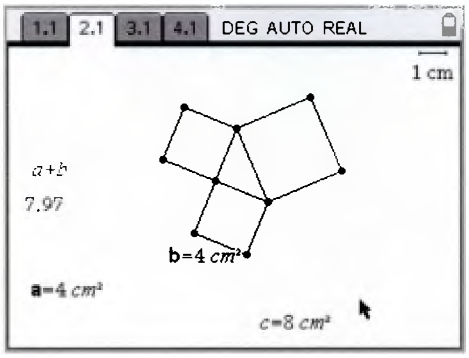


Figure 6-8 Student's handheld screen showing a response to Task 2 [STP6(tns-S22)].

Yours is quite easy to see isn't it? - that this is a right angled triangle because you've actually got a square and you can see it's a corner of a square in there – yes it is a right angled triangle. [STP6(Trans)]

Tim selected another student's screen (Figure 6-9) and talked through why it did not 'work'.

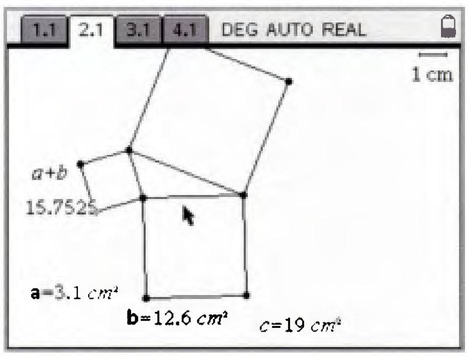


Figure 6-9 Student's handheld screen showing a response to Task 2 [STP6(tns-S16)].

Now this one here looks a little bit off, let's have a look at the numbers a add b is fifteen point seven and c is nineteen so this one is a little bit off – it is not a right angled triangle, it doesn't quite work. [STP6(Trans)]

Tim then selected a screen this did not appear to satisfy the initial task instruction, i.e. that the value of ***a*** and the value of ***b*** should sum to give the area measurement of ***c***, but did appear to work visually in that it appeared that the central triangle was right angled (Figure 6-10).

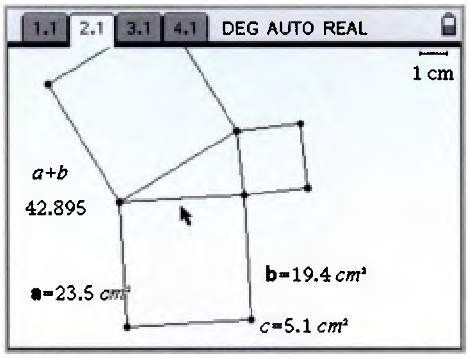


Figure 6-10 Student's handheld screen showing a response to Task 2 [STP6(tns-S20)].

> *This one, we've got a add b... doesn't quite make c either, but yours kind of works the other way round, if we look at this square here, that's five, and this square here is about nineteen and five and nineteen is about twenty four-ish and that's twenty four – So yours works a different way around – whoever this one is. [STP6(Trans)]*

Tim selected two more examples and spoke his thoughts out loud to reason through the calculation of the sum of the measured areas to verify whether they did or did not meet the initial task constraint He then asked the students to make a conjecture by saying, 'So what do you think we are learning from this then? What do you think we are noticing about the ones that work and about the ones that don't work?' [STP6(Trans)].

The following dialogue ensued:

Student:	*The more the equaller they get then... you know...*
Tim:	*The more the equaller they get then you know – okay would you like to say that mathematically?*
Student (M):	*They've all got a right angle in them*
Tim:	*They've got a right angle in them – So if the two small areas make the bigger area...*
Student (M):	*[interrupted] it makes a right angle*
Tim:	*You get a right angled triangle. Okay, so that's what we're learning here if the two smaller areas of our*

squares make the bigger area then we it's a right angled triangle. If it's a right angled triangle then the two smaller areas - of the squares - make...

Student (F): *[interrupted] the biggest area.*

Tim: *the biggest area.*

Tim then set Task 3 to the students (see page 147), asking them to move to the next page in the file and he spent a few minutes encouraging them to notice what was different about this page. These differences were concerned with the layout of the screen, for example a calculator page had been included on the right hand side; and the nature of the geometric figure itself. The construction of Task 3 differed in that the triangle had been constructed to remain right-angled when transformed by dragging and there was no square constructed on the hypotenuse of the triangle. Tim then said,

> *Now, move that [TP gestured to the triangle] if you want to, when you've settled somewhere we're trying to find how long all of the lengths are here. So we've got the length of the sides, okay? I want you to try to figure out, write it down if it helps you, how long each side is on everything. [STP6(Trans)]*

He gave out paper on which the students were asked to record their work and 'Any working out that you do, write that down as well. Okay? That will be really helpful - put the detail of your working out and thinking and how you know what it is' [STP6(Trans)] and allowed them five minutes to work on Task 3, which was the only time remaining in the lesson. In this time he also showed the students how to move to the Calculator part of the page in case they needed to use it and encouraged them to record their thinking on paper. At no time did he suggest to any students how to solve the problem, by intimating any strategy or mentioning any linked mathematical ideas such as using the square root function on the handheld to find the lengths of the sides of any of the squares from its given area. Most students began Task 3 by transferring their own selected diagram from the handheld screen onto paper, and one student added the two given areas to give the area of the square of the hypotenuse. However, Tim concluded the lesson abruptly when the end of lesson bell sounded. Before the students left the classroom, he asked the students to respond in writing to two requests, 'to write down they key facts that you learned today in mathematics - something mathematical that you learned today' and 'I want to know anything about how seeing everyone's work on the screen affected your learning in any way' [STP6(Trans)].

6.2.2.2 An analysis of the lesson activity

What follows is an analysis of the lesson using the three strata suggested by Pierce and Stacey's framework, namely the task layer, the classroom layer and the subject layer, as a means to describing the pedagogical opportunities afforded by the technology.

6.2.2.2.1 The task layer

Tim's lesson strongly evidenced an activity for which the use of the technology brought about improved speed, access and accuracy, as described by Pierce and Stacey's definition of the pedagogical opportunities that mathematics technology brings to the design of activities. Central to Tim's design of the main activity of this lesson (Task 2) was the intention to explore the regularity and generality of the mathematical context provided by a dynamic construction of squares on each of the sides of a triangle. In this task he had interpreted the notion of the variables ***a***, ***b*** and ***c*** as the registers of memory of the measured values of the areas of the squares. Tim was explicit in directing the students to change the various parameters within each of the environments, by the dragging of free vertices, with a view to students arriving at their own example that satisfied the constraint that the sum of the two areas labelled ***a*** and ***b*** should equal the measured value of the area labelled ***c***.

In preparation for the main task, Tim had included an initial task, which presented a lower level of mathematical challenge for the students but offered them the opportunity to familiarise themselves with some of the pre-requisite instrumentation skills. These included locating and opening the relevant pre-constructed file on the handhelds, moving between the pages of the file and selecting and dragging vertices of shapes. However, Tim's choice of the initial task was intentional in that there were a number of interesting strategies possible that could lead to its successful completion.

There is a sense in which this task could be interpreted as requiring students to work with real data in that the area measurements (and subsequent calculations) that they were observing were unique to each student and could appear to be 'messy'. For example, the number of decimal places displayed was floating and, depending on the vertex the students chose to drag, up to two area measurements could vary simultaneously.

Tim set a very strong expectation that the students would arrive at their own interpretations of the generality under exploration. The analysis of the lesson transcript evidenced that, on five separate occasions during the whole class discourse, he was encouraging the students to focus on aspects of the similarity and difference between the properties of the central triangle when the areas of the two smaller squares did or did not sum to equal the area of the third square. Early on in this discourse, Tim introduced the notion of 'it not quite working yet' to describe a student's screen where the condition was not met and later on in the discourse, Tim explicitly asks the students to focus on 'the ones that work' [STP6(Trans)].

There was evidence that the students needed to engage with and link multiple representations, particularly within Task 2 as indicated by the instrument utilisation scheme (Figure 6-11).

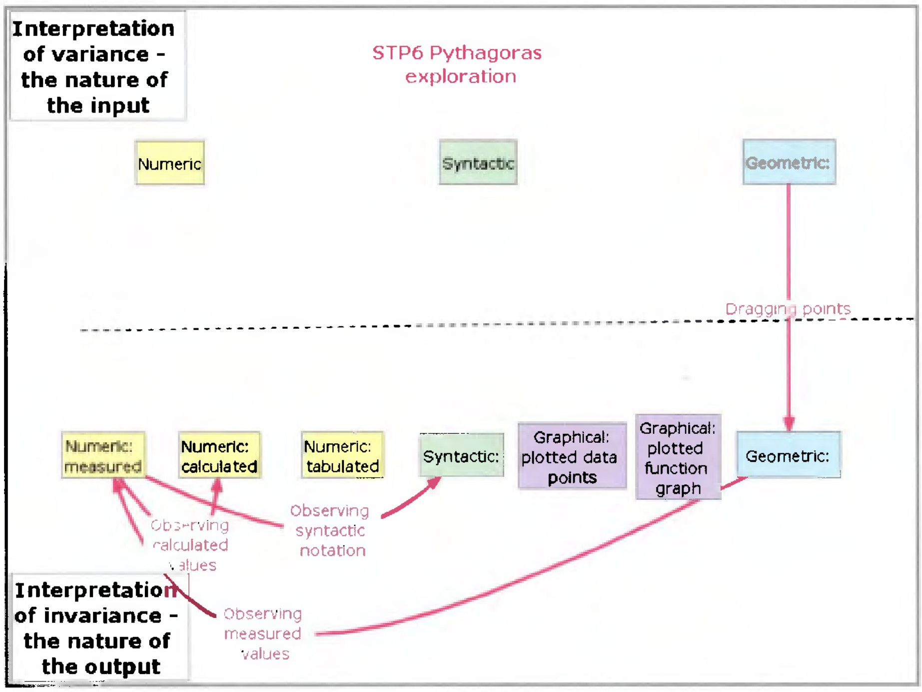


Figure 6-11 The Instrument Utilisation Scheme map for Tim's 'Pythagoras exploration [STP6].

The activity was completely mediated by the technology and, as the IUS shows, the students' interactions relied on them being to observe and make connections between the different representational outputs. This was in part due to the design of the activity whereby it was not totally clear which of the measured values (and subsequent calculations) displayed on the screen referred to which of the squares. Consequently the students needed to connect the increasing and decreasing sizes of the geometric squares with the relevant increasing and decreasing measured values.

Task 3 could be interpreted as a genuine attempt to assess the transferability of the students' newly-acquired knowledge to an associated problem, whereby they were asked to solve a more traditional Pythagoras style question in an attempt to scaffold by-hand skills.

6.2.2.2.2 The classroom layer

The analysis of Tim's lesson for the aspects that described opportunities for the use of the technology to change the social dynamics of the classroom and its didactic contract revealed some very strong evidence.

Throughout the lesson, Tim was highly explicit in suggesting expectations of how his students should be engaging with the use of the technology. Some of these expectations referred to suggested changes in the social dynamics of the classroom, such as: the public sharing of the individual students' handheld screens; student exploration of each of the environments by dragging; that the students would be working together, with the teacher's support, to arrive at a shared understanding of the mathematics under scrutiny. In addition, the acceptance by Tim of a number of spontaneous comments by students during the lesson, which had been prompted by their interactions with the technology, also suggested that in Tim's classroom each student's experience was highly valued. For example, one student spoke out freely to share his difficulty in selecting and dragging a vertex.

The way in which Tim selected and promoted what he described as 'interesting screens' as a means to reviewing particular solution strategies, also implied an emerging change to the social dynamics of the classroom. For example, the following dialogue occurred during the whole-class discourse concerning the initial task, in which the students were trying to make the areas of three given dynamic squares equal to each other. Tim had asked the class to focus on one student's response as shown in Figure 6-12.

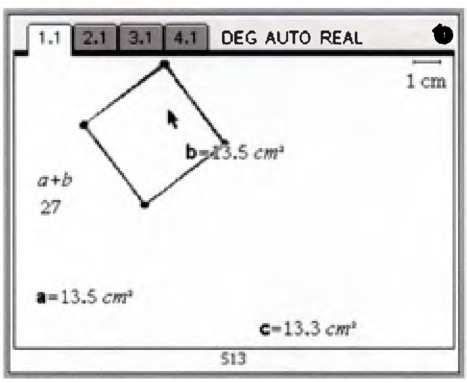


Figure 6-12 Student's handheld screen showing a response to Task 1 [STP6(tns-s13)].

Tim: *I quite like this one [Tim gestured to screen S13] - this is interesting – whose is this one?*

Sophie: *Megan's*

Tim: *What's Megan done to make all three the same? Sophie, what's Megan done to make all three the same?*

Sophie: *Put them on top of each other.*

Tim: *Put them on top of each other – How clever is that? Put them on top of each so that it's quite clear that they are the same – it's slightly off - but to put them on top of each other a very, very good idea – to make them the same size as each other.*

The students' responses to Tim's request for them to produce a written comment about the way in which their experience of the classroom network and how the shared display had affected their learning prompted an intuitive set of responses, which also evidenced the sense of a changed classroom dynamic. All of the students responded positively, with the emerging ideas relating to their increased awareness of: how to begin the activity; alternative strategies to solve the activity; their own progress relative to their peers; an awareness of correct and incorrect solutions. One student commented that she 'could compare my work with others - it showed how my work was better' [STP6(Eval-s21)]. This could be interpreted as suggesting that this students' view of success in mathematics incorporated a sense of competition with her peers or alternatively, evidence of her increased self-esteem as she was able to validate her responses alongside those of her peers. The lesson transcript revealed that this student had volunteered her handheld screen response to Task 2 to the teacher and it became the subject of the whole class discourse, with the teacher validating it as a successful response. Only one student mentioned that 'The screen helped my[*sic*] by copying other people's work' [STP6(Eval-S21)], which came from the student who had experienced difficulties selecting and dragging vertices during Task 1, and was therefore slow in getting started. On reviewing this particular student's final outcomes, it was apparent that he had managed to find successful solutions to both Task 1 and Task 2.

With respect to the renegotiation of the didactic contract between teacher and students in this lesson, as a result of the use of the technology, there were a number of notable instances. This lesson was the first occasion for all of the students in which they had experienced a shared learning space, the constituent parts of which were their own responses to a common activity, displayed through their handheld screens. The process through which Tim set his expectations of how they would engage with this new environment as a class was made clear to the students in a number of ways. As previously mentioned, Tim had told the students that they were going to be 'doing something very different today', which could actually have referred equally to the use of the classroom network and public display as it could to the use of the handheld devices, or the particular use of the MRT. Tim chose to display the whole class screen capture view publicly from the outset of the lesson. He did not comment on this decision to the students, nor did the students make any unprompted remarks about it – it just happened! The consequence of this was that Tim had begun to set the precedent for how this sort of lesson would be conducted through the subtle use of language when he addressed the students, such as always referring to the class as 'we', and by focussing on particular students' handheld screens during the whole class discourse. In return, the students were beginning to learn the rules of this new game, and were very quick to appreciate and keen to volunteer their own contributions to the classroom discourse.

6.2.2.2.3 The subject layer

Within Pierce and Stacey's framework, the 'subject layer' describes the pedagogical opportunities that concern: the exploitation of the contrast between ideal and machine mathematics; a rebalancing of the emphasis on skills, concepts and applications; building metacognition and overview in mathematics. The analysis of Tim's lesson offers a rich example of the way in which this particular MRT and the classroom system, which facilitated the shared learning space, provided evidence in all three of these areas.

Within Tim's mathematical example, the notion of 'ideal' mathematics concerned the important generalisation that, 'if and only if' a triangle contains a right angle, the sum of the areas of squares constructed on the two smaller sides will equal the area of a square constructed on its hypotenuse. This expression of Pythagoras' theorem suggests a broader interpretation than the traditional one, which makes an initial assumption that the triangle is right angled. Consequently, what is becoming a 'traditional' approach for the use of technology tends to constrain students to only exploring cases concerning right angled triangles. In this respect, it could be argued that Tim's 'ideal' mathematics involved providing his students with an opportunity to construct a broader knowledge within an appropriate mathematical environment. However, in doing so, a number of tensions came into play with respect to the 'machine mathematics'. The issue of numerical equality presented an underlying tension throughout the lesson for both the teacher and the learners. Although Tim did not explicitly say to the students that they might experience difficulty in finding situations within tasks 1 and 2 when the required values were exactly equal, he implied in his choice of language that this might be unachievable. For example, in posing Task 1, he instructed the students to try to get all three area measurements 'the same, or as close as you can get them to be the same as each other' [STP6(Trans)] and during Task 2 Tim gave validity to students responses that were almost equal by describing them as 'kind of alright' and 'close' [STP6(Trans)]. By doing this, Tim was suggesting to his students what would be an acceptable interpretation of equality within the context of the activity they had been set.

A second underlying tension, which Tim did not seem to have noticed, related to the potential conflict over his choice of algebraic labelling for this problem. He used the letters **a**, **b** and **c** as registers of memory for the measured values of the areas in each task and, although it was obvious to the students that these values were varying as vertices were dragged, the presence of the letters, and the obvious connections with students emergent understanding of algebraic representation, was not commented upon by Tim or his students. There was little likelihood of the students experiencing any cognitive conflict with respect to the traditional representation of Pythagoras' theorem as $a^2 + b^2 = c^2$ as this was an introductory lesson for them, however it did not appear that Tim had considered that this might present a problem to them in the future.

The design of this activity offered a mathematical experience to students that would be very hard to conceive without the technology. As an exploratory activity, it was attempting to focus upon the students' conceptual understanding of the properties of triangles and the particular case of the area properties of squares constructed on the three sides. Without the constraints of the English mathematics curriculum this could lead into further exploration of the relationships between the areas of the squares for particular non right angled triangles, or allow for the construction of alternative polygons, other than squares. Tim's privileging of the use of a dynamic technological tool in this way could be seen as an attempt to promote students' exploratory work within the mathematics curriculum. However, in order to comment upon if (and how) Tim built on the students' experiences in this lesson, such that they could connect it with the related skills for the application of this knowledge to new, related problems, it would be necessary to observe how this lesson was situated within subsequent lessons on this mathematical topic.

Pierce and Stacey's definition of the use of technology in order to 'build metacognition and overview' is exemplified by providing students with a 'birds-eye view' of the topic or by 'encapsulating a multi-step process by a single command' (Stacey, 2008). Within the context of Tim's lesson, the sense of a 'birds-eye view' related to Tim's decision to allow students to initially explore the more global situation of 'any' triangle, prior to focussing in on the particular situation for right-angled triangles.

6.2.3 Evidence of a hiccup

The analysis of this lesson led to the identification of five hiccups, which were broadly classified as shown in Figure 6-13.

Name	Sources	References
TP6 Hiccup1 - Difficulties over identification of dynamic objects	1	2
TP6 Hiccup2 - Students' mis-interpretations of task - 'different way around'	3	4
TP6 Hiccup3 - Instrumentation (T) 'your c has gone off the screen'	1	1
TP6 Hiccup4 - Instrumentation (S) - grabbing and dragging	1	1
TP6 Hiccup5 - Jump from MRT task to trad paper and pencil problem	1	1

Figure 6-13 Extract from Nvivo8 software showing the coded hiccups during Tim's 'Pythagoras exploration' [STP6].

What follows is the detailed analysis of one of these hiccups [TP6 Hiccup2], and an articulation of how this event may have contributed towards Tim's situated learning during and soon after the lesson.

This hiccup was observed during a point in the lesson when Tim was clearly reflecting deeply on the students' contributions to the shared learning space and 'thinking on his feet' with respect to responding to these. It coincided with his observation of an unanticipated student response. The chosen hiccup, which was previously mentioned on page 152 (see Figure 6-10), came about when a student had found a correct situation for Task 2, that is the two smaller squares' areas summed to give the area of the larger square, but the situation did not meet Tim's activity constraint of ***a* + *b* = *c*.**

Tim commented about this in his personal written reflection after the lesson,

> *One student had created a triangle for which a+b did not equal c, but (I think) a+c=b. This was also right angled. This was an interesting case because it demonstrated that the 'order' did not matter... when the sum of the smaller squares equalled that of the larger square, then the triangle became right angled.* [STP6(Journ)]

Tim revised the TI-Nspire file after the lesson, providing some convincing evidence of his learning as a result of the use of the MRT. Tim gives an insight into his learning through his suggestions as to how he thought that some of these perceived difficulties might be overcome by some amendments to the original file.

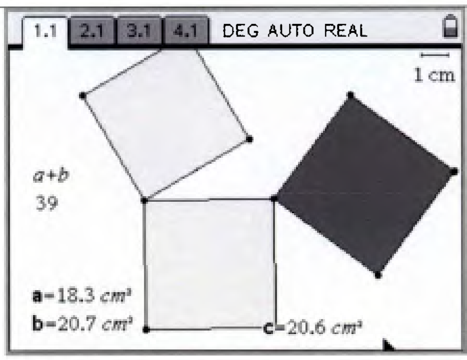
<table>
<tr><th>Opening screen</th><th>Revisions to the construction of the environment</th></tr>
<tr><td>

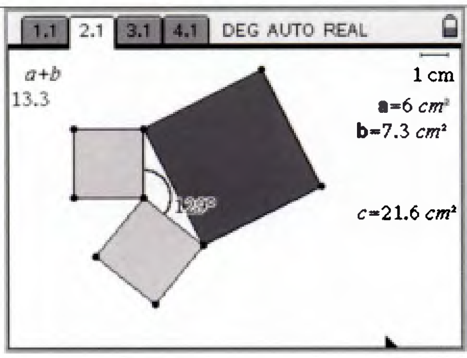
Figure 6-14 Tim's revised TI-Nspire file for Task 1 [STP6(tns-T)v.2]</td><td>Task 1 (revised): The squares whose areas were previously represented by 'a' and 'b' have been lightly shaded and the square represented by the area measurement 'c' has been darkly shaded.</td></tr>
<tr><td>

Figure 6-15 Tim's revised TI-Nspire file for Task 2 [STP6(tns-T)v.2]</td><td>Task 2 (revised): The shading from Task 1 has been replicated and Tim also added an angle measurement for the angle that is opposite the side that was intended to represent the hypotenuse.</td></tr>
</table>

Table 6-2 Tim's revisions to the .tns file for the activity 'Pythagoras exploration' [STP6].

Both of these amendments to the original file suggest that Tim wanted to direct the students' attentions more explicitly to the important representational features. He wanted to enable the students to connect the relevant squares to their area measurements and 'notice' more explicitly the condition that when the condition for the areas was met, the angle opposite the hypotenuse would be (close to) a right angle. This seemed to suggest that Tim was still trying to overcome the inherent difficulty when using mathematical software concerning the display of measured and calculated values in the hope that students would achieve an example where the areas were equal and the measured angle showed ninety degrees. This seemed to suggest a conflict with his earlier willingness to try to encourage his students to accept an element of mathematical tolerance when working with technology with respect to the concept of equality.

Finally, Tim did raise the concern that, in his design of Task 2, 'Some students struggled to get the areas of the two smaller squares to equal the area of the larger square, and this was probably because there was 'too much' possible variation here for them to be comfortable with' [STP6(Quest2)]. This provided further evidence for Tim's reflections in relation to the aims of this study, and focuses on one of the key areas of both pedagogic and mathematical knowledge construction for teachers learning to use MRT for the classroom.

6.2.4 How is the description of the lesson activity informed by the theoretical underpinning?

This study aimed to articulate more deeply the nature of, and processes involved in, teachers' learning as they introduced the MRT into their classroom practices. The identification and analysis of the classroom hiccup, and the identification of Tim's subsequent associated actions, provided evidence of his possible situated learning in relation to the use of the MRT in privileging students' explorations of variance and invariance. This learning was related to the following.

- The decision to use the MRT for this activity and display the students' results publicly resulted in an unanticipated student's responses becoming the focus within the classroom discourse. Consequently, Tim was prompted to develop a new repertoire of dialogue in response to this classroom experience that acknowledged the student's correct response within a wider mathematical sense.
- The construction of the MRT environment and the way in which its appearance would support students to notice the variant and invariant features. This included design decisions such as adopting a more subliminal approach to the labelling of objects by using shading. Tim's decision to 'rename' the squares as lighter or darker gave him an alternative means to refer to them, which avoided the need to know which square was 'a' and which square was 'b'. The important focus was the sum of the area measurements of the two lighter squares.
- The inclusion of the angle measurement seemed to be in response to Tim's perception that he needed to focus the students' attentions more explicitly on the significance of the angle measurement itself, rather than relying on their awareness that the angle appeared to be a right angle.

In developing his original instrument utilisation scheme for this activity Tim had not envisaged the scenario of the student response that led to this lesson hiccup. Tim was confident in his subject knowledge and he knew that, if the central triangle appeared to contain right-angle, the resulting student's screenshot must be an instance of Pythagoras' theorem, irrespective of the on-screen measurements and calculations. Consequently, he did not dismiss the screen as an incorrect response but sought to validate it from a wider mathematical perspective.

The analysis of this one lesson hiccup has provided an insight into the relationship between Tim's situated learning in the classroom and the potentially more epistemic learning as evidenced by his direct actions in redesigning the activity. It was anticipated that the close analysis of all of the lesson hiccups experienced by Tim during the second phase of the study would offer a valuable source of data from which to hypothesise the trajectory of his situated learning over the period of the study. The identification and analysis of all of Tim's potential lesson hiccups for each activity is detailed within Appendix 9 and the summary of Tim's learning trajectory is provided in Chapter 7.

6.3 Introducing Eleanor

Eleanor had qualified in 1987 with an ordinary Bachelor of Education degree in mathematics (11-16) from West Sussex Institute of Education and was in her early forties. She did not explicitly mention any particularly memorable experiences of using technology during this period. Eleanor recollected being involved in a project in her first few years of teaching that had involved handheld technology but commented that 'it all seemed a bit of a blur'. She said that she had 'dipped in with the use of Texas graphics calculators for many years' and that she was 'confident with most aspects but not so with the statistical side'. Eleanor thought that she tended to use technology 'as prescribed by someone else' and that she was 'not so confident in designing handheld based activities' [CELQuest1)].

For the duration of the study, Eleanor was the head of mathematics at Coastway School, a voluntary aided school for girls aged 12-16 years in a coastal town in southern England. The vast majority of the girls in the school were of white British heritage and had English as their first language. The girls had consistently achieved significantly above the national expectation in the GCSE examinations at age sixteen and the school was judged to be outstanding at its last external school inspection in 2007. The school was designated as a Technology College in 1999.

As a technology college the school was well-resourced and, in addition to several class sets of graphical calculators, the department also had its own half-suite of computers with access to a full range of mathematical software. Access to the computer suite was organised via a departmental timetable, which ensured equitable weekly access to all classes. Eleanor had an interactive whiteboard and data projector in her classroom and her own laptop, which she plugged into the school network when she was teaching. When asked about her own personal confidence with ICT for mathematics and whether she had any particular favourite resources, she commented that she 'enjoyed the geometry of GSP[6], although (I) feel weak in using this' and that she 'enjoyed the graphing elements of handheld technology' [Journ].

6.3.1 Eleanor's participation during Phase One

At the beginning of the project, Eleanor worked with her school colleague to design and plan activities and, as they both taught classes in year 8, they decided to focus on the teaching of a particular unit of work that focused on geometry. However, her colleague taught the most able students and Eleanor taught the least able students within the year group. Eleanor's first two activities closely replicated established paper and pencil approaches[7]. Towards the end of the first phase of the project Eleanor devised a lesson (CEL5) using the MRT for her more able group of 14-15 year olds. Eleanor privileged numeric and geometric representations in her early activity designs although she did involve syntactic inputs in the activity [CEL5].

The detailed description and analysis of Eleanor's activities using the MRT during the first phase is provided in Appendix 7. Each activity was analysed using the IUS map (Figure 5-2) such that the flow of representational forms was made explicit. The individual IUS maps were overlaid and Figure 6-16 shows diagrammatically the different input and output representations that Eleanor used during the first phase and her associated instrument utilisation schemes.

[6] The Geometer's Sketchpad software
[7] The second of these activities [DCH2] was described in detail in Section 4.3.4 (see Figure 4-1 and Figure 4-2).

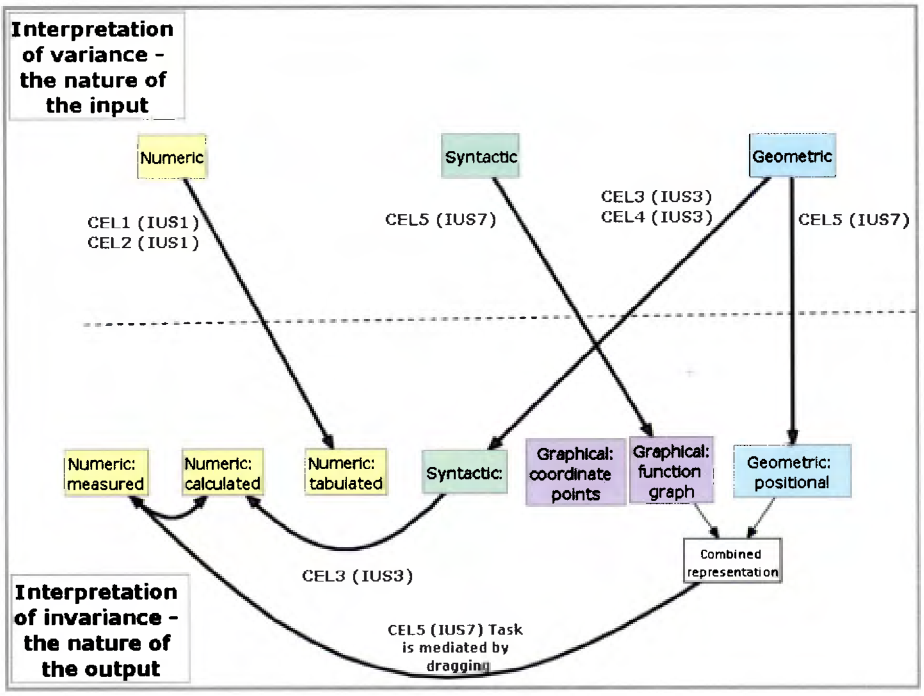


Figure 6-16 The summary of Eleanor's Instrument Utilisation Schemes produced from the analysis of her Phase One lesson data.

The first phase of the study revealed the following aspects of Eleanor's trajectory of development:

- Her activity designs had moved from being totally static to involving dynamic actions as an integral part of the conjecturing process.
- There was a shift in her pedagogic approach to include more of an explicit focus on the invariant and variant properties within activities.
- Her confidence to adapt activities devised by others to align more closely to her personal pedagogical philosophy had grown.
- She had begun to design TI-Nspire files for herself from a blank new file.
- Her activities were beginning to involve several different mathematical representations in a considered way.

6.3.2 Observing Eleanor in the classroom – The lesson introducing simultaneous equations

Following my decision to select Eleanor as one of my two case studies for further research, I arranged to observe a series of one hour lessons that she was planning for her higher achieving Year 9 students (girls aged 14-15 years), which would introduce them to the concept of simultaneous linear equations. These students were following the key stage 3 programme of study and Eleanor had assessed that they were currently working at National Curriculum level 7.

Prior to us meeting to discuss the planning of the lessons in more detail we exchanged emails and Eleanor commented,

> *'I've not used handhelds to introduce s.e. before, so a graphical approach is new to me. I usually talk about something like 2x + 4 = 6, what's x? then x + y = 6, what's x and y, eliminating one variable, then show process of solving s.e.. Obviously using nspires means solving only types that are y =, whereas I usually go to x+y types.' [Journ]*

From this brief reply I was not entirely sure whether she had used other technology to introduce simultaneous equations and that it was an approach with the handhelds that was new. However a later conversation confirmed that it was not a topic with which she had previously used technology. Eleanor and I met a few weeks before the planned lessons and we talked about her usual introductory approach alongside a wider discussion of the topic, which included a consideration of its inherent mathematical ideas and the role that different representational forms might play in supporting her students to gain a more holistic understanding. Eleanor settled on a pedagogic approach that she had learned about from another teacher in the T^3 network. This took the point of intersection as the initial premise, and ask the students to create linear functions that passed through the given point and would lead to the students generating 'families' of linear functions that contained the point as a common solution. Following our discussions Eleanor prepared the associated lesson plan, Smart Notebook file and the TI-Nspire file for the lesson (See Appendix 12). The detailed lesson analysis was completed for the second of the sequence of three lessons and, to support the contextualisation of the selected lesson, a brief summary of the first lesson is provided.

During the first lesson, Eleanor had begun by posing a mathematical problem concerning a scenario from her own real-life, which took place in a coffee shop in which the price of the purchased items was unknown (see Figure 6-17).

THIS IMAGE HAS BEEN REDACTED DUE TO THIRD PARTY RIGHTS OR OTHER LEGAL ISSUES

Figure 6-17 The opening page of Eleanor's Smart Notebook file displayed to the students to introduce the activity. [CEL9(LessStruct)]

Eleanor gave the students a few minutes to try to solve the problem both orally and using paper and pencil before informing them that, by the end of the forthcoming sequence of lessons, they would learn a number of different strategies for solving similar problems. She took some suggested answers from the students (all incorrect) but did not resolve the observable student discomfort by offering a correct solution to this problem.

Eleanor then wrote the coordinate pair (3,6) on her ordinary whiteboard. She gave the students a few minutes to discuss amongst themselves how many lines might pass through that point and then, gesturing to the three and the six, asked them to make up their own 'rule that connects those two numbers' [CEL9(Trans)]. Noticeably, she gave no indication to the students with respect to how they should express their rules. The students typed their rules into the handhelds and sent them to Eleanor's computer using the quick poll functionality. These results were displayed, revealing a range of different numeric, syntactic and text responses which Eleanor discussed with the group before resending the poll so that the students could revise their first response. The second poll occurred less than two minutes before the end of the lesson and by the time Eleanor had displayed the students' responses for public view, she had only had a limited opportunity to absorb the multiple responses. The lesson ended abruptly when the school bell sounded.

After the lesson, Eleanor and I had a brief discussion in which she said that she planned to begin the next lesson by redisplaying the results of the poll and holding a whole class discussion about the students' responses. Consequently, we saved the image of this poll onto her laptop. What follows is the detailed description and analysis of the next lesson.

The lesson was observed, audio recorded and the students' TI-Nspire files were collected at three points during the lesson to coincide with occurrences of whole class discussion. Where possible, I also saved an image of the class computer screen when it was the topic of the classroom discourse providing it did not cause an unnecessary interruption.

6.3.2.1 A description of the lesson activity

Eleanor had planned the lesson conscientiously and had prepared a formal written lesson plan [CEL9(Plan)], a set of Smart Notebook pages [CEL9(Struct)] and a TI-Nspire file [CEL9(tns-T)]. In her lesson plan she had identified her lesson objectives to be,

> *To find linear equations that have one common solution and to explore the mathematics behind these common factors. [CEL9(Plan)]*

She further expanded on this by explicitly stating that the students would,

> *...explore the intersection of linear equations graphically as a means of an introduction to simultaneous equations and to explore how the Nspire technology can enable us to do this. [CEL9(Plan)]*

The lesson contained three distinct episodes which centred on the use of the MRT technology and the descriptions of each of the associated environments are provided in Table 6-3.

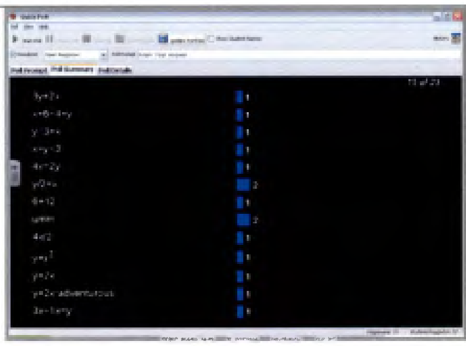
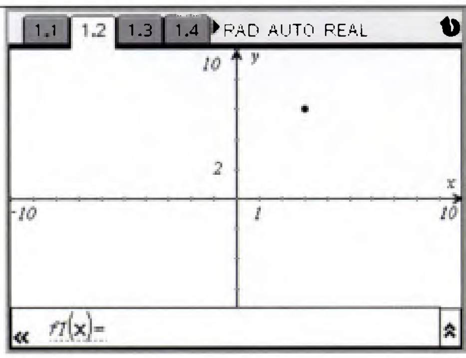
<table>
<tr><th>Opening screen</th><th>Description of the construction of the environment</th></tr>
<tr><td>
Figure 6-18 The image of the students' responses to Eleanor's initial poll [CEL9(Image1)]</td><td>Episode 1: The screen capture of the students' responses to Eleanor's request for a 'rule that connects those two numbers' when referring to the coordinate pair (3,6) [CEL9(Trans)]. This poll had been conducted at the end of the first lesson.</td></tr>
<tr><td>

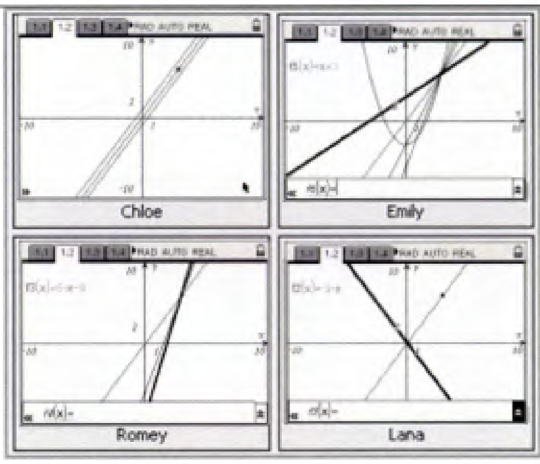
Figure 6-19 The opening screen of the .tns file for the activity 'Crossing linear graphs' [CEL9(tns-T)page2]</td><td>Episode 2: Eleanor's TI-Nspire file constructed prior to the lesson on which a single coordinate point (3,6) had been constructed on a Cartesian graphing plane.</td></tr>
<tr><td>

Figure 6-20 A close-up of four students' handheld screens showing their responses to the activity [CEL9(ScreenCapt)].</td><td>Episode 3: The third teaching episode focused on the whole class discourse concerning the screen capture of the students' responses to the previous activity. A sample of the students' handheld screen captures are shown in Figure 6-20.</td></tr>
</table>

Table 6-3 [CEL9] The lesson episodes

Eleanor began the lesson by displaying the results of the quick poll, which she had conducted at the very end of the previous lesson but had found herself with insufficient time with which to discuss the outcomes with her students in the way that she had envisaged (see Figure 6-21).

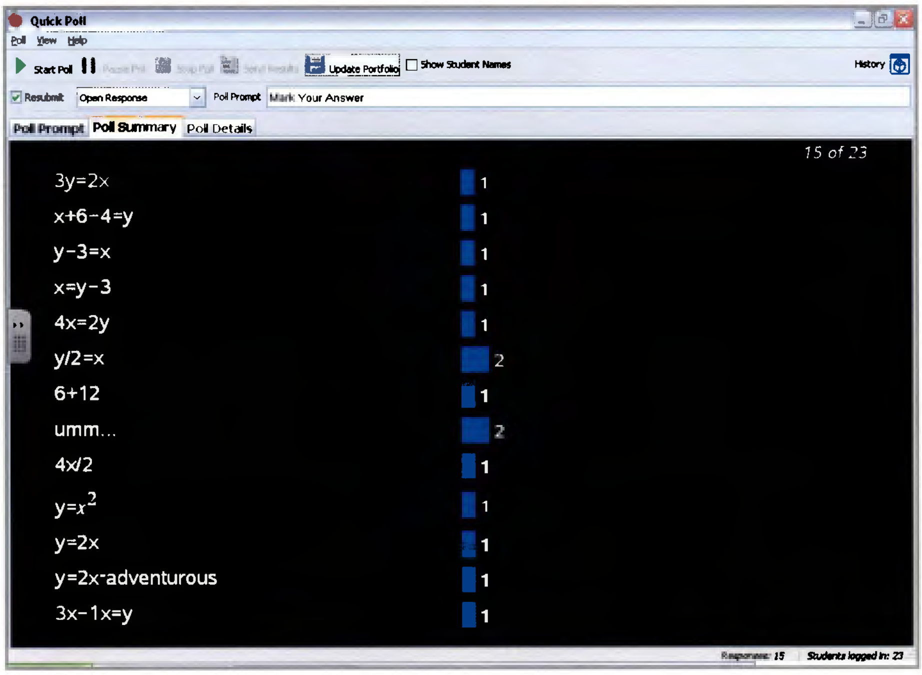


Figure 6-21 The image of the students' responses to Eleanor's initial poll which asked for a 'rule that connect the two numbers' within the coordinate pair (3,6) [CEL9(Image1)].

Eleanor reopened the discourse by saying,

Eleanor: *Alright yesterday, towards the end, I did a 'quick poll' and got you thinking about what rule you could come up with connecting, three and six. Okay? And the most common one that was spoken about was double that one [Eleanor gestured to the three, written on the whiteboard]. Okay? And what we started to do... which is why I've brought this screen up here... is to look at... oh most of you were using x and y... Why did we start using xs and ys? What was it about these numbers?*

Student: *Because of the axes?*

Eleanor: *Because of the axes, okay? Because of the axes, what else? What about this and this [Eleanor gestured to the brackets] what does that help with that... to do with the three and six? It is to do with the axes? Which is x? Which is y?*

Student: *Three is x, six is y*

Eleanor: *Okay? How come we ended up using x and y? Some of these [Eleanor gestured to the poll answers on screen], do they all work? At the end of last lesson... the first one, I think we all realised. Does this work? three y... does it equal to two xs?*

Eleanor spent the next five minutes focusing on selected students' responses and asking the class to reason whether they 'worked' within the context of the activity they were set. During this sequence, the dialogue did not include words such as, expression, equation, function, substitute or graph. Eleanor's emphasis was reliant on the students noticing the invariant property that when the values x=3 and y=6 were substituted into a selected poll response, it resulted in an equality.

The second main teaching episode of the lesson commenced when Eleanor sent the students the TI-Nspire file 'Crossing linear graphs' [CEL9(tns-T)]. She simultaneously displayed the page shown in Figure 6-22 from her own handheld device to the class using the 'live presenter' mode.

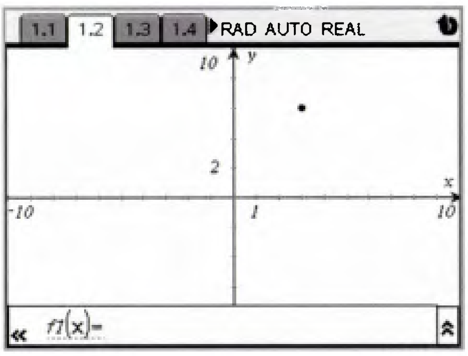


Figure 6-22 The opening screen of the .tns file for the activity 'Crossing linear graphs' [CEL9(tns-T)page2]

She instructed the class,

Eleanor: *I want you now to draw a linear graph that will go through that point... which is at what? What's that point at Stacey? What's that point at?*

Stacey: *three six*

Eleanor: *I now want you to draw a graph that goes through three six.*

Eleanor proceeded to talk the students through the instrumentation process to draw a function within the graphing application and, having chosen the function f1(x) =

2x as her example, produced the following graph.

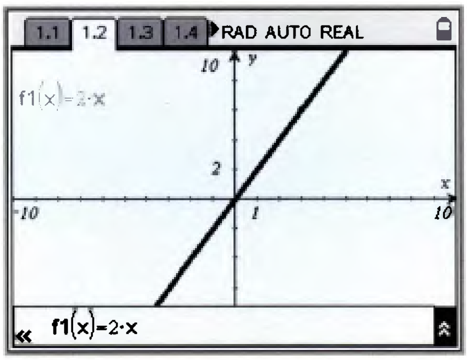


Figure 6-23 Eleanor's handheld screen after she had demonstrated how to enter the function f1(x)=2x within her introduction of the activity 'Crossing linear graphs' [CEL9(tns-T)page2]

Eleanor then set the following activity for the students,

> *Right, I want you to come up with... I've got on my notes that I want you to come up with ten different ones... I think for the moment we'll go for four? Do you reckon? But I don't want you to put two x in yours... See if you can come up with at least four... You've got paper in front of you if you need to put anything down okay? Control G will get you on and off that bottom bar.* [CEL9(Trans)]

As the students began to work on this task, Eleanor changed the class display to publicly show the screen capture view of the students' handheld screens, which was set to automatically refresh every thirty seconds. The students then worked on this challenge for just under ten minutes and, during this time, Eleanor moved around the room to support and respond to the students' questions. These included questions such as 'Can we try... Can we do two x plus one minus one?' and 'I want to get a line straight up?' [CEL9(Trans)].

This period of working was interrupted when Eleanor, on noticing Eloise's handheld screen display, made the comment, 'Eloise has got quite a few there' [CEL9(Trans)]. Making reference to Eloise's screen (see Figure 6-24), when one of the students posed the question 'How did she do it' Eleanor reposed the question to Eloise and invited Eloise to give her explanation to the class.

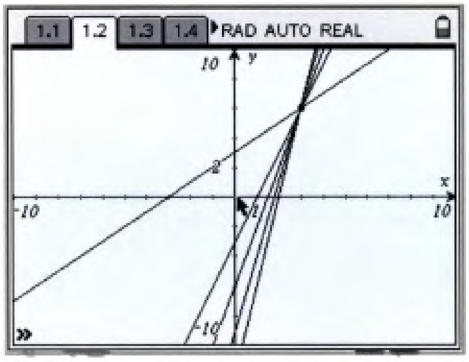


Figure 6-24 Eloise's response to the activity 'Crossing linear graphs' [CEL9(tns-S)].

Eloise responded by reading out her list of entered functions (shown in Figure 6-25), which Eleanor recorded on the ordinary whiteboard. There was no discussion of these functions and Eleanor did not ask Eloise to explain her reasoning with respect to how she had generated them.

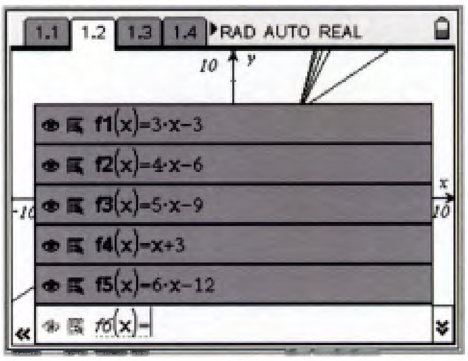


Figure 6-25 Eloise's functions, entered in response to the activity 'Crossing linear graphs' [CEL9(tns-S)].

Eleanor invited the class to contribute 'any other graphs that are different to those ones' resulting in several students' names being mentioned by the class. These included Rosie and Emily and their handheld screens are shown in Figure 6-26 and Figure 6-27.

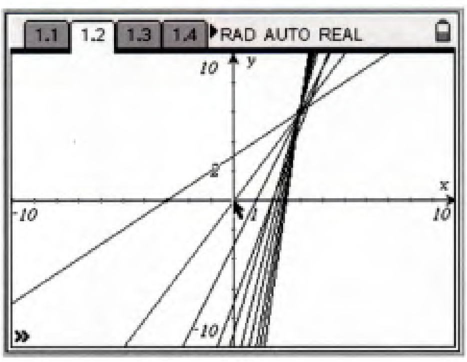


Figure 6-26 Rosie's response to the activity 'Crossing linear graphs' [CEL9(tns-S)].

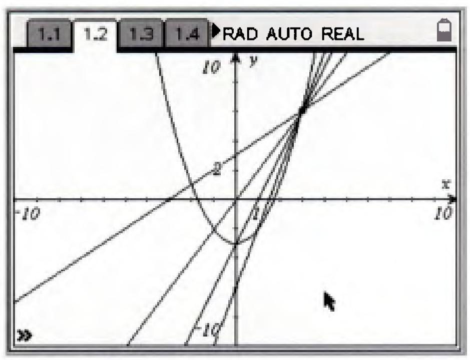


Figure 6-27 Emily's response to the activity 'Crossing linear graphs' [CEL9(tns-S)].

Eleanor took an additional contribution from Rosie, 'nine x minus twenty-one' and dismissed Emily's curved graph by saying, 'I don't want to use that for this instance yet... [pauses] it is... [pauses] I don't want to use it... [pauses] so....', Eleanor moved on to show the students how to include an additional mathematical representation, the function table alongside their graph screen [CEL9(Trans)].

At this point Eleanor appeared flustered, saying 'Girls I need you to concentrate because we're looking backwards, this is a different way of looking at this' before leading a class discourse in which, using Eloise's handheld, Eleanor talked the students through the instrumentation process to display the function table alongside their graphs. She then located the value x=3 in the first column of the table moved the cursor horizontally to enable the students to observe the corresponding values for each of their functions (see Figure 6-28 and Figure 6-29).

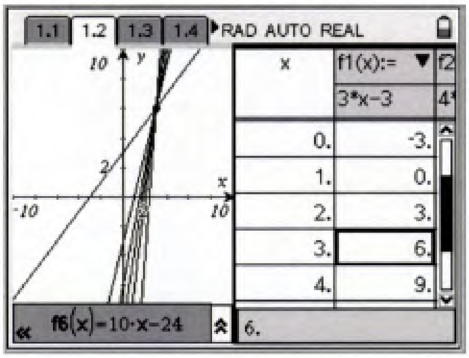


Figure 6-28 Eleanor's handheld screen displayed during her demonstration that f1(3) = 6 [CEL(tns-T)]

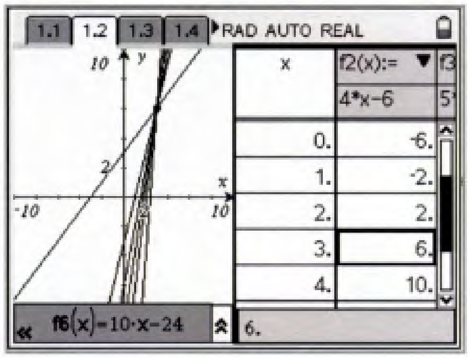


Figure 6-29 Eleanor's handheld screen displayed during her demonstration that f2(3) = 6 [CEL(tns-T)]

There was some confusion at this point as Eleanor had omitted to give the students the steps needed to make the function table the active application to enable them to move between its cells.

In the final teaching episode of the lesson, Eleanor returned to the list of Eloise's functions that had been recorded on the ordinary whiteboard, as shown in Figure 6-30.

THIS IMAGE HAS BEEN REDACTED DUE TO THIRD PARTY RIGHTS OR OTHER LEGAL ISSUES

Figure 6-30 Eleanor's whiteboard annotations during the activity 'Crossing linear graphs'. [CEL9(Image2)]

Eleanor posed the question to the class, 'How can I use that fact with any of these here [*Eleanor gestured to Eloise's functions on the board*] to prove that it goes through (3,6)?' and the following dialogue ensued,

Student: *Because... [hesitates] you... [hesitates]*

Eleanor: *What can I do?*

Student: *Errm.. because it's two x so it's.. [hesitates]*

Eleanor: *Go on...*

Student: *Because isn't like, one of them's y and one of them's x so you know that the one that's y you just work out that if it's going to be the same...*

Eleanor: *Okay you're on the right.. you're getting there... Ella?*

Ella: *But if you like, if you took ages and like worked them all out like one plus three equals and then did it up to three and you work out that all of them equal three and the answer is six*

Student: *Yeah, that's what I mean, so if x equals three, you'll have y equals six and then you know that for the first one then x equals three plus three [Eleanor annotated whiteboard] is six.*

Eleanor: *[to the class] Is that right? Is that right?*

Students: *[positively] yes, yes*

Eleanor selected a second function from Eloise's list, y = 4x – 6, saying 'What about this one' and encouraged the students to reason why they could say that it 'worked'. She completed this final phase of the lesson by saying,

> *So we know by looking at the algebra, we know by looking at the algebra that it's got to be right, they've got to be right. Okay? Would y equals two point five x work?* [CEL9(Trans)]

There was a confident response of 'No' from the class and, when Eleanor questioned why a student responded 'because it wouldn't equal six, it would equal...[Student hesitated]... ...something else' to which Eleanor responded 'something else?... that'll do at this time of day [Eleanor laughed].' Eleanor concluded the lesson by saying,

> *So we know for a fact that all those graphs all go through one unique solution which is three six. Now I want to hold that thought for tomorrow's lesson when we're gonna work backwards. Okay? And I'll leave that with you.* [CEL9(Trans)]

The end of lesson bell sounded shortly after and the class left the room.

6.3.2.2 An analysis of the lesson activity

Adopting a similar strategy to the analysis of Tim's lesson, I used the same three strata from Pierce and Stacey's framework to support the analysis of Eleanor's lesson with the aim to articulate the pedagogical opportunities afforded by the technology.

6.3.2.2.1 The task layer

Eleanor's interpretation of variance and invariance during this activity was underpinned by the assumption that, for any of the infinite number of linear functions that 'passed through' the coordinate pair (3,6), the invariant property was the equality statement produced when the values of x and y were substituted into a 'correct' function. However, from the pupils perspective there was an additional complexity that was inherent in their possible interpretation of invariance, which concerned the infinite number of possible functions than could exist in the graphical domain. This is illustrated by the student's response to the initial activity shown in Figure 6-31 where the student had entered fifteen functions and achieved only two successful outcomes[8].

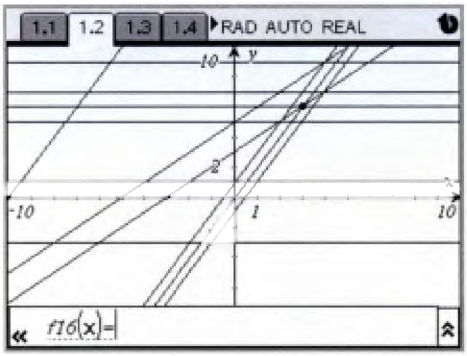


Figure 6-31 A students' handheld screen showing a search for linear functions which satisfy x=3 and y=6 [CEL9(tns-S)].

The extent of this particular student's response provides sound evidence for the way in which the technology gave the student quick access to many accurately drawn graphs. However, without Eleanor's intervention it was not obvious what this particular student was learning. At no time during the lesson did Eleanor or any other students notice this particular screen on the classroom display.

The most striking pedagogical opportunity within this lesson related to the multiple representations selected by Eleanor to support the mathematics under discussion. This was stimulated from the outset by the students' own responses to Eleanor's request for a rule that would connect the 'three and the six' within the coordinate pair (3,6). This sequence of events that followed is illustrated in Figure 6-32.

[8] This student's response was not noticed by Eleanor during or after the lesson.

THIS IMAGE HAS BEEN REDACTED DUE TO THIRD PARTY RIGHTS OR OTHER LEGAL ISSUES

Figure 6-32 A diagram showing the representational route through Eleanor's activity 'Crossing linear graphs' [CEL9].

The students' replies to this question included a range of numeric, textual and syntactic formats (see Figure 6-18 on page 169). Eleanor supported the students to translate the most common response (y = 2x) into the syntax required by the MRT in order to generate a linear function. Eleanor mediated this transition by saying, 'How do I write in, to remind you, if I wanted to draw a graph of y equals two x, for instance? So y equals is already there' [CEL9(Trans)]. In doing so she assumed that the device's syntax 'f1(x)' was equivalent to 'y=' and implied that the class were familiar with this from a previous lesson. This led to the generation of the graphical representation, which in turn led Eleanor to make the explicit connection to the 'by-hand' syntax by acting as a scribe for the students on the ordinary whiteboard. Eleanor then introduced the tabular representation of the function values before completing the lesson by verifying the validity of selected rules by substitution for the coordinate pair (3,6).

6.3.2.2.2 The classroom layer

There was a clear sense that Eleanor was renegotiating elements of the didactic contract within the lesson as on several occasions she shared her apprehension with the class, which resulted from her decision to introduce simultaneous equations by starting with the point of graphical intersection. This was exemplified by her statement 'Girls I need you to concentrate because we're looking backwards… this is a different way of looking at this' [CEL9(Trans)]. This had the potential to bemuse the students as they had not met this mathematics before so would have no idea what approaching the topic forwards might look like!

In addition, the explosion of methods and increased student agency was made public through the shared display, which prompted the students to ask their own mathematical questions of Eleanor and each other. Indeed the path of the lesson was triggered by Eloise's response to the activity, something which Eleanor could not have planned to happen in advance. Eleanor was open to the students' suggestions where they seemed to fit with her intended lesson path. However there was evidence from her reaction to Emily's handheld screen (see Figure 6-27 on page 173) of a reluctance to deviate from this path, which I probe in more detail later in this chapter.

6.3.2.2.3 The subject layer

In our discussions prior to the first lesson, Eleanor had showed me the resource that she had most recently been using to introduce simultaneous equations with another class (see Figure 6-33), which was an activity from the subscription based mathematical content provider MyMaths (Jackson and Jackson, 2010).

THIS IMAGE HAS BEEN REDACTED DUE TO THIRD PARTY RIGHTS OR OTHER LEGAL ISSUES

Figure 6-33 An extract from the 'MyMaths' resource which Eleanor normally used to introduce simultaneous equations (Jackson and Jackson, 2010).

Having reviewed this traditional approach, which would also be found in many text book schemes, it was apparent that by adopting a graphical starting point, Eleanor was reordering her teaching of the topic as a direct result of the functional opportunity that the technology had afforded. However, this decision brought some tensions into existence for Eleanor concerning the transition to the sorts of problems the students would be subsequently asked to solve within the formal examination setting. Eleanor articulated this by saying 'Obviously using nspires means solving only types that are y equals whereas I usually go to x plus y equals types' [Journ]. This discussion led to Eleanor's decision to plan a series of lessons, the aim for which was to provide the students with a meta-level view of the concept of simultaneous equations before they moved towards the more pragmatic 'drill and practice' questions.

Within the lesson, there were times when TI-Nspire was used as a checking device to verify a mathematical condition. For example, the device was viewed as the authority on the mathematics when Eleanor asked the students to bring the table of function values into view and check the calculated values of y when the x values was three. Eleanor did not conclude the dialogue at this point and she pushed the students to explain why, for the 'correct' functions, this would always be the case. This question brought the mathematics back to its starting point in the lesson, the significance of the coordinate point (3,6), and the subsequent discussion appeared to bring an element of closure to the activity for the majority of the students and the teacher, who was visibly elated at the outcomes of the lesson. She remarked immediately, 'That was lovely, that end bit was lovely... ...and I love what Kayleigh said, 'well 'cos that is what we started with?'... how obvious is that?' [(CEL9)Trans].

6.3.3 Evidence of a hiccup

There were eight identified broad classifications of hiccups during the lesson as shown in Figure 6-34.

Name	Sources	Referen
EL9 Hiccup01 - Students give diverse responses - quick poll - used positiviely by teacher	2	4
EL9 Hiccup02 - Instrumentation (S) - 'where's equals'	1	1
EL9 Hiccup03 - Instrumentation (S) 'how do you get down' - to send quick poll response	1	1
EL9 Hiccup04- 'Emily how did you do it'	1	5
EL9 Hiccup05 - 'How do you get a graph that goes straight up'	1	1
EL9 Hiccup06 - Instrumentation (S) 'How do you get onto it' - referring to full function entry line	1	1
EL9 Hiccup07 - Instrumentation(T) struggles to display table of values	1	1
EL9 Hiccup08 - Instrumentation (S) - 'How do you get that' - referring to the table	1	1

Figure 6-34 Extract from Nvivo8 software showing the coded hiccups during Eleanor's activity 'Crossing linear graphs' [CEL9].

The particular hiccup selected from Eleanor's lesson, prompted by the student questions 'Emily, how did you do it?' [CEL9Hiccup04], takes the notion of a lesson hiccup occurring as a result of an unanticipated student response within the parameters of the previously described activity that Eleanor had constructed in collaboration with me.

Less than two minutes after Eleanor had asked the students to input functions into the MRT to pass through the given co-ordinate point (3,6), Emily's screen (shown in Figure 6-35), was one of the twenty-two screens on display to the class through the whole-class digital projector.

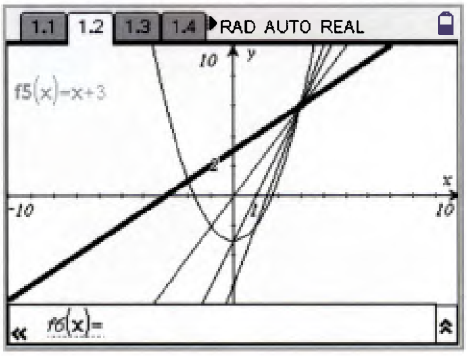


Figure 6-35 Emily's response to the activity 'Crossing linear graphs' [CEL9(tns-S)].

The following classroom conversation ensued.

Students:	*[background conversations] Is it two x equals y? No you miss out the y. What does it actually mean? Emily, how did you do it?*
Eleanor:	*[to the class] Remember you don't need to write in the y equals.*
Student:	*[background conversation] so if we've got x plus three.*
Student A:	*Emily, Emily has got a curved graph.*
Eleanor:	*Emily, Emily's got a curve there [EL pauses and turns to look at the screen].*
Student B:	*Nice*
Eleanor:	*[to Emily] Try and avoid it at the moment... [EL pauses] for what we are doing... [EL pauses] I like that... [EL pauses] Okay?*
Eleanor:	*[to the class] Emilie i-e-, where's Emilie i-e-?* [9]

[9] There were two students with similar names Emily and Emilie in the class.

Eleanor: *[to Emilie] Did you manage to get rid of that one?*

Emilie: *Yes*

Eleanor: *[to Emilie] Brilliant, well done so you've got one more - maybe adjust it - think of what we did - may be get it to move up okay?*

Eleanor: *[to Emily] Errm try not using curves - no not try don't [EL is firm] I do want just straight lines at the moment.*

Eleanor: *[to the class] Billie, where's Billie?*

No more examples of curved graphs appeared during the lesson and, at no point did Eleanor hold a dialogue with Emily, or any of the other students with respect to her rationale for the response she gave to Emily. Later in the same lesson, the following dialogue occurred when Eleanor was leading a whole-class discussion concerning the students' functions that intersected at the coordinate (3, 6).

Eleanor: *Has anybody got any other graphs different to that one? ...those ones sorry. [EL gestures to the equations on the whiteboard] Has anybody got any different ones?*

Student C: *Emily has got one.*

Eleanor: *[to Rosie] Rosie?*

Student C: *Emily's got one, Emily's*

Rosie: *nine x minus twenty-one.*

Eleanor: *Say it again nine x minus twenty-one. Emily? Was it Emily?*

Student C: *Emily's got quite a lot.*

Eleanor: *[to class] Any of yours different to any of that?*

Students: *Not really, no.*

Maria: *Do Emily, she's got a curved graph.*

Eleanor: *Yeah, I don't want to use that for this instance yet... [EL pauses] it is... [EL pauses] I don't want to use it... [EL pauses] so.... [EL pauses] Okay I'm going to stop now.*

Eleanor then moved on to the next phase of the lesson in which the students looked at the tables of values associated with each function and worked to justify why their functions had one unique solution leading to the associated algebraic proofs.

Following the lesson, I asked Eleanor for her thoughts in reaction to Emily's response to the activity to which she replied:

> *I liked it because it made me realise that my instructions had not been totally clear in only producing linear graphs and she had been quite clever in producing a rule that was quite different to what we had discussed or what I was expecting. I realised at that moment that I hadn't clarified that I wanted only linear graphs. For the direction of the work, solving simultaneous equations, in my mind it was only linear equations. I did not want to create confusion by having to deal with other equations, such as quadratic ones. I would have preferred to have made sure I had said linear in the first place which is why I was quite insistent.* [CEL9(Int)]

Eleanor's comment provides strong evidence of a possible inner conflict. Whilst she respected Emily's broader interpretation of the activity, which clearly indicated Emily's willingness to embrace a wider view of the concept of 'simultaneousness', Eleanor maintained her more 'teacherly' need to constrain the activity to, in her view, avoid confusing the wider group of students. This evidences the sort of pragmatic decisions teachers make on a daily basis in their classroom practices, whilst not knowing what the outcomes may have been had an alternative path been chosen.

6.3.4 How is the description of the lesson activity informed by the theoretical underpinning?

The development of Eleanor's knowledge concerns that of her own mathematical understandings, the MRT and her teaching approaches and classroom practices. In making sense of Eleanor's reaction to Emily's response, the driving factor seemed to be Eleanor's reluctance to deviate from her original intentions for the lesson. In her mind the lesson objectives concerned linear functions and, although she expressed both surprise and delight in seeing Emily's response, her post-lesson comment suggests that she saw this as a failure on her part in not being specific enough about the activity constraint to *only* generate linear functions. The evidence from the lesson transcript indicated that she was actually very clear in her introduction to the activity, having stated explicitly to the students that they were to generate linear graphs. Eleanor clearly appreciated Emily's divergent thinking but chose neither to hold a direct discussion with Emily nor to divert the current class activity with a discussion about Emily's solution. Eleanor's post-lesson comment implied that she saw the introduction of a quadratic solution to the problem as offering the potential for the class to be confused. So while Eleanor fully appreciated that there were many functions for which (3,6) was a coordinate pair, she had made the 'teacherly' decision to constrain the mathematical content of the lesson such that the degree of variance permitted was within the domain of linear functions. Eleanor's perception of the meaning of 'different' within this activity resulted in her not taking up opportunities to explore a number of the students' perceptions of 'different'.

To summarise, the analysis of Eleanor's lesson hiccup suggests that she may have learned that:

- The use of the MRT in this activity made Emily's unanticipated response both possible and also highly visible in the lesson.
- Emily had made a 'clever' response to the initial activity, and had provided evidence of her divergent thinking. However, as Eleanor did not challenge Emily's knowledge by asking her to explain or articulate her reasoning, she was reliant on the fact that the curve appeared to pass through (3,6)[10].
- Emily's response might have resulted from Eleanor not having strongly emphasised the requirement for linear functions. It is likely that in a future lesson of this type Eleanor would seek to clarify the activity instructions more explicitly than she did on this occasion or even highlight to the class that, although curved graphs are possible, they would not be working on them in that particular lesson.

The analysis of Eleanor's hiccup exemplifies how Verillon and Rabardel's notion of the constraints for the instrumented activity resonate with the extra information (afforded by the technology) that Eleanor was required to identify, understand and manage in reaction to Emily's response. Eleanor's pre-structuring of the activity led her to conceive her own version of the students' instrument utilisation scheme with which Emily did not comply, resulting in the occurrence of the hiccup. The fundamental assumption of Verillon and Rabardel's theory is that cognition evolves through interaction with the environment and that the genesis of cognition must take account of 'particular specific functional and structural features which characterise artefacts' (Verillon and Rabardel, 1995, p. 77). In this example, it appeared that Eleanor's knowledge was mainly *pragmatic* knowledge in relation to her role as a teacher, that is to say the way that she recognised and dealt with the hiccup. The more epistemic knowledge, which is difficult to infer from an isolated

[10] A scrutiny of Emily's TI-Nspire file revealed she had entered the function $f3(x)=x^2-3$ to generate the graph.

example such as this, might concern how this experience contributes to (or not) the process through which she pre-structures future activities, or how her previous experience in similar situations led her to act as she did in this example.

6.4 Methodological considerations for the coding of lesson hiccups

As I was analysing and coding the Phase Two lesson data, I became aware that, having identified the hiccups from the lesson data, some of the teachers' immediate responses to these hiccups indicated that they had a previously rehearsed repertoire of responses. Subsequently I also began to code each hiccup in relation to the evidence of the teacher's responses. For example, Tim's experience with the technology meant that he had a number of contingencies in response to students' basic instrumentation difficulties. He regularly promoted the key presses 'escape' and 'control z to undo' and had a number of explanation structures to support students to learn to 'grab and drag'. I devised three categories or teacher's response which were:

- No immediate response repertoire: the teacher did not appear to have any immediate contingencies, as evidenced by offering a 'holding' response such as 'we'll look at that another time'.
- A developing response repertoire: the teacher, whilst obviously acting in the moment began to offer a response that involved dialogue, use of the technology or both.
- A well-rehearsed response repertoire: the teacher responded confidently and involved dialogue, use of the technology or both.

For example, Tim's hiccup [STP6H2], that has been previously described in Section 6.2.3 would be categorised as one for which he had a developing response repertoire as he reconsidered his approach and designed a revised tns file. In contrast, Eleanor's hiccup [CEL9H4], previously described in Section 6.3.3 appears to have remained unresolved in the short term. However, an unresolved hiccup does not mean that the teacher did not learn anything from the experience in the longer term. The remainder of the thesis will focus on the outcomes of the analysis of both resolved and unresolved hiccups.

The full list of hiccups identified from the Tim and Eleanor's lesson data, and their classification according to the above set of categories is provided within Appendix 13 and Appendix 14.

6.5 Summary

In this chapter I have introduced the two case study teachers, summarised their respective outcomes for the first phase of the study and described in detail an observed lesson taught by each of them during the second phase. These lessons have been analysed with respect to the activity, classroom and subject layers. This analysis had resulted in the expansion of the notion of the lesson hiccup as the perturbation experienced by teachers during lesson, stimulated by their use of the technology. In addition I have begun to articulate the relationship between the individual hiccups and teachers' epistemological learning concerning mathematical variance and invariance within the MRT environment.

7 CHARTING TEACHERS' TRAJECTORIES: THE CASE STUDIES

'Failing to learn what is expected usually results in learning something else instead.'

(Wenger, 1998, p.8)

7.1 Introduction

This chapter analyses Eleanor and Tim's professional learning during the second phase of the study as evidenced by the analysis of their respective research data which revealed:

- The nature and evolution of their instrument utilisation schemes involving the MRT.
- The trajectories in their learning about the nature of mathematical variance and invariance within the MRT environment.
- The occurrence of their lesson hiccups leading to an articulation of how these contributed to their personal knowledge development.

The diversity of the mathematical content around which the teachers had designed their activities, coupled with the age and ability range of the students involved, did not lead to any meaningful generalisations by mathematical content in relation to any particular instrument utilisation scheme. There were insufficient lesson observations to be able to draw such conclusions. Both teachers focussed mainly on algebraic content because this was prescribed by their schemes of work at the time that the lesson observations were taking place.

In defining the hiccup, it is important to state that these perturbations were considered to be unpredicted events during the lesson. Their identification and subsequent classification constituted an objective research activity that was intended to stimulate and provoke professional conversation with the teachers involved in the research both during and beyond its timescale. They are not interpreted as a negative event and, although sometimes the teachers interpreted them as weaknesses in aspects of their activity design or pedagogical approach, I promoted their occurrence as opportunities for professional learning.

7.2 Tim's trajectory

At the end of the first phase of the study, the analysis of Tim's project data revealed the following aspects of his practice. It was evident that he was:

- committed to the design of activities which privileged his students' explorations of variance and invariance in thoughtful ways, often involving the dragging of dynamic objects;
- mindful of his students' needs with respect to the process of instrumentation;
- designing activities that adopted 'clean' representational forms in the introductory phase, but requiring careful teacher mediation to support students to embrace additional or alternative representations within the development of the activities;
- thoughtful in his consideration of the place of students' paper and pencil recordings alongside their use of the MRT.

In addition, during the first phase, Tim developed instrument utilisation schemes that that used numeric, syntactic and geometric inputs and, towards the end, he began to involve additional representations to provide further insight into the mathematics under exploration. The diagrammatic representation of his Phase One Instrument Utilisation Schemes is shown in Figure 7-1.

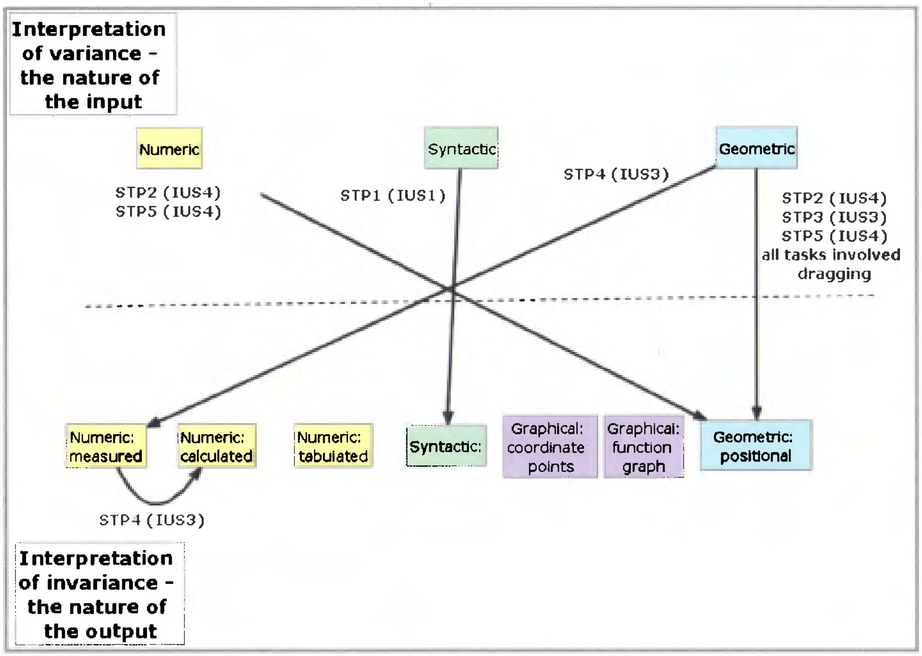


Figure 7-1 The summary of Tim's Instrument Utilisation Schemes produced from the analysis of his Phase One lesson data.

The connecting arrow between the 'Geometric' input and 'Geometric: positional' output clearly shows Tim's privileging of activities involving dragging, which were all concerned with geometric explorations. It also shows how, later in this phase, he did begin to involve other representational forms [STP4]. His schemes resonated with the classifications IUS1, IUS3 and IUS4 as described in Section 5.3.1.

7.2.1 Tim's developing conceptions of variance and invariance

One source of evidence for the evolution of Tim's conceptions of variance and invariance was the equivalent diagrammatic representation of the instrument utilisation schemes he developed during Phase Two, as shown in Figure 7-2. This overview was developed from the detailed descriptions of his Phase Two activities (Appendix 9). These schemes focus solely on Tim's intended use of the MRT by his students and, although there was an additional instrumentation 'layer' concerning his use of the classroom network, which often mediated the classroom discourse, this has been ignored at this stage of the analysis[1]. Due to the increasing complexity of these schemes, colour has been introduced to allow the path of each activity to be seen more clearly. For example, activity [STP6], which was described in full within Chapter 6, is shown in pink. In the main task within this activity, the nature of the input was the geometric representation of a triangle with squares constructed on its sides. The output representations focussed initially on the geometric positioning of the construction as free vertices were moved and then involved numeric (measured) values of areas and associated numeric (calculated) values. The related syntactical representations also formed part of the activity.

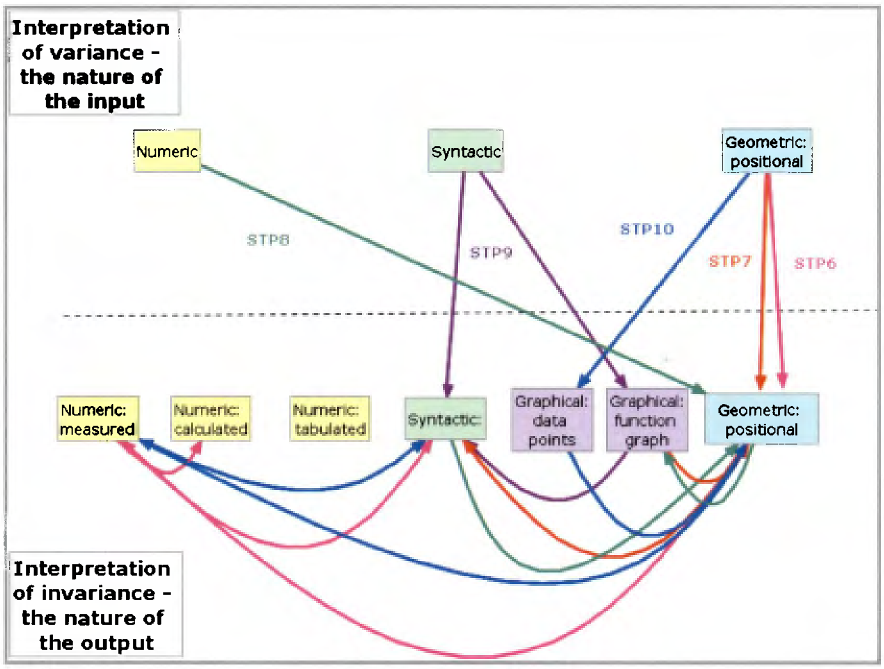


Figure 7-2 The summary of Tim's Instrument Utilisation Schemes produced from the analysis of his Phase Two lesson data.

[1] A more substantial commentary on the implications of the introduction of the classroom network is provided in Chapter 8.

Comparing this diagram with the corresponding map from the first phase (see Figure 7-1), the most obvious development was that all activities involved the output representation being examined in another representational form as an explicit component of the activity. This was clearly evidenced by the additional loops on the lower part of the diagram. This could be explained by the fact that Tim's Phase One activities were all developed for single one-hour lessons, which may have placed a time constraint on the opportunity to introduce further representations. However, as the Phase Two activities STP6, STP8 and STP10 were also developed for single lessons, this would seem to counter such a conclusion.

A deeper analysis of Tim's Phase Two schemes revealed:

- In common with the first phase Tim did not use the Numeric (tabulated) representation in any of his activities. This was surprising since four of the five activities developed during Phase Two were related to functions and graphs.
- Tim still privileged dragging as a means of interacting for two of his activities, although in one case a technical issue prevented the file from being used during the lesson by the students. In both cases the dragging was linked to other variable outputs within the activity and the students were required to actively seek mathematical connections between the different representational forms.
- Tim continued to develop innovative approaches to the creation of activities in which he looked to make use of his growing knowledge of the functionalities of the MRT. He identified subtle functionalities within the MRT that opened up new opportunities for activity design. An example of this was within [STP7] and [STP8], whereby functions of the form y=f(x) could be entered as a text object and dragged to the axes to create a 'hidden' function which was not obvious within the function stack. This enabled function matching activities to be developed.

A second source of evidence came from the analysis of his interpretation of variance and invariance within each mathematical activity. Each activity had its own 'flow' between interactions with the technology, classroom discourse and accompanying paper and pencil methods, highlighting the complexities of Tim's activity designs. There did not seem to be any one pedagogic approach. However this did illuminate the ways in which Tim was choosing to use the technology within the mediation of each of the activities. As an example, the analysis of one activity, 'Circles and lines' [STP7], is shown in Table 7-1.

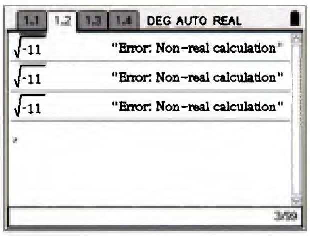
<table>
<tr><th>Mathematical objectives for the activity</th><th>How the concept of 'variance and invariance' was interpreted within the activity design.</th></tr>
<tr><td>'finding points of intersection between a straight line and a circle' [Journ]</td><td>
Initially, the invariant property was a 'given' circle ($x^2+y^2=5^2$) constructed and displayed on the handheld screen. The addition of a constructed geometric line was the variant property by nature of its infinite[2] number of positions within the view.
The position of the line was then fixed (at y=4) and the students were asked to identify the resulting invariant properties (i.e. points of intersection) using pencil and paper methods[3].
The position of the line, although still fixed as horizontal, was then allowed to vary within the paper and pencil activity and a generalisation for this situation was sought. Tim did not privilege dragging of the defined line y=4, although there is evidence that some students drew additional horizontal lines using the MRT.
The students used the calculator functionality to evaluate square roots of numbers, which provided the 'class authority' with respect to the meaning of the outputs as relevant to the original activity. i.e.


Figure 7-3 A student's handheld screen[STP7(tns-S)]
The overall intention for the activity was for the students to determine the following invariant properties:

When c>5, $x = \sqrt{(5^2-c^2)}$ was 'unsolvable', which implied 'no points of intersection'.
When c=5, $x = \sqrt{(5^2-c^2)}$ produced only one solution, which implied 'one point of intersection'.
When c<5, $x = \sqrt{(5^2-c^2)}$ produced two solutions, which implied 'two points of intersection'.

Tim reported that in the third lesson (which I did not observe) the students 'were able to check their algebraic solutions by plotting the lines on the handheld and finding points of intersection' [STP6(Journ-T)].
</td></tr>
</table>

Table 7-1 The detailed analysis of Tim's interpretation of variance and invariance within the activity 'Circles and lines' [STP7]

[2] The finite number of screen pixels would, in reality, mean that there would be a finite number of possible geometric lines.
[3] There was one pair of students who chose to input each horizontal line and construct perpendicular lines to enable them to read off the x-value from the resulting point of intersection with the x-axis. They then used paper and pencil substitution strategies to prove that they were correct.

Similar detailed descriptions were written for each of Tim's activities, the remainder of which are included within Appendix 9. A condensed version was also produced for each activity which is shown below in Table 7-2.

Activity	**Interpretation of variance**	**Interpretation of invariance i.e. the generalisation being sought**
STP6 (main activity)	The sums of the areas of two squares constructed on the sides of a triangle that were varied by dragging.	The condition that, when the two areas sum to give the area of the third side, the triangle contains a right angle.
STP7	The variation in the number and position of points of intersection between a straight line and a circle of given centre and centre.	A series of invariant properties which related the number of points of intersection between the circle and lines parallel to the x-axis to the value of the x-co-ordinate of the point of intersection.
STP8	The geometric path of the locus of a point positioned on a quadratic curve of the form y=(x-a)(x-b) where a and b were input values.	When matching functions of the type y=(x-a)(x-b) were constructed, the roots were positioned at (a,0) and (b,0).
STP9	The geometric representation of functions of the type f(x)=(x+a)(x+b) and $f(x)=x^2+cx+d$.	When a+b=c and ab=d the two functions coincide.
STP10	The geometric positioning of two pairs of coordinate points that defined two line segments.	The measured values of the gradient of each line segment. The condition that the measured value of the gradient would equal the vertical translation for every unit of horizontal translation for points along each line.

Table 7-2 A summary of Tim's interpretations of variance and invariance for his Phase Two activities.

The following findings concerning Tim's conceptions of variance and invariance have been revealed by this analysis:

- Tim stored measured values from geometric contexts as variables (registers of memory) to support generalisations from explorations that involved dragging [STP6] (Previously used in STP5)
- Tim used a geometric 'meta-level' representation to give students a broad view of the topic prior to using specific examples as a means to encourage the students to generalise [STP7].
- Tim used the MRT as the 'class authority' to provide feedback in activities concerning: calculating square roots [STP7]; 'measuring' linear equations [STP10] and matching syntactic forms for quadratic expressions [STP9].

It was important to note that in some cases the schemes were carefully pre-meditated by Tim. This was evident from his planning of the 'story' of the lesson and the requirement for the TI-Nspire file to be constructed in advance and sent to the students' handhelds at some point during the lesson ([STP6], and [STP8]). For other activities the file was constructed as part of the lesson introduction. This meant that either Tim led the construction from the front of the class, with the students simultaneously constructing their own version [STP10], or Tim and the students co-constructed the file, which Tim subsequently sent to the students' handhelds ([STP7] and [STP9]). Consequently, the diversity in these schemes led to a wide range of potential hiccups within the lessons. What follows is an overview of these lesson hiccups.

7.2.2 Tim's hiccups

Tim's lesson observation data revealed seven classifications of lesson hiccups. Several of these occurred across the different activities and, in some cases more than once within individual activities. Table 7-3 gives a summary of the hiccups alongside exemplification from the data to describe each classification. The frequency of each hiccup is made evident by the activity code accompanying the exemplification and, where a particular hiccup occurred more than once within an activity, its frequency of occurrence is shown in brackets.

Nature of the hiccup	**Exemplification**
Resulting from the initial activity design	• Issues relating to identifying and discussing objects displayed on the MRT [STP6H1] [STP9H9]. • Choice of initial examples [STP10H12]. • Sequencing of examples [STP6H5], [STP9H4], [STP9H5], [STP9H7](5).
Interpreting the mathematical generality under scrutiny.	• Students' unease over how a specific case related to the wider generality [STP7H3], [STP7H4], [STP7H6], [STP10H11](2).
Students' instrumentation issues when making inputs to the MRT. i.e. translating 'paper and pencil' mathematics into machine mathematics;	• Entering numeric and syntactic data [STP8H2], [STP8H3], [STP9H1](3), [STP9H2](6), [STP9H3](6), [STP10H3], [STP10H4]. • Plotting free coordinate points [STP10H3](3).
Students' instrumentation issues when actively engaging with the MRT. i.e. learning to use new actions...	• Grabbing and dragging dynamic objects [STP6H4(3)]. • Organising objects on the screen [STP6H3].
Students' perturbations stimulated by the representational outputs of the MRT.	• Resulting from a syntactic output [STP10H8](3). • Resulting from a geometric output [STP7H1]. • Doubts expressed about the 'authority' of the syntactic output [STP7H2], [STP9H8].
Unanticipated student responses	• Students' prior understanding is below teacher's expectation [STP10H10](3), [STP10H13](2). • Students' interpretations of activity objectives [STP6H2], [STP7H5]. • Students develop their own instrument utilisation schemes for the activity [STP7H7](2).
General technical issues[4]	• Transfer of files [STP8H1], [STP10H1](4), [STP10H2]. • Classroom display of teacher's software/handheld [STP9H6], [STP10H5], [STP10H6], [STP10H7].

Table 7-3 Summary of Tim's lesson hiccups

It was observed that, for a quarter of these hiccups, Tim responded in a way that strongly intimated that he had met that sort of hiccup before. This was evidenced by a previously rehearsed repertoire of actions and such hiccups have been

[4] All of these issues related to the use of the prototype classroom network hardware and software which were outside of the teacher's control.

indicated in Table 7-3 by underlined text. The remainder of the hiccups were classified as either, 'not appearing to have an immediate response repertoire', or 'the response repertoire was in development'.

7.2.3 Tim's situated learning

Following the identification and classification of Tim's hiccups, a scrutiny of the research data led to the identification of a series of actions by Tim that indicated that he may have become sensitised to the various hiccups. These actions were sometimes observed in the classroom, but alternatively they materialised after the lessons through his written lesson reflections, revised activity structures, post-lesson interviews and email exchanges. For example, the scrutiny of Tim's actions during the activity 'Equivalent quadratics' generated the list shown in Figure 7-4

Name	Sources	References
TP9 Action01 - Appreciated 'hiccup' over choice of expressions	1	2
TP9 Action02 - Appreciated that the MRT could have accepted any letter as variable	1	1
TP9 Action03 - Made explicit link between Medley's theory and grid multiplication	1	1
TP9 Action04 - Noticed that students appreciated being sent the same file to work on	1	1
TP9 Action05 - Observed that students were highly motivated by MRT response 'true'	1	1
TP9 Action06 - Privileged 'press control z to undo'	1	1
TP9 Action07 - Responded to students' concerns about needing to type the brackets	2	2
TP9 Action08 - Reversed second section of task 2 to force inverse strategies	1	1
TP9 Action09 - Revised language wrt needing brackets	1	1
TP9 Action10 - Revised the task to account for the L1 hiccup	1	2
TP9 Action11 - Appreciated that student had noticed 'bold' graph	1	1
TP9 Action12 - Appreciated student's connection between 'Medley's rule' and grid multiplica	1	1

Figure 7-4 The summary of Tim's actions within the activity 'Equivalent quadratics' [STP9].

The complete list of actions associated with each activity, as elicited from the research data, is fully detailed in Appendix 9. The subsequent analysis of the list of Tim's actions throughout Phase Two seemed to suggest that his situated learning related to:

- A much deeper understanding of the nature of mathematical generalisation and the way in which the related variant and invariant properties would be both conceived and mediated by the MRT within the activity design. This included an appreciation of the objects upon which the student attentions would be focused and the timings for the related whole class discourse[5].
- An appreciation of the particular difficulties that his students experienced within dynamic explorations involving multiple representations whereby their

[5] Later in the thesis I define the events where this happens as the lesson 'pinch-points'.

attentions may be drawn the simultaneous variation of several objects.

- An increased understanding of the level of detailed planning needed when designing activities involving the MRT. For each mathematical topic this included the careful consideration of: the bigger mathematical picture; the choice of the initial example(s); the mathematical progression within subsequent examples and the degree of alignment between the MRT mediated activity and traditional paper and pencil methods.
- A broader appreciation of his students' prior knowledge, understanding, attitudes and motivation towards mathematics, which had been illuminated by their engagement with the MRT, and made publicly visible by the screen capture view and quick poll responses.
- The development of a number of successful strategies in response to students' instrumentation issues concerning both the input of data to the MRT and the related mediating interactions such as dragging.
- Recognition of the emerging evidence of his students' own instrumentalisation processes whereby they had began to develop their own IUS in response to the activities and problems posed.

These outcomes cannot be interpreted as a simple cause-effect model whereby the singular occurrence of any one hiccup led to any one particular aspect of Tim's professional learning. It is hypothesised that, over time, Tim's sensitivity to some hiccups had led him to develop aspects of his pedagogy. This was certainly true of most of the hiccups that arose from the students' instrumentation issues when inputting and interacting with the MRT. Tim had a well-rehearsed set of practices to respond to these hiccups, which could be interpreted as evidence of his pragmatic knowledge development. However, the many hiccups that Tim experienced that concerned the sequencing of the examples that he selected within his activity designs ([STP6H5], [STP9H4], [STP9H5], and [STP9H7]) would indicate that a more epistemic knowledge development was required. For example, one of the more significant hiccups related to Tim's decision to choose a worksheet from a commercial scheme that used different notation within its examples to the notation with which the students had become familiar using the MRT [STP9H7]. Tim's post-lesson reflections strongly indicated that he had acknowledged that the worksheet had not allowed the students to progress from their experiences when using the technology to a more traditional paper and pencil approach. He also suggested ways in which the MRT activity might be redesigned to accommodate a wider range of mathematical notation.

An overarching outcome for Tim's involvement in this study was his much greater awareness of the importance of his choice of initial example, and the way in which this example would be expanded or developed to enable his students to arrive at common expressions of the generality in question. This often related to a smoother transition between to MRT-mediated activity and the paper and pencil approaches demanded by the current assessment regime.

7.3 Eleanor's trajectory

The analysis of Eleanor's activities, submitted during the first phase had revealed the following aspects of her practice using the MRT:

- She was designing activities which involved the students in making dynamic actions as an integral part of the conjecturing process.
- Her pedagogic approach within activities was moving towards a more explicit focus of the students' attentions on the variant and invariant properties.
- She had grown in her confidence to adapt activities to be more in line with her personal pedagogical approaches and to design her own activities from a blank page.
- She had started to include more than one mathematical representation within her activity designs.

In common with Tim, Eleanor developed activities which took numeric, syntactic and geometric inputs as their starting points. Towards the end of the first phase, Eleanor had developed a repertoire of instrument utilisation schemes that are illustrated diagrammatically in Figure 7-5.

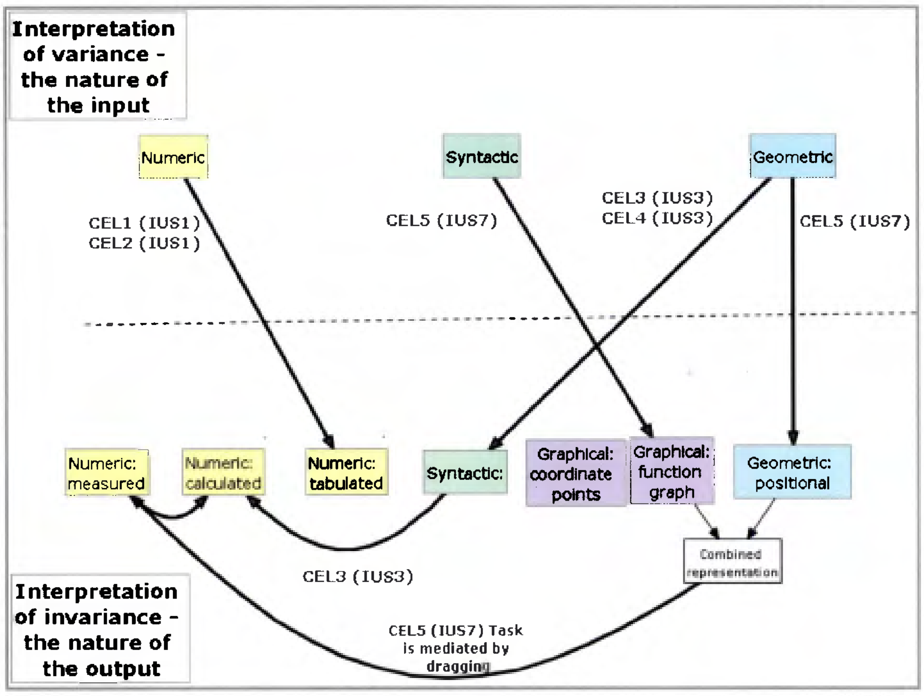


Figure 7-5 The summary of Eleanor's Instrument Utilisation Schemes produced from the analysis of her Phase One lesson data.

By the end of Phase One, Eleanor's activities were involving the dragging of dynamic objects and integrating multiple representations and her schemes resonated with IUS1 and IUS3. She accidentally developed an instrument utilisation scheme (IUS7) which took an innovative approach to superimposing freely drawn geometric lines with those defined syntactically using function notation in an activity focussing on gradients of perpendicular lines [CEL5].

7.3.1 Eleanor's conceptions of variance and invariance

What follows is a description of Eleanor's intended instrument utilisation schemes for her students' uses of the MRT during the second phase of the study. Again, it is acknowledged that there was an additional instrument utilisation scheme concerning her use of the classroom network, the impact of which will be discussed more fully in Chapter 8.

Firstly, the overview of Eleanor's instrument utilisation schemes during the second phase is shown diagrammatically in Figure 7-6. The more detailed versions of these schemes for each activity are included within Appendix 10. The colour is used to distinguish between the flows of different representations for each activity.

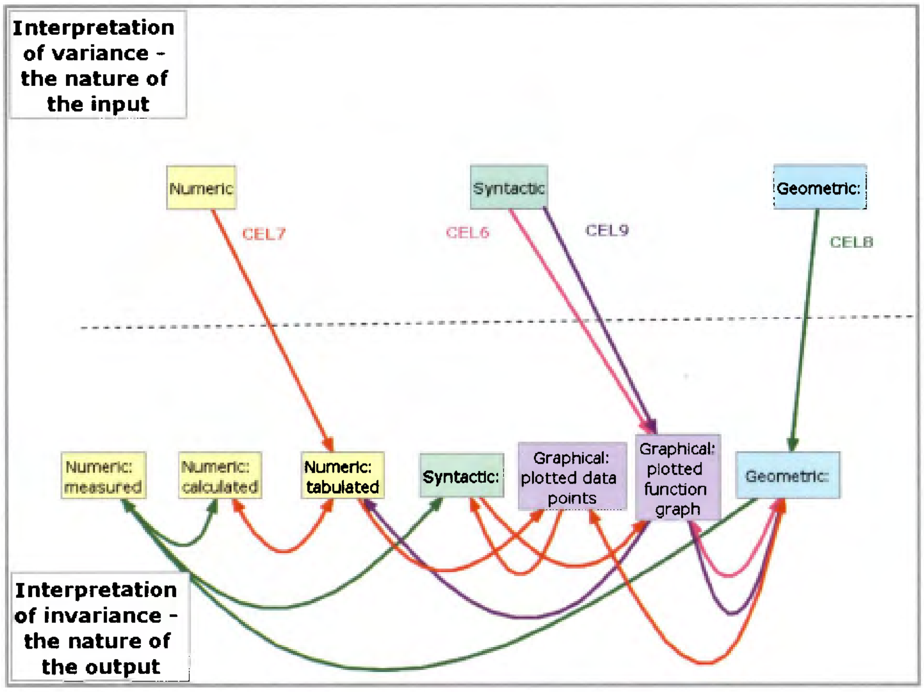


Figure 7-6 The summary of Eleanor's Instrument Utilisation Schemes produced from the analysis of her Phase Two lesson data.

It was immediately apparent that Eleanor's activities incorporated a greater diversity of representations and each activity had its own sequential flow. For example, 'Generating circles' [CEL7] began with a numeric starting point in the form of tabulated data in the spreadsheet, which included a calculated checking facility, and led to a data plot. The geometric form of the data plot informed both the numeric inputs to the spreadsheet and the algebraic interpretation that was subsequently 'tested' through a syntactic input. This 'flow' is shown in Figure 7-7.

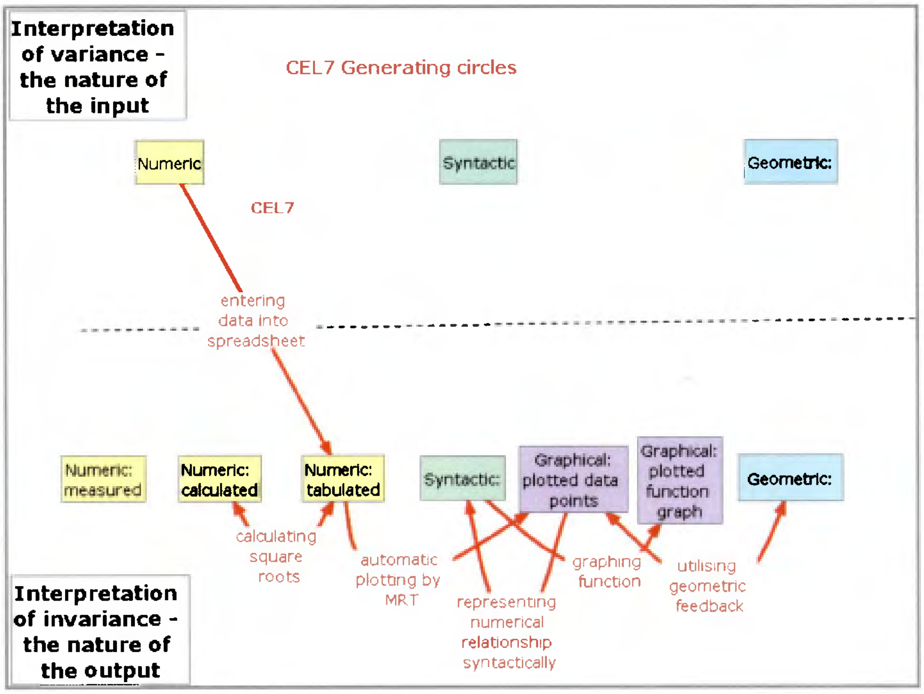


Figure 7-7 Eleanor's Instrument Utilisation Scheme map developed for the activity 'Generating circles' [CEL7].

My observation of Eleanor's pedagogic approach in this example led to my conception of the notion of 'pinch-points'. These were the points in time during the activity where Eleanor instigated a period of classroom discourse that mediated the introduction of a new representational form. These pinch-points appeared to be both thoughtfully planned and also insightful, in that they often required her to have considered in advance the necessary instrumentation skills that the students would need to master, in addition to the mathematical questions she would ask.

To arrive at a deeper understanding of Eleanor's conceptions of variance and invariance within her activities, each activity was scrutinised and a description formed. For example, Table 7-4 shows such as description for the activity 'Generating circles' [CEL7]. Similar descriptions for the remainder of Eleanor's activities are included within Appendix 10.

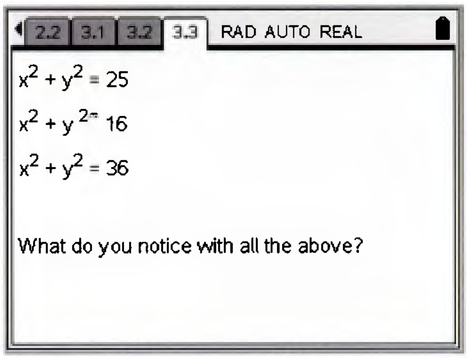
Mathematical objectives for the activity	**How the concept of 'variance and invariance' was interpreted within the activity design.**
'To establish the equation $x^2 + y^2 = r^2$	The starting point for this activity was finding values of x and y that satisfied $x^2 + y^2=25$, which Eleanor was interpreting as being in the real number domain only.
Making connections between the values placed into spreadsheet and the graph of these values.	However, as Eleanor left the definition of the domain open as a point of discussion throughout the lesson, most students began by considering positive integers and then expanded this domain to include zero and negative integers.
Making conjectures about the circle and primarily the radius to connect it to the statement of squaring 2 numbers and adding them [CEL(Plan)]	Some students extended this domain further to include positive and negative square roots of non-square numbers. The variant property was the range of permissible values for x and y, whereas the invariant property was the given initial condition.
	Although Eleanor had designed the activity to include explorations of $x^2 + y^2=16$ and $x^2 + y^2=36$, and included a page in the TI-Nspire file which encouraged the students to begin to generalise, (Figure 7-8), this part of the activity was not observed during the lesson.  Figure 7-8 [CEL7(tns-T)page3]

Table 7-4 The detailed analysis of Eleanor's interpretation of variance and invariance within the activity 'Generating circles' [CEL7].

Similar detailed descriptions of each of Eleanor's Phase Two activities are included within Appendix 10. These descriptions were then condensed to produce the summary shown in Table 7-5.

Activity	**Interpretation of variance**	**Interpretation of invariance i.e. the generalisation being sought**
CEL6	The syntactic inputs of six different sets of given functions subsequently plotted graphically on a Cartesian plane.	Comparing certain sets of functions revealed commonalties in their geometric transformations.
CEL7	The numeric input of pairs of numbers satisfying a given rule (i.e. 'square and add to give twenty-five) subsequently plotted as coordinates on a Cartesian plane.	The resulting geometric representation satisfied a defined geometric shape (i.e. a circle of centre (0,0) with radius 5). The resulting syntactic representation satisfied a defined function (a circle, $x^2 + y^2 = r^2$).
CEL8 (Main activity)	The sums of the areas of two squares constructed on the sides of a triangle that are varied by dragging.	The condition that, when the two areas sum to give the area of the square constructed on the third side, the triangle contains a right angle.
CEL9	The syntactic form of the different (linear) functions that pass through a given coordinate point, subsequently displayed on a Cartesian plane.	The condition that when the lines coincide, substituting their point of intersection into the associated linear function results in an equality statement.

Table 7-5 A summary of Eleanor's interpretations of variance and invariance for all of her Phase Two activities.

The review of Eleanor's Phase Two activities led to a number of conclusions. Eleanor described some of her activities as having adopted a 'backward' approach [CEL9(Trans)]. That is, the mathematics was being approached from the premise that when a particular condition is satisfied, then this mathematical generalisation must be true. This challenged her subject knowledge with respect to her awareness of the generalisations that were possible within the different contexts. Her pedagogical knowledge was also challenged as she began to devise new explanatory structures and representational forms to support the progress of explorations. It appeared that the process of foregrounding mathematical variance and invariance when designing activities challenged her mathematical subject and pedagogical knowledge. Eleanor also began to appreciate that, if the students were going to be able to make the mathematical generalisations that she anticipated, these generalisations needed to be manageable within the teaching timescale.

The analysis of Eleanor's lesson hiccups that follows, offers an insight into the nature of the contingent moments in the lessons that may have influenced Eleanor's professional learning over the research timescale.

7.3.2 Eleanor's hiccups

Each of Eleanor's lesson activities were analysed to reveal the hiccups. These were the notable moments during the activity when she paused or hesitated and which were triggered by her own or her students' uses of the MRT. The individual sets of hiccups are included in Eleanor's detailed activity descriptions (Appendix 11). These were then sorted, and a constant comparison method employed to reveal the following categories and associated exemplifications. For each exemplification, the relevant hiccup code is given and a number in brackets indicates the frequency of occurrence of that particular hiccup during the activity.

Nature of the hiccup	Exemplification
Resulting from the initial activity design	• Sequencing of examples [CEL6H2]. • Resulting from an unfamiliar pedagogical approach [CEL8H8].
Interpreting the mathematical generality under scrutiny.	• Students' unease over the permissible range of responses that satisfy the generality [CEL7H1], [CEL8H3]. • Students' reluctance to notice the generality [CEL6H1].
Students' instrumentation issues when making inputs to the MRT. i.e. translating 'paper and pencil' mathematics into machine mathematics;	• Entering numeric and syntactic data [CEL6H3], [CEL6H4](2), [CEL7H4], [CEL9H2].
Students' instrumentation issues when actively engaging with the MRT. i.e. learning to use new actions…	• Grabbing and dragging dynamic objects [CEL8H2](2), [CEL8H4]. • Navigating between application windows [CEL8H6], [CEL9H3], [CEL9H6], [CEL9H8]. • Student asks how to develop new instrumentation [CEL9H5]. • Accidental deletions of objects [CEL6H5], [CEL7H2].
Students' perturbations stimulated by the representational outputs of the MRT.	• Resulting from a syntactic output [CEL8H1]. • Resulting from a geometric output [CEL8H9].
Teacher's instrumentation issues when using new representation.	• Displaying the function table [CEL9H7].
Unexpected student responses	• Prompts classroom discourse triggered by the students [CEL9H4](5). • Responses are more diverse than teacher anticipated [CEL6H6], [CEL7H3], [CEL8H7], [CEL9H1](2). • Students' activity approach reveals incorrect interpretation of the mathematics [CEL8H5].

Table 7-6 Summary of Eleanor's lesson hiccups

The hiccup codes that have been underlined represent those hiccups for which I surmised that Eleanor's immediate response indicated that she had a previously rehearsed response repertoire.

7.3.3 Eleanor's situated learning

It was very clear from the analysis of Eleanor's activity hiccups, and her responses to them, that she had developed a confident set of classroom practices in response to the sort of hiccups that arose concerning students' instrumentation issues. This is indicated by the underlining of the hiccups in Table 7-6 related to the students' inputs to the MRT and in engaging actively with the functionality of the MRT. These practices included:

- careful oral phrasing of the instrumentation instructions for activities, which were often repeated several times;
- 'acting out' a physical metaphor to support students to make sense of the instrumentation process;
- instrumentation instructions displayed to the class in the Smart Notebook file for the lesson;
- interventions with individual students whereby she diagnosed their difficulties by watching them re-enact it, before talking them through the correct sequence of actions;
- encouraging and expecting the students to get together to sort out each others' difficulties during the instrumentation phase.

When Eleanor was faced with the sort of hiccups that arose as a result of an unexpected diversity in the students' responses, in some situations she seemed to have developed new pedagogic approaches. For example, in one activity she used the diversity of students publicly displayed responses to an open question to force the students to argue and agree the mathematical conventions that were acceptable as answers [CEL9]. This served both to inform Eleanor of the range of her students' existing knowledge and to widen the students' frames of understanding. Eleanor commented that this was an adaptation of an existing pedagogic approach whereby students would have previously displayed their responses to her on 'mini whiteboards[6]'. However, it appeared that the classroom argument was more strongly provoked using the technological approach because the students had wide visibility of each others' responses.

In other activities the diversity of the students' responses seemed to lead to pedagogic dilemmas for Eleanor as she looked for either commonality or diversity, as anticipated in her activity design, to support the next phase of the activity. For example, in 'Transforming graphs' [CEL6], faced with the publicly displayed set of handheld screens shown in Figure 7-9, Eleanor commented how she was unsure how to take the next phase of the classroom discourse forward as she had expected that the students would have all offered the same set of transformed functions.

[6] a 'write-on, wipe-off' board that students use to record responses and hold up for the teacher to see.

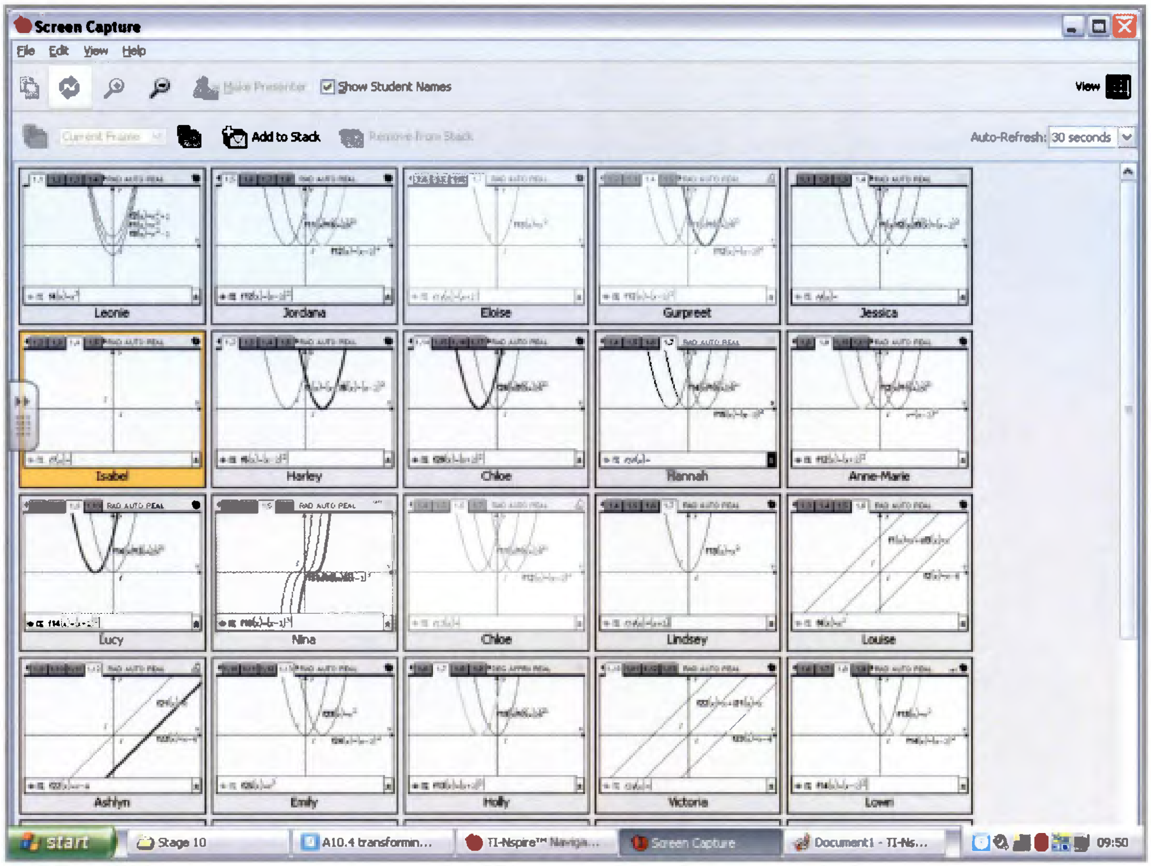


Figure 7-9 The students' handheld screens displayed as a Screen capture view during the activity 'Transforming graphs' [CEL6].

Eleanor rationalised this particular hiccup by blaming it on her original activity design saying,

> *As there were over twenty pairs of functions that the students were comparing, I realised that it was difficult to direct the students to a particular screen to compare – I needed to think carefully about how I grouped and labelled the functions when devising the worksheet next time!. Hence some of the students' screens showed vertical translations and others showed horizontal translations!* [CEL6(Journ-T)]

Her experiences in this lesson did seem to impact upon Eleanor's thinking when it came to her subsequent activity designs. She became much more focussed upon identifying the generalisable elements of each activity and how she would direct the students' attentions towards these elements as the activity progressed. In her informal planning notes she would write herself a script, which would include basics such as:

- her own instrumentation sequence to introduce the activity;
- the initial instrumentation sequence for the students;

- the key mathematical questions related to the activity;
- when she would send files to the students' handhelds;
- when she would display the screen capture view, and which page of the students' files would be displayed;
- if appropriate, the instrumentation steps for students when introducing new representations;
- if appropriate, when she would collect files from the students.

It was interesting to observe that, as Eleanor's experience with the MRT in the classroom was increasing, she did not become blasé about the process of planning activities for her students. On the contrary, her awareness of the subtle elements of activity designs that contributed to a successful experience for her students seemed to be sharpening. My observations of Eleanor's classroom practice led me to conceive the notion of 'pinch points' within activities. These were the planned moments during the activity when Eleanor directed the students to stop what they were doing, move to a particular page of the MRT activity and focus their attention towards a specific outcome. In some cases this was accompanied by the public display of the handheld screens, for example Figure 7-9. Eleanor used the screen capture view of the students' handhelds to monitor their work and would often announce an impending pinch-point to the class. It also seemed important to Eleanor that the students could place their own handheld screen within the wider class view and she was specific in her instructions to students in this respect. For example, she would say, 'Okay, I want you to concentrate on your screen first... [pauses] on your Nspire... [pauses] ...and I want you also to come to these ones [gestures to Screen capture view on public display]' [CEL8(Trans)].

Over the period of the study, Eleanor's overall approach to the design of activities had developed a greater awareness of the 'bigger mathematical picture' in relation to each topic. She had also begun to appreciate the way in which the MRT-mediated aspect of the activity would need to link with the paper and pencil approaches that the current assessment regimes demanded, and had begun to include this element within her activity designs. Her conviction that her students were developing a deeper mathematical understanding as a result of their engagement with the MRT seemed to provide an incentive for her to want to ensure smoother transitions between the two media.

7.4 The emergence of new theoretical ideas

The theoretical framework developed for this study was an adaptation of Verillon and Rabardel's *triad characteristic of instrumented activity situations* (Verillon and Rabardel, 1995) resulting in a dual layered interpretation, which considered the teacher's perspective as both a learner (with respect to mathematics, pedagogy and technology) and as a teacher (responsible for constructing situations that would lead to students' learning). This highlighted the need to develop the notion of the *object* in Verillon and Rabardel's triad and to explore the way in which the teacher's reflections *on and through* classroom practice contributed to the development of their knowledge within the domain of the study.

The research aimed to develop a language to support the description of teachers' situated learning in the classroom within the context of developing mathematics teaching and learning (concerning variance and invariance) when integrating a complex technology. The definition and further articulation of the lesson hiccup and its role within this process of teachers' knowledge development, offers a contribution to such a language as an example of an ontological theory.

The hiccups, that is to say the instances within the lesson, often indicated by clear 'pauses' or 'hesitations' on the part of the teacher, offered an *opportunity* for teachers' knowledge development in both a pragmatic and an epistemic sense. The evidence presented in this chapter sought to substantiate the emergence of the hiccup and indicate its relationship to Eleanor and Tim's situated learning. Chapter 8 will articulate the theory in more depth with a view to offering a more expansive definition of the hiccup and its relationship to teachers' situated epistemological development.

7.5 Summary

This chapter has described the outcomes of the second phase of the study with respect to Eleanor and Tim's experiences resulting from the analysis of their research data. This led to an overview of their professional learning concerning their expanding instrument utilisation schemes and the ways in which they conceptualised mathematical variance and invariance within the activities they developed. The analysis and discussion of their lesson hiccups was related to the evidence of their actions, substantiating the lesson hiccup as a notable contributory event to their learning during the period of the study.

8 Discussion

The presence of someone whose attention is differently structured, whose awareness is broader and multiply-levelled, who can direct or attract pupil attention appropriately to important features, is essential.

(Mason, 1996, p.71)

8.1 Introduction

In this chapter, the findings in relation to Eleanor and Tim's situated learning are examined as a cross case analysis relevant to the aims of the study. The discussion focuses on the nature of their situated learning by considering the evolution of their mathematical, pedagogical and technical knowledge resulting from the lesson hiccups. The chapter continues with a commentary on the nature of the observed classroom hiccups and an articulation of the hiccup theory of teachers' situated learning concerning mathematical variance and invariance in a MRT environment. It concludes by considering the contribution that the classroom network technology made to the teachers' epistemological development.

8.2 What did the teachers learn?

In this study, the technology was conceived as having dual roles in mediating both teachers' learning in the classroom and the transformation of this knowledge through the way that they reflected upon their students' learning and developed their own theoretical and practical pedagogic ideas. The data analysis revealed that the three broad areas of Eleanor and Tim's learning concerned their conceptions of variance and invariance, their pedagogical approaches and their use of the technology. These three themes, although inextricably interconnected, are now examined one by one. The key consideration is that the occurrence of lesson hiccups was a necessary contributory factor in the teachers' epistemological development.

8.2.1 Mathematical knowledge development concerning variance and invariance

A central tenet of this thesis was that activities and approaches that privileged an exploration of mathematical variance and invariance constituted a legitimate pedagogical opportunity for the use of technology in secondary mathematics classrooms. The decision to base the research on a group of teachers' developing uses of a multi-representational technology (MRT) also provided the context to explore how the teachers learned to develop activities that linked mathematical representations within the mathematical explorations. These themes were consistent with the task layer within Pierce and Stacey's pedagogical map (2008).

The teachers' evolving conceptions of variance and invariance were revealed in several ways during the study. This included how they conceptualised the initial representation of the variant property under exploration and the related representations that added insight to, or progressed, the exploration and the way in which any invariant properties would become explicit as a result of the exploration.

It is problematic to try to interpret the evolution of the teachers' conceptions of variance and invariance solely as aspects of their mathematical knowledge because the context for this development was rooted within their pedagogical decisions and actions concerning the design and evaluation of mathematical tasks. However, the teachers' choices of initial representation, as evidenced by the 'input' to the exploration, offered an insight into the mathematical content area in which the teacher perceived that the broader generality under scrutiny would reside. For example, in Tim's lesson that would eventually focus on the algebraic solution of simultaneous equations involving circles and lines, his starting point was to consider the geometric possibilities of lines crossing circles prior to introducing any algebra [STP7]. For each of the teachers' activities, this initial representation evidenced the first 'example' from which they anticipated that they would be able to guide their learners towards a specific mathematical generalisation.

Eleanor and Tim learned that there were several important aspects when making the choice of initial example within their activity design and its subsequent representation within the MRT. Firstly, it needed to <u>be</u> an example of the generalisation being sought. A hiccup that Tim experienced during the activity 'Linear equations' [STP10] led him to conclude that his choice of functions for the initial example, which was a spontaneous decision, was unhelpful. His selection of $y=1x+1$ and $y=2x$ did not support the students to begin to generalise about the gradients of the linear functions, partly because there were two digit ones inherent in $y=1x+1$, but also due to the way that the MRT displayed the measured equations[1] (see Figure 8-1).

[1] This hiccup was compounded by the output from the MRT which displayed an unhelpful 'measured' equation. (A measured equation is one which for which the MRT numerically analyses a set of points that define a geometrically constructed line, often revealing unexpected numerical errors i.e. Figure A9-41 on Page 360)

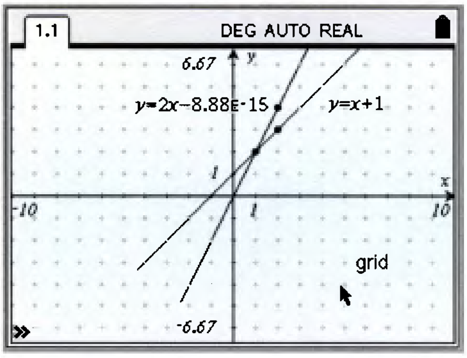


Figure 8-1 Tim's handheld screen showing the 'measured' linear functions which resulted in a lesson hiccup during the activity 'Linear graphs' [STP10(tns-T)].

However, this 'double' hiccup contributed to a development in Tim's pedagogical knowledge with respect to the importance of example choice, evidenced by his own reflective writings [STP(Journ-T)].

A second aspect of the initial task choice that may also be relevant, although there was only one instance of it resulting in a hiccup, concerned the students' articulations of their mathematical generalisations that were outside of the teacher's expectation. For example, in Task 2 of Tim's 'Pythagoras exploration' activity [STP6], one student concluded that 'a small square and a big square make a right angle triangle' and another pair of students wrote, 'to get 3 squares equal size you have to have an equilateral triangle in the middle' [STP6(Eval-S)]. Tim commented that he felt it was his role to 'find out what students think, in order to help them move forward', and that gaining an insight into the generalisations they made was an important part of this process.

Eleanor and Tim both seemed to grapple with the notion of the 'example space' in ways that were resonant with Goldenberg and Mason's elaboration of the difficulties encountered when trying to identify appropriate examples for different mathematical generalisations,

> *Sometimes there is a general 'sense' of examples without specific details, while at other times a few clear examples come to mind immediately, setting off a trail of metonymic triggers and metaphoric resonances associated with concepts, techniques, or other features, or with ways to vary or tinker with those examples.* (Goldenberg and Mason, 2008 p.188)

Tim's choice of functions in the 'Linear equations' activity could be interpreted as him having a 'general sense' of the representation of any two straight lines on a dotty grid alongside the functionality of a 'gradient measure' tool as a sufficient space in which to be able to generalise about slope. However, a more informed choice of functions, alongside the utilisation of the MRT's functionality to measure the slope of the line, rather than its equation, might be aspects of the activity that Tim would redesign if he were to use this activity again.

Having decided upon the initial input representation, another important consideration for the teachers related to their decisions about the syntactic labelling or notation of variable objects. They learned that this was a necessary element of activities when classroom discourse would be initiated to support students to notice and verbalise their generalisations. As both teachers privileged whole-class discourse focussed on key mathematical generalisations in their classroom practices, they both experienced a number of hiccups that indicated that there was insufficient notation or labelling to enable this discussion to be suitably focussed. This is an aspect that Pierce and Stacey have noted in their recent research into the way that Australian secondary mathematics teachers are developing the use of the TI-Nspire handhelds to privilege linked mathematical representations when teaching about quadratic functions (Pierce and Stacey, 2009). They have highlighted the conflict between the labelling of variables within the 'ideal' or pencil and paper mathematics alongside the conventions built into the MRT, as an important element for teachers to consider in the design of tasks.

A final consideration was the development of the teachers' knowledge concerning the grouping or sequencing of examples, that is, determining the scale and structure of the example space. This aspect of their learning not only related strongly to their personal mathematical knowledge and interpretation of the mathematical progression of ideas within the topic matter, but was also intrinsically linked to the pedagogic skill of considering the most suitable activity structure for their students. Prior to her lesson 'Transforming graphs', Eleanor was honest about her lack of confidence in the topic, saying that she needed to follow the text book quite closely when teaching standard function transformations and she would normally encourage her students to 'just learn them' [CEL6(Int-T)]. However, her experience in designing and evaluating the lesson led her to appreciate how the MRT activity could be designed to allow the students to explore <u>why</u> the functions were being transformed in a particular way by involving the Numeric (tabular) representation. This was an expansion of Eleanor's previous mathematical knowledge, which also offered her the opportunity of a new pedagogic approach. A significant aspect of her learning concerned an appreciation that if her students had a deeper understanding of why and how a particular transformation set 'worked', they would have a structure through which they could make sense of others.

In general, having selected an initial example for each activity, Eleanor and Tim expanded the example spaces to add insight to, or support the analysis of the variant and invariant properties under exploration in one, or a combination of the following approaches:

- Further examples were explored within the same mathematical representation by making new inputs in the same numeric, syntactic or geometric form [STP7], [STP8], [STP9], [STP10], [CEL6], and [CEL9].
- Further examples were explored within the same mathematical representation by dragging objects dynamically [STP6] [CEL8].
- Further examples were explored within a different mathematical representation by making new numeric, syntactic or geometric inputs [STP7], [STP8], [STP9], [STP10], and [CEL9].
- The same example was explored within a new mathematical representation [CEL7] [CEL9].

8.2.2 Pedagogic knowledge development

This study sought to build upon Rowland et al's notion of contingency, which they defined as a teacher's 'knowledge-in-interaction' that 'on occasion can be seen in the teacher's willingness to deviate from her own agenda to develop a student's unanticipated contribution' (Rowland et al., 2005, p.266). Examples from each of Eleanor and Tim's lessons of a hiccup that arose from the students' unanticipated contributions have been described in detail in Chapters 6 and categorised within Chapter 7 (see Table 7-3 and Table 7-6). The overarching categories were:

- Students' interpretations of the objectives or instructions for the MRT activity were broader than the teacher anticipated, leading to a more diverse set of students' responses.
- Students developed their own instrument utilisation schemes for the activity.
- The MRT outputs prompted classroom discourse that was initiated by the students.
- Students' approaches using the MRT reveal incorrect interpretations of the mathematics or prior understanding that were below the teacher's expectation.

The categories above were neither mutually inclusive nor single occurrences in that hiccups of this type often instigated a sequence of events in the classroom. In

Eleanor's lesson hiccup, which was described in depth in Section 6.3.3, Emily's unanticipated response as a result of her broader interpretation of the task led to the other students initiating the discourse about it. An important point to make here is that these unanticipated responses were all revealed by the interactions afforded by the MRT. This does not mean that in a non-technology mediated lesson the teacher would not experience unanticipated students' responses that would also result in situated learning. It merely highlights the nature of the opportunities that the hiccups provided for the teachers within the context of this study. The evidence from these hiccups clearly indicated that the MRT had a role to play in creating contingent moments and, as Eleanor and Tim experienced as a result of these moments, opportunities for their situated learning arose.

One element of their pedagogical development, which was illuminated by the actions Eleanor and Tim took in the classroom, concerned the process of consciously mediating whole-class discourse with a view to arriving at a consensus to new mathematical knowledge. It was obvious from the way that the majority of students responded to the teachers that both mathematics classrooms were places of enquiry and discussion, and each teacher listened carefully to the students' reactions and responses to tasks through 'mini-plenaries[2]'. Mathematical language and notation was sensitively introduced to the students as a support for both the class discourse and for developing written methods.

However, it appeared that in both teachers' activity designs these plenaries became mathematical 'pinch-points' as the teachers stopped tasks and focussed the students towards particular variant and invariant properties within the MRT explorations. Tim's approach to these pinch-points appeared to be more 'ad hoc'. He tended to move about the classroom and use the information he gained about the students' progress to decide when to stop the class. Eleanor's approach seemed more explicit as she monitored the publicly displayed screen capture view of the students' handhelds and announced to the class when a pinch-point was forthcoming.

The positioning of the initial example space within the 'wider mathematical landscape' of the topic was a second aspect to which both teachers responded following discussions and email exchanges concerning the design of activities. There was evidence that they were beginning to consider more carefully the role of the MRT within sequences of lessons and how the individual activities would support

[2] The 'mini-plenary' was a teaching approach advocated by the National Secondary Strategy in England and Wales whereby teachers would focus students on particular aspects of tasks throughout individual lessons.

their students to piece together the smaller generalisations in order to arrive at the deeper mathematical generalisation. However, this process was neither straightforward nor without its hiccups and it is most certainly an area needing further research using different methodological approaches.

The review of literature highlighted an opportunity to reflect upon whether the teachers' uses of the MRT within their designed activities in this study revealed more 'transformational' and 'meta-level' approaches to algebra, as opposed to the more common 'generational' tasks as described by Kieran (1996, 2004). Kieran's premise of 'algebra as activity' resonated with the interactions afforded by the MRT in the activities designed by Eleanor and Tim during the second phase. On reviewing Table 7-2 and Table 7-5 within Chapter 7, none of the Eleanor and Tim's activities fitted with Kieran's definition of generational tasks. Their activities fell into Kieran's remaining two categories:

- Transformational activities, mainly concerning the changing form of expressions or equations to maintain equivalence [STP8] [STP9].
- Global/meta level activities, for which algebra is a tool but are not exclusive to algebra. These would involve both generational and transformational activities [STP6] [STP7] [STP10] [CEL7] [CEL8] [CEL9].

In common with the early research into the use of technology in building students' conceptions of variables, evidence from this study revealed that the teachers had begun to create activities in which the MRT afforded 'meaningful experience to the creation of algebraic expressions' for the students (Sutherland and Rojano, 1993). In addition, Kieran expressed a note of caution regarding the possibility that the use of technology might 'make it quite easy to sidestep algebraic representation and algebraic transformation' (Kieran, 2004, p.31). However, the detailed descriptions of Eleanor and Tim's activities indicated that they both developed pedagogies that met Kieran's proposition that 'the algebra teacher has a crucial role to play in bringing algebraic representations to the fore and in making their manipulation by students a venue for epistemic growth' (ibid.).

Eleanor and Tim demonstrated approaches to their activity designs which resonated with Kaput's view that mathematics in the school curriculum was not all 'about symbols and syntax' (Kaput, 1989, pp 167-8). They faced a dilemma in needing to develop notational forms that made sense to their students alongside the requirement to use the syntactic forms designed into the MRT. So Tim's 'clean representational forms' were his response to this dilemma, although he learned that this could lead to the students' confusion over the specific objects under discussion, particularly within dynamic explorations.

This study has also provided a deep insight into Eleanor and Tim's pedagogic approaches in trying to bridge the transition from the exploratory MRT environment to incorporate the generational and transformational skills demanded by the current formal mathematics assessment regime. However, this was an element of the technology integration which they found both challenging and, in both cases, was an element of their pedagogy which they were seeking to develop further.

8.2.3 The process of instrumental genesis

As previously stated, there was a dual role for the technology within this study. On the one hand, it was mediating the teachers' learning within the mathematical domain of the research. On the other, it was mediating the transformation of this knowledge in the way that they reflected upon their students' learning and developed their own theoretical and practical pedagogic ideas. The development of the teachers' uses of the technology provided a strong indication of this process of instrumental genesis as they initially learned about the different functionalities of the MRT and subsequently developed classroom activities that incorporated chosen features.

The data analysis of the first phase of the study revealed nine broad instrument utilisation schemes that added insight to both the representational pathways of each activity alongside an indication of the nature of the intended students' interactions with each representation (see Section 5.3.1). However, the lack of classroom observation data during this phase did not enable a closer scrutiny of the way in which the representations were being used within the activity and each teacher's role in mediating the students' pathways through the representations. Phase one did reveal the trend for teachers to initially adopt IUS 1:*Vary a numeric and syntactic input and use the instrument's functionality to observe the resulting output in numeric, geometric, tabular or graphical form*. Whilst some teachers did not progress beyond this scheme, others diversified to schemes that began with geometric inputs and involved dragging (IUS3) or those which extended IUS1 by using another representational form to re-examine the initial output. A few teachers developed more diverse schemes which involved more advanced aspects of the MRT's functionality.

In common with the students in Guin and Trouche's study, Eleanor and Tim underwent an initial phase of discovery followed by an organisational phase whereby mathematical consistency between different representational outputs were sought (Guin and Trouche, 1999). Their experiences during the first phase, in which they became familiar with the individual MRT applications and their instrumentation, contributed to the development of their later activities. On the other hand, there were often two levels of instrumentation for them to consider. The first related to their own instrumentation schemes which were necessary for the conception and design of the activities. The second concerned their awareness of the students' instrumentation process, to which both Eleanor and Tim were noticeably sensitised.

Eleanor and Tim's very different starting points with respect to their previous knowledge and experience with handheld technology did not seem to impact greatly upon their perceptions of the success of their classroom experiences. On the one hand, Eleanor's activities were less technically demanding in their design, requiring basic instrumentation skills such as entering functions syntactically [CEL6, CEL9] and linking spreadsheet data to a scatter plot to produce a coordinate graph [CEL7]. Eleanor expressed that she did feel 'frustrations at not always having the technical know-how' [CEL(Journ)]. However, she did appreciate that her technical knowledge was developing as a result of her involvement with both the research project and the wider community of TI-Nspire teacher-users[3]. Tim's activities were more technically ambitious and there were several examples of how he had moved from his own instrumentation phase, whereby he learned about particular functionalities and began to instrumentalise these functionalities within his activity designs. Whilst Tim did experience one hiccup that was a direct consequence of a more complex activity design, when it lost functionality during the wireless transfer to the students' handhelds, he concluded that the underlying principle was one that he would use again.

In the second phase of the study, the evidence from the diagrammatic versions of the utilisation schemes seemed to suggest strongly that Eleanor and Tim were moving more confidently between different representations of the mathematics within each activity. This did not mean that all activities involved vastly different representational outputs of the MRT that would be evidenced by movement between the different TI-Nspire applications. It was due to the teachers' more subtle interpretation of the different representations. For example, in Tim's activity 'Circles and lines' [STP7], in which the students explored the possible points of intersections of lines with the circle $x^2 + y^2 = 5^2$, the activity involved a geometric

[3] Eleanor had become an active member of the Texas Instruments' sponsored Teachers Teaching with Technology (T^3) community, regularly contributing to meetings and conferences.

representation as its starting point, to which an algebraic interpretation was introduced.

There were some representational outputs that the teachers rarely used. For example, displaying the 'function table' (showing coordinate values) alongside the graph of a defined function was a representational output that was only used once by Eleanor in her final activity [CEL9]. This lack of use could result from the design of the MRT as the function table can only be revealed alongside a 'Graphing' page after it has been defined and graphed. This is unlike other representational forms within the MRT that can exist as their own 'page' and could be pre-designed into the activity file. On the other hand, there is some evidence in the research that little is known about how teachers integrate tables of functions within the teaching of function and graphs with technology, suggesting that this might be a widely underused aspect (Goulding and Kyriacou, 2008).

Finally, although the methodological design of this study does not allow for any conclusive remarks to be made concerning the emergence of social utilisation schemes for the MRT within the individual schools where Eleanor and Tim taught, there was emerging evidence of these within the wider community of TI-Nspire users. Eleanor had joined Tim in becoming an active member of the T^3 community in the United Kingdom. Eleanor presented her lesson 'Generating circles' on a couple of occasions and it stimulated wide discussion amongst other teachers about the way that she constructed the activity in the classroom and how she had involved the multiple representations in moving from the numeric, to the geometric and syntactic forms. Similarly, Tim's 'Pythagoras exploration' had also been taken up by other teachers.

8.3 How did the teachers learn? - The emergence of the hiccup as an ontological innovation

The evidence from the study strongly supports the thesis that teachers were engaged in substantial situated learning, which was prompted by their experiences of lesson hiccups, as they designed and evaluated activities using the MRT. These activity designs privileged explorations of variance and invariance in some way and most also involved multiple mathematical representations. In this sense the hiccup is considered to be an epistemological phenomenon, that is, a rupture in the fabric of the teacher's knowledge.

The hiccup is defined as a perturbation experienced by the teachers during lessons that is stimulated by their use of the technology and which illuminates discontinuities in their knowledge. This definition requires the hiccup to have been 'noticed' by the teacher and the evidence for this noticing was interpreted as a clear pause or hesitation on their part. In some cases, the teachers responded to the hiccup in a way that suggested that they had a previously rehearsed response repertoire. In others cases, the teachers offered a response, although it appeared to be 'in formation' or a holding response. Some hiccups were ignored by the teacher within the classroom setting in that no direct response was made.

All of these hiccups provided opportunities for the teachers to at least interrogate, if not develop their knowledge. It is not suggested that all hiccups would lead to a clear learning outcome for the teachers. However, the research evidence from my study is rich with examples of how individual hiccups (and combinations of hiccups) have prompted the teachers to rethink the subtle aspects of their activity designs.

There were of course many other types of hiccups that occurred during lessons other than those prompted by the technology. These concerned general classroom management issues, for example, resulting from students' off-task behaviour. However, these were outside of the domain of the study.

If the hiccup is being interpreted as a vital contributory element of the teachers' situated learning, it is necessary to articulate how the existence of the hiccup prompted the teachers' learning (and the nature of this learning) as a means to understand how the process worked. What follows is a detailed discussion of the different types of hiccups that emerged from the study, followed by a more substantial articulation of their role in the teachers' epistemological development.

8.3.1 Hiccups – the emergent classifications

The cross-case analysis revealed that the hiccups were attributed to the following eight considerations:

1. Aspects of the initial activity design:
   - Choice of initial examples.
   - Sequencing of examples.
   - Identifying and discussing objects displayed on the MRT.
   - Unfamiliar pedagogical approach.

2. Interpreting the mathematical generality under scrutiny:
   - Relating specific cases to the wider generality.
   - Appreciating the permissible range of responses that satisfy the generality.
   - Failing to notice the generality.

3. Unanticipated student responses as a result of using the MRT:	• Students' prior understanding is below teacher's expectation. • Students' interpretations of activity objectives. • Students develop their own instrument utilisation schemes for the activity.
4. Perturbations experienced by students as a result of the representational outputs of the MRT:	• Resulting from a syntactic output. • Resulting from a geometric output. • Doubting the 'authority' of the syntactic output.
5. Instrumentation issues experienced by students when making inputs to the MRT and whilst actively engaging with the MRT:	• Entering numeric and syntactic data. • Plotting free coordinate points. • Grabbing and dragging dynamic objects. • Organising on-screen objects. • Navigating between application windows. • Enquiring about a new instrumentation. • Deleting objects accidentally.
6. Instrumentation issue experienced by one teacher whilst actively engaging with the MRT:	• Displaying the function table.
7. Unavoidable technical issues[4]:	• Transferring files to students' handhelds. • Displaying teacher's software or handheld screen to the class.

Of these, hiccup types 6 and 7 were considered to be trivial or unavoidable and these were discarded from the discussion that follows. Consequently, only the first five types are discussed in detail.

[4] As previously mentioned, the teachers were using prototype classroom network technology which did result in some equipment failures during some lessons. Although these occurrences were definitely classed as hiccups, they were considered to be outside of the domain of the research study.

8.3.1.1 Hiccups arising from the initial activity design

The importance of the teachers' choices of initial examples (and to some extent the subsequent sequencing of examples), in relation to the contribution that the related hiccups had to their epistemological development, has already been discussed at length in Sections 8.2.1 and 8.2.2. This section focuses on a discussion of the nature of the remaining hiccups that fell into this broad category.

The sequencing of examples within the MRT activities was a common hiccup that both Eleanor and Tim experienced a number of times. These could be considered to be the most tangible type of hiccup because they sometimes recurred during the lesson as the students arrived at a particular point within an activity. Hence they were often very obvious to the teacher. Tim experienced this hiccup in a highly visible way when his choice of worksheet in the 'Equivalent quadratics' activity caused students to question the conflict between the notation used by the MRT and that of the accompanying paper and pencil activity [STP9]. Eleanor concluded that her lesson 'Transforming graphs' was very unsuccessful because she had been over-ambitious in her expectations of the way that the sequence of examples she chose would enable the students to appreciate the generalisation as intended.

Within the initial design of tasks, the implicit or explicit notation that was attributed to the variant objects, either within the actual syntactic notation within the MRT or as part of the accompanying classroom discourse, was a source of hiccup. This type of hiccup revealed itself when there were dynamic objects involved (i.e. in activity [STP6]) as the teacher was challenged to direct the students' attentions to the particular variant properties.

Eleanor experienced a hiccup when she introduced a new pedagogical approach to encourage her students to consider the generalisation they were beginning to make within the activity 'Triangles and squares' [CEL8 Hiccup08]. She used an 'always-sometimes-never' embedded question stem within the TI-Nspire file that gave the following mathematical statement and asked the students to respond with a single choice (Figure 8-2).

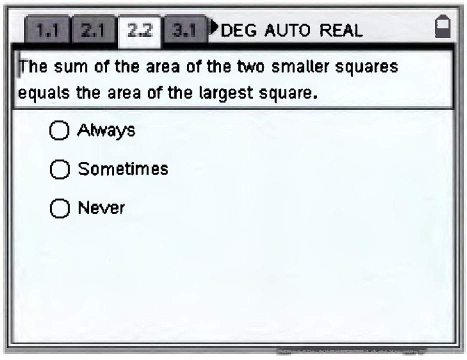


Figure 8-2 The 'always-sometimes-never' question included within Eleanor's activity 'Triangles and squares' [CEL8(tns-T)page2.2]

Eleanor collected their responses, displayed them to the class and initiated the following discourse:

Eleanor: *So one person said 'always', twenty two of you said 'sometimes' and one said 'never'. Why have we got those answers? Can anyone who said 'always' want to respond to the person who said 'never'.*

Student: *I put 'never' because every time I tried to it wouldn't go, like it wouldn't go exactly.*

Eleanor: *Okay, right well that's maybe to do with the software that we're using, okay? What about sometimes? Staci?*

Staci: *I managed to get it, but then it might not work with every single way. If the squares were like, one was smaller and one was slightly bigger, then it might not work.*

Eleanor: *Okay. But you did manage to get it, when did you manage to get it?*

Staci: *Like two seconds before I moved them*

Eleanor: *What was special about the arrangement that you had?*

[Staci is unsure how to respond]

Don't worry, alright... Bryony?

Bryony: *I put sometimes because erm it was sometimes out and sometimes if you tried to get it to that number it wouldn't work and it would just go to the decimal point after.*

Eleanor *So slightly with the accuracy. Anybody else sometimes?*

[CEL(Trans)]

After the lesson, and in her written lesson reflection, Eleanor commented on how her initial response on seeing the majority of students answer 'sometimes' was that they had fully understood that 'sometimes' was conditional on the enclosed triangle containing a right angle. However, her questioning of the students led her to conclude,

The results of this were totally unexpected as most selected sometimes

> *(which is what was expected) but on further investigation the reasons behind this were unusual. It was not because of being right angled or not, it was to do with the accuracy of the technology and that it didn't allow results to always be spot on.* [CEL8(Journ-T)]

This hiccup prompted Eleanor to reflect upon how her students were interpreting 'always-sometimes-never' within the context of both the mathematics of the task itself and the way in which the students' experiences with the technology had influenced their decisions. She concluded that her students needed more experience of this type of question within non-technology mediated mathematical activities before they would be able to see past the 'machine mathematics' to the 'ideal mathematics' that she was seeking.

8.3.1.2 Hiccups arising from the students' different interpretations of the mathematical generality under scrutiny

There were three sorts of hiccups which evidenced that the students had interpreted the mathematical generality in a different way to that intended by the teacher in their activity design. In some cases, these hiccups were related to the students attempting to 'see beyond' the example space in which the teacher's activity design resided. In others, the students were challenged by the range of permissible values within the defined example space and in a number of cases, the students failed to notice the generality at all.

Tim experienced the first of these when a student expressed unease over how a particular case that she was exploring related to the wider generality during the activity 'Circles and lines' [STP7]. She was working on a task which required her to justify the points of intersection between horizontal lines of the type $y=c$, where c was a positive or negative number and the given circle $x^2+y^2=5^2$.

Student(f):	*What if the circle... Does the circle have to be from this point [gestures to the origin] for it to work?*
Tim:	*No it does not have to be from that point? [hesitation] but the equation for a circle I have given you there is quite a nice equation, x squared plus y squared equals five squared, yes?*
Student(f):	*yes, it is*
Tim:	*The equation for a circle that's not centred on the origin is more of a complicated equation.*
Student(f):	*Oh well*
Tim:	*It does have an equation, but it's nastier.*
	Just to show you that, let's try and see if it does it... I've not tried this before... let's draw a circle anywhere, let's

just draw a circle there, okay, that's a circle somewhere else, then if I ask the computer to give me the equation for that one, there you go

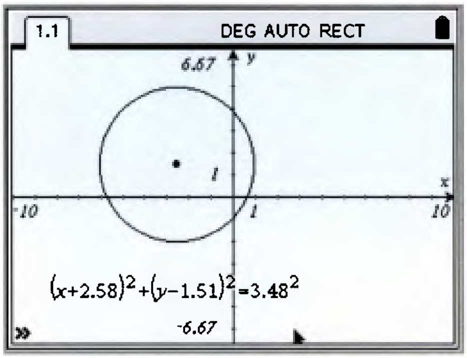


Figure 8-3 Tim's measured circle function used in discussion with a student during the activity 'Circles and lines' [STP7(tns-T)v2]

It's x plus two point five eight squared plus y minus one point five one squared equals three point four eight squared [student laughs] It's a bit messier can you see? Yes?

Student(f): *Yes*

Tim: *Now actually it's not that complicated because it's gone... It's moved two point five eight one way and one point five one away from there and the radius of that is three point four eight...*

But can you see that the maths involved would be a bit more complicated there.

Student(f): *Yes*

Tim: *That's why I've kept it fairly straight forward today and stuck with those, which for GCSE is about all you need to be able to do, okay?* [5]

[STP7(Trans)]

Tim responded to the student's question by using the MRT to draw and measure the equation of a circle that had its centre positioned away from the origin was his first attempt at this, something that he acknowledged to the student. His willingness to explore how the MRT would respond, and his satisfaction at being able to justify how the MRT had displayed the resulting equation, had expanded his knowledge of the technology in this instance. Knowing Tim as I now did, I would expect him to see an opportunity to further develop this sequence of lessons to encompass this functionality, albeit with his older students within the advanced level mathematics courses.

The hiccups which evidenced that the students' were uneasy about the permissible

[5] Solving equations of this type was beyond the expectation of the mathematics course that this group of students were following.

range of responses satisfying the generality under scrutiny, were particularly interesting in that there was evidence that teachers might be able to capitalise on this as a pedagogical approach, as Eleanor had done within the activity 'Generating circles' [CEL7]. She began the lesson with a quick poll in which she asked the students to give two numbers, which when individually squared and summed equalled twenty five. She shared the students' responses publicly, which initially included only the values 3 and 4. Eleanor re-sent the poll, asking the students to think of a different pair of numbers that satisfied the condition. The next set of responses included a zero, which prompted the hiccup when a student questioned Eleanor about whether zero was 'allowed'. Eleanor used this hiccup very productively to expand the range and type of permissible variables that the students were allowed to include. Later in the activity, this range was expanded again as students began to contribute square roots of non square numbers and negative numbers.

There were many occasions throughout the lesson observations when it was obvious to both me and the teacher that the students were failing to notice the mathematical generality in the way that the teacher had anticipated in their activity design. Indeed, as all of the activity designs had this as a fundamental aim, many of our associated discussions focussed on this aspect. Therefore it was very surprising when the analysis of the lesson hiccups revealed only one example where a noticeable hiccup had occurred. This was in Eleanor's lesson 'Transforming graphs' [CEL6] as she displayed the screen capture view of the students' handheld screens. (This example had been previously discussed in section 7.3.3).

I would hypothesise that very few actual hiccups were observed of this type because, as a unit of analysis, it was possibly too big to be useful. That is, the opportunities for the teacher to learn about the 'global view' of the students' appreciation of the generality were elements within other hiccup types. For example, a previously described hiccup [CEL6 Hiccup09], in which Eleanor used the 'Always-Sometimes-Never' question stem to find out if the students had appreciated the intended generality within the activity 'Triangles and squares' [CEL8]. The students' responses, when probed, seemed to reveal that they had not appreciated the generality under scrutiny, even though the vast majority responded 'correctly' to the question. On the other hand, the use of the screen capture view and quick poll did offer pedagogic strategies for teachers to assess the whole class perceptions of the generalities being sought. It would be likely that, as the teachers developed more use of these functionalities, this type of hiccup might become more frequent.

8.3.1.3 Hiccups arising from unanticipated student responses as a result of using the MRT

Whilst it could be argued that any unanticipated student responses could be a direct outcome of the task design, I am adopting the perspective that it was unrealistic for teachers to have considered all possible lines of enquiry in an activity's initial design, particularly when it involved a new technological tool. From this perspective, all of the activity designs were initially theoretical, although as the teachers built their history of experiences, so they were better equipped to consider possible students' reactions and build their repertoires of responses. Within the context of a constructivist mathematics classroom, the notion of 'student-proof' activities in which all possible deviations from the designed task were eliminated, was considered undesirable. My perception of Eleanor and Tim's activity designs were that they wanted to design tasks in which alternative lines of enquiry were encouraged as a means to promoting and extending their students' thinking. In classifying hiccups of this type, I considered the incident to be a hiccup if the teacher's immediate reaction was one that expressed surprise (or sometimes dismay) at the outcome.

The hiccups of this type have been previously discussed in Section 8.2.2 within the context of the teachers' pedagogical development and, to summarise, they concerned:

- students' interpretations of the objectives or instructions for the MRT activity that were broader than the teacher anticipated, sometimes leading to a more diverse set of students' responses;
- students developing their own instrument utilisation schemes for the activity;
- the MRT outputs prompting classroom discourse that was initiated by the students;
- students' approaches when using the MRT revealing incorrect interpretations of the mathematics or their prior understanding that was below the teacher's expectation.

8.3.1.4 Hiccups arising from students' perturbations concerning the representational outputs of the technology

The hiccups that concerned the students' perturbations triggered by the representational outputs of the technology related to several aspects, which included making sense of particular geometric and syntactic outputs or a doubting of the 'authority' of the MRT.

The hiccups that related to making sense of geometric and syntactic outputs were directly connected with the activity demanding too high a cognitive load for the students. In all cases this was due to the number of variables that were changing within the activity. Tim acknowledged this issue within the activity 'Pythagoras exploration' [STP6], concluding that there was 'too much variation' for his students to be able to draw purposeful mathematical conclusions' [STP6(Trans)]. Eleanor's students responded similarly when faced with too many different types of function transformations to explore within her activity 'Function transformation' [CEL7]. Both Eleanor and Tim seemed to really appreciate the implications of this particular type of hiccup, possibly because it acted as a barrier to many of the students appreciating the generalities in the way that they had anticipated.

A second interpretation of the hiccups resulting from students not being able to make sense of the syntactic output from the MRT concerned the nature of the syntactic output itself, a direct function of the MRT. For example, within Tim's activity 'Linear graphs' [STP10], described earlier in section 8.2.1, where the measured equation of the line joining the points (1,2) and (2,4) was displayed as y=2x-8.88E-15.

Finally, there were a couple of hiccups within two of Tim's activities where the students expressed doubt over the role of MRT as an 'authority' in the classroom. Within the activity 'Circles and lines' [STP7] the students used the calculator functionality to evaluate square roots of numbers and there was evidence of a student's repeated inputs as he tried to make sense of the output. This led to the student seeking an explanation of the MRT output from Tim (Figure 8-4).

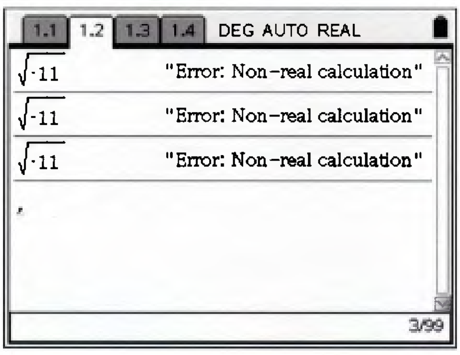


Figure 8-4 An example of a student's handheld screen in which the MRT's authority appears to be doubted during the activity 'Circles and lines' [STP7(tns-S)].

The other hiccup occurred during the activity 'Equivalent quadratics' [STP9] when one student questioned the response from the MRT after Tim had entered a mathematical statement (Figure 8-5).

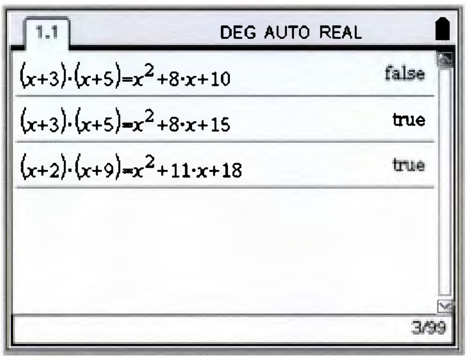


Figure 8-5 Tim's handheld screen that led to a student asking 'Why didn't it say false?' within the activity 'Equivalent quadratics' [STP9(tns-T)].

Student(m): *Why didn't it say false?*

Tim: *Sorry?*

Student(m): *Why didn't it say false?*

Tim: *Because we got it right.*

If I'd done... and I can copy this... [Tim copies line and pastes it below]

Let's say I didn't put eighteen, let's say I put seventeen, instead of eighteen, that would have been wrong... 'cause we worked out it was eighteen yeah?

[Tim changes the eighteen to seventeen]

If I put seventeen in instead, it says false.

Alright? You get the idea?
[STP9(Trans)]

In the second hiccup example, Tim's technical knowledge and confidence enabled him to respond quickly to the student's question. It was interesting in that Tim's immediate response was to provide another example, whereas the deeper interpretation of the student's question may not have related to the MRT's authoritative output at all. He may have been addressing a far more fundamental issue concerning why $(x+2)(x+9)= x^2+11x+18$ was true.

8.3.1.5 Hiccups arising from students' inputs and active engagement with the technology

An unsurprising (in my view) aspect of the research findings related to the occurrence of hiccups resulting from the students' instrumentation of the MRT. This finding resonated with the extensive body of research concerning the difficulties that students experience when learning to use a new technology for mathematics. The nature of the hiccups was also unsurprising in that they concerned: entering numeric and syntactic data; plotting free coordinate points; grabbing and dragging dynamic objects; organising on-screen objects; navigating between application windows; enquiring about new instrumentations and deleting objects accidentally. It should be stressed that the range of classes that Eleanor and Tim selected to use the MRT varied greatly in their mathematical abilities and experience with the MRT. This study was not designed to draw any conclusions concerning the frequency and nature of this particular type of hiccup as the primary research subjects were the teachers.

However, a noticeable outcome of the analysis of this type of hiccup was related to Eleanor and Tim's skill in responding to them. They demonstrated that they understood the hiccups immediately and exhibited a range of confident response repertoires. Over the course of the whole study, there was significant evidence to suggest that Eleanor, in particular, had evolved a very successful set of classroom strategies to support the students' instrumentation phase (see section 7.3.3).

8.3.2 Hiccups – their role in teacher learning

This section seeks to expand upon the way in which the hiccup can be described as an ontological innovation to support an articulation of how Eleanor and Tim appropriated the mathematical, pedagogical and technical knowledge through their design and evaluation of activities using the MRT, which privileged variance and invariance.

An important aspect of the hiccup is that it provided the opportunity for the teacher to rethink and reflect upon the conditions for its occurrence. In trying to understand how any particular hiccup may have contributed to the teachers' epistemological development, two things needed to have happened. Firstly, the hiccup needed to have been noticed by the teacher, which by definition was indicated by a noticeable pause or hesitation. Secondly, the teacher needed to have rationalised or reflected upon the hiccup in some way, such that their subsequent actions indicated that an aspect of situated learning may have resulted. It would be naïve to suggest that there was a direct causal link between <u>all</u> hiccups <u>all</u> subsequent actions, but there is enough evidence in this study to suggest that, over time, the teachers involved developed their mathematical, pedagogical and technical knowledge in response to their hiccups. It would also be naïve to suggest that my exchanges with the teachers (and indeed their involvement in the study) had not also influenced opportunities for the teachers' situated learning during the second phase. Whilst I cannot surmise what might have happened had I not been there, it is highly likely that their learning was accelerated by having a 'significant other' with whom to discuss activity designs, classroom experiences and, most importantly, reflect upon the hiccups.

Mason and Spence argue that the importance of 'knowing-to act' is an essential aspect of teachers' pedagogic knowledge that is not just dependent upon the situation, but the psychological stance of the person in that situation (Mason and Spence, 1999, p.141). Taking the premise that the hiccup presents the teacher with an unanticipated situation, Mason and Spence claim that each situation 'triggers and resonates access to acts' and that 'these triggers and resonances can be enhanced and sensitised'. This study revealed that both Eleanor and Tim had become sensitised to the hiccups that related to their students' instrumentation skills with the MRT. However, there were notable aspects of their subject and pedagogical knowledge concerning the design and structuring of tasks exploring variance and invariance, for which their sensitivities were less acute. The notion that the teachers might not 'know-to' act in the moment is a realistic one that could be explained by them just not thinking about it or, as in Eleanor's case, when faced with the Emily's curved graph, her reaction suggested that she may have exhibited 'cognitive dissonance' (Festinger, 1957). The combination of her praise to Emily followed by a direct instruction not to 'do curved graphs' suggest that Eleanor was holding conflicting ideas regarding her admiration of Emily's creative response to the task alongside her mathematical boundaries for the topic.

There was a need for the teachers to have understood the hiccup, something that was not always possible in 'real-time' during the lesson. This was when the post-lesson discussions and emails were important, particularly if my own systematic analysis of the research data had provided opportunities to revisit certain hiccups with the teachers, some time after the lessons had taken place. For example, Eleanor and I discussed a student's screen which was captured halfway through the second lesson when she was working on the activity 'Crossing linear graphs' [STP9]. Eleanor had introduced the task that required the students to use the MRT to generate linear functions other than $y=2x$ that would pass through the coordinate (3,6). Immediately after the task introduction, Maria asked Eleanor a question.

Maria:	*Miss, can we do two x plus one minus one?*
Eleanor:	*[Hesitated and looked at the graph on the board]*
	No. [firmly]
	Well you can... [questioningly]
Maria:	*Is that cheating?*
Eleanor:	*Well you can - yeah, but it's not exactly very exciting is it.* [CEL7(Trans)]

Eleanor then moved away to talk with another pair of students and Maria had no more direct discourse with her during the lesson, nor did Maria's handheld screen feature in any whole class discourse.

A few weeks later, after I had transcribed the lesson, Eleanor and I met to discuss the lessons. I had selected the above extract from the transcript and we reviewed Maria's TI-Nspire file (Figure 8-6 and Figure 8-7).

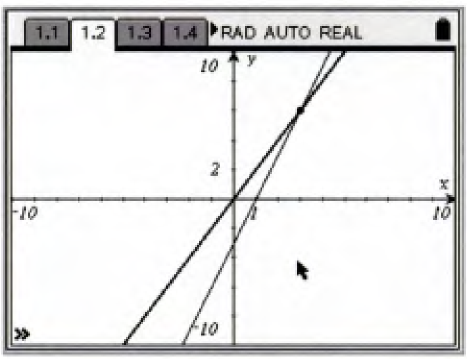


Figure 8-6 Maria's handheld screen showing her response to the activity 'Crossing linear functions' [CEL9(tns-S)1].

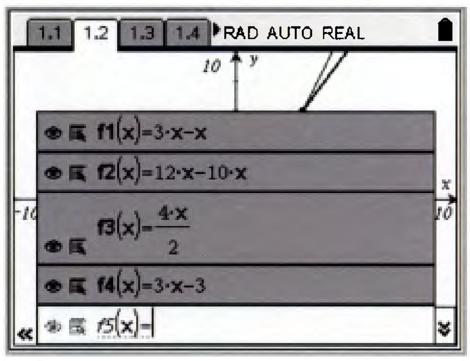


Figure 8-7 Maria's corresponding functions for the graphs displayed in the previous figure [CEL9(tns-S)1].

Eleanor's immediate reaction was one of hesitation as she tried to make sense of what Maria had done. She then expressed some surprise and immediately appreciated the mathematical implication of her giving Maria permission to enter equivalent functions. Eleanor admitted that she had never thought about this before and, although she then realised it was obvious, she also saw an opportunity to develop a task for her younger students that would specifically explore equivalent expressions. Eleanor also suggested that she could use such an approach in an activity to check algebraic expansions graphically.

Eleanor was intrigued about Maria's subsequent progress during that activity and we reviewed her TI-Nspire file, which was captured at the end of the lesson.

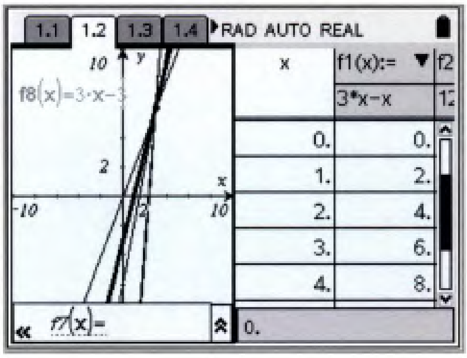


Figure 8-8 Maria's handheld screen showing her response to the activity 'Crossing linear functions' later in the same lesson [CEL9(tns-S)2].

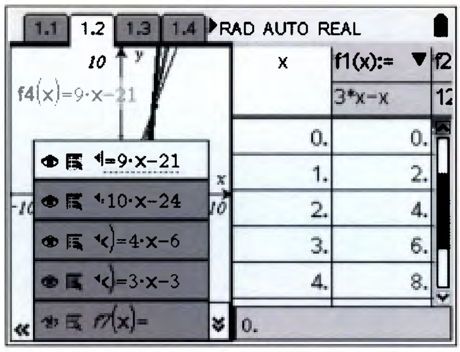


Figure 8-9 Maria's corresponding functions for the graphs displayed in the previous figure [CEL9(tns-S)2].

Eleanor was relieved to see that Maria had broadened the type of function that she was entering and moved away from equivalent functions to include some of the functions that had featured during the intervening whole-class discourse. Eleanor commented that adopting a more systematic approach to reviewing the students' files, as a means of gaining an insight into their mathematical thinking was something that she felt would be useful to add to her practice.

Both teachers experienced a certain level of anxiety concerning their personal aspirations to be able to respond to everything that was now visible. In the short term, that is during individual activities, this concerned identifying the focus for and the timing of the activity 'pinch-points'. In the medium term, this was more focussed towards smoother progressions for subsequent activities. There were times when it appeared that the more the teacher knew the harder it became to know how best to guide the learners.

Within the study, there was little evidence of the teachers beginning to plan for particular hiccups to happen as an integral part of an activity's design, although I would hypothesise that this would begin to happen as they revisited familiar activities with new classes.

8.4 The impact of the classroom network technology on Eleanor and Tim's learning

This study was not specifically designed to examine the role of the classroom network technology, which both Eleanor and Tim began to use during the second phase. However, as it became part of the technical setup in their classrooms, it had an obvious and observable impact on several aspects of their professional learning.

An immediate impact resulted from the functionality that enabled TI-Nspire files to be sent from the teacher's handheld device or computer to the students' handhelds at any time during the lesson. This meant that the files could be developed and amended, moments before the beginning of a lesson, or developed collaboratively with the students during the lesson. In both teachers' cases, the teachers developed pedagogical approaches which involved the students in co-constructing the software files with which they would be interacting. It appeared that this process of co-construction had a positive impact upon the students' willingness to participate in the activities, as they anticipated the files being sent and were generally motivated to open the file to begin the exploration. It also meant that the students did not necessarily need all of the instrumentation skills to construct each MRT task, although they were aware of its underlying construction. Both Eleanor and Tim commented that this was a distinct advantage of the classroom network technology.

The screen capture functionality, whereby all of the students' handheld screens were simultaneously displayed on public view in the classroom, appeared to have had a noticeable impact on a number of aspects of Eleanor and Tim's learning. This shared display prompted just over a quarter of the total number of lesson hiccups (n=63) which were mostly concerning:

- unanticipated students' responses [CEL9H4], [CEL6H6], [CEL7H3], [CEL8H7], [CEL9H1], [CEL8H5], [STPH10], [STPH13], [STP7H1].
- interpreting the mathematical generality under scrutiny [CEL7H1], [CEL8H3], [STP10H11] ;
- issues concerning the initial activity design [STPH1], [STPH9], [STP10H12];
- students' perturbations stimulated by the representational outputs of the multi-representational technology [TP10H8], [TP7H1].

For both teachers the hiccups that resulted from unanticipated students' responses were the type of hiccup that was made most visible by the screen capture view. Whilst it could have been possible that the teachers would have experienced the same hiccups whilst moving around the classroom and supporting individual students, it seems more likely that the simultaneous display of all of the students screens vastly increased the probability of the hiccup occurring.

Finally, both teachers used the quick poll (or the embedded questioning facility[6]) at

[6] The classroom network system enabled the teacher to embed questions within the TI-Nspire file, which could then be collected, displayed and analysed.

least once during the second phase (Eleanor n=3, Tim n=2). In all of these uses except one [CEL7], it resulted in a lesson hiccup. As the students' answers were on public display using screen capture view, these have been included in the analysis above.

Eleanor and Tim both commented that having the screen capture functionality improved their awareness of their students' mathematical knowledge and understanding. Tim commented that 'the maths that my students struggle with is not always what I expect' (STPJourn-T).

I interpreted this in two ways. There was an immediate pragmatic knowledge that would influence their decisions about the next episode in that lesson or a subsequent lesson. However, there was the potential for more epistemic knowledge, whereby their teaching practices would evolve to use the diversity of the students' responses as part of their pedagogy. Eleanor had begun to do this in her lesson 'Generating circles' [CEL7] when she used the diversity of the students responses to force them to decide upon the numerical domain for the problem.

Tim captured the dilemmas that teachers face in this new pedagogic setting when he said 'I will not always be able to respond to everything that I am bombarded with... I have to make choices' (Journ-T). Both Eleanor and Tim had learned about the diversity of students responses within different activities and this learning was informing their subsequent activity designs. In addition they were evolving strategies for responding to this diversity in the classroom setting as it happened. This new knowledge could be interpreted as Mason and Spence's 'knowing-to act in the moment' which requires 'some degree of sensitivity to features of a situation' and 'a degree of readiness as a result of what is being attended to' (Mason and Spence, 1999, p. 151). The ways that Eleanor and Tim acted in response to the various hiccups they experienced gave an insight into their different forms of this knowledge. There was only one example when Eleanor and Tim both experienced an identical hiccup in a lesson as a result of the shared public view. Tim's response to an unanticipated response by a student during the 'Pythagoras exploration' was very different response to Eleanor's when faced with the same situation (see Figure 8-10 and Figure 8-11). Tim was confident in the mathematics such that he was able to justify the correctness of the situation to the students, whereas Eleanor was more hesitant and offered no immediate explanation.

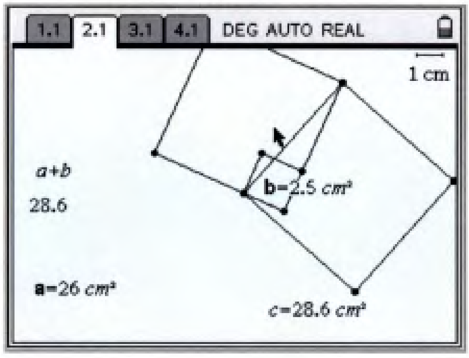


Figure 8-10 A student's response to Task 2 within Tim's lesson 'Pythagoras exploration' [STP6(tns-S)]

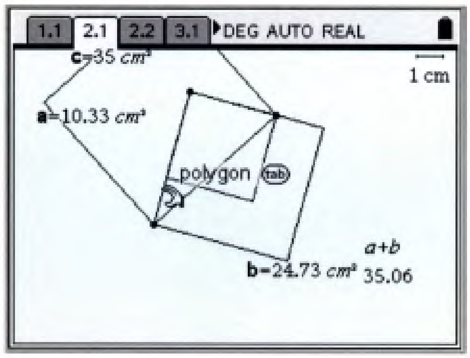


Figure 8-11 A student's response to Task 2 within Eleanor's lesson 'Triangles and squares' [CEL8(tns-S].

8.5 Summary

This chapter discussed the findings in relation to Eleanor and Tim's situated learning as a cross-case analysis as relevant to the aim for the research. This learning was categorised with respect to the evolution of their mathematical, pedagogical and technical knowledge that resulted from their lesson hiccups. This was followed by a commentary of the nature of the lesson hiccups and an articulation of the hiccup theory of teachers' situated learning. The chapter concluded with a discussion of the contribution that the networked classroom technology made to the teachers epistemological development.

9 CONCLUSIONS AND IMPLICATIONS

The contingent moments or 'hiccups' that teachers experience when integrating a multi-representational technology into their classroom practice provide both rich contexts for their situated learning and fruitful foci for professional discourse.

9.1 Introduction

This chapter summarises what has been achieved by this research project and articulates how the findings contribute to the research field within the domain of the study. In doing so, it makes links between the research findings and those of the key studies outlined within the review of literature. In addition, it suggests ways in which the process of mathematics teacher development concerning multi-representational technologies might be reconceived. The limitations of the study are discussed and some directions for further research are identified.

9.2 Summary of the findings

The findings of the study are organised as follows:

- The theoretical construct of the 'triad of instrumented activity' (within the context of the study) is expanded upon as a direct outcome of the research.
- The way in which the teachers' ideas about mathematical variance and invariance *shaped* and *were shaped by* their use of the technology is described.
- Some important considerations for activity designs that involve the exploration of mathematical variance and invariance within MRT environments are concluded.

9.2.1 Reconsidering the triad of instrumented activity

This study sought to develop a clearer articulation of the nature of the 'object' within Verillon and Rabardel's triad of instrumented activity, which had been adapted for the research as shown in Figure 9-1.

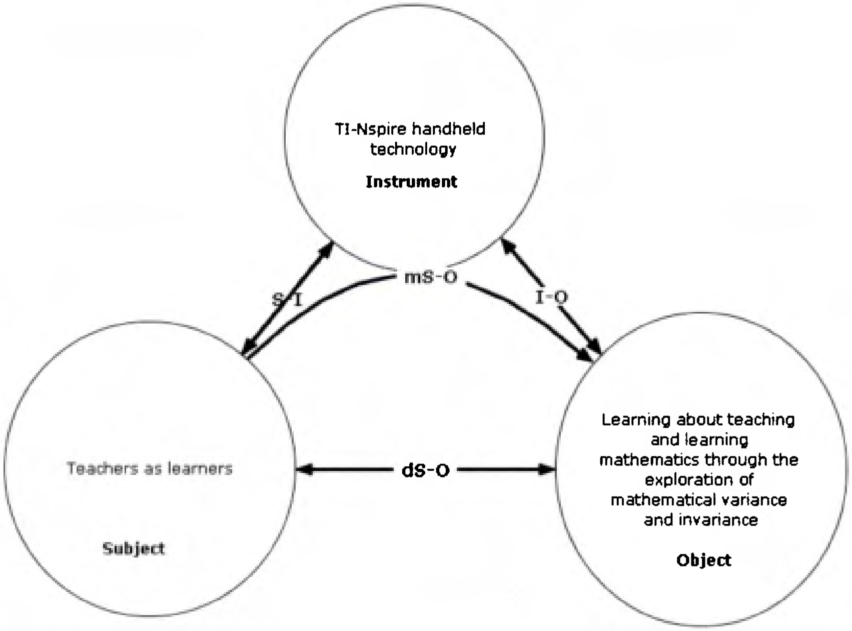


Figure 9-1 My adaptation of Verillon and Rabardel's triad of instrumented activity in which the instrument is the TI-Nspire handheld, the subject is 'teachers as learners' and the object is teachers' learning about the teaching and learning of mathematics through the exploration of mathematical variance and invariance.

The methodological approach developed for the study was designed to illuminate the trajectory of the teachers' epistemological development, as evidenced by the evolution of their instrument utilisation schemes alongside their personal mathematical and pedagogical knowledge development as they designed and evaluated classroom activities. The longitudinal case study approach provided rich data that illuminated the nature of the different interactions within the triad. Each of the interaction types within the triad is now considered individually with respect to *how* each type of interaction contributed to the teachers' learning within the domain of the study

9.2.1.1 Direct Subject-Object (dS-O) interactions

This element of the triad concerns the non-technology mediated ways that teachers developed their ideas and approaches to incorporate an expanded perspective of mathematics and its pedagogy. It is therefore very difficult to identify the learning that took place as a result this type of interaction because the context for this study concerned the use of technology. I did not have lesson observation data to be able to compare the teachers' approaches to the exploration of variance and invariance in the non-technology settings. However, the teachers' personal reflections on the outcomes of their technology mediated learning (mS-O) provided an insight into their broader thinking. They encountered the following aspects of their learning within the technology setting that could be 'transferable' to the non-technology setting:

- A broader conception of mathematical variance and invariance.
- An appreciation of different forms of expression of mathematical generality.
- The importance of the initial choice and expandability of the example space.
- An appreciation of the teachers' role in supporting students transitions between different mathematical representations within paper and pencil approaches.

However, particularly in Eleanor's case, there was strong evidence to suggest that the use of the technology had enhanced her own mathematical understanding which would impact upon her confidence to teach certain topics within the non-technology setting if she so chose.

9.2.1.2 Direct Subject-Instrument (S-I) interactions

This type of interaction was predominantly concerned with development of the individual teachers' instrumentation of the MRT. For Eleanor and Tim, this development spanned Phase One and Phase Two of the study, which was a period of two years and five months. There was a visible trajectory of development in both the range of representations that the teachers privileged and their competence and confidence to move between the different representations. To some extent, the teachers mirrored the students' behaviour in Guin and Trouche's research in that they were initially in a phase of discovery as they realised the affordances of each application in the MRT, and moved into a phase where they sought mathematical consistency between representations (Guin and Trouche, 1999). However, their underlying motives for these actions were very different to those of the students in Guin and Trouche's study. As they were learning to use the various applications, the teachers engaged in meta-level thinking, which concerned the accessibility of the related instrumentation steps for their own students. For the teachers, it was difficult for them to separate these two trains of thought. In Eleanor's case she would note down her own instrumentation steps and later incorporate these into her own lesson plans and structures for her students. Tim was more confident of his own instrumentation skills with the MRT, appearing to internalise his new technological knowledge more quickly and he tended to support his students verbally in the classroom.

9.2.1.3 Direct Instrument-Object (I-O) interactions

This element of the triad concerns the 'cold' interactions between the teachers' learning and the technology, which was triggered by the outputs from the MRT in the classroom setting. For example, Tim learned about the 'unhelpful' format of the measured linear equation in his activity 'Linear graphs' [STP10]. The research revealed that, although the teachers may have conceptualised the instrumentation utilisation scheme for an activity at the planning stage, they did not always rehearse all of the instrumentation steps in advance of the lesson. Hence a number of hiccups occurred in the classroom situation that would provide evidence for the way that these forms of interaction contributed to the teachers' overarching epistemological development.

These interactions were highly significant in the teachers' epistemological development as in some ways the teachers were more willing to rethink the MRT activity as a tangible response. The TI-Nspire file was an artefact that could be 'blamed' and redesigned, whereas remembering the human classroom interactions that may have instigated a particular hiccup and reconsidering the pedagogical approach was more difficult. The teachers needed both time and the systematic evidence that the study produced to support this deeper reflective process to happen. However, I would consider that it is essential that teachers have an opportunity to make sense of the hiccups that were revealed as a result of the unanticipated outputs from the technology itself. There is a danger that, without such opportunities, teachers would reject the use of technology completely because it 'throws up' occasional responses which place them in situations in the classroom that they lack the confidence and knowledge to question or explain.

9.2.1.4 Mediated Subject-Object (mS-O) interactions

This particular interaction was of predominant interest to my study as I sought to understand the processes through which the technology supported the trajectory of the teachers' epistemological development concerning variance and invariance. Consequently, sections 9.2.2 and 9.2.4 articulate this in more detail by considering how the teachers' mathematical ideas *shaped* and *were shaped by* their interactions with the technology and the role that the lesson hiccups had in stimulating this development.

9.2.2 How teachers' mathematical ideas about variance and invariance *shaped* and *were shaped by* their uses of the technology

There was a two-way consideration of the nature of the way in which the mathematical terrain was being reconceptualised with respect to its content and pedagogies. The teachers' mathematical ideas of variance and invariance *shaped* how they used the technology, which was evident from the activities they designed and the representations that they privileged. The converse was also true as their activity designs were shaped as a result of their increased knowledge about the affordances of the MRT.

The study revealed a number of ways in which the teachers' mathematical ideas concerning explorations of variance and invariance *shaped* their use of the technology. The data from the first phase revealed teachers' initial tendencies to develop instrument utilisation schemes that relied on single inputs which were examined in a single output representation (IUS1). Their choice of representations gave an indication of how they initially conceived the variant and invariant properties and it was understandable that the teachers would take it slowly in becoming familiar with different representational forms (and the associated instrumentations) before they were in a position to begin to consider these alongside each other. The significance of Eleanor and Tim's positioning of this initial example space became more apparent as the project progressed and, irrespective of their increased knowledge of the affordances of the MRT, they began to use familiar representations to develop more substantial activities. Tim's activity 'Circles and lines' [STP7] was a good example of this as it required little more than a geometric figure displayed by the MRT as a stimulus for the activity that followed.

Over time, Eleanor and Tim's increasing knowledge of the technology and its functionalities provided substantial evidence for how the teachers' mathematical ideas concerning explorations of variance and invariance were being *shaped by* their use of the technology. In his final reflection, Tim wrote,

> *During the original TI-Nspire project I had a very distinct moment when all of a sudden I recognised the importance of variation (in its many forms) to focus student attention. This led to the two-part article that I wrote for ATM. In it I recognised the need to identify the mathematical property (generalisation?) that I am trying to draw attention to... and then identify the variables within the situation. Then by using ICT, vary that which will draw attention to the mathematical property. I mention*

> *this because after I had this 'moment'... my thinking processes in task design began to change. There was greater clarity in my head for the need to consider what I wanted to vary as I planned the Mathematics-ICT task. Up until this point I designed tasks that did involve variation... but I don't think that I was as aware of what I was doing in the task design.* [STPJourn-T]

In Tim's case, the focus on lesson design that formed a large part of the professional activity during the first phase of the project had stimulated him to reflect upon the importance of a deep consideration of the variant and invariant properties within his task design. His Phase Two activities clearly evidenced how he was continuing to do this, and still learning about how complex a process this was.

Although Eleanor had not verbally articulated her personal reflections on her epistemological development as deeply as Tim had, the research data was rich with evidence of how her thinking and planning had been significantly influenced by the affordances of the MRT. In her case, several seeds of ideas stayed dormant within her until she saw an opportunity to use them in her own classroom and she was openly influenced by the ideas of others involved in the project. On several occasions Eleanor referred to her lesson activities as having 'a backward approach', in that the students were being given the opportunity to experience a mathematical situation first before being presented with the underlying theory.

A number of activities developed by Eleanor and Tim could be considered to be offering new approaches to the mathematics that are not evident in existing schemes and resources. For example, Tim's 'Pythagoras exploration' [STP6] took as its starting point an example space that was much broader than that evident within the current legitimate curriculum and Eleanor's 'Generating circles' [CEL7] offered an insight into how the integration of multiple representations might be manifested in accessible activities for secondary students.

There was very clear evidence that Eleanor and Tim had developed 'warranted practice' in their use of the MRT, as described by Ruthven, in which they had articulated explicit rationales for their 'practice in action', analysed the processes in operation, assessed the impact on their students' development and sought to refine their practices accordingly (Ruthven, 1999). In developing this practice, they both developed the language through which they could better articulate the tacit knowledge that Polanyi and Schön describe as an integral part of human thinking and practice (Polanyi, 1966, Schön, 1987).

9.2.3 Designing activities that privilege the exploration of variance and invariance in a MRT environment

The scrutiny of over eighty classroom activities within this study, the majority of which emphasised the exploration of variance and invariance, has provided a substantial data set on which to base a response to Stacey's call for further research, 'What are the major considerations when exploiting each of the pedagogical opportunities' within the pedagogical map (Stacey, 2008).

Mason et al's premise that 'a lesson without the opportunity for learners to express a generality is not in fact a mathematics lesson' could be considered to have been an initial constraint for the teachers involved in this study when designing their activities to explore variance and invariance (Mason et al., 2005, p.297). However, an element of the teachers' epistemological development was related to their realisation that expressing generality was a very important aspect of the activities that they went on to design. The teachers' increasing attentions to the way that the MRT environment supported or hindered this, and the design of the associated supporting resources and their role in mediating the associated classroom discourse, was another element of their professional development.

Evidence from the study suggests that the process of designing tasks that utilise the MRT to privilege explorations of variance and invariance is a highly complex process which requires teachers to carefully consider how variance and invariance might manifest itself within any given mathematical topic. The relevance and importance of the initial example space, and how this might be productively expanded to support learners towards the desired generalisation is a crucial aspect of activity design.

The starting point for any classroom activity is its initial design and the following set of questions, generated as a result of this study offer a research-informed approach:

- What is the generalisable property within the mathematics topic under investigation?
- How might this property manifest itself within the multi-representational technological environment – and which of these manifestations is at an accessible level for the students concerned?
- What forms of interaction with the MRT will reveal the desired manifestation?
- What labelling and referencing notations will support the articulation and

communication of the generalisation that is being sought?

- What might the 'flow' of mathematical representations (with and without technology) look like as a means to illuminate and make sense of the generalisation?
- What forms of interaction between the students and teacher will support the generalisation to be more widely communicated?
- How might the original example space be expanded to incorporate broader related generalisations?

There is a degree of resonance between the questions posed as a result of this study (concerning the design of activities privileging explorations of variance and invariance) with the principles of lesson design that Pierce and Stacey concluded following an analysis of the use of TI-Nspire handhelds to privilege the use of multiple representations within the teaching of quadratic functions (Pierce and Stacey, 2009). Their principles included:

> *focus on the main goal for that lesson (despite the possibilities offered by having many representations available); identify different purposes for using different representations to maintain engagement; establish naming protocols for variables that are treated differently when working with pen and paper and within a machine; and reduce any sources of cognitive load that are not essential.* [ibid, p.228]

Responses to the questions just posed uncover a very generic 'top level' of thinking which makes little sense in the absence of a clear mathematical context. The next level of thinking becomes closely related to the topic of the mathematics itself for which a number of existing structures and approaches can support teachers to develop further this aspect of their practice (Watson and Mason, 1998, Mason and Johnston-Wilder, 2005, Mason et al., 2005).

9.2.4 The significance of the hiccup in understanding and articulating pedagogy

The need for mathematics teachers to have contingent knowledge that enables them to respond in-action is a firmly established aspect of the research literature (Shulman, 1986, Rowland et al., 2005, Mason and Spence, 1999). However, the research literature concerning how this contingent knowledge developed through classroom practice is under developed. This thesis makes an original contribution to understanding *what* and *how* teachers learn about the concept of mathematical variance and invariance within a technological setting through the construct of the lesson hiccup.

Hiccups, by their definition, only occurred if the teacher noticed them. This implies that the teacher needed to have an existing knowledge of the mathematics and the technology, which sensitised them to the hiccup. Their associated knowledge would be an influence on the degree of this sensitivity. For example, when Eleanor noticed that her structuring of the students' worksheet within the lesson 'Transforming graphs' [CEL6] made it difficult for her to develop the classroom discourse in the way she had envisaged, her existing mathematical knowledge-in-action was such that she could not fully appreciated the meaning behind the hiccup.

The teachers did appreciate that some hiccups, although possibly unwelcome when they were first experienced, might actually be desirable and *designed into* future activities. For example, in some situations a diversity of students responses might make it difficult for the teacher to support the class towards particular generalisations, whereas in others activities it might expand the example space. This element of knowing *when* hiccups might be reconceived as a teaching approach is an element of the new pedagogies that became possible for the teachers working in the MRT environment with access to the shared public view. Indeed Eleanor developed this pedagogical approach within her activity 'Generating circles' [CEL7] when she used the students' diversity of numerical responses to expand their notion of the acceptable number range.

An important appreciation of the lesson hiccup is that, although they might sometimes be unwelcome or unhelpful, they undoubtedly provided an opportunity for the teachers to reflect on the experience and to develop their mathematical, pedagogical and technological knowledge. In many cases the teachers' gained a deeper insight into their students' prior knowledge and understanding.

A number of the hiccups instigated by the students' own mathematical questions developed into mathematical pinch-points during the activities in which the teachers saw validity in the question that was worthy of a wider classroom discourse. Again, the teachers' sensitivity to these questions was crucial in taking up the opportunities motivated by the students' own ideas. However, in some cases the teacher's own subject knowledge and confidence could hinder productive lines of enquiry. Alternatively the teacher feels pressured to rationalise the students' question alongside the legitimate curriculum.

The process through which teachers mediate whole class discourse with a view to arriving at a consensus with respect to new mathematical knowledge when the 'substance' is provided by a multi-representational technology is another important element of their developing pedagogy, an element highlighted in other studies (Guin and Trouche, 1999, Drijvers et al., 2010).

9.2.5 Peripheral findings from the research

A number of peripheral findings emerged from the study and, although in some cases there was insufficient or incomplete data to support any bold claims from the research, it did signify that future research in these areas would be desirable.

9.2.5.1 The impact of the classroom network technology

The discussion provided in section 8.4 highlighted a number of ways in which the network classroom technology provided a significant window onto the teachers' professional learning. This was evidenced by the proportion of lesson hiccups that were made visible through its functionalities, in particular the screen capture view. It follows therefore that, if the central tenet for this thesis is that lesson hiccups form an important contributory element of teachers' situated learning within the domain of the study, increasing the visibility of these hiccups could increase the rate of teaching learning within MRT settings.

However, both teachers expressed doubt over the demands that this widely expanded view of their students' thinking placed on their abilities to steer the mathematical path within lessons. This led to a focussing of their activity designs as they very quickly began to appreciate important aspects such as the example space, levels of cognitive load and the labelling and notation of variant objects.

There was no doubt that the classroom network technology proved to be a significant research tool as a wealth of data was made available to me that may otherwise had remained hidden. The ability to capture students' work (files and screen capture views) during mathematical 'pinch-points' and analyse these alongside the lesson transcripts was an important element of the professional discussions I had with the teachers, in addition to the obvious advantage as a data collection method.

There was one highly visible outcome of the use of the classroom network technology, which concerned my own and the teachers' observations of the students' level of engagement with the mathematical activities in both Eleanor and Tim's classrooms. We noticed that the students appreciated working on the same file as a class and anticipated receiving it on their own handheld to be able to continue their own explorations. The students found the screen capture view a motivating environment and actively sought similarities and differences between their own screens and those of their peers. In many cases, the students instigated the mathematical pinch-points in lessons by announcing their own theories or questioning other students' responses. In this sense the classroom dynamic was visibly placing students' work at the heart of the classroom discourse in the way that Pierce and Stacey described (Pierce and Stacey, 2008). However, there is a need for longitudinal systematic studies which explore the students' experiences in such classrooms as a means to understanding how the nature of the mathematics curriculum and its pedagogies might be reconceptualised in such environments.

9.2.5.2 The design of the TI-Nspire handheld

This study was not designed to evaluate the individual functionalities of the TI-Nspire handheld and, as the operating software was revised several times during the period of the study there were aspects of its instrumentation that continuously evolved. However, my research did reveal that the subtleties of this evolution were an aspect of the teacher's technological knowledge development that resulted in both successes and frustrations. For example, functionality that was previously identified by the teachers as being pedagogically helpful disappeared. For example, during Phase One all of the teachers responded positively to the appearance of the actual names of the variables within the equation of the measured 'movable line' within the activity 'Weighing sweets' [GBA3/GAS1].

There were some aspects of the functionality that were underused in both Phase One and Phase Two of the study, which might be a result of their location within the menu structure. The function table was an example of this and I have suggested possible reasons for this within section 8.2.3.

9.3 Rethinking mathematics teachers' professional development concerning multi-representational technology

The longitudinal nature of this study has enabled a number of recommendations to be made concerning the nature, focus and timescale of teachers' professional development activities involving the MRT. In addition, prompted by my reflections of my role during the second phase, the importance of a 'significant other' in supporting teachers' development is also considered.

9.3.1 The need for a clear mathematical focus

It is very common for teacher development projects and courses that are concerned with technology to focus mainly on developing teacher's instrumentation skills at the expense the important discussions concerning the nature of the mathematics under consideration and the associated pedagogical considerations. During the second phase of the study, the mathematical focus for the research 'variance and invariance' provided a challenging and relevant focus to support both Eleanor and Tim to design, evaluate, (and redesign) their classroom activities. Whilst it could be argued that any individual mathematical topic might suffice, indeed each teacher needed to have a mathematical topic in mind as they began to design each activity, it was the overarching principle of 'what will change and what will stay the same' that prompted the actual activity designs. Having adopted Pierce and Stacey's pedagogical map as an analytical tool when analysing the observed lessons, I would conclude that any of the pedagogical opportunities contained within would offer a substantial starting focus for teachers engaged in such projects (Pierce and Stacey, 2008).

9.3.2 The role of the 'significant other' in supporting teachers' situated learning

Reflecting on my role as researcher in this study, in which I worked closely with Eleanor and Tim, in both cases I can identify examples where conversations and email exchanges led them to take risks in their activity designs by adopting new technological and pedagogical approaches. Whilst I cannot possibly comment upon what might have happened if my role had been totally non-interventionalist, my presence as a 'significant other' stimulated mutually beneficial professional conversations and supported both teachers to have the confidence to take risks in their classroom when adopting new approaches. In some sense, knowing that I would be observing their lessons triggered them to think afresh about their teaching. An example of this was an email exchange I had with Tim the evening before I was due to observe him with one of his classes. Tim had written a synopsis of his previous lesson with this class, saying that they had been exploring equations of circles using the MRT and, by the end of the lesson, they were 'generally happy' $x^2+y^2 = r^2$, where r was the radius of a circle, with centre at the origin. He continued,

> *They didn't have a lesson today (another exam)... but I was thinking that I would like to look at finding points of intersection between a*

straight line and a circle. Am struggling at the moment to find a way to use the Nspire for this other than to 'check' answers using the co-ordinate tool... any ideas? [Journ]

My response posed a series of questions to Tim, in which I was trying to help him to think deeply about his underlying aims for the activity,

What is the 'big picture' here?

Ultimately I expect it is to solve GCSE exam style questions... but en route is it?

- appreciate when the line will cross the circle, 0, 1 2 times (I can see everyone with a common circle and then each students adding a chosen straight line to this...)

- getting a sense of where the line and circle might cross - i.e. Can they get it (by dragging the line) to cross in a given quadrant too?

- exploring the algebra to arrive at how you would prove that the line and circle crossed at a certain place... You could approach this backwards by giving the students (an) incorrect point(s) of intersection and proving it doesn't by substitution...

- leading into finding algebraically where the line crosses - several approaches here - do you lead the class through one or allow some to come up with this for themselves (with some clues)

- offering an extension activity of 'finding formulae' for x and y in terms of m and c where y = mx+c crosses y^2+x^2=r^2

- identifying the 'conditions' for the circle and line to intersect 0, 1 and 2 times (extension) [Journ]

It was evident from the activity that Tim subsequently designed that he had thought about and responded to a number of the questions that I had posed. However, reflecting on this exchange at the end of the study, I realised that in my response I had merely 'laid bare' my own thinking about how I would design the activity based on the information that Tim had provided about the topic. My own approach was to stand back from the very specific mathematical objective of the lesson and to attempt to identify the more fundamental conceptual knowledge that underpins with a view to designing activities where students can have an opportunity to appreciate where, how and why facts, concepts and skills fit together. My own deep understanding of the functionality of the MRT, gained though my personal research and experience in this area enabled me to bridge gaps in the teachers' knowledge in the sense that Shulman advocates (Shulman, 1986).

It is perfectly reasonable to suggest that a 'significant other', that is somebody to co-design and co-evaluate with, would enhance all aspects of mathematics teachers' practices. However, the nature of the multiple representations of the mathematics within the activities developed for this study triggered many hiccups. My presence as a non-judgemental observer in the classroom with mathematical, technical and pedagogical insight enabled the teachers to reflect on aspects of their practice in a highly supported way and acted to accelerate their epistemological development.

9.3.3 The professional development timeline

The evidence from the first phase of the study indicated that, for the activities that focussed on explorations of mathematical variance and invariance, all but two of the fifteen teachers began by developing instrument utilisation schemes that used numeric and syntactic inputs and explored the associated outputs in numeric, syntactic and graphical forms (IUS1). In all nine different instrument utilisation schemes were defined of which five were used by individual teachers or pairs of teachers working together (IUS2, IUS4, IUS6, IUS7 and IUS9). This left two schemes that were used more widely (IUS3 and IUS5). Eight of the teachers progressed to adopt schemes that resonated with IUS3, in which geometric input representations were mediated by dragging, with the output representations explored in numeric, syntactic and geometric forms. Eight teachers extended IUS1 to include the introduction of another representational output and this was defined as IUS5. Within the timescale of the first phase (nine months), this gave an indication of the breadth of schemes that were possible as the teachers learned to use a complex MRT with associated professional development support.

Eleanor and Tim's involvement spanned eighteen months and the analysis of their Phase Two instrument utilisation schemes did not lead to a large number of new schemes. However, the evidence of the way that their use of multi-representational outputs developed within their activity designs suggested that the number of schemes they had created was not actually important. It was the way in which their familiarity within each representational form enabled them to develop tasks that linked these in productive ways that took the time. An important aspect of this was the consideration of the flow of representations within activities and the teacher's role in mediating their introduction and the transitions between representations.

9.4 Limitations of the research design and methodology

The nature of this 'naturalistic' research study involving teachers in the secondary school resulted in no control over how the teachers involved in Phase One and Phase Two would respond. Also, I was not in a position to specify that they used the MRT to focus upon any particular class, topic or activity. Consequently the study cannot offer any comparative conclusions in relation to any particular age range, ability range or curricular area. The outcomes were skewed towards algebraic contexts as a result of the mathematical focus on variance and invariance and the opportunities that the Phase Two teachers saw in their schemes of work that related to this.

The methodological decision to select the teachers for Phase Two of the study who were technologically more confident and who demonstrated a more socio-constructive approach to their activity design and pedagogical practices, whilst enabling a rich set of research data to be gathered, does limit the validity of the research findings for the broad range of secondary mathematics teachers.

9.5 Directions for further research

There are a number of potentially productive lines of enquiry for further research. The first two would involve a secondary analysis of my existing research data and the remainder are posed to the wider mathematics education community.

Hoyles et al (2004) set a challenge to the mathematics education research community for studies that illuminated the roles that mathematics teachers play in technology-mediated classrooms. The substantive data that this study has collected, particularly concerning teachers actions in the classroom is worthy of re-examination in response to this challenge. In particular a deeper examination of their mediating actions during classroom activities would lead to a further articulation of the role that they play within exploratory activities using multi-representational technologies.

My experiences of this research project have led me to question how professional development activities for teachers concerning the use of multi-representational technologies (and the associated wireless network technology) might be best structured to complement or enhance the situated learning that is natural to a teacher's personal knowledge development. I would be interested to research how the identification and systematic analysis of lesson hiccups by teachers might form an integral element of such professional development. This might provide an opportunity to validate the hiccup theory of teacher learning within a wider community of mathematics teachers and serve to expand both its definitions and exemplifications.

As previously indicated, further studies of the students' experiences within the wirelessly connected handheld classroom are needed to widen our understanding of how the nature of the mathematics curriculum and its pedagogies might be reconceptualised in such environments.

Finally, I offer the hiccup theory of teacher learning to the mathematics education research community as a construct to support research into situated teacher learning more widely than the domain of the study.

10 References

AHMED, A. 1987. Better mathematics. London: Her Majesty's Stationery Office.

AHMED, A. & WILLIAMS, H. 1997. Numeracy project: a catalyst for teacher development and teachers researching. *Teacher Development,* 1**,** pp. 357-84.

ANDERSON, G. & ARSENAULT, N. 2002. *Fundamentals of educational research,* London, Routledge-Falmer.

ARTIGUE, M. 1998. Teacher training as a key issue for the integration of computer technologies. *In:* TINSLEY, D. & JOHNSON, D. (eds.) *Vol. 119, Proceedings of the IFIP TC3/WG3.1 Working Conference on Secondary School Mathematics in the World of Communication Technology: Learning, Teaching, and the Curriculum: Information and Communications Technologies in School Mathematics.* London Chapman and Hall.

ARTIGUE, M. 2001. Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *Computer Algebra in Mathematics Education Symposium.* Utrecht University, Netherlands.

ARZARELLO, F. & ROBUTTI, O. 2008. Multi-representations in TI-Nspire and TI-Navigator environments. *Paper presented to Sharing Inspiration.* Berlin, 16-18 May 2008.

BARDINI, C., RADFORD, L. & SABENA, C. Year. Struggling with variables, parameters and indeterminate objects or how to go insane in mathematics. *In:* CHICK, H. L. & VINCENT, J. L., eds. 29th Conference of the International Group for the Psychology of Mathematics Education, 2005 Australia. University of Melbourne, pp. 129-136.

BAULAC, Y., BELLEMAIN, F. & LABORDE, J. M. 1988. Cabri Géomètre. Paris: Cedic-Nathan.

BEDNARZ, N., KIERAN, C. & LEE, L. (eds.) 1996. *Approaches to algebra: Perspectives for research and teaching,* Dordrecht: Kluwer.

BRITISH EDUCATIONAL COMMUNICATION AND TECHNOLOGY AGENCY 2007. Harnessing Technology Review 2007. Coventry: British Educational Communication and Technology Agency,.

BRITISH EDUCATIONAL RESEARCH ASSOCIATION 2004. *Revised ethical guidelines for educational research,* Macclesfield, UK, British Educational Research Association.

BRITT, M., IRWIN, K. & RITCHIE, G. 2001. Professional conversations and professional growth. *Journal of Mathematics Teacher Education,* 4**,** pp. 29-53.

BROWN, M., JONES, K. & TAYLOR, R. 2003. Developing geometrical reasoning in the secondary school: outcomes of trialling teaching activities in classrooms. A report from the Southampton/Hampshire group to the Qualifications and Curriculum Authority. Southampton: University of Southampton.

CLARK-WILSON, A. 2008. Evaluating TI-Nspire™ in secondary mathematics classrooms: Research Report. Chichester, UK: University of Chichester.

CLARK-WILSON, A. 2009. Connecting mathematics in the connected classroom: TI-Nspire™ Navigator™. Chichester, UK: University of Chichester.

COBB, P., CONFREY, J., DISESSA, A., LEHRER, R. & SCHAUBLE, L. 2003. Design experiments in educational research. *Educational Researcher,* 32**,** pp. 9-13.

CONFREY, J. 1990. Student conceptions, representations, and the design of software for mathematics education. *In:* BOWEN, B. (ed.) *Design for learning.* Cupertino, CA: Apple Computer, External Research.

COONEY, T. & KRAINER, K. (eds.) 1996. *Inservice mathematics teacher education: The importance of listening,* Dordrecht: Kluwer Academic.

DENSCOMBE, M. 2003. *The good research guide,* Maidenhead, Oxford University Press.

DEPARTMENT FOR CHILDREN SCHOOLS AND FAMILIES 2007. National Curriculum for mathematics. London, UK: Qualifications and Curriculum Agency.

DISESSA, A. A. & COBB, P. 2004. Ontological Innovation and the Role of Theory in Design Experiments. *Journal of the Learning Sciences,* 13**,** pp. 77-103.

DRIJVERS, P., DOORMAN, M., BOON, P., REED, H. & GRAVEMEIJER, K. 2010. The teacher and the tool: instrumental orchestrations in the technology-rich mathematics classroom. *Educational Studies in Mathematics***,** pp. 1-22.

DRIJVERS, P. & TROUCHE, L. 2008. From Artifacts to Instruments: A Theoretical Framework behind the Orchestra Metaphor. *In:* BLUME, G. & HEID, K. (eds.) *Research on technology in the learning and teaching of mathematics: Syntheses, cases and perspectives. Vol 2 Cases and perspectives.* Charlotte: National Council of Teachers of Mathematics/Information Age Publishing.

ERNEST, P. 1991. The impacts of beliefs on the teaching of mathematics. Available: http://www.ex.ac.uk/~PErnest/impact.htm [Accessed 02-12-04].

ERNEST, P. 1996. The nature of mathematics and teaching. Available: http://www.ex.ac.uk/~PErnest/pome/pompart7.htm [Accessed 02-12-04].

FESTINGER, L. 1957. *A theory of cognitive dissonance,* Evanston, IL, Row, Peterson and Company.

FEY, J. 1989. Technology and mathematics education: a survey of recent developments and important problems. *Educational Studies in Mathematics,* 20**,** pp. 237-272.

FINZER, W. 2007. Fathom. Emeryville, CA: Key Curriculum Press.

FISHER EDUCATIONAL LTD 2009. *Ten Ticks Level 6 Pack 7,* Bolton.

GODWIN, S. & SUTHERLAND, R. 2004. Whole-class technology for learning mathematics: The case of functions and graphs. *Education Communication and Information,* 4**,** pp. 131-152.

GOLDENBERG, P. & MASON, J. 2008. Shedding light on and with example spaces. *Educational Studies in Mathematics,* 69**,** pp. 183-194.

GOULDING, M. & KYRIACOU, C. 2008. A systematic review of the use of ICTs in developing pupil's understanding of algebraic ideas. Report. London: EPPI-Centre, Institute of Education.

GUIN, D. & TROUCHE, L. 1999. The complex process of converting tools into mathematical instruments: The case of calculators. *International Journal of Computers for Mathematical Learning,* 3**,** pp. 195-227.

HENNESSY, S. 2000. Graphing investigations using portable (palmtop) technology. *Journal of Computer Assisted Learning,* 16**,** pp. 243-258.

HER MAJESTY'S INSPECTORATE 1985. Curriculum matters 3: Mathematics from 5 to 16. London: HMSO.

HOYLES, C. & LAGRANGE, J. B. (eds.) 2009. *Mathematics Education and Technology - Rethinking the Terrain: The 17th ICMI Study,* Berlin: Springer.

HOYLES, C., NOSS, R. & KENT, P. 2004. On the Integration of Digital Technologies into Mathematics Classrooms. *International Journal of Computers for Mathematical Learning,* 9, pp. 309-326.

HYDE, R. 2004. A snapshot of practice: Views of teachers on the use and impact of technology on secondary mathematics classrooms. *Paper presented to Topic Study Group 22 at the International Congress on Mathematics Education 10, Copenhagen.*

INSTITUTE OF EDUCATION 2006. Ethics Approval for Doctoral Student Research Projects v1.0 London: Institute of Education.

JACKSON, A. & JACKSON, C. 2010. *My Maths* [Online]. Staffordshire: My Maths Ltd. Available: http://www.mymaths.co.uk/ [Accessed 13th May 2010].

JANVIER, C. (ed.) 1987. *Problems of representation in the teaching and learning of mathematics,* Hillsdale, NJ: Lawrence Erlbaum Associates.

JAWORSKI, B. 1994. Investigating mathematics teaching. London: Falmer.

KAPUT, J. 1986. Information technology and mathematics: Opening new representational windows. *Journal of Mathematical Behavior,* 5, pp. 187 - 207.

KAPUT, J. 1989. Linking Representations in the Symbol Systems of Algebra. *In:* KIERAN, C. & WAGNER, S. (eds.) *Research Issues in the Learning and Teaching of Algebra* Reston, VA: Lawrence Erlbaum/NCTM.

KAPUT, J. 1992. Technology and mathematics education. *In:* GROUWS, D. (ed.) *Handbook of research on mathematics teaching and learning.* New York: Macmillan.

KAPUT, J. 1998. Transforming algebra from an engine of inequity to an engine of mathematical power by 'algebrafying' the K-12 curriculum. *In:* NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS & MATHEMATICAL SCIENCES EDUCATION BOARD (eds.) *Proceedings of a national symposium.* Washington, DC: National Research Council, National Academy Press.

KEY CURRICULUM PRESS 2003. The Geometer's Sketchpad v.4. CA: Key Curriculum Press.

KIERAN, C. Year. The changing face of school algebra. *In:* ALSINA, C., ALVAREZ, J. M., HODGSON, B., LABORDE, C. & PEREZ, A., eds. Selected Lectures from the 8th International Congress on Mathematics Education, 1996 Seville, Spain. SAEM Thales, pp 271-290.

KIERAN, C. 2004. The core of algebra: Reflections on its main activities. *In:* STACEY, K., CHICK, H. & KENDAL, M. (eds.) *The future of the teaching and learning of algebra: The 12th ICMI Study.* Dordrecht: Kluwer Academic Publishers.

KIERAN, C. & WAGNER, S. (eds.) 1989. *Research Issues in the Learning and Teaching of Algebra* Reston, VA: Lawrence Erlbaum/NCTM.

KIERAN, C. & YERUSHALMY, M. 2004. Research on the Role of Technological Environments in Algebra Learning and Teaching. *In:* STACEY, K., CHICK, H. & KENDAL, M. (eds.) *The future of the teaching and learning of algebra: The 12th ICMI Study.* Dordrecht: Kluwer Academic Publishers.

KNIGHTS, C. 2009. The perceived impact of ICT on mathematical learning by mathematics teachers in the UK. *Teaching Mathematics and its Applications,* 28, pp. 212-215.

LABORDE, C. 2001. The use of new technologies as a vehicle for restructuring teachers' mathematics. *In:* LIN, F. & COONEY, T. (eds.) *Making Sense of Mathematics Teacher Education.* Dordrecht: Kluwer Academic.

LAKATOS, I. 1976. *Proof and refutations,* Cambridge, Cambridge University Press.

LERMAN, S. 1983. Problem solving or knowledge centred: The influence of philosophy on mathematics teaching. *International Journal of Mathematical Education in Science and Technology,* 14, 7.

MALISANI, E. & SPAGNOLO, F. 2009. From arithmetical thought to algebraic thought: The role of the 'variable'. *Educational Studies in Mathematics,* 71, pp. 19-42.

MASON, J. 1996. Expressing generality and roots of algebra. *In:* BEDNARZ, N., KIERAN, C. & LEE, L. (eds.) *Approaches to algebra: Perspectives for research and teaching.* Dordrecht: Kluwer.

MASON, J. 2002. *Researching your own practice: The discipline of noticing,* London, Routledge-Falmer.

MASON, J., GRAHAM, A. & JOHNSON-WILDER, S. (eds.) 2005. *Developing thinking in algebra,* London: Open University Press.

MASON, J. & JOHNSTON-WILDER, S. (eds.) 2005. *Developing thinking in geometry,* London: Open University Press.

MASON, J. & SPENCE, M. 1999. Beyond mere knowledge of mathematics: The importance of knowing-to act in the moment. *Educational Studies in Mathematics,* 38, pp. 135-161.

MONAGHAN, J. Year. Teachers' classroom interactions in ICT-based mathematics lessons. *In:* VAN DEN HEUVEL-PANHUIZEN, M., ed. Proceedings of the 25th Conference of the International Groups for the Psychology of Mathematics Education, 2001 2001a Utrecht. pp. 383-390.

MONAGHAN, J. 2001b. Teachers' classroom interactions in ICT-based mathematics lessons. *In:* VAN DEN HEUVEL, M. & PANHUIZEN, M. (eds.) *Proceedings of the 25th Conference of the International Groups for the Psychology of Mathematics Education.* Netherlands: Freudenthal Institute.

MONAGHAN, J. 2004. Teachers' activities in technology-based mathematics lessons. *International Journal of Computers for Mathematical Learning,* 9, pp. 327-357.

MORENO-ARMELLA, L., HEGEDUS, S. J. & KAPUT, J. J. 2008. From static to dynamic mathematics: historical and representational perspectives. *Educational Studies in Mathematics,* 68, p. 99-111.

MOUSLEY, J., LAMBDIN, D. & KOC, Y. (eds.) 2003. *Mathematics teacher education and technology,* Dordrecht: Kluwer Academic.

NATIONAL COUNCIL FOR EDUCATIONAL TECHNOLOGY 1998. Data capture and modelling in mathematics and science. *In:* OLDKNOW, A. & TAYLOR, R. (eds.). Coventry: National Council for Educational Technology.

NORTON, S., MCROBBIE, C. & COOPER, T. 2000. Exploring mathematics teachers' reasons for not using computers in their teaching: Five case studies. *Journal of Research in Computing Education,* 33, pp. 87-105.

NOSS, R. & HOYLES, C. 1996. *Windows on mathematical meanings: Learning cultures and computers,* Dordrecht, Kluwer Academic.

NOSS, R., SUTHERLAND, R. & HOYLES, C. 1991. Final Report of the Microworlds Project Vol. II: Teacher attitudes and interactions. London: Institute of Education.

OFFICE FOR STANDARDS IN EDUCATION 2006. Evaluating mathematics provision for 14–19-year-olds. London: Her Majesty's Inspectorate.

OFFICE FOR STANDARDS IN EDUCATION 2008. Mathematics: Understanding the Score. London: Department for Children, Schools and Families.

OLDKNOW, A. 2009. Plenary: Their world, Our world - Bridging the divide. *In:* BARDINI, C. (ed.) *Proceedings of the International Conference for Technology in Mathematics Teaching 9.* Metz, France: University of Metz.

OLDKNOW, A. & TAYLOR, R. 2003. *Teaching mathematics using information and communications technology,* London, Continuum.

PAPERT, S. 1980. *Mindstorms: Children, computers, and powerful Ideas,* New York, Basic Books.

PIERCE, R. & STACEY, K. 2008. Using pedagogical maps to show the opportunities afforded by CAS for improving the teaching of mathematics. *Australian Senior Mathematics Journal,* 22**,** pp. 6-12.

PIERCE, R. & STACEY, K. 2009. Researching principles of lesson design to realize the pedagogical opportunities of mathematics analysis software. *Teaching Mathematics and its Applications,* 28**,** pp. 228-233.

POLANYI, M. 1962. *Personal knowledge: Towards a post-critical philosophy,* Chicago, The University of Chicago Press.

POLANYI, M. 1966. *The tacit dimension,* Chicago, The Universty of Chicago Press.

QSR INTERNATIONAL 2008. Nvivo8. Melbourne: QSR International.

QUALIFICATIONS AND CURRICULUM AUTHORITY. Year. Reasoning, explanation and proof in school mathematics and their place in the intended curriculum. *In:* ABRAMSKY, J., ed. QCA International Seminar, 2001 London. Qualifications and Curriculum Authority.

QUALIFICATIONS AND CURRICULUM AUTHORITY 2004. *Developing reasoning through algebra and geometry,* London, Department for Education and Skills.

QUALIFICATIONS AND CURRICULUM AUTHORITY 2007. Draft National Curriculum for Mathematics. London: Qualifications and Curriculum Authority.

RADFORD, L. 1996. The roles of geometry and arithmetic in the development of algebra: Historical remarks from a didactic perspective. *In:* BEDNARZ, N., KIERAN, C. & LEE, L. (eds.) *Approaches to algebra: Perspectives for research and teaching.* Dordrecht: Kluwer.

RIDGEWAY, J. 1997. Vygotsky, informatics capability and professional development. *In:* PASSEY, D. & SAMWAYS, B. (eds.) *Information technology supporting change through teacher education.* London: Chapman Hall.

ROBSON, C. 2001. *Real world research,* London, Blackwell.

RODD, M. & MONAGHAN, J. 2002. Graphic calculator use in Leeds schools: Fragments of practice. *Journal of Information Technology for Teacher Education,* 11**,** pp. 93-108.

ROWLAND, T., HUCKSTEP, P. & THWAITES, A. 2005. Elementary teachers' mathematics subject knowledge: The knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education,* 8**,** pp. 255-281.

RUTHVEN, K. 1999. Reconstructing professional judgement in mathematics education: From good practice to warranted practice. *In:* HOYLES, C. (ed.) *Rethinking the mathematics curriculum.* London: Falmer Press.

RUTHVEN, K. 2003. Linking algebraic and geometric reasoning with dynamic geometry software: Final report to the Qualifications and Curriculum Authority. Cambridge: University of Cambridge, Faculty of Education.

RUTHVEN, K. 2008. The Interpretative Flexibility, Instrumental Evolution, and Institutional Adoption of Mathematical Software in Educational Practice: The Examples of Computer Algebra and Dynamic Geometry. *Journal of Educational Computing Research,* 39**,** pp. 379-394.

RUTHVEN, K., DEANEY, R. & HENNESSY, S. 2009. Using graphing software to teach about algebraic forms: a study of technology-supported practice in secondary-school mathematics. *Educational Studies in Mathematics,* 71**,** pp. 279-297.

RUTHVEN, K. & HENNESSY, S. 2002. A practitioner model for the use of computer-based tools and resources to support mathematics teaching and learning. *Educational Studies in Mathematics,* 49**,** pp. 47-88.

RUTHVEN, K., HENNESSY, S. & DEANEY, R. 2004. Incorporating dynamic geometry systems into secondary mathematics education: Didactical perspectives and practices of teachers. *British Educational Research Association Conference.* Manchester: BERA.

RUTHVEN, K., HENNESSY, S. & DEANEY, R. 2008. Constructions of dynamic geometry: A study of the interpretative flexibility of educational software in classroom practice. *Computers and Education,* 51**,** pp. 297-317.

SCHIELACK, J. & CHANCELLOR, D. (eds.) 1994. *Stop, look listen: Building reflection into continuing professional development,* Reston VA: National Council for Teachers of Mathematics.

SCHÖN, D. 1984. *The Reflective Practitioner: How professionals think in action,* New York, Basic Books.

SCHÖN, D. 1987. *Educating the reflective practitioner,* London, Jossey-Bass.

SCHWARTZ, J. & YERUSHALMY, M. 1985. The geometric supposers. New York: Sunburst Communications.

SHAFFER, D. W. & KAPUT, J. 1998. Mathematics and virtual culture: An evolutionary perspective on technology and mathematics education. *Educational Studies in Mathematics,* 37**,** pp. 97-119.

SHULMAN, L. 1986. Those Who Understand: Knowledge Growth in Teaching. *Educational Researcher,* 15**,** pp. 4-14.

SOFT WAREHOUSE 1988. Derive. Honolulu HA.

STACEY, K. 2008. Pedagogical Maps for Describing Teaching with Technology. *Paper presented at Sharing Inspiration Conference.* Berlin 16-18 May 2008.

STRADLING, B., SIMS, D. & JAMISON, J. 1994. Portable computers pilot evaluation summary. Coventry,UK: National Council for Educational Technology.

SUTHERLAND, R. 2007. *Teaching for learning mathematics,* Maidenhead, Open University Press.

SUTHERLAND, R. & MASON, J. 1995. *Exploiting Mental Imagery with Computers in Mathematics Education,* Berlin, Springer-Verlag.

SUTHERLAND, R. & ROJANO, T. 1993. A Spreadsheet Approach to Solving Algebra

Problems. *Journal of Mathematical Behavior,* 12, p. 351-383.

TEXAS INSTRUMENTS 2007. TI-Nspire™. Dallas, TX: Texas Instruments Inc.

TEXAS INSTRUMENTS 2007a. *TI-Nspire™ Reference Guide,* Dallas, TX, Texas Instruments.

TEXAS INSTRUMENTS 2007b. *TI-Nspire™ Software Guide,* Dallas, TX, Texas Instruments.

TEXAS INSTRUMENTS 2009. TI-Nspire™ Navigator™. Dallas, TX: Texas Instruments Inc.

THOMPSON, A. 1984. The relationship of teachers' conceptions of mathematics and mathematics teaching to instructional practice. *Educational Studies in Mathematics,* 15, pp. 105-127.

THOMPSON, A. (ed.) 1992a. *Teachers' beliefs and conceptions; a synthesis of the research,* New York: Macmillan.

THOMPSON, A. 1992b. Teachers' beliefs and conceptions; a synthesis of the research. *In:* GROUWS, D. (ed.) *Handbook of research on mathematics teaching and learning.* New York: Macmillan.

THOMPSON, D. A. 1917. *On growth and form,* Cambridge, Cambridge University Press.

TROUCHE, L. 2004. Managing the complexity of human/machine interactions in computerized learning environments: Guiding students' command process through instrumental orchestrations. *International Journal of Computers for Mathematical Learning,* 9, pp. 281-307.

VANDENBERG, D. 1997. *Phenomenology and educational discourse,* Sandton, South Africa, Heinemann.

VERILLON, P. & RABARDEL, P. 1995. Cognition and Artefacts: A contribution to the study of thought in relation to instrumented activity. *European Journal of Psychology of Education,* 10, pp. 77-102.

VYGOTSKY, L. 1981. The instrumental method in psychology. *In:* WERTSCH, J. & COLE, M. (eds.) *The Concept of Activity in Soviet Psychology.* New York: ME Sharpe.

WATSON, A., DE GEEST, E. & PRESTAGE, S. 2003. Deep Progress in Mathematics: The Improving Attainment in Mathematics Project. Oxford: University of Oxford, Department of Educational Studies.

WATSON, A. & MASON, J. 1998. *Questions and Prompts for Mathematical Thinking,* Derby, Association of Teachers of Mathematics.

WENGER, E. 1998. *Communities of practice. Learning, meaning and identity,* Cambridge, Cambridge University Press.

WERTSCH, J. 1981. *The concept of activity in Soviet psychology,* New York, ME Sharpe.

WILSON, K., AINLEY, J. & BILLS, L. 2005. Spreadsheets, pedagogic strategies and the evolution of meaning for variable. *In:* CHICK, H. & VINCENT, J. (eds.) *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education,*. Melbourne: University of Melbourne.

ZODIK, I. & ZASLAVSKY, O. 2008. Characteristics of teachers' choice of examples in and for the mathematics classroom. *Educational Studies in Mathematics,* 69, pp. 165-182.

11 APPENDICES

APPENDIX 1: PROJECT AGREEMENT

Project schools will:

- release teachers to attend project conferences and meetings;
- have a satisfactory procedure for securing equipment;
- support teachers to participate fully in the evaluation project by providing time and resources, as appropriate;
- return the loan kit at the end of the project. Schools continuing to Phase Two of the pilot will keep the loan kit.

Project teachers will:

- attend project conferences to familiarise themselves with the handheld and associated software;
- use, adapt (and develop) resources for use in KS3 and 4 mathematics classrooms;
- maintain a log of their project activity and associated reflections;
- use the evaluation tools provided to deeply evaluate the impact of the handheld and software in the classroom;
- submit all project data by agreed deadlines.

APPENDIX 2: ETHICAL AGREEMENT

Main theme: The nature of teachers' learning about mathematical variables within a multi-representational technological environment.

Aim: To learn more about how secondary mathematics teachers develop their understanding of the concept of a variable within mathematics through developing their use of a multi-representational technological tool, in this case TI-Nspire.

Timescale: July 2007 – June 2009

This research project will be carried out in accordance with the British Educational Research Association's *Revised ethical guidelines for educational research* (BERA 2004) and I am therefore seeking your voluntary informed consent to be included in the study. The nature of your involvement is:

- ☐ Participation in discussions and interviews with the researcher, which may be video or audio recorded;
- ☐ Sharing written notes, lesson plans and evaluations with the researcher;
- ☐ Allow the researcher to observe lessons and talk with students.

All permissions would be sought prior to any publication and you will be offered the opportunity to verify statements/data and the researcher's interpretation of the data throughout the period of the project.

Please indicate if you give your consent to your name and affiliation being used in the final Thesis.

- ☐ I consent to my name and affiliation being included in the Thesis.
- ☐ I do not consent to my name and affiliation being included in the Thesis.

Any audio or video recordings made as part of the research process will not be used in any other context without your specific consent and any personal data will be stored and used in accordance with the Data Protection Act (1989).

I agree to the terms of this ethical agreement.

Signature ______________________________

Name ______________________________

Affiliation ______________________________

APPENDIX 3: SUMMARY OF PHASE ONE LESSON DATA

A summary of the research data each of the sixty-one lessons on which the conclusions and recommendations are based is provided below.

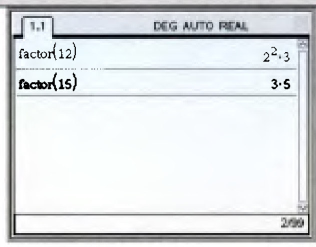
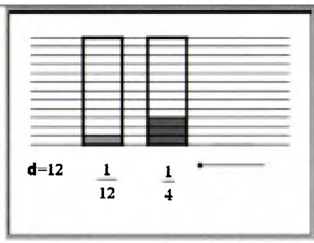
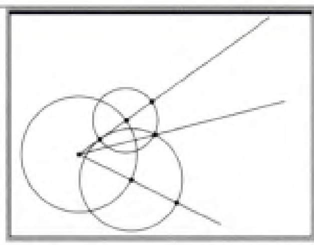
Lesson code	**Lesson title**	Lesson Plan (Plan)	Lesson Structure (Struct)	Teacher's .tns file (tns-T)	Students' .tns files (tns-S)	Students' Activity Sheet (Activity)	TI-Nspire Help Sheet (Help)	Lesson Evaluation (Teacher) (Quest2)	Lesson Evaluation (Student) (LEval-S)	Teachers ongoing reflection (Journ-T)	Teacher's Presentation (Pres-T)
STP1	Prime factorisation										
STP2	Visual fractions										
STP3	Angle bisectors										
STP4	Circle circumferences										
STP5	Enlargements										
SJK1	Prime factorisation										
SJK2	Polygon angles										
SJK3	n^{th} terms of sequences										
SJK4	Straight line graphs										
SJK5	Scattergraphs										
CEL1	Quadrilateral angles										
CEL2	Triangle angles										
CEL3	Intersecting lines										
CEL4	All angles recap										
CEL5	Perpendicular lines										
CHS1	Four fours										

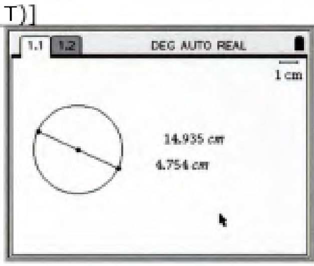
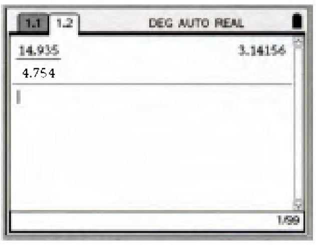
Lesson code	Lesson title	Lesson Plan (Plan)	Lesson Structure (Struct)	Teacher's .tns file (tns-T)	Students' .tns files (tns-S)	Students' Activity Sheet (Activity)	TI-Nspire Help Sheet (Help)	Lesson Evaluation (Teacher) (Quest2)	Lesson Evaluation (Student) (LEval-S)	Teachers ongoing reflection (Journ-T)	Teacher's Presentation (Pres-T)
CHS2	Triangle angles										
CHS3	Dice										
CHS4	Intersecting lines										
LSM1	Sums of sequences										
LSM3	Mystery Factors										
LSM4	Trial and improvement										
LMF1 **LSM2**	Digit sums										
LMF2	Four fours										
LMF3	Trial and improvement										
LMF4	Mystery factors										
LMF5	Circles and Pi										
LMF6	Box plots										
HAH1	Four fours										
HAH2	Temperature conversion										
HAH3	Further Conversion										
HAH4	Adding fractions										
HAH5	Data handling coursework										
HRG1	Standard form										
HRG2	Straight line graphs										
HRG3	Data handling coursework										

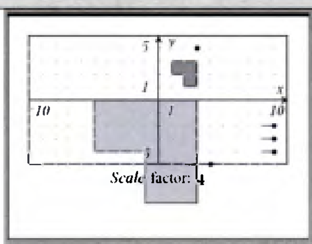
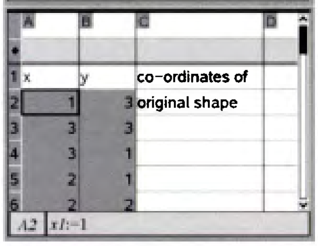
Lesson code	Lesson title	Lesson Plan (Plan)	Lesson Structure (Struct)	Teacher's .tns file (tns-T)	Students' .tns files (tns-S)	Students' Activity Sheet (Activity)	TI-Nspire Help Sheet (Help)	Lesson Evaluation (Teacher) (Quest2)	Lesson Evaluation (Student) (LEval-S)	Teachers ongoing reflection (Journ-T)	Teacher's Presentation (Pres-T)
PCT1	Four fours										
PCT2	Reflections										
PCT3	Angles lessons										
PCT4	Solving simultaneous equations 1										
PCT5	Solving simultaneous equations 2										
PSH1	Rich Aunt problem										
PSH2	Circles and tangents										
PSH3	Parallel and perpendicular lines										
PSH4	Exponential growth and decay										
PSH5	Enlargements										
BAK1 BJJ1	Four fours										
BAK10 BJJ10	Constructing equilateral triangles										
BAK2	Percentage multipliers										
BAK3 BJJ3	Investigating straight lines										
BAK4 BJJ4	Exploring angles										
BAK5 BJJ5	Comparing box plots										
BAK6 BJJ6	Weighing sweets										
BAK8	Pi Day										
BAK9 BJJ9	Investigating quadrilaterals										
BJJ7	Angle bisectors										

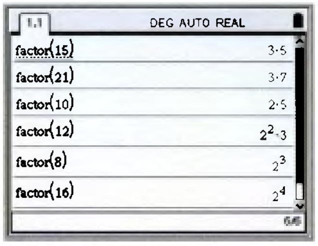
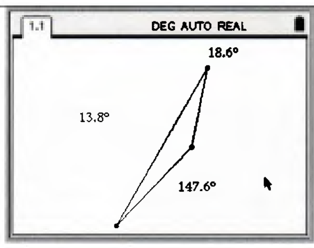
Lesson code	Lesson title	Lesson Plan (Plan)	Lesson Structure (Struct)	Teacher's .tns file (tns-T)	Students' .tns files (tns-S)	Students' Activity Sheet (Activity)	TI-Nspire Help Sheet (Help)	Lesson Evaluation (Teacher) (Quest2)	Lesson Evaluation (Student) (LEval-S)	Teachers ongoing reflection (Journ-T)	Teacher's Presentation (Pres-T)
GBA1	Box plots comparisons										
GBA2	Vitruvian man										
GBA3	Weighing sweets										
GBA4	Introducing trigonometry										
GBA5	Circles 1 and 2										
GRE1	Four fours										
GRE2	Vitruvian man										

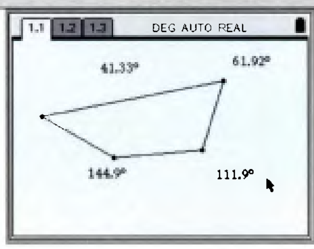
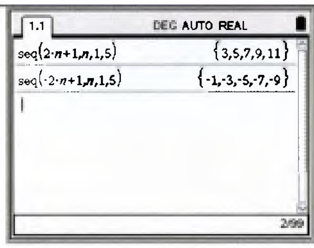
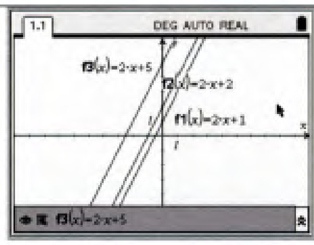
APPENDIX 4: SUMMARY OF ACTIVITIES SUBMITTED DURING PHASE ONE

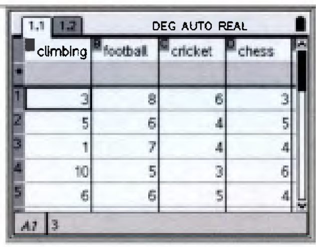
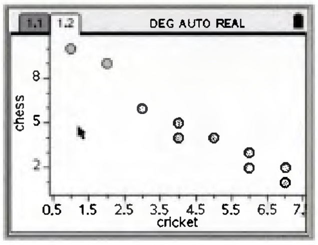
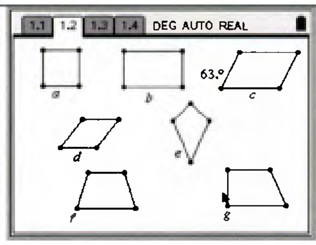
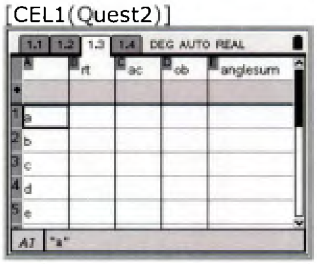
Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
STP1 Prime factorisation Students created a new file and used the factor() command within the Calculator application to explore different inputs and outputs to encourage generalisation. Students recorded their work in exercise books. Student self-evaluation supported the teacher assessment.	 Teacher reflection after the lesson implied a greater appreciation of how the initial choice of input numbers might have affected the students' abilities to see generalities.	**Variance** = changing the input number (manual text/numeric input to Calculator application) **Invariance** = 'the prime factorisation of a number is unique. Also that all prime numbers have only two factors, square numbers have an odd number of factors.' [STP1(Quest2)].	**IUS1**: *Vary a numeric or syntactic input and use the instrument's functionality to observe the resulting output in numeric, syntactic, tabular or graphical form.*
STP2 Visual fractions Students opened a pre-constructed file which had already been loaded onto the handhelds by the teacher. It used a dynamic environment to explore adding fractions to encourage generalisation. Students recorded their work in exercise books.		**Variance** = the individual values of the numerators and denominators of the two fractions to be added and the value of the 'divisor' (***d***). The position of the point on the line segment as it is dragged resulting in the 'emptying of the LH bar into the RH bar. **Invariance** = the dimensions of the fraction bars.	**IUS4:** *Vary a numeric input and drag an object within a related mathematical environment and observe the resulting visual output.*
STP3 Angle bisectors Students created a new file and used the Graphs and Geometry application to explore the construction of angle bisectors leading to the construction of an angle fan. Students saved their work.	 *As they did this, they were getting immediate visual feedback. When students constructed their own bisectors they were able to see if they were constructing correctly by trying to move the lines dynamically.* *Students made connections by having to relate the properties of the circle to*	**Variance** = the size of the circles drawn to construct the angle bisector - offering a more 'global' approach to the construction of angle bisectors. **Invariance** = the condition when the two circles constructed on the rays are equal, resulting in an angle bisector.	**IUS3:** *Vary the position of an object (by dragging) that has been constructed in accordance with a conventional mathematical constraint and observe the resulting changes.*

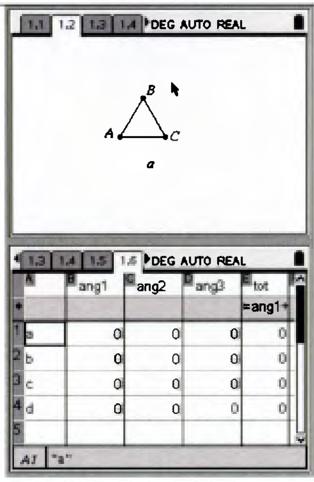
Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
	consideration of equal lengths. *Students were able to articulate reasons for the symmetry of the construction and saw that the equal radii proved that the construction was symmetrical.* [STP3(Quest4-T)]		
STP4 Circle circumferences Students created a new file and used the Graphs and Geometry application to construct a dynamic circle, measure its properties and tabulate the data in their exercise books. They then used the Calculator application to find the ratio of the circumference to the diameter and discussed their findings as a class. Students saved their work.	 	**Variance** = change the diameter and observe the change in the circumference of the constructed circle. **Invariance** = the numeric ratio between the diameter and circumference of the circle visible as a linear relationship when plotted graphically.	**IUS3:** *Vary the position of an object (by dragging) that has been constructed in accordance with a conventional mathematical constraint and observe the resulting changes. Use another representational form to add insight to or justify/prove any invariant properties.* **New possibilities for the action**: The use of dragging to offer quasi empirical approach to gathering 'real' data for numerical analysis. **Private utilisation scheme**: Geometric construction of a circle by centre and radius. Measurement of diameter and circumference. Data capture of diameter and circumference within Spreadsheet. Observation of linear relationship. Modelling linear function to coincide with collected data values. Testing of hypothesis in Spreadsheet. **Social utilisation scheme**: This approach has been documented in other research studies (Arzarello and Robutti, 2008).

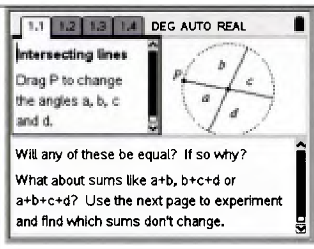
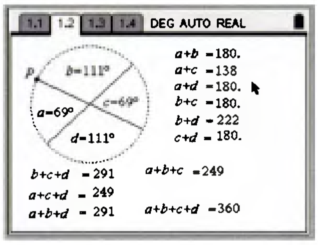
Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
STP5 Enlargements (Two lessons) [The first lesson was not reported in detail. In it the students begun by creating a new file and used the Graphs and Geometry application to construct shapes, enlarge them and vary the centre of enlargement and position.] In the second lesson the students worked from a pre-constructed file provided by the teacher which focused their attentions on the variant properties within the construction.	  *At the end of the first lesson, students were able to describe quite accurately where the centre of enlargement would be.* *I found it helpful to have both the approaches of (a) an enlargement that students created themselves and (b) a prepared file where a slider controlled the enlargement as this gave students the opportunity to see the enlargement in small increments and help them create a mental picture of the process of enlargement and its relationship to the centre of enlargement.* *I was pleasantly surprised at how comfortable the students were at creating a shape and enlarging it.* [STP4(Quest2)]	Variance= position of shape, appearance of shape, position of centre of enlargement, scale factor. Invariance = the similarity of the two shapes.	**IUS4:** *Vary a numeric input and drag an object within a related mathematical environment and observe the resulting visual output.* **New possibilities for the action**: The students had 'constrained freedom' to explore a concept that is difficult to mediate without the use of technology. Multiple representations were incorporated within the activity. The use of dragging to observe the effect of moving the position of the centre of enlargement. **New conditions for organising the action**: Many choices available to the teacher with respect to how they direct the students' attentions during the activity. **Private utilisation scheme**: Students chose whether to vary the scale factor (by changing its numeric input), the centre of enlargement (by dragging its position) or the appearance of the original shape (by changing its coordinates). **Social utilisation scheme**: A similar lesson was developed by RD (PSH5) although it is not evident from the available research data whether this

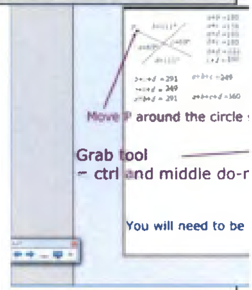
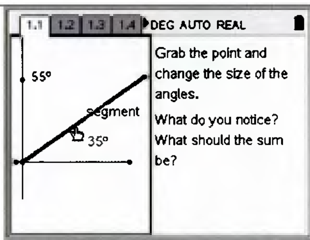
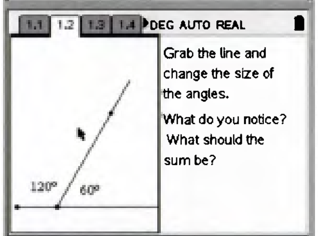
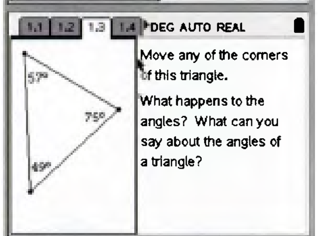
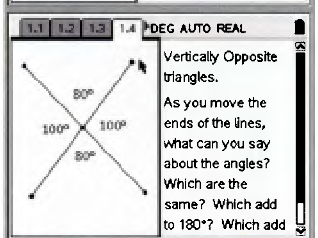
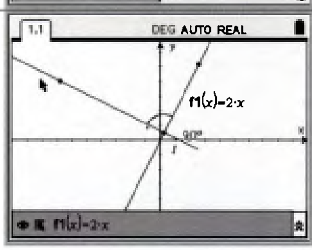
Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
			was as a direct result of any discussion between the two teachers.
SJK1 Prime factorisation Students created a new file and used the *factor()* command within the Calculator application to explore different inputs and outputs to encourage generalisation. Students recorded their work on a worksheet prepared by the teacher.	 Q. What aspect(s) of the idea would you use again? *Definitely the worksheet idea as this enables pupils to work at their own pace. I needed the worksheet for me as well as for them. I was able to refer to the sheet and that helped my confidence. The sheet also allowed pupils to continue with the work whilst I went round to help students with a problem. BUT - the sheet could have been more structured i.e. not jump around haphazardly but be more systematic. Factor (1), Factor (2), Factor (3), etc... I liked Exercise 1 but questions 1 and 2 were too hard for this group. I was nervous to use the device even though I am a very experienced teacher of maths. I needed the worksheet as support. Having done one lesson I would now be confident to try again. The worksheet could have been more interesting. Pupils seemed to enjoy the lesson.* [SJK1(Quest2)]	Teacher had constructed a worksheet (with support from her mentor) that did lead students through a set of suggested input numbers that progressed in their level of complexity. **Variance** = changing the input number (manual text input to calculator application using factor () syntax) **Invariance** = all prime numbers had only two factors.	**IUS1**: *Vary a numeric or syntactic input and use the instrument's functionality to observe the resulting output in numeric, syntactic, tabular or graphical form.* **Private utilisation scheme**: Students worked with a numerical input and output within one application responding to the same set of numbers, as provided by the teacher. No use of multiple representations using the technology. **Social utilisation scheme**: Whilst JK developed this lesson in close collaboration with TP, who taught a similar lesson [STP1]. The lessons had very different utilisation schemes. JK provided a structured worksheet for the students that contained a variety of questions that did not appear to follow any conceptual progression.
SJK2 Polygon angles Students opened a pre-constructed file that had been distributed to them during the lesson using the link cables. The		**Variance** = the position of the vertices (and therefore the values of the interior angles) and also the number of sides of each polygon. **Invariance** = the angles sum for all	**IUS3:** *Vary the position of an object (by dragging) that has been constructed in accordance with a conventional mathematical constraint and observe the resulting changes.*

Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
students dragged the vertices of the polygons to different positions, summed the values of the angles in the calculator application and recorded the measurements and totals in their exercise books.		polygons with the same number of sides is constant.	*Use another representational form to add insight to or justify/prove any invariant properties.*
SJK3 n^{th} terms of sequences Students created a new file and used the *Sequence* command within the calculator application to generate different arithmetic sequences and explore the output. They were then given 5 terms of an arithmetic sequence and asked to predict expression for the n^{th} term. Students recorded their work on a worksheet prepared by the teacher.	 *Students worked well but were a lot slower than I expected!* [SJK3(Quest2)] *When typing seq(2n+1,n,1,5) pupils mis-typed, missed out commas or added spaces.* [SJK3(Quest2)]	**Variance** = the expression for the nth term that is input to the MRT. **Invariance** = the resulting output of the first five values of the sequence.	**IUS1**: *Vary a numeric or syntactic input and use the instrument's functionality to observe the resulting output in numeric, syntactic, tabular or graphical form.*
SJK4 Straight line graphs Students begun by plotting the graphs of y=2x+1 followed by y= 2x+2 in the exercise books. They were then asked to predict where the line y=2x+5 would be positioned. They then used the MRT to create a new file and used the graphs and geometry application to explore drawing parallel lines of the form f(x) = 2x+c. Midway through the lesson, the teacher asked them to keep the value of c constant and vary the value of the gradient, m.	 *It was very difficult to get a generalisation from the pupils. I feel perhaps it was the way I questioned them. When looking at the 'fan' going through the point (0,5) there was little that the students spotted apart from the fact that it looked like a fan! Perhaps they were more concerned with the 'm' changing than the 'c' remaining constant. I shall try to put more thought into my questioning in future!* [SJK4(Quest2)]	**Variance** = the values of 'm' and 'c; in the linear equations that were entered. **Invariance** = the position of the linear function on the axes as determined by its gradient and intercept. *What effect changing the 'm' has on the straight line , keeping 'c' constant and what effect changing the 'c' has on the straight line, keeping the 'm' constant. Trying to get from them what the 'm' is and what the 'c' is.* [SJK4(Quest2)]	**IUS1**: *Vary a numeric or syntactic input and use the instrument's functionality to observe the resulting output in numeric, syntactic, tabular or graphical form.*

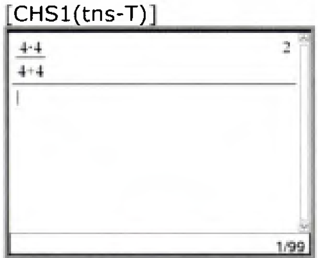
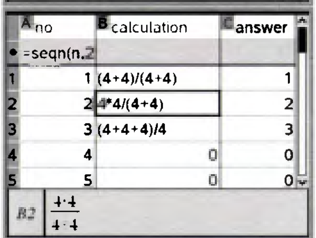
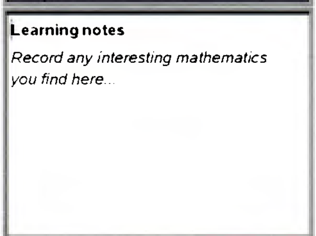
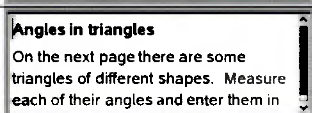
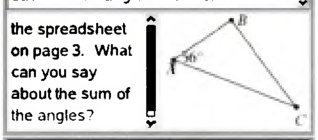
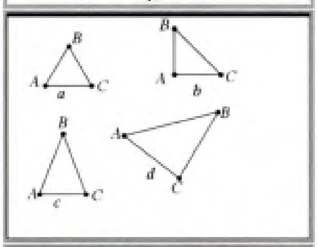
Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
SJK5 Scatter graphs Students created a new .tns file and used the spreadsheet application to enter a given set of data and produce scatter plots within the data and statistics application in order to respond to a set of questions relating to correlation. Students recorded their findings on an accompanying worksheet. The teacher reminded them how to identify different types of correlation.	 	**Variance** = the data set (provided by the teacher) that was entered. **Invariance** = the conclusion that, for some sets of data there is an approximate proportional relationship between the two sets.	**IUS1**: *Vary a numeric or syntactic input and use the instrument's functionality to observe the resulting output in numeric, syntactic, tabular or graphical form.*
CEL1 Quadrilateral angles (2 lessons) Students opened a pre-constructed file that had already been loaded onto the handhelds by the teacher. They recorded the types of angles within some different quadrilaterals and measured the interior angles to find the angle sum. This data was entered manually into the spreadsheet. They did not drag any of the vertices. Students were asked to comment on their findings using a Notes page.	 *Have bigger pictures drawn, even as one per page would aid students' ability (sic) to measure more accurately and not end up moving shapes by mistake.* [CEL1(Quest2)] 	**Variance** = measured values of interior angles of given polygons. **Invariance** = the angles sum for all polygons with the same number of sides is constant.	**IUS2**: *From a given set of static geometric objects, make measurements and tabulate data to explore variance and invariance within the measured data in numeric and tabular forms.* **New possibilities for the action**: As the shapes were constructed within the dynamic geometry application, it would have been possible to drag them to observe changes in angles. **Private utilisation scheme**: Measurement of pre-constructed angles by selecting three vertices in appropriate order and placing measurement on the page. Moving to new page in document Entering measured

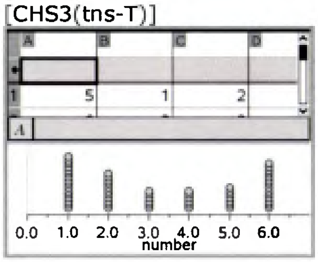
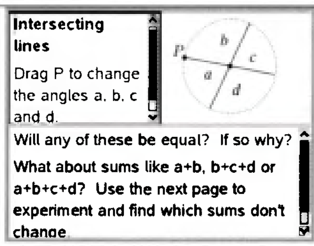
Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
			values in spreadsheet and observing angle sum (pre-entered formula in spreadsheet). **Social utilisation scheme**: This activity was developed during the first project meeting and the idea was picked up and redesigned by four other teachers.
CEL2 Triangle angles Students opened a pre-constructed file that had already been loaded onto the handhelds by the teacher. They measured the interior angles of several polygons using the graphs and geometry application. They did not drag any of the vertices. They entered angle measurements manually into a Spreadsheet page and commented on their findings using a Notes page.		**Variance** = measured values of individual angles and subsequent calculated totals. **Invariance** = displayed total is in range 179 -181	**IUS2**: *From a given set of static geometric objects, make measurements and tabulate data to explore variance and invariance within the measured data in numeric and tabular forms.* **Private utilisation scheme**: Measurement of pre-constructed angles by selecting three vertices in appropriate order. Entering data into spreadsheet page with automatically calculated total. EL prescribed the instrumentation for the students with respect to measuring each of the angles. *To measure angles: menu* *7: measurement* *4: angle* *(Highlight the 3 corners in order)* *enter* *move measurement to correct point (hand opens and closes (ctrl middle doughnut))* [CEL2(Struct)] **Social utilisation scheme**: This activity was a revised version of the lesson that was jointly developed

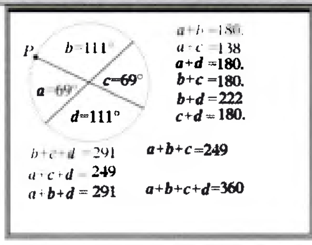
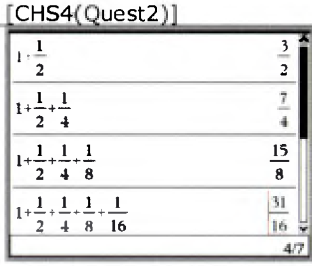
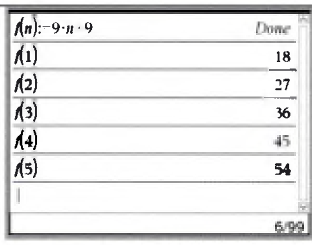
Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
			by EL and LL. As LL taught and evaluated the activity first, EL made changes with support from CK. Noticeably, this activity has not developed from the previous one in which students worked with static shapes.
CEL3 Intersecting lines Students opened a pre-constructed file that had already been loaded onto the handhelds by the teacher. They dragged various points to observe changes in angle properties using the Graphs and Geometry application and commented on their findings using a Notes page. Students saved their work.	  Questions on Notes page: Which of the sums add up to 180° Why? Which of the sums add up to 360° Why?	**Variance** = the positioning of the vertices that were already fixed on the circumference of the circle. **Invariance** =the inherent angle properties (equality of vertically opposite angles; sum of angles around the centre); the length of the lines (all diameters); the size of the initial circle.	**IUS3:** *Vary the position of an object (by dragging) that has been constructed in accordance with a conventional mathematical constraint and observe the resulting changes.* **New possibilities for the action:** First use of dragging. First used of saved variables in order to recalculate angles as the geometric objects are dragged. **New conditions for organising the action**: Possibility of displaying the measured and saved values in a number of ways. The teacher opted to display all of the possible angle calculations. **Private utilisation scheme**: This focussed on supporting the students to 'grab and drag'.

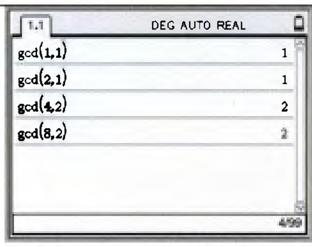
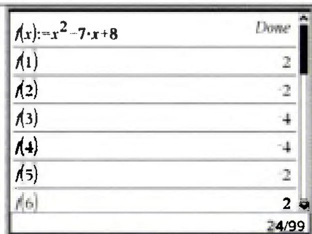
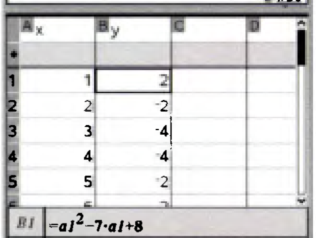
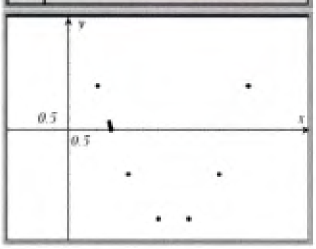
Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
			 **Social utilisation scheme**: Similarities with the approach developed by KS later in the project (GBA5).
CEL4 All angles recap Students opened a pre-constructed file that had already been loaded onto the handhelds by the teacher. They reviewed the mathematics from the other reported TI-Nspire lessons (CEL1 Quadrilateral angles, CEL2 Triangle angles and CEL3 Intersecting lines) using the Graphs and Geometry application but using the drag facility to vary the angles and recorded their own comments using the Notes application. Students saved their work.	   	**Variance** = positions of geometric objects and resulting angle measurements. **Invariance** = Activity 1.1: angle sum of a right angle (a problematic concept as the students have no given information to suggest that it 'should' be a right angle; Activity 1.2: sum of the 'angles on a straight line' Activity 1.3: Sum of the interior angles of a triangle Activity 1.4: Angle properties of crossing lines.	**IUS3:** *Vary the position of an object (by dragging) that has been constructed in accordance with a conventional mathematical constraint and observe the resulting changes.* **New possibilities for the action**: Involving dragging as the main activity. Inviting students to add comments to their work. **New conditions for organising the action**: Private utilisation scheme: 'Grab and drag' central to each of the interactives. **Social utilisation scheme**: Beginning to emerge within the classroom.
CEL5 Perpendicular functions Students created a new file and used the Graphs and Geometry application to define a linear function and draw a	 *If the perpendicular value*	**Variance** = initial definition (positioning?) of linear function, 'by eye' positioning of geometric line, value of measured angle, the gradients of the two lines (by measurement or by definition).	**IUS7:** *Construct a graphical and geometric scenario and then vary the position of geometric objects by dragging to satisfy a specified mathematical condition. Input functions syntactically to*

Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
freehand 'perpendicular line'. They used the angle measure facility to check their accuracy, dragged the geometric line until it was perpendicular and then generated parallel functions to this geometric line. They used a Notes page to record their findings. Students saved their work.	*did not appear to fit the rule we checked angles were 90°.* [CEL5(Quest2)] Q. What changes would you make? *To reinforce looking for connections between perpendicular gradients as written for the value of m. To encourage more formal methods of recording results. To introduce a spreadsheet page to establish that the product of perpendicular gradients is -1. Slope can be measured. Ensuring all hand-held are in degree mode and all with a float of 3.* *Students were very intuitive with using the new technology. Would need to reinforce girls to do more mathematical thinking and reflection and to view the technology as a means to do this.* [CEL5(Quest2)]	**Invariance** = the condition that, when the lines were perpendicular, the products of their gradients would equal -1.	*observe invariant properties.* **New possibilities for the action**: This was a very innovative use of the technology – it combined defining functions with dragging and measuring angle. EL also suggested develops that would integrate the Spreadsheet page as a means for collating results and checking conjectures. **New conditions for organising the action**: **Private utilisation scheme**: *To draw a simple linear function on a graphs page. To construct a line that crosses this at 90o. To measure this angle and draw a selection of parallel lines to this drawn line. To record findings.* [CEL5(Quest2)] *Results were recorded on a notes page and we brought together the many concepts related to gradients, parallelness, perpendicular and y=mx+c.* [CEL5(Quest2)] **Social utilisation scheme**: This idea was 'mis-remembered' by EL from a lesson developed by RD and reported at the third project meeting [PSH3].
CHS1 Four fours Students opened a pre-constructed file that had already	Activity introduction on page 1.1 *Name:* *Date:* *1.1 Four fours* *Using only four fours, how can you calculate all of the*	**Variance** = the inputs that the students provide that met the activity constraints with respect to 'four fours' and the	**IUS1**: *Vary a numeric or syntactic input and use the instrument's functionality to observe the*

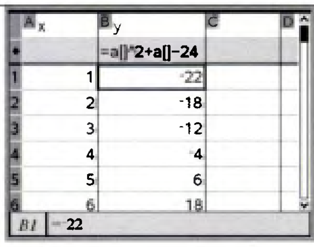
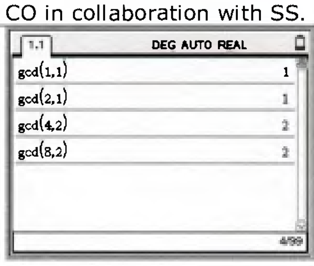
Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
been loaded onto the handhelds by the teacher. They used the Calculator application to try to use four fours and any combination of operations, including brackets, to generate the numbers from one to a hundred. They recorded their results in the Spreadsheet application and used a Notes page to record their findings. Students saved their work.	*whole numbers from 1 to 100?* *1.2 An example (4+4)/4+4=6. Try others here.* *1.3 Fill in your calculations here* *1.4 Learning notes* [CHS1(tns-T)]   	permitted mathematical operations. **Invariance** = the machine constraints with respect to how the input was interpreted.	*resulting output in numeric or syntactic form.*
CHS2 Triangle angles Students opened a pre-constructed file that was distributed to them during the lesson using the link cables. They measured the interior angles of several polygons using the Graphs and Geometry application. They did not drag any of the vertices. They entered angle measurements manually into a Spreadsheet page and commented on their findings using a Notes page. Students saved their work.	    *(I) Liked the idea of the*	**Variance** = measured values of individual angles. **Invariance** = displayed total is in range 179 -181.	**IUS2**: *From a given set of static geometric objects, make measurements and tabulate data to explore variance and invariance within the measured data in numeric and tabular forms.*

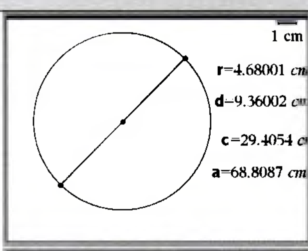
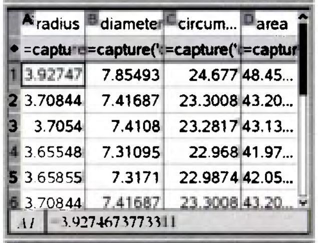
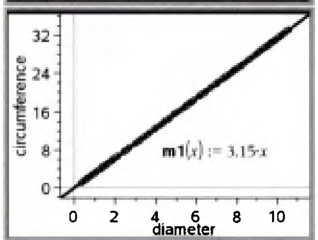
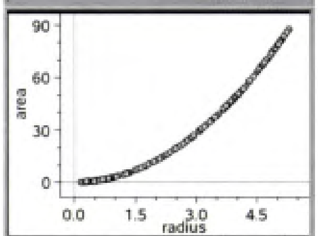
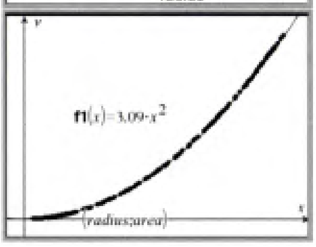
Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
	measuring having to be done in a precise order as this reinforces the conventions of reading angles which in the past many students have not appreciated. As a result of the difficulties some of the girls had with the activity, which we felt was as a result of their being too much on one page we changed the layout of the pages before the activity was used with a lower ability group. The change meant there was only one triangle to a page. This also allowed us to make each triangle bigger and therefore easier for the students to 'pick up' the bit that was required. I would keep this change if the activity was to be used again with any other group. [CHS2(Quest2)]		
CHS3 Dice Students opened a pre-constructed file that had already been loaded onto the handhelds by the teacher. They used a straight dice simulation in the Spreadsheet application to generate sixty dice throws and observe the frequency graph in the Data and Statistics application. They then opened a second .tns file that included loaded dice and tried to ascertain which dice were loaded. Students saved their work.	Activity instructions on page 1.1 *A simple dice.* *A dice is rolled 60 times.* *The next page shows the numbers rolled and works out how many times each number was rolled.* [CHS3(tns-T)]  	**Variance** = the output of the dataset as a frequency plot. **Invariance** = the bias applied to the mathematical model that generated the dataset.	**IUS1**: *Vary a syntactic input and use the instrument's functionality to observe the resulting output in graphical form.*
CHS4 Intersecting lines Students opened a pre-constructed .tns file that had already been loaded onto the handhelds by the teacher. They dragged various	 	**Variance** = the positioning of the vertices that were already fixed on the circumference of the circle. **Invariance** =the inherent angle properties (equality of vertically opposite	**IUS3:** *Vary the position of an object (by dragging) that has been constructed in accordance with a conventional mathematical constraint and observe the resulting changes.*

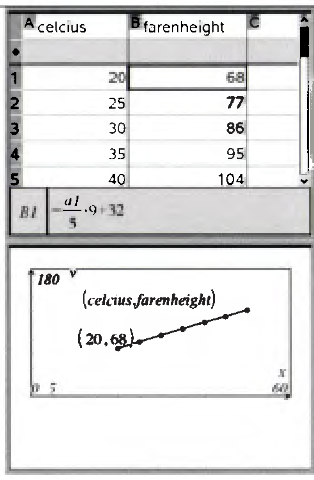
Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
points to observe changes in angle properties using the Graphs and Geometry application and commented on their findings using a Notes page. Students saved their TI-Nspire work.	 What aspect(s) of the idea would you use again? *All of it* What changes would you make? *Make the shape bigger to try to help the girls pick up the points more easily* [CHS4(Quest2)]	angles; sum of angles around the centre); the length of the lines (all diameters); the size of the initial circle.	
LSM1 Sums of series Students were shown how to enter fractions using the library key. They were also shown how to copy and paste the previous sum. Students recorded their calculations in their exercise books. The teacher demonstrated to the students how to graph the sequence and plot its function – it is not clear from the questionnaire how this was done.	 *Instead of writing the answers in fractions, I should have introduced the decimal. This would have help them to draw conclusion without can prompt.* [LSM1(Quest2)]	**Variance** = initially - numerical inputs, the terms of the sequence. **Invariance** = the total for each calculation, the limit of the sequence.	**IUS1**: *Vary a numeric input and use the instrument's functionality to observe the resulting output in numeric form.*
LSM2 **LMF1** Digit sums Students began generating two digit numbers that were a multiple of the sum of the two digits and used the Calculator application to verify results. They then identified the sequences of solutions that were multiples of 2, 3, 4 etc. and made conjectures about their findings. The teacher showed	 *One student typed the following into a calculator page: 12=1+2x4* *The Nspire reported 'true' which the student was very keen to share (because he wanted to know what was going on).* [LSM1(Quest2)].	**Variance** = the initial choice of number, the chosen multiple, the sequence definition as input to the handheld, the term number. **Invariance** = for any defined sequence, the value of the defined sequence for any a given term number.	**IUS1**: *Vary a numeric or syntactic input and use the instrument's functionality to observe the resulting output in numeric, syntactic, tabular or graphical form.*

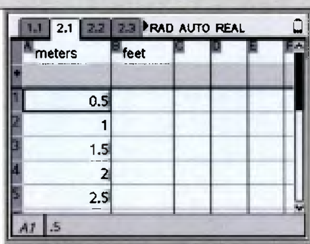
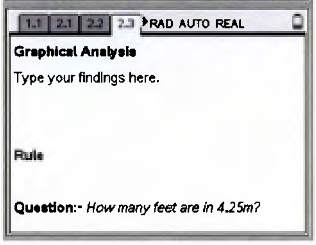
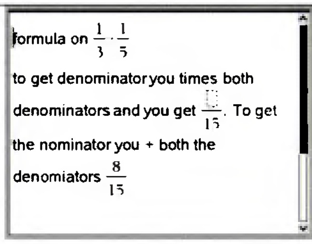
Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
the students how to generate sequences and calculate terms – it is not clear from the questionnaire exactly how this was done.			
LSM3 Mystery factors Students created a new file and used the Calculator application to explore the outcomes of inputting GCD(n,N) to explore highest common factor and LCM(n,N) to explore lowest common multiple. Students recorded their findings in their exercise books.	 *Gaining more confidence in supporting students to structure their work with little guidelines.* [LSM3(Quest2)]	**Variance** = choice of input numbers. **Invariance** = output in accordance with machine's algorithm for calculating GCD and LCM.	**IUS1**: *Vary a numeric input and use the instrument's functionality to observe the resulting output in numeric form.*
LSM4 Trial and Improvement Students created a new file and used the Calculator application to define a given quadratic function using function notation. They then evaluate the function by substituting different values of x to try to find the solutions of the equation. Students then used a Spreadsheet page to automate the process and observed the scatter plot of the values of x and f(x). Students recorded their observations in their exercise books.	   *When students started to look for values to fill into their tables, it would have been better to tell them to use f(x) notation (which these students are familiar with from other Inspire (sic) work). In fact it was left open to students to find a method, but the f(x) idea was explained to a few and*	**Variance** = the change in the evaluated y-values for different inputted x-values for a defined function. This was also shown in a tabulated form and graphical form. **Invariance** = that the x-value for the particular case where the y-value equalled zero – is called the 'solution'. This was observable as the point of intersection between the function and the x-axis within the graphing representation.	**IUS5:** *Vary a numeric or syntactic input and use the instrument's functionality to observe the resulting output in numeric, syntactic, tabular or graphical form. Use another representational form to add insight to or justify/prove any invariant properties.*

Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
	spread, which was inefficient. It could be that not suggesting anything would have led to interesting discussion about the efficiency of different methods, but in this case it would be important for the teacher to avoid giving any method. *The starter problem by inspection was too easy (surprisingly for one student, who insisted on sharing the answers very quickly). This spoilt the search. This student and one or two others needed to be given a much harder example to start with.* *It was difficult to explain how to set up the spreadsheet in the middle of the lesson. The lack of desktop software and/or a viewscreen meant more teacher direction to ensure absolute attention was needed at this stage.* *Student data collection is vital – the table in the book ensured students could reflect on their values.* *Creating a new file and saving it at the start of the activity prevented change of focus later on (saving caused a nuisance for some if they hadn't done this).* *Very clear activity instructions are vital for the smooth running of the lesson.* *A teacher projection to explain short practical sequences is very important. Especially for the spreadsheet.* [LSM4(Quest2)]		
LMF2 Four fours Students opened a pre-constructed file that had already been loaded onto the handhelds by the teacher. They used the Calculator application to try to use four fours and any combination of operations, including brackets, to generate the	Activity introduction and screenshots as shown in CHS1 *Initial introduction should also be done on the handheld instead of doing that in their books* *The use of pupils presenting their feed back to the whole class* [LMF2(Quest2)]	**Variance** = the inputs that the students provide that met the activity constraints with respect to 'four fours' and the permitted mathematical operations. **Invariance** = the machine constraints with respect to how the input was interpreted.	**IUS1**: *Vary a numeric or syntactic input and use the instrument's functionality to observe the resulting output in numeric, syntactic, tabular or graphical form.*

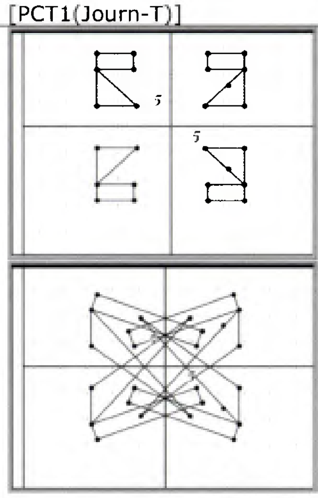
Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
numbers from one to a hundred. They recorded their results in the Spreadsheet application and used a Notes page to record their findings. Students saved their work.			
LMF3 Trial and Improvement Students began by estimating the solution to √32 and using the calculator to check and refine their estimates to one decimal place. Prompted by the teacher the students moved to discuss the solution to x^2 −x +24 =0 and again, tried some examples using the calculator. The teacher showed them how to enter a formula into the spreadsheet page and use 'fill down' to generate 10 terms. Students were asked to decide if the results were too high or too low and record this in their exercise books.	 *Make the instructions for using the spreadsheet more similar to the written table used in the starter.* *Get students to type L or H in the third column.* *Give more explicit instructions for setting up the spreadsheet.* *Students were very well behaved and focused on the work and used the handheld very sensibly.* *Students were very fluent in their functional operation, they clearly were familiar with how it works. Simple instructions were given for operational tasks and the vast majority were able to do them.* *Students just got on with the activity as if this was utterly normal!* [LMF3(Quest2)] Questionnaire completed by CO in collaboration with SS.	**Variance** = the defined function, the x-value. **Invariance** = the calculated values of the defined function.	**IUS1**: *Vary a numeric or syntactic input and use the instrument's functionality to observe the resulting output in numeric, syntactic, tabular or graphical form.*
LMF4 Mystery factors Students created a new file and used the Calculator application to explore the outcomes of inputting GCD(n,N) to explore highest common factor and LCM(n,N) to explore lowest common multiple. Students recorded their findings in their exercise books.		**Variance** = choice of input numbers. **Invariance** = output in accordance with machine's algorithm for calculating GCD and LCM.	**IUS1**: *Vary a numeric or syntactic input and use the instrument's functionality to observe the resulting output in numeric, syntactic, tabular or graphical form.*

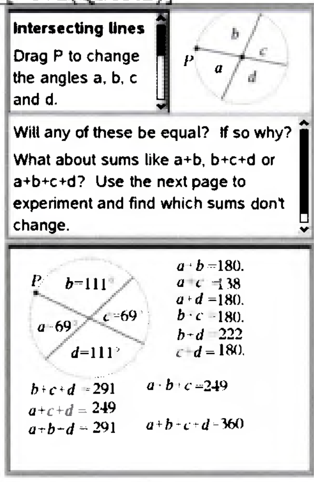
Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
LMF5 Circles and Pi Students created a new file and the Graphs and Geometry application and used a step by step set of instructions prepared by the mentor to construct a circle with a diameter, measure its properties and save them as variables. Students then opened a Spreadsheet page to perform a data capture of the radius, diameter, circumference and area. They then added a Data and Statistics page and produced a scatter plot of the diameter and circumference.	 *What aspect(s) of the idea would you use again?* *The transfer from the graphical to spreadsheet side to see linkages.* [LMF4(Quest2)] *Students very please and surprised by the automatic data collection* [LMF4(Quest2)] *Yes but it is too much for one lesson, so may be better to be delivered in two lessons.* [LMF4(Quest3)]	**Variance** = positioning of circle, radius of circle, positioning of movable line (and resulting gradient). **Invariance** = sampling rate of data, appearance of graphs (subject to initial scale on axes), position of curve $y=x^2$.	**IUS8:** *(Construct a geometric scenario and then) vary the position of objects (by dragging) and automatically capture measured data. Use the numeric, syntactic, graphical and tabular forms to explore, justify (and prove) invariant properties.*
LMF6 Box plots *analysing the salaries of*	**Insufficient data submitted**	**Insufficient data submitted**	**Rejected from Phase One data analysis.**

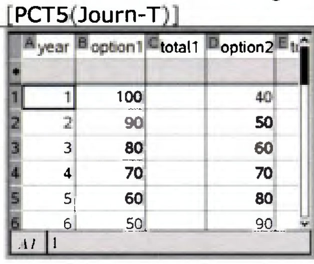
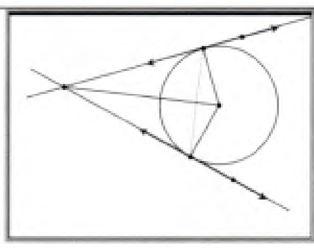
Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
graduates and non graduates using box plots [LMF6Quest5)]			
HAH1 Four fours Students opened a pre-constructed file that had already been loaded onto the handhelds by the teacher. They used the Calculator application to try to use four fours and any combination of operations, including brackets, to generate the numbers from one to a hundred. They recorded their results in the Spreadsheet application and used a Notes page to record their findings. Students saved their work.	Activity introduction and screenshots as shown in **CHS1** *Due to personal lack of knowledge I became very prescriptive in terms of the step by step tasks that I set out, for example – "Press the 'on' button and no more!" Fantastic start to the project as it enabled me to see how much I knew and what I needed to work on before I went any further.* [HAH1(Quest2)] *by me doing the activity first I was confident to show the pupils.* [HAH1(Quest3)]	**Variance** = the inputs that the students provide that met the activity constraints with respect to 'four fours' and the permitted mathematical operations. **Invariance** = the machine constraints with respect to how the input was interpreted and its resulting display.	**IUS1**: *Vary a numeric or syntactic input and use the instrument's functionality to observe the resulting output in numeric, syntactic, tabular or graphical form.*
HAH2 Temperature conversion Students created a new file and used a step by step set of instructions created by the teacher to open a Lists and Spreadsheet page and input a set of values to represent temperatures in Celsius and type the given formula (=a1/5X9-32) to generate a set of temperatures in Fahrenheit. Students then were instructed to insert a Graphs and Geometry page and told how to produce a scatter plot of temperatures in °F against °C. They were instructed to join the points by using the Attribute 'Points are connected'.		**No evidence of students' exploration of variance and invariance within the lesson.**	**Rejected from Phase One data analysis.**

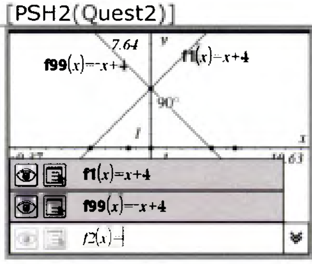
Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
HAH3 Further Conversions Students opened a pre-constructed file that was distributed to them during the lesson using the link cables. They used activity instructions written in the Notes pages to draw scatter plots of various conversion data using Graphs and Geometry pages. The students had to deduce the conversion relationships in order to generate the spreadsheet data and respond to questions within the Notes pages.		**Variance** = The student's choice of initial data set. The student's choice of function to replicate the conversion. **Invariance** = the resulting scattergraph.	**IUS1**: *Vary a numeric or syntactic input and use the instrument's functionality to observe the resulting output in numeric, syntactic, tabular or graphical form.*
HAH4 Adding fractions Students created a new file and used the Calculator application to add fractions of three types (same denominators, co-prime denominators, any denominator). The students recorded their own generalisations in the Notes application. The students shared these during a whole class plenary mediated by the teacher with the support of the project mentor. Students saved their work.	 *Much more discussion work – really got the boys involved in the mathematical theories and how the operations work. The theories really 'stuck' with the boys because they discovered it!* [HAH4(Quest3)]	**Variance** = The fractions chosen by the students as inputs. The fraction operation. **Invariance** = The calculated output,	**IUS1**: *Vary a numeric or syntactic input and use the instrument's functionality to observe the resulting output in numeric, syntactic, tabular or graphical form.*
HAH5 Data Handling Coursework Students chose whether to use TI-Nspire or Excel to carry out their GCSE Statistics Data Handling project.	**Insufficient data submitted** – it is not clear how the technology was used in this activity.	**Insufficient data submitted**	**Rejected from Phase One data analysis.**

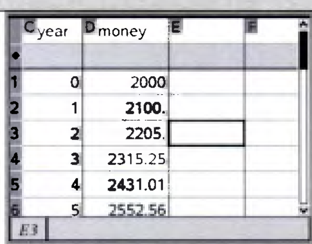
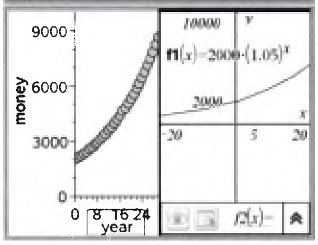
Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
HRG1 Standard Form Students chose whether to use their TI-Nspire handheld to solve a series of problems that led to Standard Form notation.	**Insufficient data submitted** – it is not clear how the technology was used in this activity.	**Insufficient data submitted**	**Rejected from Phase One data analysis.**
HRG2 Equations of Straight Line Students created a new file and used the Graphs and Geometry application to investigate equations that produce parallel lines or lines that go through a particular point on the y-axis. Students then tested their predictions for the position of some given linear functions.	No teacher file or students' work submitted. *I led the lesson with my handheld hooked up to the OHP. We investigated parallel lines (same gradient) – the number in front of the x and how we could measure it directly from the graph. Also we thought about how we could now easily plot straight lines by hand – something we would try next lesson! We moved on to test the effect of the intercept – what it meant and whether it was just true for straight lines.* (HRG2(Journ-T)].	**Variance** = The values of m and c for input linear equations. The appearance of the linear functions on the screen, their lengths and orientation. **Invariance** = the definition of gradient and intercept.	**IUS1**: *Vary a numeric or syntactic input and use the instrument's functionality to observe the resulting output in numeric, syntactic, tabular or graphical form.*
HRG3 Data Handling Coursework Students chose whether to use TI-Nspire or Excel to carry out their GCSE Statistics Data Handling project.	**Insufficient data submitted** – it is not clear how the technology was used in this activity.	**Insufficient data submitted**	**Rejected from Phase One data analysis.**
PCT1 Four fours Students opened a pre-constructed file that had already been loaded onto the handhelds by the teacher. They used the Calculator application to try to use four fours and any combination of operations, including brackets, to generate the numbers from one to a hundred. They recorded their results in the Spreadsheet application and used a Notes page to record their	Activity introduction and screenshots as shown in CHS1 *A syntax error was produced by adding = to the end of the calculation which promoted a discussion on the use of the handheld enter button.* *The use of higher level keys on the calculator e.g. square root button promoted experiment and challenged previous knowledge.* *Squaring and square roots were discovered* *Making zero and 1 helped make other numbers* *Allowed pupils to choose their own style of calculations, not teacher led.*	**Variance** = the inputs that the students provide that met the activity constraints with respect to 'four fours' and the permitted mathematical operations. **Invariance** = the machine constraints with respect to how the input was interpreted.	**IUS1**: *Vary a numeric or syntactic input and use the instrument's functionality to observe the resulting output in numeric, syntactic, tabular or graphical form.*

Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
findings. Students saved their work.	*The handheld allowed the pupils to use functions beyond their previous knowledge and gave confidence to experiment.* *The pupils were able to learn from errors.* *The handheld provided a clear record with correct notation.* *The record was electronic and teachers had a permanent record of their work. The facility of being able to save work allows for the pupils to build on previous work and knowledge. The file could be returned to and added to at a later date.* *The use of the handheld encouraged accuracy.* *Working in pairs promoted mathematical discussion of operations, for instance squaring numbers. Also discussion from pair to pair.* [PCT1(Journ-T)]		
PCT2 Reflection Students created a new and used the Graphs and Geometry application to construct a freehand 2-D shape and reflect it in both axes. They then deformed their original shape by dragging vertices and observed the resulting transformations.	 *During the introduction the feedback from the software supported students to learn from their predictions, where the outcomes differed.* *Students were able to visualise where the image was going to end up and the dynamic nature of the software supported the boys to learn quickly. They also discussed their mathematical ideas with each other.* [PCT2(Quest2)] *The boys were more engaged than usual with all students engaged in the activity and keen to share their outcomes with others*	**Variance** = The position and appearances of the initial objects on the coordinate plane. The selection of the lines of symmetry. **Invariance** = Resulting line symmetry (in the axes) as defined by the machine constraints.	**IUS3:** *Vary the position of an object (by dragging) that has been constructed in accordance with a conventional mathematical constraint and observe the resulting changes.*

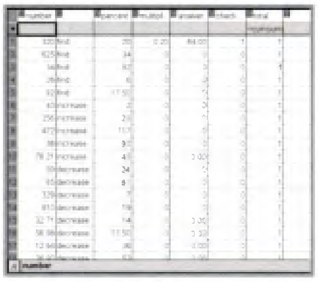
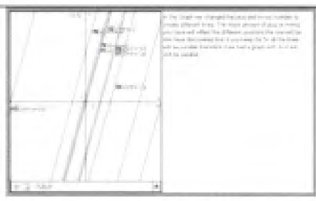
Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
	– something that is not normal for this group. They were particularly intrigued when we begun to drag the shapes and use the dynamic nature of the software. [PCT2(Quest2)]		
PCT3 Intersecting lines Students opened a pre-constructed file that had already been loaded onto the handhelds by the teacher. They dragged various points to observe changes in angle properties using the Graphs and Geometry application and commented on their findings using a Notes page. Students saved their TI-Nspire work.	 *Sequence of lessons written already.* *We felt we would like to adjust the activities to make them more simplistic. (One page to one diagram).* [PCT3(Quest2)] *Some pages were too full and 'froze' when moving points.* *By 3rd lesson most pupils could navigate and use 'doughnut' much more effectively.* [PCT3(Quest2)] *The lesson files were too prescriptive – we had requested a lesson to use geometry menu – to allow pupils to measure angles accurately and spot themselves which were equal/add to 180 or 360 etc.* [PCT3(Quest3)]	**Variance** = the positioning of the vertices that were already fixed on the circumference of the circle. **Invariance** =the inherent angle properties (equality of vertically opposite angles; sum of angles around the centre); the length of the lines (all diameters); the size of the initial circle.	**IUS3:** *Vary the position of an object (by dragging) that has been constructed in accordance with a conventional mathematical constraint and observe the resulting changes. (Use another representational form to add insight to or justify/prove any invariant properties).*
PCT4 Solving simultaneous equations (Linear and quadratic) Students created a new file and used the Graphs and Geometry application to define two functions and use the Construction menu to find the intersection points.	*It would be possible to then move on to solving the pairs of equations algebraically with a clear picture in the pupils' minds of what the algebraic manipulation achieves/means.* *It would also be possible to use the calculator page to store a value of x then define a function and check that the equations have the same values at the point of intersection(s).* [PCT4(Quest2)] (What changes would you make?)	**Variance** = the given pairs of functions. **Invariance** = the point of intersection as observed (or calculated) using the technology.	**IUS5:** *Vary a numeric or syntactic input and use the instrument's functionality to observe the resulting output in numeric, syntactic, tabular or graphical form. Use another representational form to add insight to or justify/prove any invariant properties.*

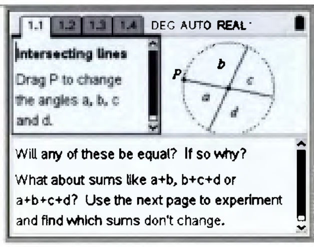
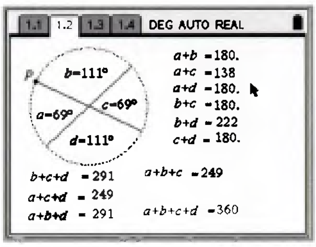
Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
	See above re using store value of x and define function to check [PCT4(Quest2)] *Without the need to plot graphs by hand they were able to work at mathematics beyond this.* *They were able to work through the file at their own speed and understanding.* [PCT4(Quest2)] *The split screens allowed them to be systematic in their working and recording (Changes next time) checking simultaneous equations intersected! could allow for them to create files - examples of their own.* [PCT4(Quest3)]		
PCT5 Sequences	No teacher file or students' work submitted. *This lesson highlights the progression that needs to be made by the teacher in their own learning of the use of a new ICT tool, in that, as their capabilities increase they are able to integrate its' use into their teaching.* [PCT5(Journ-T)]	Insufficient data submitted	Insufficient data submitted
PSH1 Rich Aunt problem Students created a new file and used the Spreadsheet application to model the results of four different investment options aver different lengths of time. They input formulae to the spreadsheet and used the fill down functionality to generate lists of data and answer a set of related questions.	 *After teaching this one I decided a pre-constructed spreadsheet would avoid the time consuming in the lesson e.g. column headings* [PSH1(Quest3)] *Pupils may have progressed further in the activity if a lesson on formulae in excel had been taught previously.* [PSH1(Quest3)]	**Variance** = The four different investment options. The spreadsheet formula input to 'match' each model. **Invariance** = the resulting data generated by the Spreadsheet formula.	**IUS1**: *Vary a numeric or syntactic input and use the instrument's functionality to observe the resulting output in numeric, syntactic, tabular or graphical form.*
PSH2 Angles in circles and tangents Students opened a pre-constructed file that had already been loaded onto the handhelds by the teacher. They used a geometric	 *Pupils got on with the syntax needed really well -*	**Variance** = Position of construction on the page. Dimensions of circle. Position of point on circumference. **Invariance** = Position of symmetrical point on circumference.	**IUS3:** *Vary the position of an object (by dragging) that has been constructed in accordance with a conventional mathematical constraint and observe the resulting changes.*

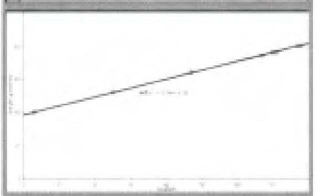
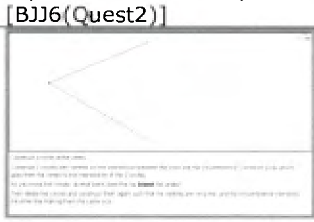
Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
figure in the Graphs and Geometry application to make generalisations about angle and line properties of a circle's radius and tangent.	*much better than younger pupils did.* [PSH2(Quest2)] *I know feel pupils could construct some of their own diagrams to aid their learning of the properties of shapes, lines and angles.* [PSH2(Quest2)] *I was surprised at how positive the students were and how easily they could use the handheld e.g. opening files, selecting, using menus.* [PSH2(Quest2)] *Avoided the need for the pupils to be able to draw accurate diagrams.* *Gave pupils more opportunity to 'play' with the maths.* *Highlighted the importance of correct notation.* [PSH2(Quest2)]	Point of intersection of tangents.	
PSH3 Equations of parallel and perpendicular lines Students created a blank file and used the Graphs and Geometry application to generate linear functions of parallel lines. They then explored how to generate perpendicular lines, by measuring the angle between them and adjusting it to 90 degrees by dragging. They made and tested conjectures about the gradients of pairs of perpendicular lines. During the plenary they collated their findings resulting in a generalisation.	 'YOU HAVE TO DIVIDE 1 BY THE AMOUNT X'S.' [PSH3(tns-S)] *Pupils really appreciated the ability to move the line on the screen and then be told the equation of the new line.* [PSH3(Quest2)] *It would have been useful to have accompanying resources for pupils to record their graphs and note findings.* [PSH3(Quest2)] *it allowed pupils to explore the problem for themselves quickly and efficiently so the focus was on the intended learning rather than issues with drawing graphs that could have occurred otherwise.* [PSH3(Quest2)] *Measurements not always exact e.g. 90.34...° between y=2x and y=-0.49x* [PSH3(Quest2)]	**Variance** = The initial linear function (entered syntactically). The second linear function. The positioning of the points used to define the angle between the two lines. The measured angle between the two linear functions. **Invariance** = The mathematical condition - when the lines are perpendicular, the product of the gradients is negative one.	**IUS6:** *Vary the position of an object that has previously been defined syntactically (by dragging) to satisfy a specified mathematical condition.*

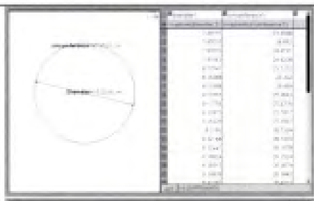
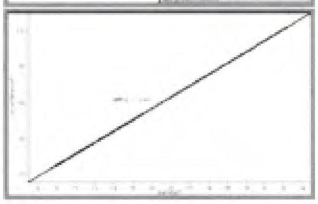
Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
PSH4 Exponential growth and decay Students used 25 real dice each to obtain data to simulate an exponential decay. The group's results were collated by the teacher using the Spreadsheet application and a Scatter plot of the results drawn in the Data and Statistics page. Students then worked on similar problems from a text book scheme, to model growth and decay problems and fitted algebraic functions to the data.	*Pupils could recognise the shape of an exponential graph and most could describe what is happening in that graph – phrases such as "the more there is the more it increases" led to connection with a multiplier.* [PSH4(Quest2)] *I would like to try more experimental lessons – especially with connections to other subjects. I would also like to try and introduce the Nspire to some of the science department so they could utilise it within lessons.* [PSH4(Quest2)] *It was powerful for the pupils to be able to connect the data they worked out for themselves to a function drawn on a graph. Collecting data on the spreadsheet function to be able to represent it in graphical form.* [PSH4(Quest2)] *It would have been nice to try and assign a function to the dice/atom activity and compare this using split screen view on the nspire. I have since found out that you can add an exponential regression line which gives you the equation of the line as well – this could have been used as a check to the one they found themselves.* [PSH4(Quest2)]	**Variance** = The data input to the spreadsheet. The function input to the graphical application. **Invariance** =The resulting scattergraph of the data in the Data and Statistics application. The resulting function plot in the Graphs application.	**IUS5:** *Vary a numeric or syntactic input and use the instrument's functionality to observe the resulting output in numeric, syntactic, tabular or graphical form. Use another representational form to add insight to or justify/prove any invariant properties.*

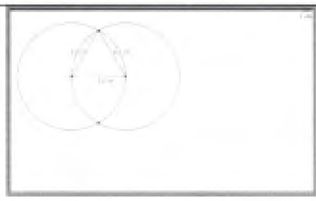
Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
PSH5 Enlargements Students opened a pre-constructed file that was distributed to them during the lesson using the link cables. They worked in the Graphs and Geometry application to enlarge a given shape by changing the scale factor using integer values. Students observed the resulting perimeter and area measurements and entered these into a Spreadsheet application. The teacher led a whole class plenary to discuss findings. The teacher did not review the students' files.	No teacher file or students' work submitted. *To an extent, I did as much as I my technical abilities would allow in the time that I had.* *If I had asked for technical help earlier I could have transferred the doc. prior to the lesson.* [PSH5(Quest3)] No – *I wanted to capture data from the enlargements to the spreadsheet but neither Adrian or I could do this – pupils had to enter data manually*. [PSH5(Quest3)] *Pupils realised the need for centre of enlargement as the Nspire needed that info to complete the enlargement.* *& pupils found they ran out of space and needed to change the centre of enlargement to fit it.* [PSH5(Quest3)]	**Variance** = The appearance and the position of the given shape. The original perimeter and area of the given shape. The scale factor. **Invariance** = The appearance and position of the enlarged shape. The resulting perimeter and area of the enlarged shape.	**IUS2:** *From a given set of static geometric objects, make measurements and tabulate data to explore variance and invariance within the measured data in numeric and tabular forms.*
BAK1 **BJJ1** Four fours Students downloaded a pre-constructed file from the school virtual learning environment to their laptop computers. They used the Calculator application to try to use four fours and any combination of operations, including brackets, to generate the numbers from one to a hundred. They recorded their results in the Spreadsheet application and used a Notes page to record their findings. Students continued the activity for homework and uploaded their files to the school virtual learning environment at the	Activity introduction and screenshots as shown in CHS1 *Also, the division sums all changed to fractions – again they found this confusing. Some students were really happy to try things out and enter calculations to investigate what was going on. Others were more reluctant and needed some prompting just to have a go. Many wanted to key straight into the spreadsheet, which meant them having to know the answer before they had tried the question, these pupils were pointed to the calculator page and encouraged to type in any sum and see what happened.* [BAK1(Quest2)]	**Variance** = the inputs that the students provide that met the activity constraints with respect to 'four fours' and the permitted mathematical operations. **Invariance** = the machine constraints with respect to how the input was interpreted.	**IUS1**: *Vary a numeric or syntactic input and use the instrument's functionality to observe the resulting output in numeric, syntactic, tabular or graphical form.*

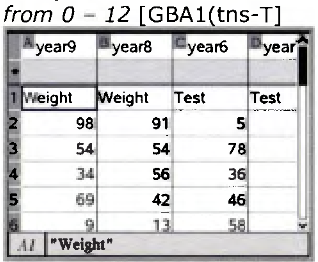
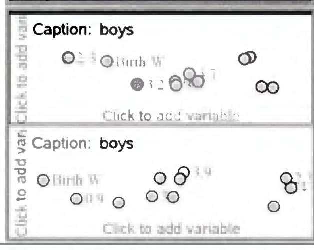
Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
end of the lesson.			
BAK2 Percentage multipliers Students downloaded a pre-constructed file from the school virtual learning environment to their laptop computers. Students solved some problems relating to comparing decimals and percentages in the Spreadsheet application and used a Calculator page to check for validity. In the second part of the lesson they used the Spreadsheet application to solve numerical problems relating to percentage increases and decreases of given amounts. The students uploaded their files to the school virtual learning environment at the end of the lesson.	*On each line on the following spreadsheet page there is a question.* *For example:* *35% < = > 1/3* *Delete the two incorrect symbols from "<", "=" and ">" so that the line reads correctly e.g.* *35% > 1/3* *If you make a mistake use the "undo" button - the blue arrow.* *You can check your results using the calculator page 3.* [BAK2(tns-T)]. 	**No evidence of students' exploration of variance and invariance within the lesson.**	**Activity rejected from data analysis process.**
BAK3 **BJJ3** Investigating straight lines Students created a new file and used the Graphs and Geometry application to explore equations of straight lines of the form y = mx + c by varying m and c. They were encouraged to split their page and write about their observations using the Notes application. Students uploaded their .tns files to the school virtual learning		**Variance** = The values of m and c for input linear equations. The appearance of the linear functions on the screen, their lengths and orientation. **Invariance** = the definition of gradient and intercept.	**IUS1**: *Vary a numeric or syntactic input and use the instrument's functionality to observe the resulting output in numeric, syntactic, tabular or graphical form.*

Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
environment at the end of the lesson.			
BAK4 **BJJ4** Exploring angles Students downloaded a pre-constructed file from the school virtual learning environment to their laptop computers. Students opened a pre-constructed file that had already been loaded onto the handhelds by the teacher. They dragged various points to observe changes in angle properties using the Graphs and Geometry application and made comments in response to questions on a Notes page. Students uploaded their files to the school virtual learning environment at the end of the lesson.	  Questions on Notes page: Which of the sums add up to 180° Why? Which of the sums add up to 360° Why? *Able to move and use dynamic diagrams and to make conjectures about vertically opposite angles.* [BAK4(Quest2)]	**Variance** = the positioning of the vertices that were already fixed on the circumference of the circle. **Invariance** =the inherent angle properties (equality of vertically opposite angles; sum of angles around the centre); the length of the lines (all diameters); the size of the initial circle.	**IUS3:** *Vary the position of an object (by dragging) that has been constructed in accordance with a conventional mathematical constraint and observe the resulting changes.*
BAK5 **BJJ5** Comparing box plots Students downloaded a pre-constructed .tns from the school virtual learning environment to their laptop computers. They used a simple data set in the Spreadsheet application and used the defined functions mean and median to calculate these values. Some students also used formulae to calculate the range. They then inserted at Data and Statistics page and drew Box plots of	*On the following page you will find the scores of 4 students from a series of maths test over the past 10 weeks. The tests are marked out of 10.* *By working out the four averages, decide on who is the best student.* *Write on the third page who is the best student.* [BJJ5(tns-T)]	**Variance** = Four different data sets already input into the spreadsheet. **Invariance** = Calculated Mean, Mode Median and Range for each set of data. Displayed Box Plots.	**IUS1**: *Vary a numeric or syntactic input and use the instrument's functionality to observe the resulting output in numeric, syntactic, tabular or graphical form.*

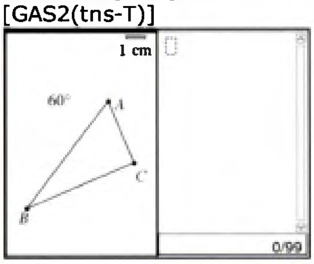
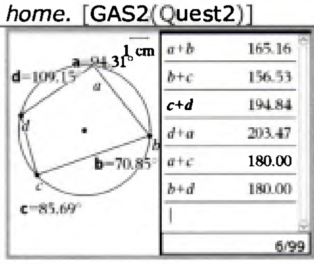
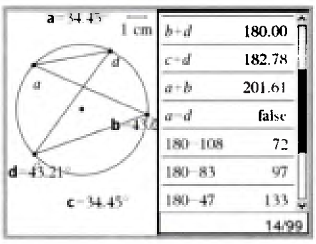
Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
the data to make comparative statements. Students uploaded their .tns files to the school virtual learning environment at the end of the lesson.			
BAK6 **BJJ6** Weighing sweets Students downloaded a pre-constructed .tns from the school virtual learning environment to their laptop computers. They posed their own questions using a Notes page and then carried out a practical activity. Students input data into the Spreadsheet application and observed the data plotted as a scatter plot in the Data and Statistics application. They used a movable line to model the algebraic link between the data sets. Students uploaded their .tns files to the school virtual learning environment at the end of the lesson.	*In this activity you will try to find out some missing information about an opened container of sweets. Page 1.2 is a blank table for recording useful results - for example the number of sweets eaten and weight. Page 1.3 shows a scattergraph of the data in the table.* *Use pages 1.4 onwards to record the questions you came up with and how you solved them.* [BJJ6(tns-T)]    *Students saw the difference between groups with different weighted sweets and were able to make explanations as to why.* [BJJ6(Quest2)]	**Variance** = The data generated by the students according to the strategy they devised for weighing the sweets. The equation of the movable line to fit the data. Different sweets. **Invariance** = The resulting scatterplot for the data collection. The resulting function from the movable line.	**IUS5:** *Vary a numeric input and use the instrument's functionality to observe the resulting output in tabular or graphical form. Use another representational form to add insight to or justify/prove any invariant properties.*
BJJ7 Angle bisectors Students created a new .tns file and used the Graphs and Geometry application and were shown how to construct angle bisectors leading to the construction of an 'Angle fan'. Students saved	 NOTE: KC had taken out the aspect of Tim's lesson that led to the need to have circles of equal radii in this construction.	**No evidence of students' exploration of variance and invariance within the lesson.**	**Rejected from Phase One data analysis process.**

Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
their TI-Nspire work. Students uploaded their .tns files to the school virtual learning environment at the end of the lesson.			
BAK8 Pi Day Students created a new .tns file and used the Graphs and Geometry application with a set of instructions to construct a circle with a diameter, measure its diameter and circumference and save them as variables. Students then opened a Spreadsheet page to perform automated data capture of the diameter and circumference. They then added a Data and Statistics page and produced a scatter plot of the diameter and circumference. They added a movable line to find the algebraic relationship between the variables. Students uploaded their .tns files to the school virtual learning environment at the end of the lesson.	  *They drew a segment relatively easily and also a circle, but found it quite difficult to attach the two together so that they were linked. A lot of them did not appreciate the importance of them being linked together until we tried to drag them bigger and smaller and collect the data. It would be good to use alongside measuring real objects. It is a nice idea to plot diameter against circumference and use the moveable line to find the value of pi. We need to change the fix function to get the value to more decimal places – I forgot to do this before the start or tell the kids to do it.* [BAK8(Quest2)]	**Variance** = Positioning of circle. Radius of circle. Positioning of movable line. **Invariance** = Sampling rate of data. Appearance of graphs (subject to initial scale on axes). Resulting gradient of movable line.	**IUS8:** *(Construct a geometric scenario and then) vary the position of objects (by dragging) and automatically capture measured data. Use the numeric, syntactic, graphical and tabular forms to explore, justify (and prove) invariant properties.*
BAK9 Investigating quadrilaterals Students downloaded a pre-constructed .tns from the school virtual learning environment to their laptop computers. They opened a Graphs and Geometry page and followed a set of instructions in the Notes page to construct a rectangle. They	 Activity instructions On the following pages are different quadrilaterals. Measure the length of each line. Which lines are the same length? Which lines are parallel? Measure some angles. Which angles are the same? Do any angles add up to	**Variance** = the chosen quadrilateral, the construction technique, the measurements taken, **Invariance** =	**IUS3:** *Vary the position of an object (by dragging) that has been constructed in accordance with a conventional mathematical constraint and observe the resulting changes. Use another representational form to add insight to or justify/prove any invariant properties.*

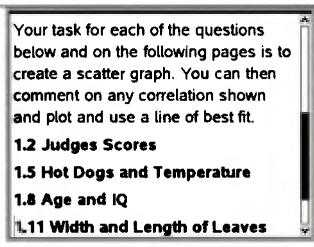
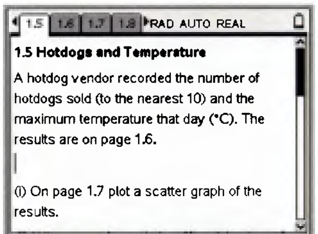
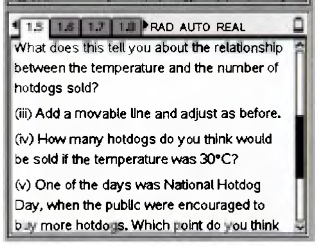
Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
then moved on to measure the interior angles and side lengths for a given set of quadrilaterals (square rhombus, parallelogram, trapezium, isosceles trapezium, kite and arrowhead). They were encouraged to drag the vertices and record what they noticed using the Notes application. Students uploaded their .tns files to the school virtual learning environment at the end of the lesson.	180°? Record your finding in the space provided *Being able to move the shapes around and see what they could change them into and which observations remained true was very useful to the students. The accuracy of the measurements helped them as well – they could see that their lines were exactly the same length, rather than just approximately as it might have been if we'd drawn them by hand.* [BAK9(Quest2)]		
BAK10 **BJJ10** Constructing equilateral triangles Students created a new file and used the Graphs and Geometry application to explore how to construct an equilateral triangle using circles alongside making the same constructions using a straight edge compasses. Students uploaded their files to the school virtual learning environment at the end of the lesson.		**No evidence of students' exploration of variance and invariance within the lesson.**	**Rejected from Phase One data analysis process.**
GBA1 Box plots comparison Students opened a pre-constructed .tns file that had already been loaded onto the handhelds by the teacher. They used given sets in the Spreadsheet application to draw parallel box plots in the Data and Statistics application and compare sets of	*Box and whisker diagrams are use to compare sets of data. Your job is to compare the sets of data given to you on the spread sheet, draw the box plots and use the notes page to comment on the differences.* *Question 1: Year 9 & year 8 weights* *Question 2: Year 6 & year 7 test scores* *Question 3: Hours of sun in resort 1 & resort 2* *Question 4: Birth weights boys & girls* *Question 5: Goals scored by a team in 2006 & 2007*	**Variance** =The two data sets selected by the students. **Invariance** = The resulting Box plots and statistical summary data.	**IUS1**: *Vary a numeric or syntactic input and use the instrument's functionality to observe the resulting output in numeric, syntactic, tabular or graphical form.* **New possibilities**

Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
data. They recorded their comments using the Notes page. Students saved their work.	*For Questions 1 & 2 set scale from 0 - 100* *For Question 3 set scale from 0 - 15* *For Question 4 set scale from 0 - 5* *For Question 5 set scale from 0 – 12* [GBA1(tns-T]  A year9 / B year8 / C year6 / D year 1: Weight, Weight, Test, Test 2: 98, 91, 5 3: 54, 54, 78 4: 34, 56, 36 5: 69, 42, 46 6: 9, 13, 58 A1 "Weight"  		
GBA2 Vitruvian man Students opened a pre-constructed .tns file that had already been loaded onto the handhelds by the teacher. They had previously collected a set of heights and arm spans sixty people that had been entered into the Spreadsheet application. The students made and tested hypothesis by using the Data and Statistics application to draw scatter plots of specific data. They recorded their findings using the Notes application. Students saved their work, which was reviewed by the teacher.	*The Ancient Roman architect Vitruvius considered proportion when designing buildings and this led to him defining his Vitruvian Man. He wrote that for a human man:* • *the length of a man's outspread arms is equal to his height* • *the distance from the top of the head to the bottom of the chin is one-eighth of a man's height* • *the maximum width of the shoulders is a quarter of a man's height* • *the distance from the elbow to the tip of the hand is one-fifth of a man's height* • *the length of the hand is one-tenth of a man's height* • *the distance from the bottom of the chin to the nose is one-third of the length of the head* • *the length of the ear is one-third of the length of the face* *Your activity is to investigate the connection between a man's arm span and height together with one other statement from*	**Variance** =The data sets selected by the students for comparison. **Invariance** = The resulting statistical plots summary data.	**IUS1**: *Vary a numeric or syntactic input and use the instrument's functionality to observe the resulting output in numeric, syntactic, tabular or graphical form.*

Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
	the list above. Is this also true for women?		
GBA3 **GAS1** Weighing sweets Students opened a pre-constructed .tns file that was distributed to them during the lesson using the link cables. They carried out a practical activity and input data into the Spreadsheet application and observed the data plotted as a scatter plot in the Data and Statistics application. Students used a movable line to model the algebraic link between the data sets.	**Activity as described BAK6/BJJ6** *Students chose between a table and a graph to answer their questions, e.g. used a table for weight of container and graph to compare data. Students came up with their own questions to answer about problem and discussed their solutions throughout. Students found loopholes in rules and weighed two sweets and halved to get the weight of one sweet.* *Students justified conjectures and generalisations when deciding how heavy and individual sweet was and weight of the pot.* *Students evaluated results from moveable line to judge links with container of sweets.* *Students began to appreciate that there are a number of different techniques used to use to analyse a situation.* *Students began to reason inductively (some loads and some held back by handset errors).* *Students examined patterns in sweet weight.* *Students used knowledge of 1st 2 sweets to make deduction about 2nd 2 sweets and work backwards.* *Students recognised the constraints of the scales only measuring to the nearest gram. Students linked graphs to correlation.* [GBA3/GAS1Quest4)]	**Variance** = The data generated by the students according to the strategy they devised for weighing the sweets. The equation of the movable line to fit the data. Different sweets. **Invariance** = The resulting scatterplot for the data collection. The resulting function from the movable line.	**IUS5:** *Vary a numeric input and use the instrument's functionality to observe the resulting output in tabular or graphical form. Use another representational form to add insight to or justify/prove any invariant properties.*
GBA4 **GAS2** Introducing trigonometry Students opened a pre-constructed .tns file that was distributed to them during the lesson using the link cables. They used the Graphs and Geometry	*Triangle Investigation* *On page 1.2 you will find a triangle which you are to investigate. You can click and drag point B to resize the triangle.* *Activity 1* *Measure each side of the triangle by using the ctrl+menu function and choosing length. Then store each measurement as the side name AB, BC or AC by using the ctrl+menu*	**Variance** = The size and orientation of the given right angled triangle. The measured lengths of the triangles sides. **Invariance** = The fixed angle size within the right angled triangle. The calculated values of the ratios of the sides.	**IUS9:** *Vary the position of a geometric object by dragging to observe the resulting changes. Save measurements as variables and test conjectures using a syntactic form.*

Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
application to measure the sides of given right-angled triangles and saved each measurement as a variable. They used a Calculator page to calculate various ratios. Groups of students focused on a different set of triangles and these groups were mixed up to support the class to generalise about trigonometric rations during the whole-class plenary.	*function and choosing store.* *Activity 2* *Investigate the result of *((BC)/(AB)), ((AC)/(AB)), or ((BC)/(AC)) using the calculator page.* *Answer the questions on page 1.3.* *Activity 3* *Record your results of activity 3 in page 1.4, please note which sides you are investigating.* [GAS2(tns-T)]  *Surprise that the ratio remained constant for a given angle – even though they had just worked on identifying similar triangles, this fact had clearly not hit home.* [GAS2(Quest2)]		
GBA5 Circles theorems Students created a new .tns file and inserted a split page with a Graphs and Geometry (analytic window) alongside a Calculator page. They constructed a circle and an inscribed quadrilateral, measuring the four interior angles and saving the measurements explicitly as variables. They used the calculator page to check angle sums using the variable notation and made generalisations relating to their explorations. (In the subsequent lesson they explored the angle properties of a quadrilateral with three vertices on the circle's	 [GBA5 (tns-S2)]  [GBA5 (tns-S3)] *Getting the pupils to construct the situation and play about with it themselves is definitely a valuable activity that I would use again.* [GBA5(Quest2)] *I feel that my teaching of this topic has been enhanced by this activity. Although it could have been done on a geometry program in a computer*	**Variance** = the size of the circle, the position of the vertices of the quadrilateral or 'bow' shape. **Invariance** = opposite angles of a cyclic quadrilateral sum to 180°; angles subtended on the same arc are equal.	**IUS9:** *Construct a graphical or geometric scenario and then vary the position of a geometric object by dragging to observe the resulting changes. Save measurements as variables and test conjectures using a syntactic form.*

Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
circumference and the fourth vertex positioned at the centre of the circle.)	*room, the use of the TI-nspire was less disruptive and as this class have used them before it was easier for them to access.* *I feel that the pupils will remember these circle theorems as they have had a practical lesson to discover them rather than being shown or told them and then simply applying them.* [GBA5(Quest2)] *Having discovered one circle theorem in the previous lesson they now seemed more confident in making generalisations about the angles that they had measured and any connections they were noticing. They were again able to move things around and see what happened to the angles.* [GBA5(Quest2)]		
GRE1 Four fours Students opened a pre-constructed .tns file that had already been loaded onto the handhelds by the teacher. They used the Calculator application to try to use four fours and any combination of operations, including brackets, to generate the numbers from one to a hundred. They recorded their results in the Spreadsheet application and used a Notes page to record their findings.	Activity introduction and screenshots as shown in CHS1 *The visual representation of the problem on screen helped to differentiate between different 'sums'.* [GRE1(Quest2)]	**Variance** = the inputs that the students provide that met the activity constraints with respect to 'four fours' and the permitted mathematical operations. **Invariance** = the machine constraints with respect to how the input was interpreted.	**IUS1**: *Vary a numeric or syntactic input and use the instrument's functionality to observe the resulting output in numeric, syntactic, tabular or graphical form.*
GRE2 Vitruvian man The students had previously collected a set of heights and arm spans sixty year 9 students. They opened a pre-constructed .tns file that included the data within the Spreadsheet application.	Activity as described in GBA2	**Variance** =The data sets selected by the students for comparison. **Invariance** = The resulting statistical plots and summary data.	**IUS1**: *Vary a numeric or syntactic input and use the instrument's functionality to observe the resulting output in numeric, syntactic, tabular or graphical form.* The students made and tested hypotheses by using the Data and

Lesson code, title and activity description	Relevant screenshots and implied evidence of teacher learning	Interpretation of variance and invariance	Implied Instrument Utilisation Scheme
What I am really trying to do is to get pupils to focus on the analysis of results and to try and get them to see that drawing relevant conclusions from statistical data is as important as calculating from the data. [GRE3 (Journ-T)]			Statistics application to draw scatter plots of specific data. They recorded their findings using the Notes application. Students saved their work, which was reviewed by the teacher
GRE3 Scatter plots Students opened a pre-constructed .tns file that had already been loaded onto the handhelds by the teacher. They used given data sets in the Spreadsheet application to draw scattergraphs in the Data and Statistics application. They commented on the resulting correlation, added a movable line and responded to questions inferred from the data. They recorded their comments using the Notes page. Students saved their TI-Nspire work.	 Your task for each of the questions below and on the following pages is to create a scatter graph. You can then comment on any correlation shown and plot and use a line of best fit. **1.2 Judges Scores** **1.5 Hot Dogs and Temperature** **1.8 Age and IQ** **1.11 Width and Length of Leaves**  1.5 1.6 1.7 1.8 RAD AUTO REAL **1.5 Hotdogs and Temperature** A hotdog vendor recorded the number of hotdogs sold (to the nearest 10) and the maximum temperature that day (°C). The results are on page 1.6. (i) On page 1.7 plot a scatter graph of the results.  1.5 1.6 1.7 1.8 RAD AUTO REAL What does this tell you about the relationship between the temperature and the number of hotdogs sold? (iii) Add a movable line and adjust as before. (iv) How many hotdogs do you think would be sold if the temperature was 30°C? (v) One of the days was National Hotdog Day, when the public were encouraged to buy more hotdogs. Which point do you think	**Variance** = inherent in the data sets within the activity. **Invariance** = The text 'variable' appears when the students move the cursor towards the axes in order to select the data for the scatter plot.	**IUS5**: *Vary a numeric input and use the instrument's functionality to observe the resulting output in numeric and graphical form. Use another representational form to add insight to or justify/prove any invariant properties.*

APPENDIX 5: CODING CATEGORIES AND SUB-CATEGORIES FOR TEACHER LEARNING IN PHASE ONE

Category of teacher learning	Name
Activity design	TL - Amendment to teacher's intro IUS
	TL - Balance of construction to exploration within IUS
	TL - Better initial examples
	TL - 'blend' of ICT and paper-pencil approach
	TL - Good 'first' activity
	TL - Need for supporting resources for teachers and pupils
	TL - New ways of organising classes for mathematical purpose
	TL - Trying a new approach
	TL - Whole class mediation – IUS
	TL - Appreciation of the need for a tighter focus on generalisation
Expectations of students	TL - Pupils enjoyed the activity
	TL - Pupils working faster
	TL - Student reluctance to use technology
	TL - Students' instrumentation issues
	TL - Students' instrumentation successes
	TL - Students use of technical language beyond teachers' expectation
	TL - Surprise at students; ability to use technology
	TL - Teacher observation of genuine student surprise and intrigue
	TL - Speed of student uptake
Instrument utilisation schemes	TL - Amendment to students' IUS
	TL - Classroom management issues
	TL - Importance of teacher demonstration re IUS
	TL - Need to discuss machine v real mathematics
Meta level ideas	TL - Appreciation of 'fundamental' aspect of mathematics
	TL - Fundamental change to teacher pedagogy
	TL - Need for teacher to practise own skills - IUS
	TL - Software supporting classroom discourse
	TL - Students' need for a big picture view of learning
	TL - Students' need for teacher to renegotiate the didactic contract
	TL - Students working at own pace
	TL - Teacher realisation of another way of learning
	TL - Teachers' appreciation of improved student accuracy
	TL - When not to use technology
	TL - Visual imagery provided scaffolding

APPENDIX 6: ANALYSIS OF TIM'S PHASE ONE LESSON DATA

What follows is an account of Tim's learning trajectory as evidenced by the analysis of his lesson data from the first phase. This provides an insight into his emergent instrument utilisation schemes and some evidence for his learning during this period of time.

I have already commented that Tim's starting point with respect to his technical and pedagogic knowledge relating to mathematics education technology was noticeably more advanced that the other participants in the project. He had been personally acquainted with handheld technology for over seventeen years and in that period he had been active in the authoring of teaching materials and the professional development of other teachers. However, his school context, and in particular, his concerns over supporting his learners through the initial instrumentation period led him to begin quite tentatively, using an activity of the type IUS1.

STP1 Prime factorisation

Tim's first lesson using the MRT was with his Year 8 class, a set 1 (of 6)[1] working at National Curriculum levels 5-6. Tim showed his students how to manually entered a text and numeric input to the calculator's 'factor()' function with the aim of observing the invariant properties: 'the prime factorisation of a number is unique; prime numbers have only two factors; square numbers have an odd number of factors' [STP1(Quest2)]. In his activity design Tim intended that, by changing the input number, the students would gain an appreciation of the aforementioned invariant properties through their interpretations of the numerical outputs (Figure A6-1).

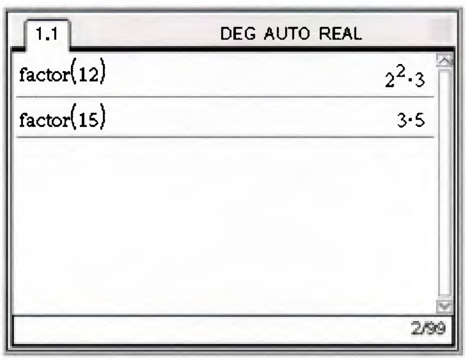


Figure A6-1 [STP1(tns-T)]

[1] Each year group at Blue Coat School was divided into two bands. These were then sub-divided into mathematics sets according to their mathematical attainment. Set 1 would have included the most able students in that half of the year group.

In addition to his mathematics objectives for the activity, he had also identified two objectives that concerned the students' instrumentation phase, 'to learn how to use the factor() command and also how to copy and paste items from the history on the calculator page' [STP1(Quest2)]. This reflected Tim's existing knowledge in that he knew that his lesson introduction would need to mediate these instrumental techniques as an explicit act. The students recorded their work in their exercise books. Tim's post-lesson evaluation suggested that his classroom experience had led him to think more carefully about his introduction to the activity, implying his increased appreciation of how his initial choice of input numbers might have affected the students' abilities to see the intended generalities. Tim went as far as to suggest an alternative sequencing for the activity thus:

> *I would probably start by deliberately choosing numbers that are a product of exactly two primes (each to the power 1) in order to encourage students to begin to generalise.*
>
> *factor(6)=2x3*
>
> *factor(35)=5x7*
>
> *factor(14)=2x7*
>
> *I would then ask students what they thought the output of, say factor(21) and factor(55) might be.*
>
> *Just when the students thought they were getting the hang of this I would throw in a question where the prime factorisation contained an index, such as factor(16). I would look surprised at the outcome and ask them to investigate further. [STP1(Quest2)]*

Tim's evaluation of the activity was in part triangulated by a sample of his students' written work that he had photocopied and submitted to the study. This showed the sequences of numbers that the students had explored, which appeared to be initially random choices, although later into the activity, some students began to work more systematically. Already there was evidence to suggest that Tim was reflecting on the requirement for him to think carefully about structuring the activity to enable him to support his students to 'see' the invariant properties in the way that he envisaged.

STP2 Visual fractions

The second activity that Tim submitted to the project reflected the fact that he had now had ownership of his handheld for over five months. He had clearly been exploring the opportunities to define variables and link these to the geometric representations on the screen, which resulted in his creation of a set of two activity resources that he called 'visual fractions'. In these he allowed the user to change five integer values that represented the 'divisor' (***d***) and the numerators and denominators of two fractions. Both fractions had to be less than one. Having input the five values, the user was expected to drag a point on a line segment from left to right, the outcome of which was to perform a fraction addition by appearing to pour the contents of the first fraction bar into the second one (Figure A6-2).

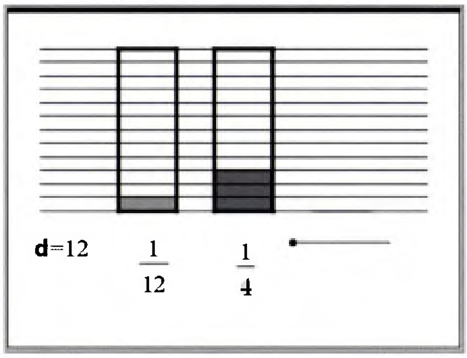


Figure A6-2 [STP2(tns-T)]

The instrument utilisation scheme that Tim developed was an enhanced version of IUS1, which I labelled IUS5 (Vary a numeric input and drag an object within a related mathematical environment and observe the resulting visual output). It differed in that Tim had introduced an additional requirement to select and drag a point fixed on a line segment from left to right, which served to enact the fraction calculation as a visual representation. Tim commented that he felt that this dragging added the dimension of time to the process.

Tim used this activity with the same year 8 class that had worked on the Factor activity. There was a particular subtlety relating to the invariant property that Tim was keen for the students to observe through this activity, which concerned the value the students assigned to ***d*** the divisor. He was hoping that they would notice that, when ***d*** bore a particular relationship to the denominators of the fractions, the positions of the horizontal lines of division made it possible for his students to predict and 'read off' the outcome of their fraction additions.

In designing this activity Tim was offering a dynamic visual representation of fraction addition that he hoped would provide a bridge to later pencil-and-paper methods. However, Tim also observed that not all of his students were receptive to this dynamic approach and noted,

> *Some students already had a strong mental image of fractions and/or were already capable of working with the addition and subtraction of*

fractions quite confidently. These students were reluctant to use the TI-nspire to explore fractions, seemingly because they did not want to have their understanding of fractions challenged. They made comments such as, 'It's easier without the TI-nspire,' or, 'I can do fractions already'. [STP2(Quest2)]

On the other hand Tim reported that there were some students for whom the activity using the technology had influenced their subsequent paper and pencil work when they were practising fraction addition calculations without access to the technology. Tim included the extract from one of his students' exercise books as evidence for this (Figure A6-3).

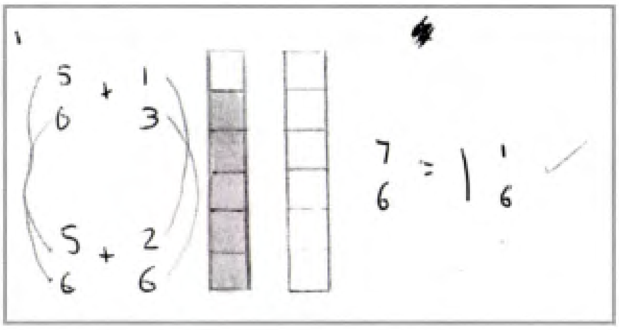


Figure A6-3 STP2(StudWork)

An interesting observation of this student's work is that, in addition to drawing the fraction bars, the student has omitted to write a conventional horizontal line between the numerator and denominator of all of the fractions. This may well be a result of the student's attention being focused on the importance of the values of the numerators and denominators within the technological setting and failing to notice the conventional syntax for fractions.

STP3 Angle bisectors

The third lesson that Tim submitted during Phase One was a geometric activity concerning the conventional Euclidian construction of an angle bisector and he created an approach that used only a geometric representation. He took a wide perspective on the mathematics concerning this construction and, rather than teaching his students a practical algorithm that would lead to a successful construction he maintained his desire to allow his students to explore the possible constructions with a view to them arriving at their own interpretation of the conditions needed for a successful angle bisector. Tim again chose to work with the same year 8 class as previously and he began the lesson by working with them to create a dynamic construction on the board using the TI-Nspire software displayed on the class interactive whiteboard (see Figure A6-4).

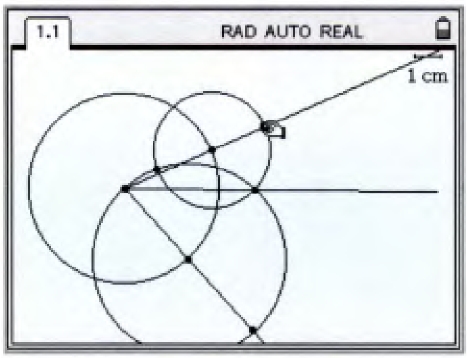


Figure A6-4 [STP3(tns-T)]

Tim used the instrument utilisation scheme that I classified as IUS3: vary the position of an object (by dragging) that has been constructed in accordance with a conventional mathematical constraint and observe the resulting changes. In his scheme the conventional mathematical constraint was the dynamic behaviour of circles of different radii that had been centred on the points of intersection of a circle constructed the vertex of two rays.

Tim justified his approach by saying 'rather than being given a set of instructions to follow, students made sense of a moving diagram in which the angle bisector exists as a special case. Students were able to understand what it is about the construction that makes it work' [STP3(Quest2)]. His deliberate 'misconstruction' seemed to motivate his students to want to know why it was that when the radii of the two variable circles were equal, an angle bisector was produced. Tim commented that as a result 'Students were able to articulate reasons for the symmetry of the construction and saw that the equal radii proved[2] that the construction was symmetrical' [STP3Quest4)]. However, Tim implied in his lesson evaluation that he still had doubts concerning some of his students' appreciation of the need for equal radii and commented that, if he were to repeat the activity he would probably change his approach thus,

> *In my initial whole class lesson working with the students on the construction of the angle bisector I would draw in some lines to measure the length of the radii to emphasise their equality. [STP3(Quest2)]*

This suggested adaptation resonates with Ruthven et al's (2004) research findings in which they observed that the teachers in their study used the dragging of dynamic variation to look at the effect on numeric measures. However Tim's suggested amendment seems to imply that he would use the numeric measurement to add weight to the students' conjecture that the radii were the

[2] Tim's emphasis.

same, rather than as the initial purpose for the process of dragging.

Tim reported that, for some students, their experiences with the technology in the angle bisectors activity bore a similarity to the fraction addition activity in that the representational form within the technological environment impacted upon their paper and pencil methods. He made the comment that 'Students created accurate angle bisectors, interestingly, they drew circles not arcs in their books' [STP3(Quest2)], which resonates with the example of student's work shown previously in Figure A6-3. In the two lessons that followed this activity, Tim reported that he gave his students the opportunity to practice constructing angle bisectors for themselves using the technology by creating their own angle fan. This required them to construct nested angle bisectors and hide the construction circles such that 'they were able to see if they were constructing correctly by trying to move the lines dynamically' [STP3Quest4)]. A successful example of such a construction is shown in Figure A6-5 and Figure A6-6.

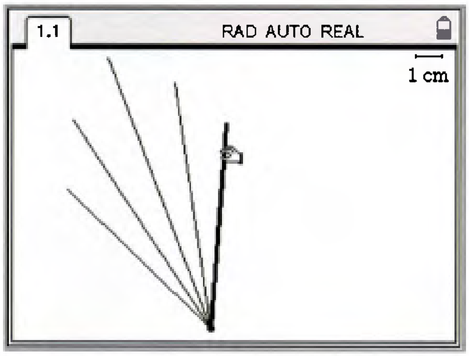


Figure A6-5 [STP3(tns-S)1]

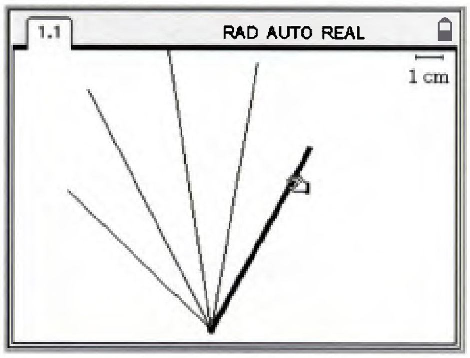


Figure A6-6 [STP3(tns-S)2]

Reflecting on Verillon and Rabardel's theory of instrumented activity, Tim's approach had clearly offered a 'new condition for organising the action' concerning the construction of an angle bisector that resonated with his preferred teaching approach and that offered his students an opportunity to arrive at their own justification for the validity of the method. The lessons did highlight somewhat predictable issues concerning students' instrumentations and Tim reported that they experienced difficulties in successfully placing new points onto objects and successfully locating points of intersection. However, the desire to produce a successful dynamic fan seemed to offer a genuine motivation to the students to overcome these issues.

STP4 Circle circumferences

Tim submitted his fourth activity seven months into the project and, yet again it used a different utilisation scheme, which I have already defined in Section 5.3.1 as

IUS3: *Vary the position of an object (by dragging) that has been constructed in accordance with a conventional mathematical constraint and observe the resulting* changes. *Use another representational form to add insight to or justify/prove any invariant properties*. This activity was developed for use with a year 9 (set 2) group working at around National Curriculum level 6.

Tim designed the activity 'Circle circumferences' in which his mathematical objective for the students was 'to introduce students to the relationship between the circumference and the diameter of a circle' [STP4(Quest2)]. In his lesson evaluation, Tim wrote,

> *I started with a lesson starter in which I gave students several questions of the form a÷8=20 (where 8 and 20 were replaced with different numbers) and asked them to find out the value of a. This led into some work with the class on inverse operations... and they were able to say that if a÷b=c then a=c x b. I did this because I was going to use the Nspires to establish that Circumference ÷ Diameter= π, and wanted them to be able to make the connection that Circumference=Diameter x π. [STP4(Quest2)].*

In the subsequent activity, which was mediated by the technology, Tim led the students through the construction of a circle and its diameter and how to measure the resulting circumference and diameter. He instructed his students to draw a table in their exercise books with three columns labelled 'circumference', 'diameter' and 'circumference ÷ diameter'. They were then shown how to drag the circle to vary the dimensions and each set of instantaneous results was recorded in the table in their exercise books. The students were then asked to move to the calculator application on the next page to divide the circumference by the diameter and also to record this result in the table (see Figure A6-7 and Figure A6-8).

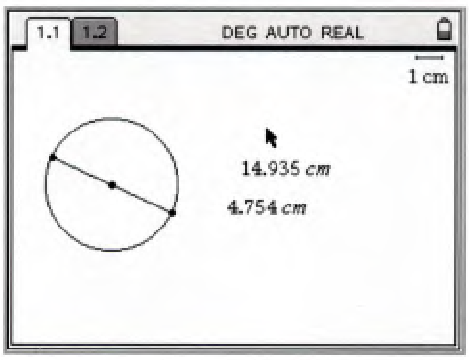


Figure A6-7 [STP4(tns-T)page1]

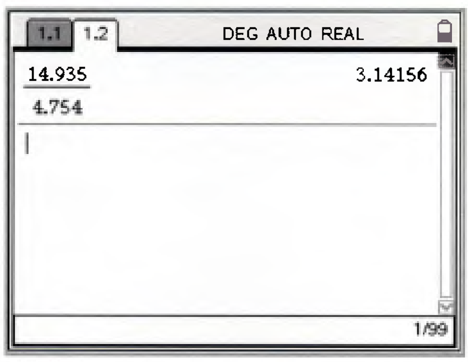


Figure A6-8 [STP4(tns-T)page2]

Tim concluded that,

> *In many ways this activity was quite 'controlled' by me to lead to the conclusion that I wanted them to achieve... however, the students felt in control because they could vary the size of the circle as they wanted – rather than me giving them a specific size of circle to use. [STP4(Quest2)]*

This indicated Tim's awareness of the need to focus his students' attentions towards the specific invariant property c ÷ d = π. At the final project conference, Tim selected this lesson as his most successful of the first phase, citing the reason, 'there was a definite 'aha' moment for the students when they realised that c/d always equalled 3.14159...' [STPQuest5)]. He also commented that 'constructing the circle itself gave students a 'sense of' how the construction was likely to behave when dragged' [STPQuest6)].

In Tim's evaluation he also suggested an alternative instrument utilisation scheme for this activity, 'If time allowed (and it is time consuming working with students to construct circle diameters) I would have linked the c and d variables to a spreadsheet and used an automatic data capture.' [STP(Quest2)]. That Tim was open to consider alternate utilisation schemes to approach the same mathematical content was an indication of his ability to consider the merits of different instrumentation approaches.

STP6 Enlargements

Nine months into the first phase Tim submitted his final activity, which he had developed for use with his year 8 group (set 2 of 7) who were working at National Curriculum level 6. The activity 'Enlargements' was designed in two parts and it spanned two fifty-five minute lessons. Tim only reported briefly on the outcomes of the first lesson, saying,

> *Use handhelds to enlarge shapes – students 'play around' with moving the centre of enlargement to see what happens to the image. Discussion about how the centre affects the position of the image. [STP5(Quest2)]*

During the second lesson, the students worked from a file that Tim had created and transferred to the handhelds before the lesson (see Figure A6-9 to Figure A6-11).

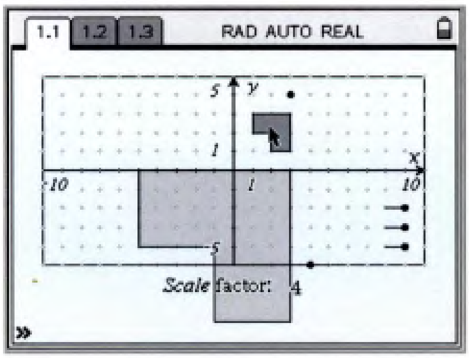


Figure A6-9 [STP5(tns-T)page1]

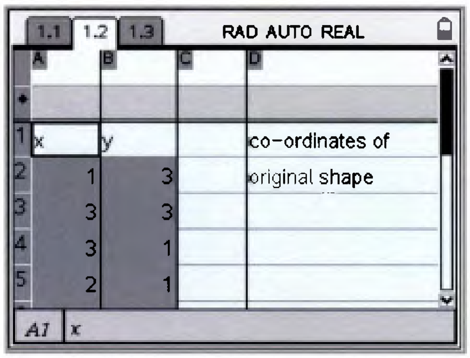


Figure A6-10 [STP5(tns-T)page2]

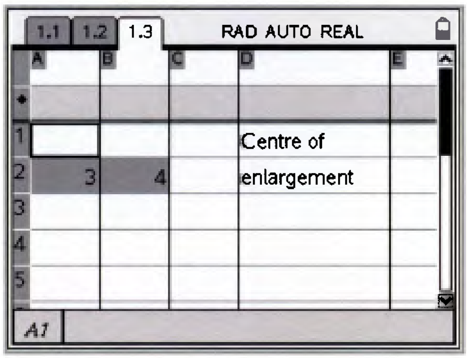


Figure A6-11 [STP5(tns-T)page3]

In this activity, there were many opportunities for variation. These included the position and appearance of the original shape, as determined by its coordinates in the second (spreadsheet) page of the file (Figure A6-10). Similarly the position of the image shape was determined by the values input for the 'Centre of enlargement' on the third page (Figure A6-11). The relative sizes of the image and enlargement were determined by numerically editing the scale factor on page 1 (Figure A6-11). Finally the three line segments on the first page acted as switches, enabling the user to switch on and off the appearance of the object, the image and the centre of enlargement. These switches had the effect of slowing down the activity in a similar way to Tim's previous use of a slider in the Fractions activity [STP2]. In addition these switches provided an opportunity for Tim to focus the students' minds towards particular aspects of variance and invariance by literally switching on and off their attentions. So it would be possible to pose questions such as 'how does changing the centre of enlargement affect the relative sizes of the shapes?'. Despite the complexities involved with this activity, it fitted the instrument utilisation classification **IUS4**: *Vary a numeric input and drag an object within a related mathematical environment and observe the resulting visual output*.

In his lesson evaluation, Tim concluded,

At the end of the first lesson, students were able to describe quite *accurately where the centre of enlargement would be. I found it helpful to have both the approaches of (a) an enlargement that students created themselves and (b) a prepared file where a slider controlled the enlargement as this gave students the opportunity to see the enlargement in small increments and help them create a mental picture of the process of enlargement and its relationship to the centre of enlargement. I was pleasantly surprised at how comfortable the students were at creating a shape and enlarging it. [STP4(Quest2)]*

This activity presented Tim with many choices with respect to how he mediated the students' activities. There was a sense of 'constrained freedom' in that there were many opportunities for exploration within an area of mathematics that is difficult to teach without the use of technology. The inclusion of the multiple representations of the object and centre of enlargement as coordinate values also added another dimension to the exploration.

APPENDIX 7: ANALYSIS OF ELEANOR'S PHASE ONE LESSON DATA

CEL1 Quadrilateral angles

Eleanor planned her first activity carefully 'Quadrilateral angles' [CEL1] and she used her school's standard lesson planning template in addition to the production of a set of Smart Notebook pages, which she used with the students to introduce the activity.

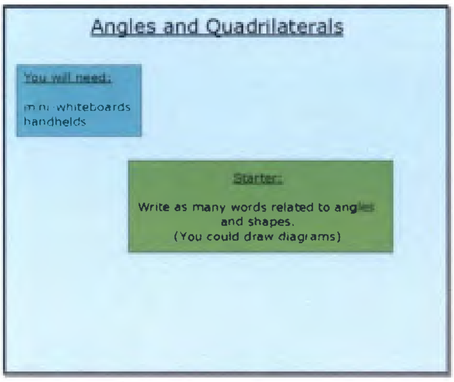
Lesson plan [CEL1(LessPlan)]:

LESSON PLAN

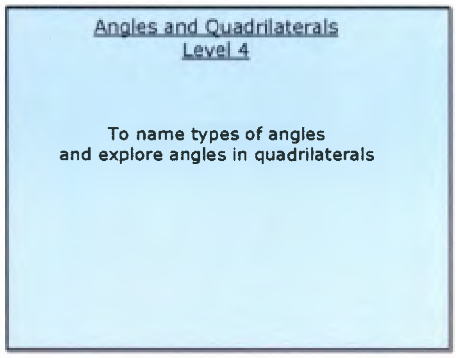
CLASS [redacted]	TOPIC Shape and space: angles	
DATE 25.07.09	LESSON 2	DAY Tuesday
Learning Objectives To name types of angles and explore angles in quadrilaterals		
Vocabulary Angles, acute, obtuse, right angles, quadrilaterals		
Starter First slide: In pairs write as many words related to angles and shapes on mini-whiteboards. Use diagrams if need be.		Resources Slide 1
Introduction Pull out ideas from class relating to acute, obtuse, and right angles. Use of conventions to label and measure angles.		Slide 2/3
Main Activity Use handheld to carry out a small activity of measuring angles on prepared quadrilaterals. Need to pass around the file from 1 handheld to another. Need to demo the use of the handheld to measure angles. Leave slide 7 up for instructions. NEW SKILL: LOADS OF RUNNING AROUND EXPECTED!		Slide 4-9
Differentiation		
SEN Pupils		
Plenary Review learning objectives have been achieved and share pupil successes. What have I learnt today? What skills have I learnt today? GAR		
Homework		
Assessment Individual and whole class questions and answers and work produced.		

CLASS [redacted]	TOPIC Shape and space: angles	
DATE 27.07.09	LESSON 2	DAY Thursday
Learning Objectives To name types of angles and explore angles in quadrilaterals		
Vocabulary Angles, acute, obtuse, right angles, quadrilaterals		
Starter Journal logs from last lesson		Resources Slide 2
Introduction Recap over any ideas of using the handheld from last lesson.		Slides
Main Activity Use handheld to carry out a small activity of measuring angles on prepared quadrilaterals. Need to demo the use of the handheld to measure angles. Leave slide 7 up for instructions. NEW SKILL: LOADS OF RUNNING AROUND EXPECTED! Last 15 mins need to get feedback from lessons for our report back.		Slides Form to fill in and journals to complete.
Differentiation		
SEN Pupils		
Plenary • Review learning objectives have been achieved and share pupil successes. • What have I learnt today? What skills have I learnt today? GAR		
Homework		
Assessment • Individual and whole class questions and answers and work produced. •		

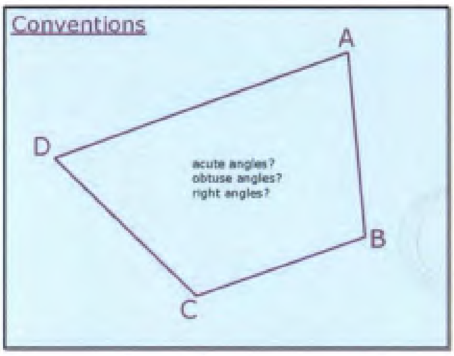
Smart Notebook Lesson Structure [CEL1 (LessStruct)]:



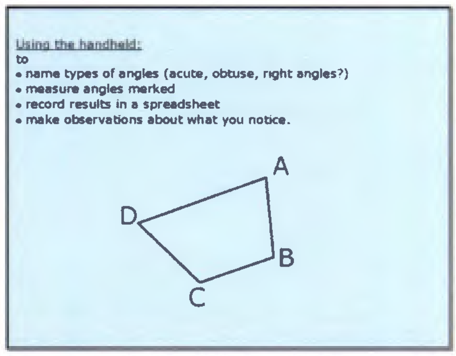
Sep 23-16:03



Sep 23-16:03

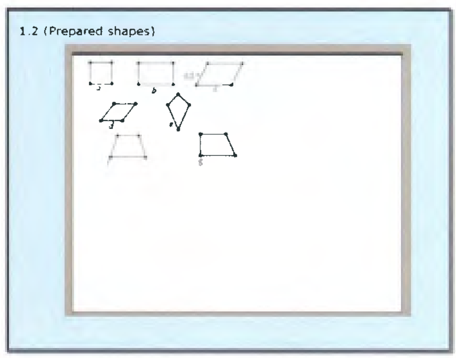


Sep 23-16:03



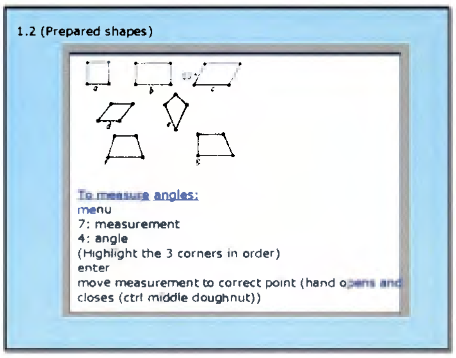
Sep 23-16:03

Sep 23-16:03

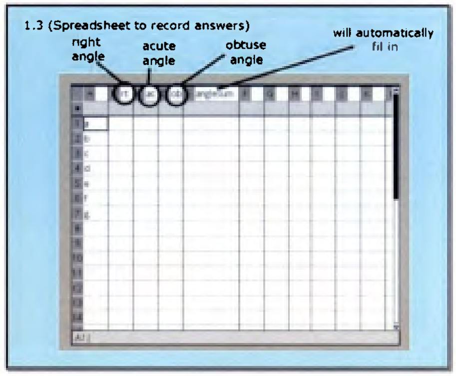


Sep 23-16:03

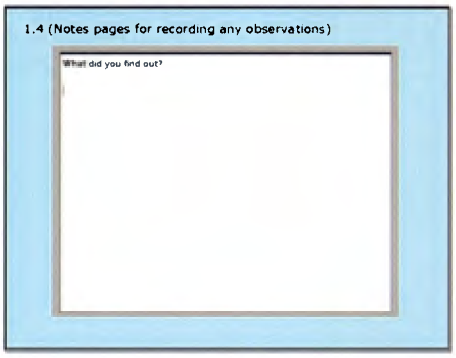
1



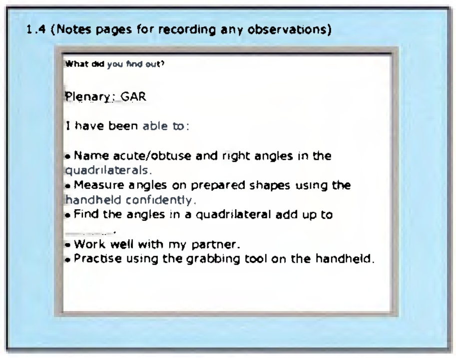
Sep 23-16:03



Sep 23-16:04



Sep 23-16:04



Sep 23-16:04

The activity was designed to span two one hour lessons which were a couple of days apart. In creating the Smart Notebook pages she had already worked out how to use the screen capture facility within the teacher's software to enable her to support the students' instrumentation phase by including screen shots of their handhelds. The TI-Nspire activity was one that Eleanor and Helena had requested from a member of the wider project team members at the first project conference as neither of them felt they had the TI-Nspire instrumentation skills needed to produce it themselves. They asked for a set of different quadrilaterals from which the students could note down the number of right, acute and obtuse angles, and then check by measuring them[1]. There was a column for angle sum, into which the students were expected to enter their calculated angle sum, although it was not clear from Eleanor's lesson plan exactly how the students would do this. For

[1] Eleanor did not appear to have noticed that the underlying construction of the set of quadrilaterals provided in this activity was such that they were geometrically square, rectangular, rhomboid etc. This had been done, partly to determine accurate angle measurements, but also to facilitate the opportunity for her students to observe the invariant properties of each type of quadrilateral by dragging.

example, whether they would be expected to: enter the calculation into the relevant spreadsheet cell using the = key; insert a calculator page and sum the four angles; add their four measurements using a paper and pencil method. However, as Eleanor submitted all of the students' .tns files as part of her project data, I was able to ascertain that none of the students progressed beyond measuring some of the interior for some of the shapes, so any activity or discussion relating to the resulting angles sums seems unlikely to have occurred during the lessons.

Eleanor's lesson plan indicated that her objectives for the first of the two lessons was to 'Pull out ideas from class relating to acute, obtuse, and right angles' and to focus on the 'Use of conventions to label and measure angles' [CEL1(Plan)]. Her lesson structure indicated that the students should first measure the angles in each quadrilateral (Figure A7-1) before recording the 'count' for each type of angle (right, acute and obtuse) into the spreadsheet (Figure A7-2).

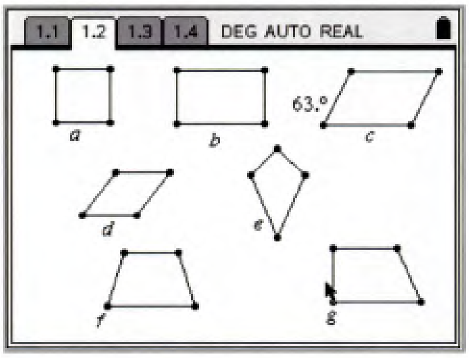


Figure A7-1 [CEL1(tns-T)page1]

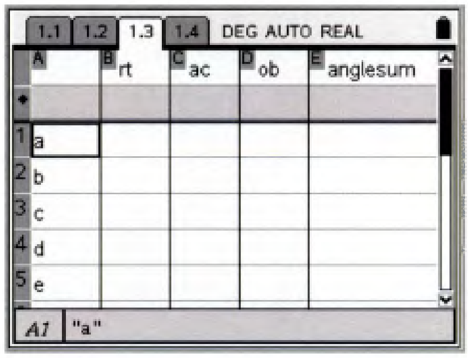


Figure A7-2 [CEL1(tns-T)page2]

Within Eleanor's lesson structure, she appears to have a misunderstanding about the inclusion of the spreadsheet column labelled 'anglesum' as she indicates to the students that it would 'automatically fill in'. As the students were actually entering the 'counts' for each type of angle in each quadrilateral in the three columns headed 'rt', 'ac' and 'ob' it is not clear how she envisaged that this would work (see Figure A7-3).

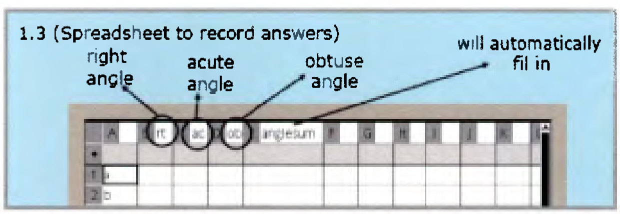


Figure A7-3 [CEL1(Struct)] Eleanor's supporting instructions for students

In Eleanor's lesson evaluation, she commented mainly on the students' instrumentation issues concerning the selection of points when measuring each of the angles and the subsequent placing of measured values of the page. She suggested how she would amend the activity for future use, saying she would 'Have bigger pictures drawn - even as one per page would aid students ability to measure more accurately and not end up moving shapes by mistake' [CEL1(Quest2)].

CEL2 Triangle angles

Eleanor's second activity was prepared for the same class of year 8 students three weeks after the first lesson. It followed a similar instrumentation scheme to the first lesson except this time the focus was on the angle properties of triangles and the emphasis was on measuring the interior angles of a given set of shapes (Figure A7-4) and summing these measurements within a spreadsheet (Figure A7-5).

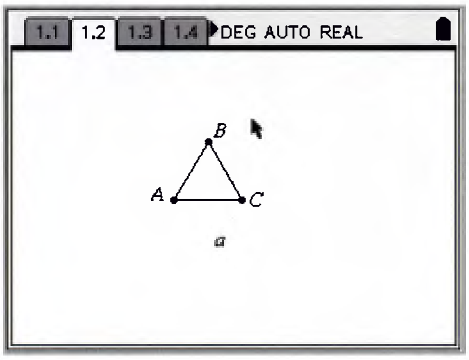


Figure A7-4 [CEL1(tns-T)page1]

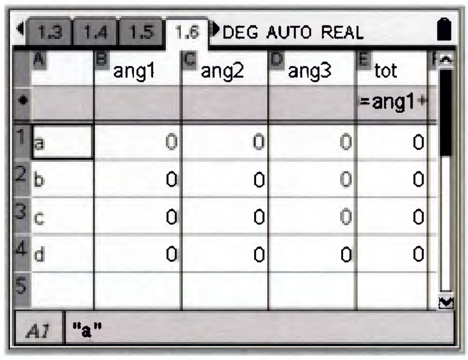


Figure A7-5 [CEL1(tns-T)page2]

Immediately it was obvious that Eleanor had learned from her experiences during the first lesson as she had adapted the approach and separated the individual shapes onto different pages. As with her first lesson she produced a detailed lesson plan [CEL2(Plan)] and accompanying Smart Notebook structure [CEL2(Struct)], which included the instrumentation steps needed for her students to remind them how to measure an angle (Figure A7-6).

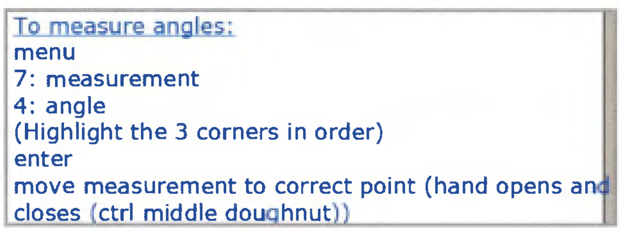


Figure A7-6 [CEL2(Struct)] Eleanor's activity instructions for students

An interesting observation of these instructions is the emergence of a classroom vocabulary concerning the instrumentation of the technology with 'control middle doughnut' being offered by Eleanor as a legitimate phrase concerning the selection of an object on the screen!

Both of Eleanor's first activities were classified as IUS2: *From a given set of static geometric objects, make measurements and tabulate data to explore variance and invariance within the measured data in numeric and tabular forms*. Whilst there was a progression in both the presentation of the activity and the level of mathematics with which the students had engaged, there did not appear to be an explicit emphasis on variance and invariance, although there were opportunities for this within both activities.

CEL3 Intersecting lines

Eleanor's third activity was a departure from her previous approaches when, still working with the same year 8 group of students, she used an activity that required them to drag objects within a geometric environment for the first time. It was now five months into the project and Helena and Eleanor had again requested a 'bespoke' design that would enable the students to explore the angle properties of intersecting lines [CEL3]. The resulting activity 'intersecting lines' maximised the functionalities afforded by the MRT in that it would use the measured values of the angles, rounded to the nearest degree to test a set of conjectures.

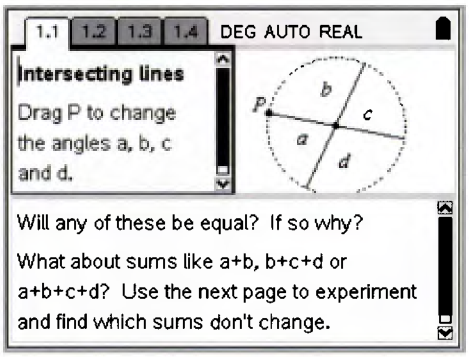


Figure A7-7 [CEL3(tns-T)page1]

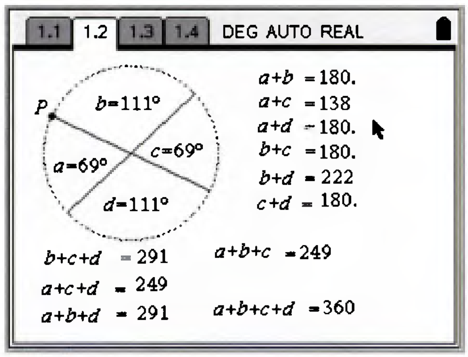


Figure A7-8 [CEL3(tns-T)page2]

This activity offered the students a single movable point, P that was constrained on the circumference of the circle (Figure A7-8). The four angles a, b, c and d had been pre-measured and stored as variables. All of the possible angle sums had been evaluated and were displayed on the handheld screen. Consequently, as the point P was dragged, each sum was recalculated.

Eleanor's written lesson reflection reported that the students had encountered many difficulties overcoming the instrumentation needed to be able to select and drag the point P. The technique required by the handheld was to use the doughnut key to move the cursor near to the point P. When the cursor arrow changed to a 'grab hand' the students had one of two instrumentation schemes to select the point: press the control key followed by the 'middle doughnut' to close the hand or simply to hold down the 'middle doughnut' key for a few seconds. Having grabbed the point the dragging was effected by holding down the up, down right or left doughnut keys to move the point in the desired direction.

Eleanor also commented that, 'the file was running very slow because their (sic) were too many inter-connecting calculations. We wanted to make it more open-ended.' [CEL3Quest3)]. Eleanor worked with another project team member to redesign the activity, which offered a much more open format (see Figure A7-9 to Figure A7.13). The angles were no longer saved as variables and the point was no longer constrained onto the circumference of a circle. Also, the numerous angle calculations on the screen had been removed, making the screen much less busy. When justifying these amendments, Eleanor reported, 'There needed to be a balance between having enough data evident for pupils to feel a sense of ownership of the activity through having a choice about what to look at and making it too overwhelming.' [CEL3(Quest2)]. These amendments gave an indication of her preferred approach to the topic for this particular group. It also indicated that Eleanor was shifting her instrumentation scheme towards IUS3: *Vary the position of an object (by dragging) that has been constructed in accordance with a conventional mathematical constraint and observe the resulting changes*.

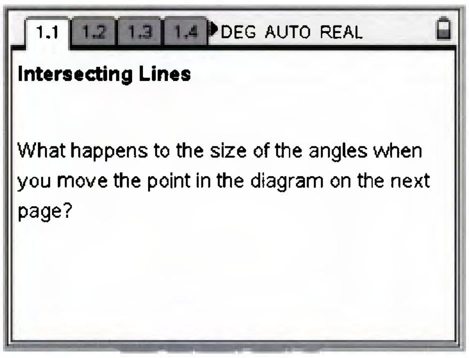


Figure A7-9 [CEL3(tns-T)v2page1]

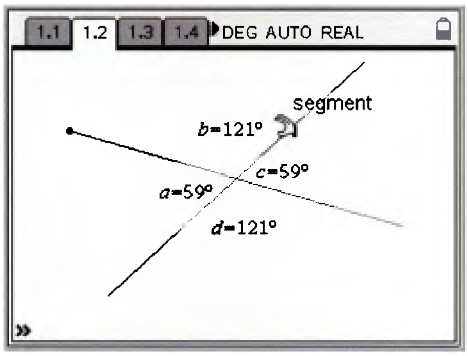


Figure A7-10 [CEL3(tns-T)v2page2]

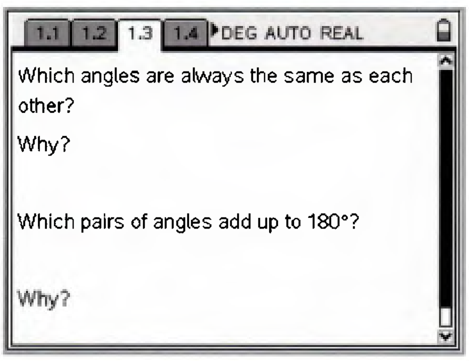


Figure A7-11 [CEL3(tns-T)v2page3]

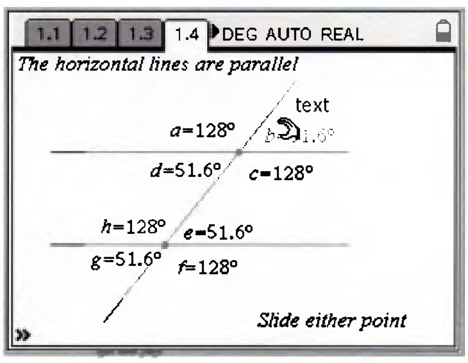


Figure A7-12 [CEL3(tns-T)v2page4]

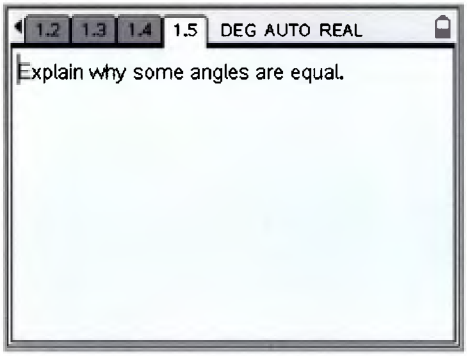


Figure A7.13 [CEL3(tns-T)v2page5]

CEL4 All angles recap

About a week after this lesson, Eleanor planned an activity for which she prepared the TI-Nspire file on her own for the first time. This was significant in Eleanor's development as she indicated that she now felt that she had sufficient understanding of the functionality of the Geometry application to confidently create her own activity from a new blank file. The resulting activity, which she called 'All angles recap' was designed to give this same year 8 class an opportunity to bring together their understanding of a number of the angle properties that they had been learning over the previous few weeks [CEL4]. There were four pages to the file and each page had been split with a dynamic geometric construction on the left hand side and a Notes page on the right hand side (see Figure A7-14 to Figure A7-17).

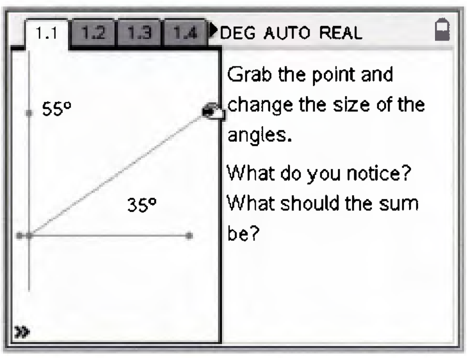


Figure A7-14 [CEL4(tns-T)page1]

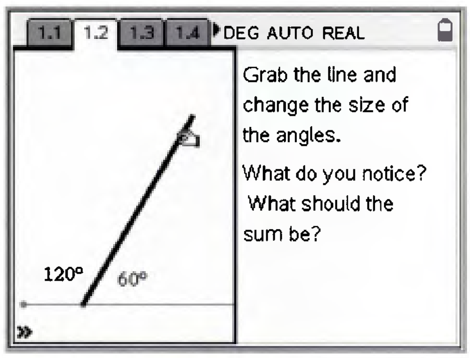


Figure A7-15 [CEL4(tns-T)page2]

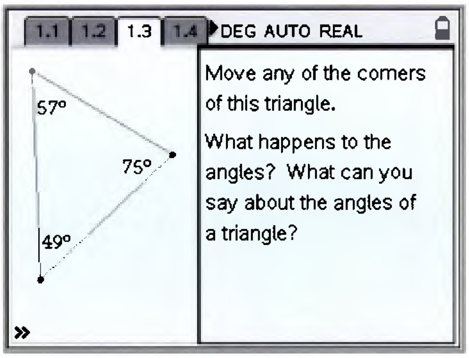


Figure A7-16 [CEL4(tns-T)page3]

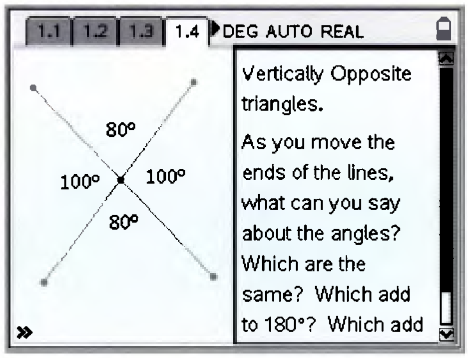


Figure A7-17[CEL4(tns-T)page4]

What was evident from this activity was that Eleanor now saw dragging as the main student activity and, by inviting the students to write their own comments on the notes pages, she was encouraging the students to share their responses. Each page offered a different geometric environment and required the students to explore each situation moving points and reporting their observations and conjectures. Eleanor had also pre-measured the angles, which she had initially dragged such that there were 'nice numbers' on display. Eleanor reported all of the students' .tns files to the project, and the students' screens shown in Figure A7-18 and Figure A7-19 were selected by Eleanor as evidence of her perception of a successful outcome for these students in the activities. In these files, there was evidence that the student had engaged in dragging and had been prepared to note her observations. In response to the activity on the first page (Figure A7-18) the student uses the expression '...even when you do something like...' [CEL4(tns-S4)page1], which offers an insight into the student's thinking in that she was beginning to generalise her findings.

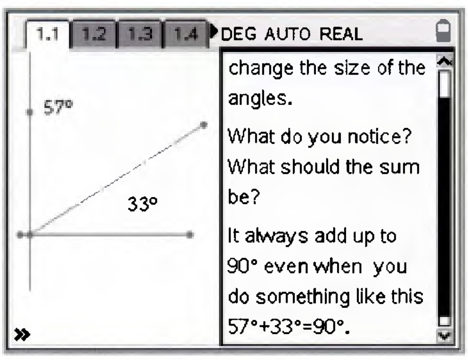


Figure A7-18 [CEL4(tns-S4)page1]

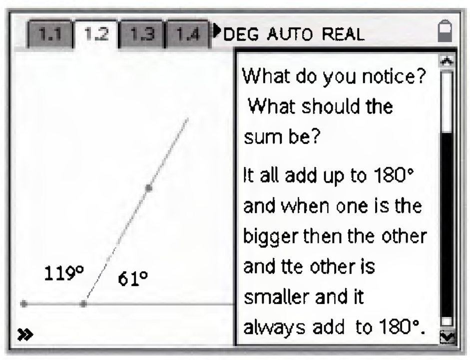


Figure A7-19 [CEL4(tns-S4)page2]

However, a contrasting response from another student evidenced the fact that not all of the students did drag any of the points or lines, however still recorded their observations which could be interpreted as correct. In Figure A7-21, it may well be that the 'nice numbers' that Eleanor presented within the activity acted to distract the students if they had 'spotted' the connection with 180° and therefore saw no need to drag anything on the diagram.

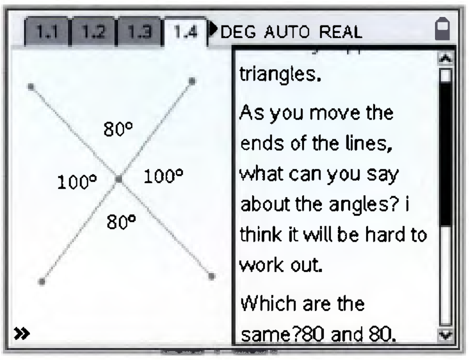


Figure A7-20 [CEL4(tns-S2)page4]

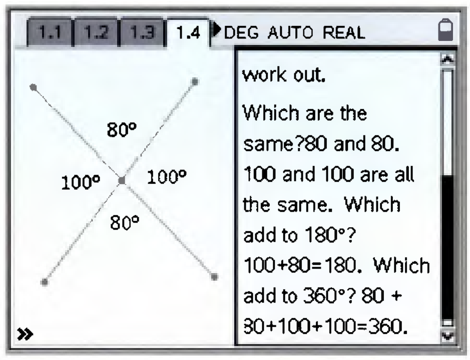


Figure A7-21[CEL4(tns-S2)page4] contd.

During the first phase Eleanor was invited by the SSAT to contribute an article to their monthly 'Maths and Computing Update' that was sent to all specialist schools. In her article she wrote about this lesson, and noted that 'my bottom set [*of students*] were beginning to use generalisations!' [CEL4(Journ-T)].

CEL5 Perpendicular functions

The final activity that Eleanor reported to the first phase of the project was an obvious development from her previous approaches. She had already commented informally that she had felt mathematically constrained by her decision to focus her project activity on her lower ability year eight group. The project meetings had led to a number of activities within other areas of the curriculum being shared and discussed by the teachers and Eleanor took up an idea for an activity that Sophie had shared at the third project meeting in January 2008. (Sophie's activity, 'Perpendicular lines' [PSH4] was described in detail in Section 5.3.1 on page 118).

However, Eleanor 'misremembered' Sophie's instrumentation of the activity and as a result, devised a very similar activity, which also contained some subtle differences. In Sophie's activity, all of the lines were generated by defining linear functions. When Eleanor came to recreate the activity, although she generated the initial line by means of a linear function, Eleanor then asked her students to add a geometric line to the figure that they judged to be perpendicular to the function according to their 'eye'. Eleanor described how the activity progressed by, as Sophie had done, measuring the angle between the two lines and dragging the geometric line such that the measured angle was approximately 90°. An example of this is shown in Figure A7-22.

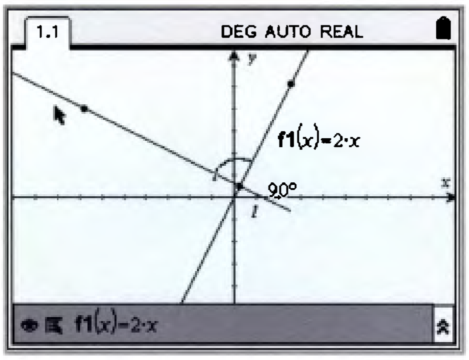


Figure A7-22 [CEL5(tns-T)]

Eleanor was using this activity with an older, more mathematically able class, a Year 10 set who were working towards the higher tier GCSE examination and she judged their current attainment to be at about National Curriculum level 7. She had identified the lesson goals saying, 'I wanted to enable them to find their own relationships between gradients of linear graphs, parallel lines and ultimately perpendicular lines' [CEL5(Quest2)].

Eleanor then developed the activity by asking the students to generate further linear functions that were parallel to the original function, using the measured geometric line as a guide. Eleanor was using the technology to try many examples quickly so as to provide immediate feedback to the students. Eleanor supported this by saying,

> *The time saved in not having to draw out any graphs is the main factor when using this technology. Trial and improving different functions enables girls to experiment and by establishing their own rules/findings they have a better understanding of what the conventions are. [CEL5(Quest2)]*

The students' work that Eleanor submitted in support of her evaluation of this lesson showed a diversity of responses. The most apparent observation of their responses was that the 'foci' of their individual explorations were more diverse than in Eleanor's previous lessons (see Figure A7-23 to Figure A7-27). This may be because she felt that this more able group would be more capable of independent exploration. However her evaluation did suggest that the majority of the students' difficulties still related to aspects of the instrumentation.

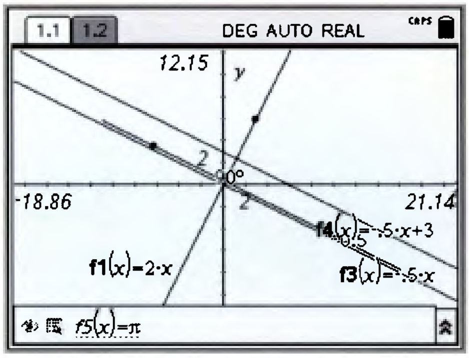


Figure A7-23 [CEL5(tns-S1)]

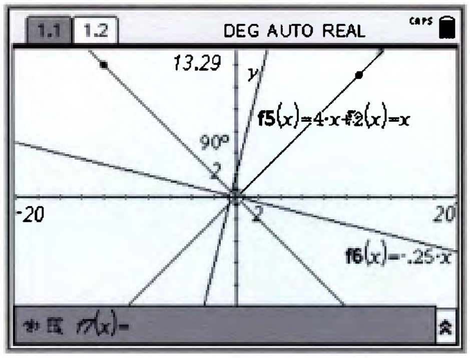


Figure A7-24 [CEL5(tns-S2)]

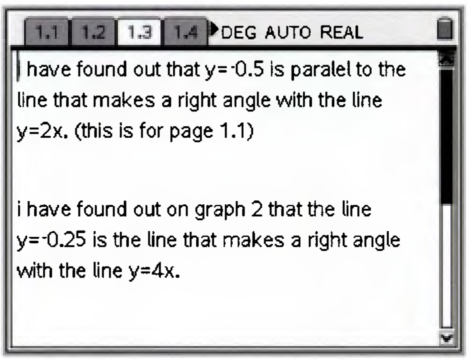


Figure A7-25 [CEL5(tns-S5)]

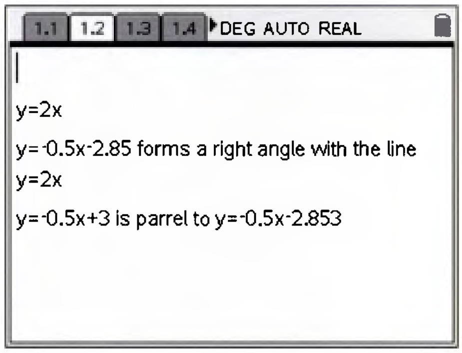


Figure A7-26 [CEL5(tns-S6)]

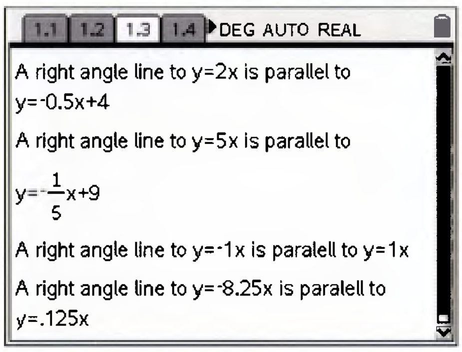


Figure A7-27 [CEL5(tns-S7)]

Eleanor made some overall comments in response in which included:

- the students seemed 'au fais (sic) with many graphs on one page';
- there were lots of 'comments and mistakes and not quite rights';
- the students had been 'looking at parallel as well as perpendicular' [CEL5(Quest2)].

The instrument utilisation scheme that Eleanor developed for this activity was unique within the first phase of the project. I classified it as IUS7: *Construct a graphical and geometric scenario and then vary the position of geometric objects by* *dragging to satisfy a specified mathematical condition. Input functions syntactically to observe invariant properties.*

When Eleanor spoke of this lesson, she expressed surprise by the number of parallel linear functions the students seemed to need to input in order to be convinced that the gradient properties they had focussed upon held up to their scrutiny (see Figure A7-28 and Figure A7-29).

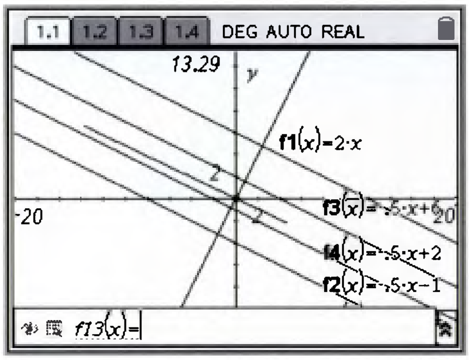


Figure A7-28 [CEL5(tns-S3)]

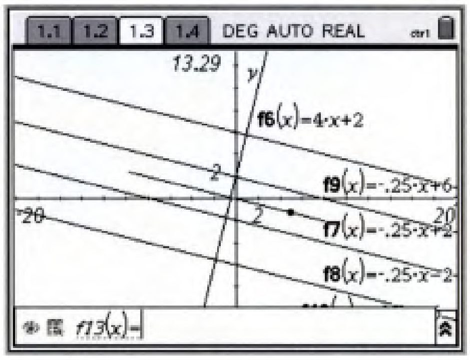


Figure A7-29 [CEL5(tns-S4)]

By generating many perpendicular functions, Eleanor said that she felt that some students had reached a deeper understanding of the generalisation in that by seeing that all lines parallel to y = mx+c would have a gradient of -1/m. In this respect Eleanor's accidental amendments to Sophie's original instrumentation scheme had led Eleanor to an unanticipated and surprising lesson outcome.

In her overall evaluations Eleanor made several comments relating to how she would amend her instrument utilisation scheme if she were to develop this activity further, which were:

- To introduce a spreadsheet page to establish that the product of perpendicular gradients is -1;
- Use of slope measurement;
- Ensuring all hand-held are in degree mode and all with a float of 3. [CEL5(Quest2)]

This indicates that she felt it important to bring in another representation in which the students could collate results and check their conjectures. Her appreciation that the slopes of the functions could be measured, saved as variables and hence 'captured' would offer an additional environment for the students' explorations.

Eleanor also considered how she would refine aspects of the her original teaching and learning approach by stating that her revised intentions would be: 'to reinforce looking for connections between perpendicular gradients as written for the value of m; to encourage more formal methods of recording results' [CEL5(Quest2)]. This lesson indicated an acceleration point in Eleanor's development and it appeared that she had now taken personal ownership of the technology. She had overcome her initial instrumentation difficulties and was now embarking on activity designs that fitted with her personal view of what constituted valid mathematical activity for her students in accordance with their ages and abilities.

APPENDIX 8: SUMMARY OF TIM'S PHASE TWO DATA

STP6 Pythagoras theorem

Data	**Description**
[Journ]	Lesson observation notes.
[STP6(tns-T)]	Tim's tns file.
[STP6(Quest2)]	Tim's written lesson evaluation.
[STP6(Trans)]	Lesson transcript.
[STP6(Eval-S)]	Students' written evaluations.
[STP6(ScreenCapt-Activity1)]	Digital screen capture of displayed handheld screens.
[STP6(Video-Activity1)]	Video sequence of Tim's introduction to activity 1.
[STP6(Video-Activity2)]	Video sequence of Tim's introduction to activity 2.
[STP6(Journ-T)]	Tim's written lesson reflection.
[STP6(tns-S)]	Full set of students' tns files
[STP6(tns-Tv2)]	Tim's tns file (revised after the lesson).

STP7 Circles and lines

Data	**Description**
[Journ]	Email exchange with Tim prior to lesson. Lesson observation notes (lessons 1 and 2).
[STP7(Trans)L1]	Transcript of lesson 1.
[STP7(Image1)L1]	Image of Tim's computer screen capture of the quick poll responses (lesson 1).
[STP7(Video)L1]	Video of Tim's introduction to the main activity (lesson 1).
[STP7(tns-S)L1]	Folder of students' .tns files captured at the end of lesson 1.
[STP7(Image1a)L1]	Image of Tim's handwritten whiteboard work – *$\sqrt{25}$ and $\sqrt{-25}$.*
[STP7(Image1b)L1]	Image of Tim's handwritten whiteboard work – *substituting into $x^2 + y^2 = 25$.*

Data	Description
[STP7(Int-T)L1]	Transcript of post-lesson interview with Tim, which included the planning discussion for Lesson 2.
[STP7(Trans)L2]	Complete transcript of lesson 2.
[STP7(tns-S)L2a]	Folder of students' .tns files captured 25 minutes into the 55 minute lesson.
[STP7(tns-S)L2b]	Folder of students' .tns files captured at the end of the lesson.
[STP7(Image1)L2]	Image of Tim's handwritten whiteboard work – *'proving the points of intersection'*
[STP7(Journ-T)]	Tim's written lesson reflection.

STP8 Quadratic curves

Data	Description
[Journ]	Email exchange with Tim prior to lesson. Lesson observation notes.
[STP8(Int-T)]	Pre-lesson discussion with Tim.
[STP8(Trans)]	Transcript of lesson.
[STP8(tns-S)1]	Folder of students' .tns files captured 55 minutes into the 55 minute lesson.
[STP8(tns-S)2]	Folder of students' .tns files captured at the end of the lesson.

STP9 Equivalent quadratics

Data	Description
[Journ]	Email exchange with Tim prior to lesson. Lesson observation notes (all lessons).
[STP9(Int)L1]	Pre-lesson discussion Post lesson discussion
[STP9(tns-T)L1]	Tim's tns file.
[STP9(Activity)L1]	Students' activity sheet
[STP9(Trans)L1]	Transcript of lesson 1.
[STP9(Video)L1]	Video of Tim's whole class plenary discussions.
[STP9(StudWork)L1]	Paper copies of students' written work
[STP9(Trans)L2]	Transcript of lesson 2.

Data	**Description**
[STP9(Activity)L2]	Students' activity sheet
[STP9(StudWork)L2]	Paper copies of students' written work
[STP9(Int)L3]	Pre-lesson discussion Post lesson discussion
[STP9(tns-T)L3]	Tim's tns file.
[STP9(Activity)L3]	Students' activity sheet
[STP9(Trans)L3]	Transcript of lesson 3.
[STP9(Video)L3]	Video of Tim's whole class plenary discussions.
[STP9(tns-S)L3]	Set of class files *brackets1.tns* – collected midway during lesson 3
[STP9(tns-S)L3a]	Set of class files *brackets2.tns* – collected at end of lesson 3
[STP9(StudWork)L3]	Paper copies of students' written work

STP10 Linear graphs

Data	**Description**
[STP10(Int-T)]	Pre-and post-lesson discussions with Tim
[STP10(Trans)]	Transcript of lesson
[Journ]	Lesson observation notes.
[STP10(Video)]	Video of Tim's final whole class teaching episode.
[STP10(tns-T)1]	Tim's .tns file to introduce activity 1.
[STP10(tns-T)1a]	Tim's .tns file displayed to students at the end activity 1.
[STP10(tns-T)2]	Tim's .tns file to introduce activity 2.

APPENDIX 9: DETAILED DESCRIPTIONS AND ANALYSES OF TIM'S PHASE TWO ACTIVITIES

STP6 Pythagoras exploration

The detailed description of this lesson is provided within the main thesis (Section 6.2.2.1).

Instrument utilisation scheme: The IUS developed by Tim is shown in Figure A9-1. The three tasks which comprised the activity were each mediated by a geometric representation and the students were required to drag points and observe the resulting syntactic and numeric (measured and calculated) representations. This IUS seems consistent with IUS3, as previously defined in Section 5.3.1.

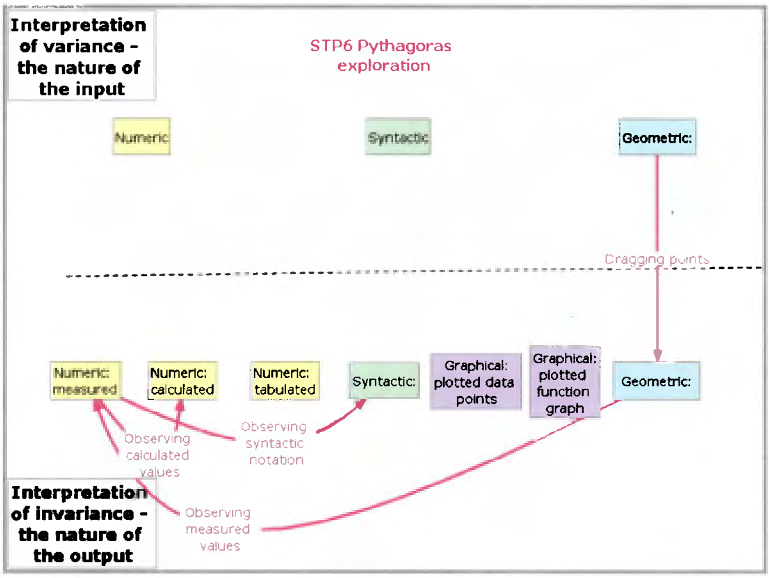


Figure A9-1 IUS [STP6] Pythagoras exploration

The students' main interaction with the MRT was by dragging and following this, they were required to closely observe the resulting changes in the geometric orientation of the diagram, its geometric properties and the changes in the measured and calculated values. All of these are considered to be dynamic objects. Consequently, whilst engaging with the second (and main) task, the students' attentions were being drawn to four different dynamic objects within the representation.

Hiccups identified from the lesson data: There were five broad classifications of hiccups identified from the lesson data as shown in Figure A9-2.

Name	Sources	References
TP6 Hiccup1 - Difficulties over identification of dynamic objects	1	2
TP6 Hiccup2 - Students' mis-interpretations of task - 'different way around'	3	4
TP6 Hiccup3 - Instrumentation (T) 'your c has gone off the screen'	1	1
TP6 Hiccup4 - Instrumentation (S) - grabbing and dragging	1	1
TP6 Hiccup5 - Jump from MRT task to trad paper and pencil problem	1	1

Figure A9-2 [STP6] Activity hiccups

These categories were more fully described as:

Instrumentation issues experienced by the students, which concerned grabbing and dragging on-screen objects and dealing with displayed measured values that 'disappeared' from the screen as other objects were dragged.

Difficulties concerning the whole class discourse as a result of a lack of clarity in the naming and referencing of the multiple dynamic objects under discussion.

Unexpected student responses as a result of their different interpretations of the instructions that Tim provided for the second activity. These included: vertices being dragged so that the squares were displayed on the 'interior' of the triangle's sides; reproducing the first activity (all areas equal) and producing the 'correct' right angled triangle but ignoring the calculated values.

Possible evidence of situated learning: Evidence from the various data sources led to a list of eight actions by Tim that might provide evidence of his situated learning during this activity. These are shown in Figure A9-3.

Name	Sources	References
TP6 Action1 - Appreciated that further bridging needed from the dynamic exploration to the	1	1
TP6 Action2 - Appreciated the range of strategies the students come up with	3	6
TP6 Action3 - Appreciated the task had 'Too much possible variation' for the students' comf	1	1
TP6 Action4 - Commented that he liked the 'Empirical sense of generality' that the approach	1	1
TP6 Action5 - Redesigned the task to respond to hiccup	2	4
TP6 Action6 - Suggested the advantage that colour coding would bring when grouping the s	0	0
TP6 Action7 - Appreciated that the progression to the trad problems was not quite right	1	1
TP6 Action8 - Noticing students' interpretations of the generality	1	1

Figure A9-3 [STP6] Evidence of teacher's actions

Tim commented on the nature of the dynamic variation within the activity and in particular what constituted 'too much variation' for the students to be able to draw purposeful mathematical conclusions [STP6(Trans)]. Tim spoke of the lack of an 'aha' moment when the students realised that a right angled triangle was formed as an outcome of the second activity. He reflected on this point thus:

> *I think this was to do with the difficulty in getting the areas to 'exactly' match. I think that this was a problem with the activity – and one possible improvement would be to construct the activity so that the vertices of the triangle 'snapped' to a grid. Another possibility would be*

to create a message that appeared when a+b was quite close to c, if not 'equal' to c. [Journ]

The occurrence of Hiccup2 provided evidence for Tim's heightened awareness of his students' thinking. In this example, a pair of students had continued to work on the first activity (to produce three squares with equal areas), but within the environment for Activity 2, producing the following screen. Tim selected this screen during a period of whole class discourse when discussing Activity 2 (Figure A9-4).

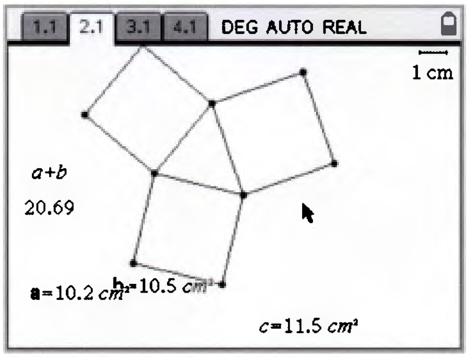


Figure A9-4 [STP6(tns-s19)] Student's response

TP: Let's look at the ones that don't work - let's go back to the bottom ones that don't work, like that one for example, now that one is clearly not a right angled triangle is it?

Student: It's equilateral

TP: It's just about an equilateral triangle and we've got a add b is about twenty one, a is ten, b is ten, so that's about twenty one isn't it? Ah! But the c is eleven point five, so they're all about the same area aren't they - a is about ten, b is about ten and c is about eleven, twelve.

In Tim's personal notes that he made after the lesson, he commented about his students' responses to Activity 1 (to form squares of equal areas) saying,

One student overlaid all three squares to accomplish this. This was an 'interesting' way of solving the problem and I was able to comment on this. A couple of other students formed an equilateral triangle. [STP6(Journ-T)]

This seemed to indicate that Tim had appreciated that these students had not actually progressed to the second activity, and found an opportunity to value effort, even though it was not seen as a successful outcome for the lesson. One of the other students in the class commented in his evaluation of the lesson about this particular screen, saying that 'I learned that to get 3 squares equal size you have to have an equilateral triangle in the middle' STP6(Eval-s22).

Alongside this, Tim seemed to have recognised the diversity of the students' abilities to articulate the generalisation on which the second (and main) activity had focussed.

- An understanding of the issues within the original activity design that had led students to misunderstand the activity objectives.
- A greater understanding of the possible student progression from the dynamic exploration to more traditional paper and pencil strategies when solving Pythagoras problems.
- The difficulties experienced by himself and the students when trying to clearly identify the mathematical objects under discussion. Tim's redesign of the activity showed an awareness of the need to label and colour code particular objects more obviously.
- His developing thinking about the role that the shared class view of the handheld screens might have within the design of mathematical activities. For example, Tim commented:

> *Students are naturally interested in their own screen when all screens are displayed. In working with all screens in a 'shared space', I am more interested in the 'whole'... the 'big picture'. I think this can affect the motivation and interest of individual students. If I happen to be commending certain screens, those students feel pleased. If I am moving screens to the bottom because they have not managed to make (in this instance) a+b=c, then those children can become less engaged. These issues are worth considering in future activity design, i.e. How can students be kept interested in the 'whole picture' when their individual screen is only a small part of that 'whole picture'? [STP6(Quest2)]*

In addition, Tim suggested developments to the classroom display technology that would support the grouping and sorting of the students' handheld screens. In overall conclusion, Tim said that he liked the 'empirical sense of generality' that this activity approach had afforded.

STP7 Circles and lines

A series of tasks offered over a sequence of two lessons to a year 11 (set 1) group of twenty nine students working towards the GCSE higher tier examination. The two lessons were conducted on the same day.

Tim reported that, in the preceding lesson he and the class had,

> *explored the equation of a circle by drawing circles centred on the origin and then getting the Nspire to find the equation of the circle.... Students were generally happy by the end of the lesson that x^2+y^2 = r^2, where r is the radius of the circle. They didn't have a lesson today (another exam)... but I was thinking that I would like to look at finding points of intersection between a straight line and a circle. Am struggling at the moment to find a way to use the Nspire for this other than to 'check' answers using the co-ordinate tool... any ideas? [Journ].*

I responded with the following 'ideas'.

> *What is the 'big picture' here? Ultimately I expect it is to solve GCSE exam style questions.. but en route is it to?*
>
> *- appreciate when the line will cross the circle, 0, 1 2 times (I can see everyone with a common circle and then each student adding a chosen straight line to this...)*
>
> *- getting a sense of where the line and circle might cross - i.e. Can they get it (by dragging the line) to cross in a given quadrant too?*
>
> *- exploring the algebra to arrive at how you would prove that the line and circle crossed at a certain place... You could approach this backwards by giving the students (an) incorrect point(s) of intersection and proving it doesn't by substitution...*
>
> *- leading into finding algebraically where the line crosses - several approaches here - do you lead the class through one or allow some to come up with this for themselves (with some clues)*
>
> *- offering an extension activity of 'finding formulae' for x and y in terms of m and c where y = mx+c crosses y^2+x^2=r^2*
>
> *- identifying the 'conditions' for the circle and line to intersect 0, 1 and 2 times (extension)* [Journ]

Lesson one: In the first of the observed lessons Tim began by posing two questions 'what's the square root of twenty-five?' and 'what's the square root of minus twenty-five?'. The questions promoted much discussion between students and Tim conducted a poll to ascertain the students' responses, the results of which are shown in Figure A9-5. Tim did not resolve the students' conflicting ideas, but told them that they would come back to this problem later in the lesson.

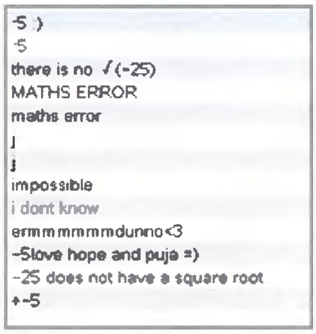


Figure A9-5 [STP7] Students' responses to 'What is √(-25)?'

Following this initial activity, Tim opened a blank page within the Graph and Geometry application and constructed a geometric circle with its centre positioned at the origin and its circumference at the coordinate (0, 5). He then 'measured' its equation (displayed by the MRT as $x^2+y^2=5^2$) and sent the resulting file to the students' handhelds (Figure A9-6).

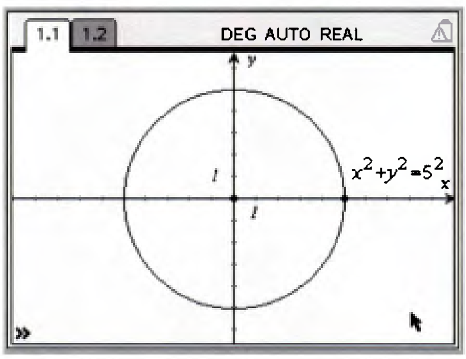


Figure A9-6 [STP7(tns-T)]

Tim asked the students to draw a geometric line anywhere on the page. He then displayed their handheld screens publicly in the classroom, revealing a diversity of responses which included the screens shown in Figure A9-7 to Figure A9-10.

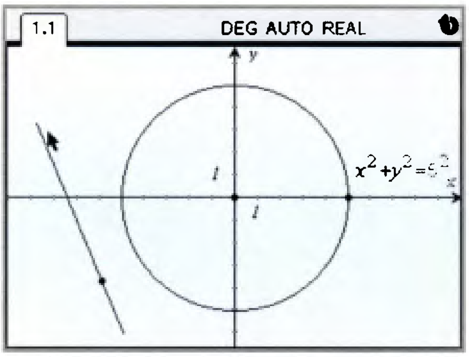


Figure A9-7 [STP7(tns-S1)]

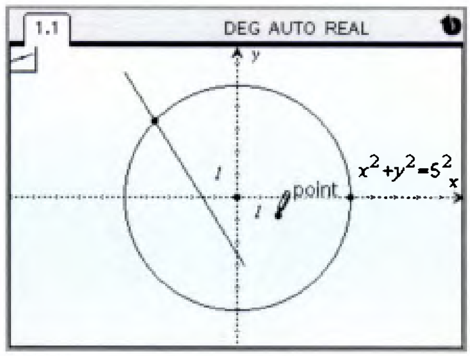


Figure A9-8 [STP7(tns-S2)]

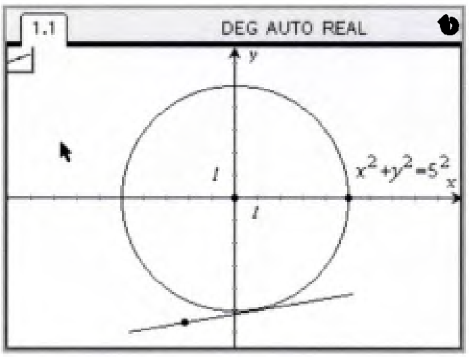


Figure A9-9 [STP7(tns-S3)]

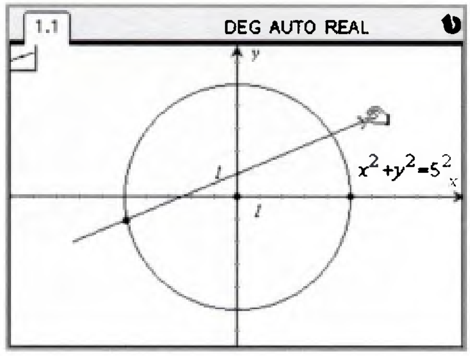


Figure A9-10 [STP7(tns-S4)]

This led to a period of classroom discourse concerning the number of possible points of intersection between a circle and a straight line. The limitations of the technology led to some unanticipated screens (for example, Figure A9-8). Tim included the phrase 'if the line was to continue' to encourage the students to imagine that their lines were continuous when considering the number of points of intersection in each case.

In the final phase of his whole-class introduction, Tim added a second line (y=4) to the diagram by typing y=4 as a text object and dragging it to the x-axis. He sent this file to the students' handhelds (see Figure A9-11).

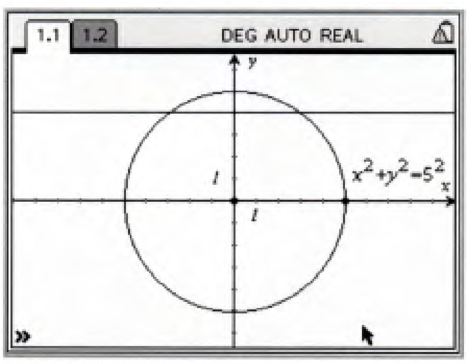
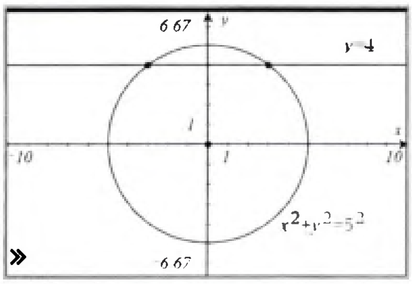


Figure A9-11 [STP7(tns-T)]

Tim asked the students to try to work out 'where it crosses' and also said 'I'm hoping that some algebra takes place at some point' [STP7(Trans)]. He distributed the activity sheet (Figure A9-12), which outlined 'Challenge one'.

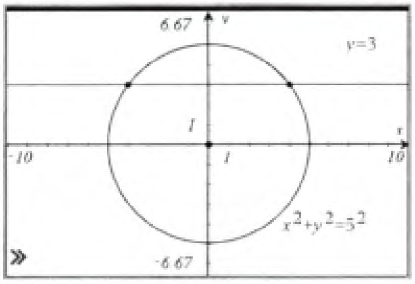
Circles and Lines... Challenge One



OK so here's the first challenge:

The straight line crosses the circle at two points.

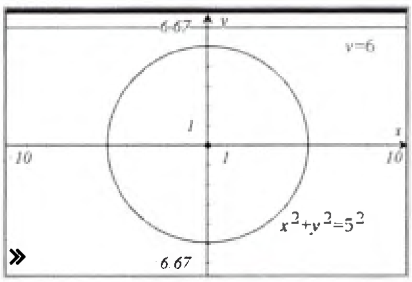
What are the co-ordinates of the two points?



Now the line has changed to y=3.

What are the co-ordinates of the intersection points now?

What of y=2, y=-2... is there a general rule?



Hopefully you will by now have a method (using algebra) to help you find the points of intersection.

Have a look at this situation...
Do the line and circle intersect?
What happens if you try your algebra method on this one?

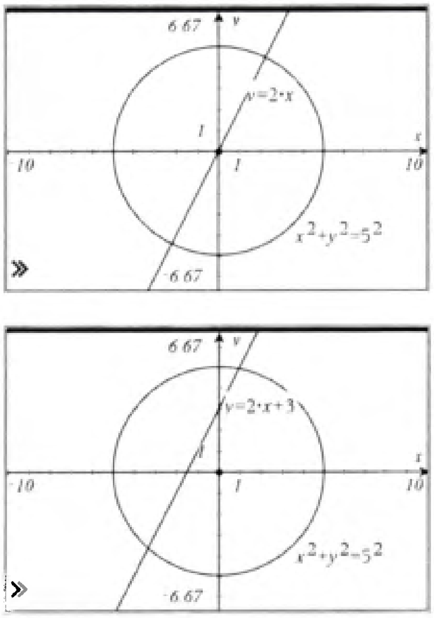
Figure A9-12 [STP7(Activity1)]

The students worked in groups of two to four and recorded their work on large A1 paper. They were encouraged to annotate the worksheet and record their written strategies. Tim concluded the first lesson by working through the substitution of y=4 into $x^2+y^2=5^2$ on the class whiteboard, which resulted in the class appreciating that the coordinates of the points of intersection would be (3,4) and (-3,4).

Lesson two: The second lesson was a direct continuation of the first and it took place later the same day. Tim redistributed the handhelds, the students' written work and the first activity sheet containing 'Challenge one' [STP7(Activity)1]. The students were asked to continue working independently in the same groups. Throughout this phase the students had access to the MRT, although they were not instructed to use it in any particular way.

Towards the end of the second lesson a group of four students had completed the first challenge. Tim encouraged them to look at the second activity sheet (Figure A9-13). As before, the students had access to the technology.

Circles and Lines... Challenges Two and Three



Here is another line that intersects the circle.

The equation this time is y=2x.

Try to find the points of intersection now.

What is different about this line?

Try to find the points of intersection.

Do you have to change your approach? How?

Figure A9-13 [STP7(Activity2)]

Tim continued to move around the classroom, supporting small groups and individual students, often in response to their questions, but otherwise to ask them how they were progressing and what they were finding out.

Fifteen minutes before the end of the lesson, Tim stopped the class and led a period of whole-class discourse at the ordinary whiteboard without using the technology. He revisited the algebraic solution to $x^2 + y^2 = 25$ and $y=4$ and contrasted this with the algebraic solution to $x^2 + y^2 = 25$ and $y=6$, highlighted to the students that they were 'not getting an answer, we're getting a negative square root and that's impossible' [STP6(Trans)]. Tim then asked the students to consider what the value of y would need to be such that horizontal line would touch the circle at only one point and supported the students to test their conjecture that the y-value would equal five.

Interpretation of variance and invariance: Initially, the invariant property was a 'given' circle ($x^2+y^2=5^2$) constructed and displayed on the handheld screen. The addition of a constructed geometric line was the variant property by nature of its infinite number of positions within the view. The position of the line was then fixed (at y=4) and the students were asked to identify the resulting invariant properties that concerned points of intersection using pencil and paper methods .

The position of the line, although still fixed as horizontal, was then allowed to vary within the paper and pencil activity and a generalisation for this situation was sought. Tim did not privilege dragging of the defined line y=4, although there is evidence that some students drew additional horizontal lines using the MRT.

The students used the calculator functionality to calculate square roots of numbers, which provided the 'class authority' with respect to the meaning of the outputs in relation to the original activity. (See example in Figure A9-14)

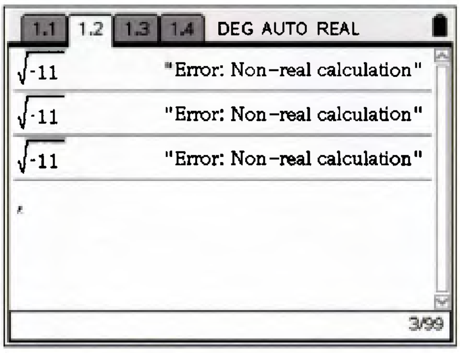


Figure A9-14 [STP7(tns-S)]

The overall intention for the activity was for the students to determine the following invariant properties:

- When c>5, x = $\sqrt{(5^2-c^2)}$ was 'unsolvable', which meant no points of intersection within the context of the original activity.
- When c=5, x = $\sqrt{(5^2-c^2)}$ produced only one solution, which meant one point of intersection within the context of the original activity.
- When c<5, x = $\sqrt{(5^2-c^2)}$ produced two solutions, which meant two points of intersection within the context of the original activity.

Tim reported that in the third lesson (which I did not observe) the students 'were able to check their algebraic solutions by plotting the lines on the handheld and finding points of intersection' [STP6(Journ-T)].

Implied instrument utilisation scheme: The IUS developed for this lesson was unusual in that, although the starting point was geometric, this was mainly to support the students' initial engagement with the activity. There was no dragging involved. The use of the whole class display had a significant part to play in mediating the phase of the lesson where the students' individual responses were quickly and efficiently compared and discussed. Tim's addition of the line y=3 to the original diagram was sent to the students handhelds. However, as they were not instructed to interact with this file in any way, most students used paper and pencil methods to work on the activities. There was one pair of students who were the exception to this and they did choose to use the MRT as part of their response. The IUS suggested by Tim's activity design is shown diagrammatically in Figure A9-15.

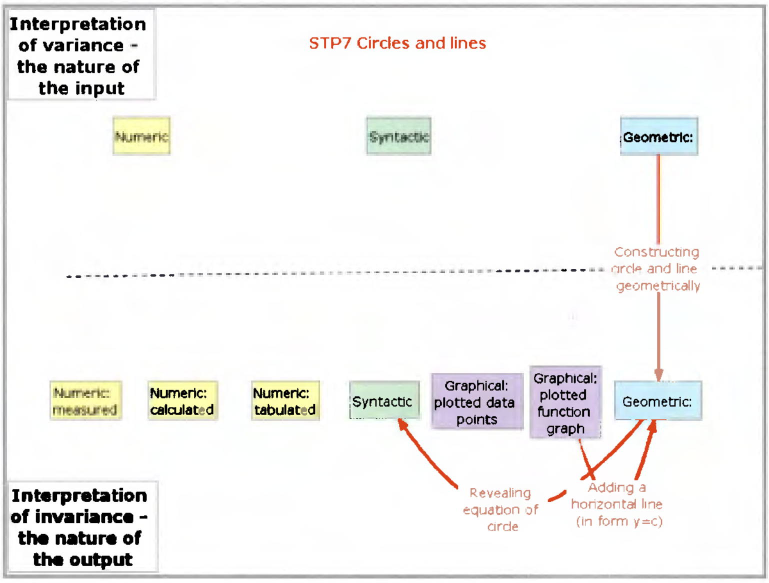


Figure A9-15 IUS [STP7] Circles and lines

Hiccups identified from the lesson data: There were seven hiccups identified from the lesson data, which were individually coded into the categories shown in Figure A9-16.

Name	Sources	References
TP7 Hiccup1 - Instrumentation (T) - Lines crossing circle only once	1	1
TP7 Hiccup2 - Student not believing the MRT response to sqrt(-11)	1	1
TP7 Hiccup3 - Student questioned the robustness of the generality	1	1
TP7 Hiccup4 - Student's dissatisfaction about line and circle only touching once	1	1
TP7 Hiccup5 - Students struggled to see the purpose for the generalisation	1	1
TP7 Hiccup6 - Umprompted student question - responded with MRT (moving centre of circl	1	1
TP7 Hiccup7 - Students own IUS is in conflict with the task design	2	2

Figure A9-16 [STP7] Activity hiccups

These categories can be more fully described as:

- Instrumentation issues experienced by the teacher, which concerned unanticipated responses from some students' constructions of a line that would be expected to intersect the circle twice, but only appeared to intersect once (see Figure A9-8).
- Highlighting students' disbeliefs concerning: the robustness of the generality that they were beginning to articulate; the technology's response to √(-11);

and whether it was possible for the line to touch the circle only once.

- Unexpected students' responses arising from the students own instrumentalisation within the activity, suggesting that they had not appreciated the need to use the algebra to prove the points of intersection.

Possible evidence of situated learning: Evidence from the various data sources led to a list of five actions by Tim that might provide evidence of his situated learning during this activity. These are shown in Figure A9-17.

Name	Sources	References
TP7 Action1 - Apprecated that student identified the mathematics topic	1	2
TP7 Action2 - Appreciated how the diversity of students' responses supported their abilities	1	1
TP7 Action3 - Appreciated students' abilities to recognise the bigger mathematical picture	1	2
TP7 Action4 - Appreciated that a wide diversity of responses might yield a desired response	1	1
TP7 Action5 - Responded to 'lines only crossing once'	1	1

Figure A9-17 [STP7] Evidence of teacher's actions

Tim's choice of questions within [STP7(Activity)2] indicated his sense of progression with respect to developing the students' knowledge and confidence to solve more complex activities relating to the intersection of circles and lines.

The analysis of the lesson transcript evidenced a number of conversations that Tim had held with students, which indicated a wide range of possible learning sites for him in relation to:

- The students own instrumentalisation of the activity. (Cameron geometrically constructed a perpendicular line through the point of intersection and read off where it crossed the x-axis). Tim mentioned '...and with Cameron, and it will be on its work, he did something I think to do with intersections and coordinates, so that's quite interesting' [STP7(Int)] (see Figure A7-17).

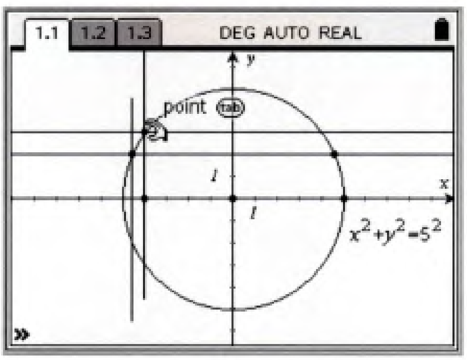


Figure A9-18 [STP7(tns-S)] Cameron's response to the activity

- The students' abilities to appreciate the position of the specific topic of the lesson within a larger mathematical landscape. This was in relation to a recognition of the more general topic ('Is this simultaneous equations, sir?' [STP7(trans)]) and with respect to the limitations of the generality being pursued within the lesson (i.e. non-horizontal lines and tangents).

- My post-lesson discussion with Tim revealed that he was pleasantly surprised by the students' responses to the initial activity.

and I though with a circle of radius five, it's not.. it' so big that no-one would come up with one that didn't intersect the circle I thought, it's not going to happen. And there they were... there were these three that didn't intersect the circle... and that was quite lovely.. and even the tangent... just about...and I wasn't expecting that.. I though they'll all go through twice and I'll have to tweak this now because my circle is too big... but no, it was quite nice that. But there's no way of knowing how that would have gone. Yes that was lovely actually, that was really nice. [STP7(Int)]

Reviewing Tim's overall approach for this activity it seemed that he had developed the following mathematical progression:

- Explore the 'big picture' first, how many times might it be possible for any line to cross a circle. (Although, by defining the initial circle as $x^2 + y^2 = 5^2$, rather than leaving the circle's equation 'undefined', this imposed a scale on the 'big picture').
- Consider where horizontal lines of the form y = c (integer values of c) integer values) might cross, using the symmetry of the problem to support the students to identify two (integer) solutions.
- Consider where horizontal lines of the form y = c (decimal values of c) integer values) might cross, using the symmetry of the problem to support the students to identify two (decimal) solutions.
- Extend the problem to consider intersection of lines of the form y = mx.
- Extend the problem to consider intersection of lines of the form y = mx+c.

In my post lesson interview with Tim I commented on this progression and the following conversation ensued,

ACW: that's a really nice way into that [topic] because taking the horizontal line is so... Had you seen that anywhere?

TP: No... ...and what I'm doing now with going through the y equals line and then going to a diagonal line, I've not seen that anywhere either - it *goes straight into the hardest case. But actually that is quite nice, and for some reason I hadn't tweaked that the answer was an integer - it was a three four five triangle, I should have recognised that but.. erm, and erm*

ACW: Well that's nice too, because you can now take it on to think about well what if it was a little bit lower?

TP: and I had planned that anyway... well I don't think that will as nice a number... I need to check that before the lesson [TP laughs].

STP8 Quadratic curves

A series of tasks offered during a single lesson with twenty six middle ability 14-15 year old students working towards the GCSE higher tier examination. In his communication to me prior to the lesson Tim reported,

They have had a couple of TI-Nav lessons this week. They started with a graph of y=(x+4)(x-4) and then changed the +4 and -4 to other numbers in order to get a pattern of evenly spaced overlapping quadratic curves. They did this quite successfully. Interesting at the end of the first lesson, I asked the students (using quick poll) where the graph y=(x+2)(x-6) would cut the x-axis and more students thought that 7 students thought it would cut at +2 and -6 while only 5 thought it would cut at -2 and +6 (a number didn't vote or chose another option). I want to explore the links between the factored and un-factored quadratic expressions and also explore the solutions of the equations and consider why they are the 'opposite' signs to that expected. [Journ]

The students were sent a file in which it was intended that they would enter two values into a spreadsheet that represented 'a' and 'b' within a quadratic of the form y = (x-a)(x-b) (see Figure A9-19). They would then be asked to drag a related point on a Cartesian plane (see Figure A9-20) and make conjectures about the nature of the point's locus (which traced the path of the related quadratic function). The students were then expected to predict the factorised form of the quadratic function and, by entering this as a function 'match' the locus of the moving point (Figure A9-21).

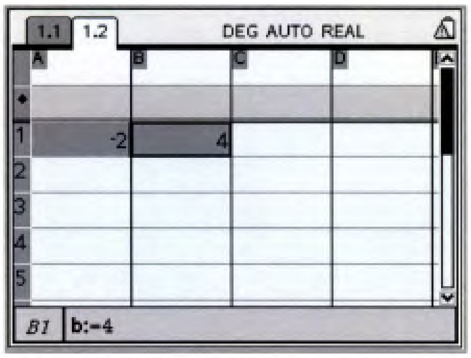


Figure A9-19 [STP8(tns-T)] Spreadsheet page

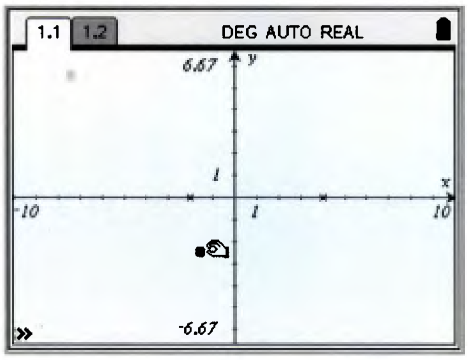


Figure A9-20 [STP8(tns-T)] Graphing page

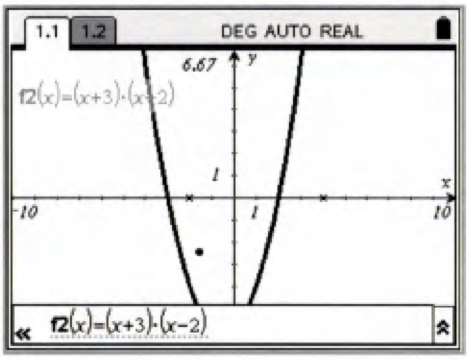


Figure A9-21 [STP8(tns-T)] Graphing page with an incorrect 'function match' displayed

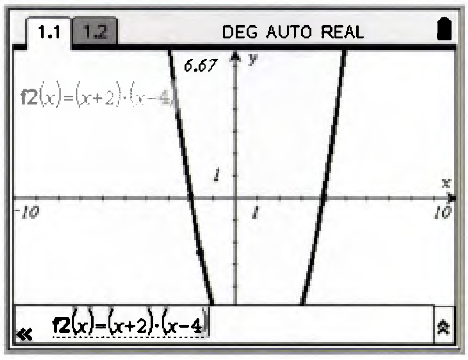


Figure A9-22 [STP8(tns-T)] Graphing page with the correct 'function match' displayed

However a technological issue occurred when the file was transferred, which resulted in the movable point losing its connection with the underlying quadratic curve on which it had been constructed. Consequently, having introduced the initial activity from the front of the class, Tim had to quickly amend the activity.

The students used the same TI-Nspire file but Tim then showed them how to unhide the quadratic graph. Therefore, the focus of the amended activity became one of conjecturing the quadratic function (in its factorised form) that would match the quadratic graph that had been generated by the spreadsheet values. The students then checked this by observing whether the graphs coincided. The earlier disruption to the lesson meant that the students had only a few minutes to work on the amended activity.

Interpretation of variance and invariance: The activity required the students to 'notice' the effects on the appearance of both the position of two points on the x-axis (marked with crosses) and the locus of a draggable point as they varied the two numeric values within the spreadsheet. Tim was hoping that the students would notice the invariant property that, when the two functions coincided, the numeric values within the spreadsheet carried the opposite signs to the values of a and b within the entered functions.

Implied instrument utilisation scheme: In Tim's intended activity, he was conceiving an IUS which did not resonate with any seen previously in the study. Indeed, when Tim first suggested the activity approach, I had noted in my research journal, 'Neat idea – plot the function, put a point on it, hide the function... Drag it so that it moves – Can you work out what the function is by observing the crossing points?' [Journ].

This was the first time that I had seen a teacher develop a pedagogic approach that required the students to interact by dragging to try to relate the path of the moving point to their existing knowledge about the shape of quadratic functions. By requiring the students to try to conjecture the function and enter it syntactically, they were being encouraged to notice the geometric form of the graph and connect them to the syntactic form. In addition, the notion of relative speed would also become a consideration for students interacting in this way.

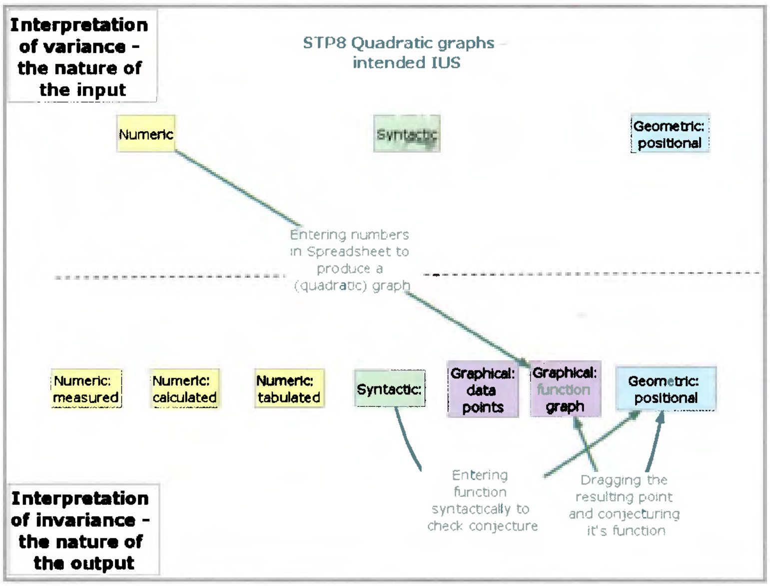


Figure A9-23 Intended IUS [STP8] Quadratic graphs

Due to the technical issues experienced, Tim's adapted activity reverted to the more familiar IUS1 (Section 5.3.1.), whereby direct numeric and syntactic inputs were made, and the outcomes compared (Figure A9-24).

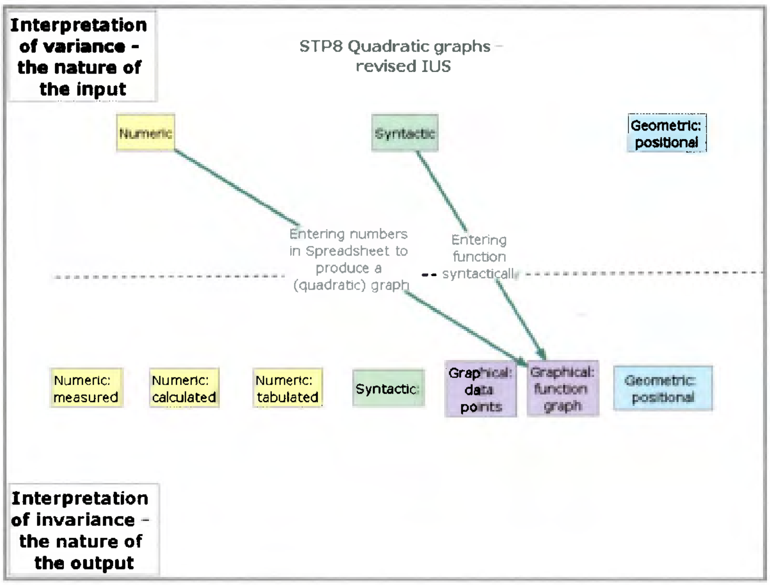


Figure A9-24 Revised IUS [STP8] Quadratic graphs

Hiccups identified from the lesson data: Three hiccups were observed in this lesson, and they fell into three distinct categories as shown in Figure A9-25.

Name	Sources	References
TP8 Hiccup1 - Instrumentation (T) - Transfer of file resulted in lost functionality	1	1
TP8 Hiccup2 - Students appear to enter 'crazy' functions	1	1
TP8 Hiccup3 - Instrumentation (S) - Not understanding how MRT file is working	1	1

Figure A9-25 [STP8] Activity hiccups

The most obvious (and time consuming) hiccup in this lesson resulted from the unexpected technical failure when there was a loss of functionality within the transferred file. Consequently, the students received a new set of activity instructions and this could partly explain the remaining hiccups, which may be attributable to their confusions about what they were supposed to be doing.

For example, one pair of students has not realised that the highlighted cells in the spreadsheet dictated the function that they were attempting to match. Having successfully matched the function f1(x)=(x-2)(x+4), they entered a second set of values into the spreadsheet with the expectation that this would produce a second quadratic graph (Figure A9-26 and Figure A9-27). They successfully entered a second function to match their new choice of values, however received no feedback from the MRT in this respect (Figure A9-28).

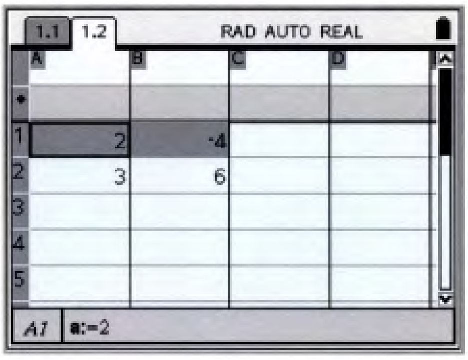


Figure A9-26 [STP8(tns-S)page2] Spreadsheet values

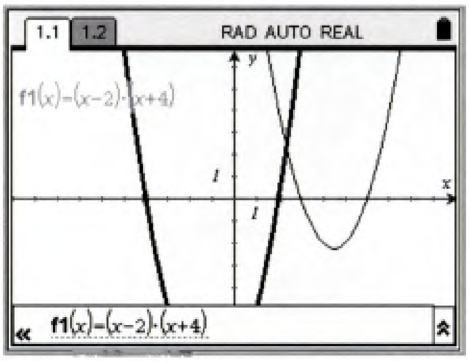


Figure A9-27 [STP8(tns-S)page1] A correctly matched function

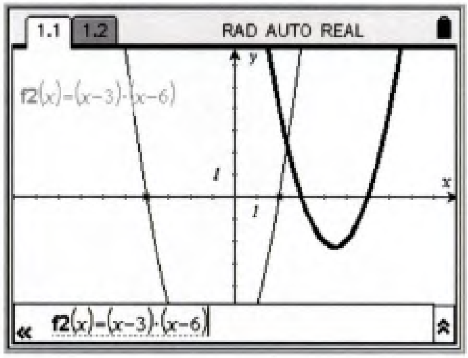


Figure A9-28 [STP8(tns-S)page2] An unresolved attempt

Possible evidence of situated learning: Evidence from the various data sources led to a list of two actions by Tim that might provided evidence of his situated learning during this activity. These are shown in Figure A9-29.

Name	Sources	References
TP8 Action1 - Highlighting roots as crosses to aid discussion	1	1
TP8 Action2 - Student asks exact question teacher wants to probe	1	1

Figure A9-29 [STP8] Evidence of teacher's actions

In our pre-lesson discussion, Tim spoke in an animated way about his design of the intended activity,

> *TP: Okay, if we set one of these files up so that you've got two points and you can move them around by changing the values on the spreadsheet... ...If we can put two in there and negative four there and we can store that as b perhaps and store that one as a... and over here I could have x minus a and x minus b, is that logical? [TP draws function]... ...So there's our roots, minus two and four and if we hide that [referring to the curve] and if I hide that and if I've intersection points between my axes and the equation Da Da!... ...So I now want to hide this don't I? [TP is refers the quadratic graph] Do I want to leave those*

> *points on?*
>
> *ACW: No leave the points there, that's nice, if you leave the points there it puts a focus on them.*
>
> *TP [TP puts a point on the function] We're not interested in its coordinates, hide those, okay so now hide that [referring to the quadratic graph itself]*
>
> *ACW: I think leave the points there - it's a nice focus...*
>
> *TP: Okay have we got this now? Now we can pick that up, [referring to the point on the now hidden function] and we can drag it and it goes 'ding' and that's quite nice. [STP8(Int)]*

Early on in the lesson, soon after Tim had introduced the amended activity, one student asked the question 'why is the sign the opposite?' causing Tim to hesitate and then respond with the statement, 'That's what we're trying to find out' [STP8(Trans)]. It was noticeable that partly due to the earlier disruptions, which limited the lesson time, Tim did not return to this question, which referring back to our pre-lesson correspondence, seemed to have been a main objective for his activity design.

STP9 Equivalent quadratic equations

A series of tasks offered during a sequence of three lessons with a year 10 group working towards the GCSE foundation tier examination. Between 13 and 16 students were present in each of the lessons and Tim had assessed that they were currently working at between GCSE Grades D and E. In the preceding lessons the students had plotted quadratic graphs using paper and pencil methods. Tim had said that in the observed lessons he wanted to 'spend the lessons with this group looking at multiplying out quadratics.... and comparing graphs of non-multiplied out and multiplied out quadratic expressions. Possibly extend to factorising...' [Journ].

In our pre-lesson discussion, Tim expressed his concern about the more algorithmic approaches to teaching the expansion and factorisation of quadratic expressions to (particularly) less able students

> *TP: so with this group I've got today, I'm wondering whether I move on this afternoon to just multiplying out brackets, so they'll be quicker at doing it properly and to get the process in their heads, [leading to] 'however if I had two x plus three it wouldn't work. So there's an issue there'. [STP9(Int)]*

Tim had also revealed his normal approach when teaching students to expand and factorise quadratic expressions thus:

> *What I do or have done is I'll do up to fifteen 15 minutes on multiplying out brackets and then I'll say, 'okay let's cover this up and work backwards'. And after a couple more examples I use a highlighter pen and say like [writes $(x+3)(x+5)=x^2+8x+15$] five three eight five three. And then six two twelve, what's going on here? And I do it that way with several examples and try and force them to recognise the pattern and then we go backwards from there... [STP9(Int)]*

We discussed an alternative teaching approach, which used a grid method of multiplication and results in an alternative visual representation of the products. Tim mentioned that he had not used this strategy when teaching expanding brackets as he felt that his students needed to appreciate the link between multiplication and calculating areas before they could fully understand the strategy. Tim felt that, as expanding brackets was a topic within algebra, this could confuse many of his students.

Lesson 1: In the first lesson, Tim gave the students a set of quadratic expressions[1] that he had prepared (see Figure A9-30) and asked the students to try find pairs that matched by entering selected pairs into the TI-Nspire as functions and identifying when the graphs coincided.

[1] These were described throughout the lesson as 'expressions with brackets' and 'expressions with squareds'.

(x+2)(x+5)	
(x+3)(x+6)	
(x+2)(x+7)	
(x+4)(x+5)	
(x+3)(x+5	
(x+7)(x+1)	
(x+2)(x+3)	
(x+3)(x+1)	

These expressions match the expressions in the boxes. By drawing graphs on your TI-Nspire, find out how they match:

$x^2 + 9x + 18$ $x^2 + 8x + 15$ $x^2 + 9x + 14$ $x^2 + 4x + 3$

$x^2 + 7x + 10$ $x^2 + 9x + 20$ $x^2 + 5x + 6$ $x^2 + 8x + 7$

Figure A9-30 [STP9(Activity)] Matching quadratic expressions[2]

Using his own handheld in 'live presenter' mode, Tim showed the students how to enter the first factorised expression, (x+2)(x+5), as a function to produce a quadratic graph. Tim then sent this version of the TI-Nspire file to the students' handhelds (see Figure A9-31).

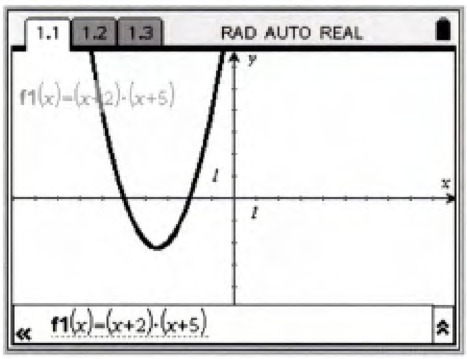


Figure A9-31 [STP9(tns-T)] Graphing f1(x)=(x+2)(x+5)

He then asked a student to choose one of the expanded expressions from the activity sheet (x^2+9x+18 was selected) and entered this as a second function on

[2] Tim noticed his accidental omission of a bracket in the fifth expression and asked the students to write it onto their worksheets.

the handheld, which resulted in a second quadratic graph that differed from the first. He explained to the students that their activity objective was to enter one of the expanded expressions such that the two graphs coincided and record this on the activity sheet. Tim then selected a second expanded expression (x^2+7x+10), showed the students how to clear the previously entered function and replace it with the new function, which coincided with (x+2)(x+5). He encouraged the students to notice that the new function briefly appeared as a bold line as it was drawn.

The students proceeded to work at the activity and Tim moved around the classroom responding to students' difficulties and encouraging the students to come up with their own theory about what was happening. Several students began to notice a connection between the integer term within the expanded expression and the two integer terms within the factorised expression. Tim did not enter into discussions with the students about their theories, but asked the students to record them on their work. The lesson ended abruptly with Tim informing the students that they would 'look at how you did this morning this afternoon and compare what you though the answers were... ...There were a few good theories floating around this morning which I want to explore this afternoon' [STP9 (Trans)].

Lesson 2: The second lesson took place later the same day, immediately after the midday break. The beginning of the lesson was disrupted as a result of a lunchtime incident outside the school to which Tim had been required to respond. During the break Tim had prepared an activity sheet to follow on from the previous lesson's work (Figure A9-32).

More Matching Quadratic Graphs

(x+2)(x+9)	
(x+3)(x+6)	
(x+1)(x+18)	

These expressions match the expressions in the boxes.
By drawing graphs on your TI-Nspire, find out how they match:

$x^2 + 9x + 18$ $x^2 + 19x + 18$ $x^2 + 11x + 18$

$x^2 + 3x - 18$	$x^2 - 7x - 18$
$x^2 + 7x - 18$	$x^2 + x - 12$
$x^2 - x - 12$	$x^2 - 4x - 12$
$x^2 - 3x - 18$	$x^2 + 4x - 18$

(x+9)(x-2) (x+4)(x-3) (x+6)(x-3) (x-6)(x+3)
(x-6)(x+2) (x-9)(x+2) (x+6)(x-2) (x-4)(x+3)

Figure A9-32 [STP9(Activity)L2] More matching quadratic graphs

His choice of initial functions clearly evidenced that he had appreciated that the majority of the students had concluded a limited theory concerning the connection between the quadratics in their factorised and expanded forms. By selecting quadratic functions with a common integer term, he was aiming to force the students to expand their theories and look at the coefficient of x within the expanded expressions. The use of the technology in the lesson was identical to that of the previous lesson. The students entered expressions to check their equivalence by plotting the graphs. Towards the end of the lesson, Tim held a plenary discussion in which he asked a student to offer is theory and, having established the 'sum- product' connection between the factorised and expanded expressions for the example $(x+3)(x+6)=x^2+9x+18$, Tim proceeded to show the students what this would look like using the 'grid multiplication' representation, which the students were subsequently asked to record (Figure A9-33).

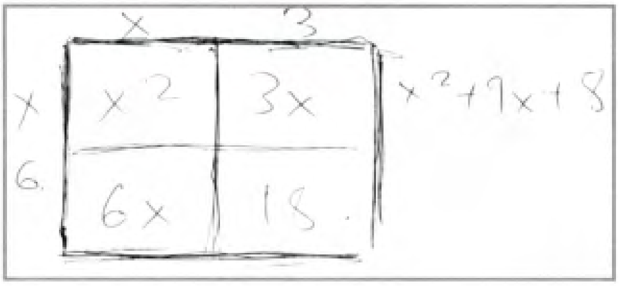


Figure A9-33 [STP9(StudWork)L2] Grid multiplication strategy

There were far fewer observable hiccups during the second lesson, which I would surmise to be as a result of the similarity in the approach with that of the preceding lesson.

Lesson 3: The final lesson in this sequence of three lessons took place the following morning. In our very brief discussion before the lesson, Tim had said,

> *I was planning quite late last night and I couldn't come up with something interesting until this morning when I got to work. I'm going to do quite a, in a sense, quite a tedious lesson. If I was to store some random decimal number into the value for x, you can then type in x plus three, x plus four equals whatever seven x plus twelve and it would say if it was true and if it was wrong it would say false. So we can use it as a checking tool which is not a Navigator lesson but it's a good use of the Nspires to make a dull exercise a bit more interesting. So it's going to be quite a dull lesson in a sense but... ...they need that and they'll get some success out of that. So I'm going to print off a worksheet for them and see if that will work. [STP9(Int)]*

The worksheet Tim selected was one of the National Curriculum Level 6 resources from 'TenTicks' (Fisher Educational Ltd, 2009), a web-based resource bank to which the school subscribed (see Figure A9-34).

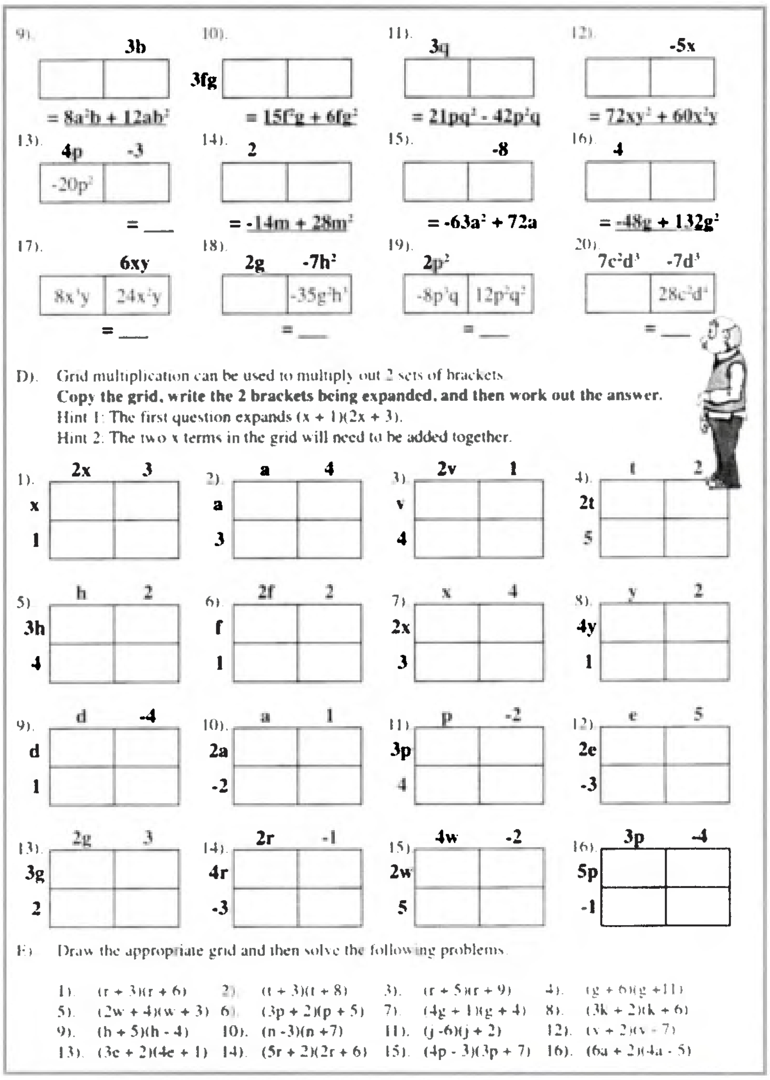


Figure A9-34 [STP9(Activity)L3] TenTicks worksheet

Tim began the lesson by sending a pre-prepared TI-Nspire file to the students' handhelds, although he did not direct the students to the file at this point (see Figure A9-35).

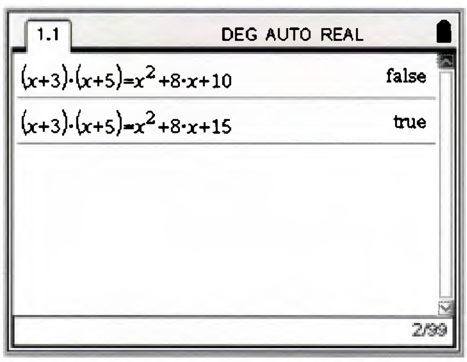


Figure A9-35 [STP9(tns-T)L3] Equivalent quadratic expressions

Tim then reminded the students how to expand a factorised quadratic expression using the grid multiplication method that he had introduced at the end of the previous lesson using an example from the previous lesson's activity (x+2)(x+9). The students were asked to record this on the blank side of the worksheet he had just distributed.

Tim opened the TI-Nspire file and demonstrated to the students how to enter the statement $(x+2)(x+9)=x^2+11x+18$ into a Calculator page to check its validity. Some students also followed this using their own handhelds.

The students were then asked to work independently, using the grid method to answer the questions on the worksheet beginning with Section D, halfway down the page. They were asked to enter the correct terms in each grid and then sum the terms to produce the correct quadratic expansion. Following this, they were encouraged to check their statements using the MRT. The students found this highly motivating and there were shouts of delight from several students when they gained positive feedback from the MRT.

Interpretation of variance and invariance: The generality that Tim wanted the students to explore within the first and second lesson concerned the identity $(x+a)(x+b)=x^2+(a+b)x+ab$. By giving them a range of different factorised and expanded expressions to match he intended for them to make the connection between the sum and product in each case. During the first lesson, Tim's examples included positive integers only. All of the expanded expressions had different integer terms and three of them contained the term 9x [STP9(Activity)1]. Consequently the product relationship was the most obvious invariant property for the students to notice.

The first part of the second activity sheet was obviously designed to expose this assumption, by including three examples with an integer term of eighteen, forcing the students to consider the relationship between the numbers in the brackets and the coefficient of x [STP6(Activity)2].

In the third lesson, the expectations with respect to variance and invariance were at a much higher mathematical level in that the questions posed required students to appreciate that $(ax+b)(cx+d)= acx^2+(bc+ad)x+bd$ for positive and negative integer values of b and d.

Instrument utilisation scheme: The IUS that Tim developed for this series of lessons resonated with IUS1, as the use of the MRT required students to input their conjectures as syntactic expressions or statements and use the feedback to decide upon the validity of these conjectures. This is shown diagrammatically in Figure

A9-36.

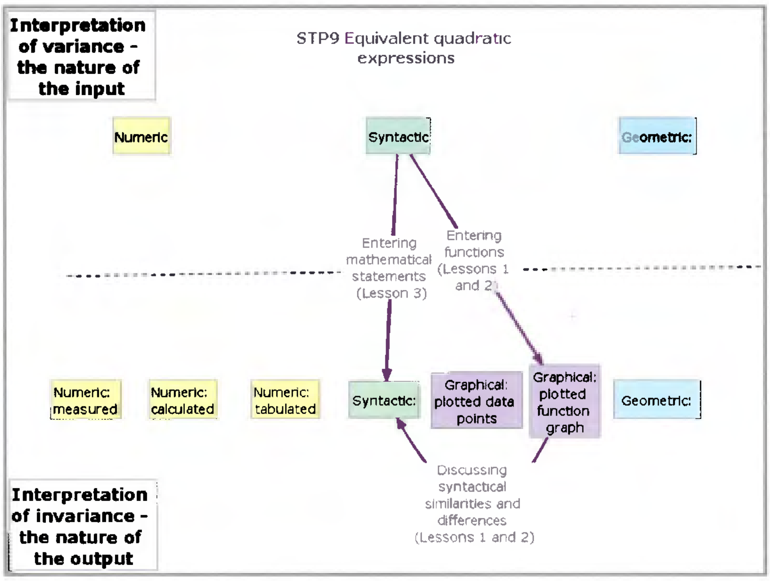


Figure A9-36 IUS [STP9] Equivalent quadratic expressions

Hiccups identified from the lesson data: Nineteen hiccups were observed in this series of three lessons and they were grouped into eight categories as shown in Figure A9-37.

Name	Sources	References
TP9 Hiccup01 - Student's instrumentation issue entering X^2	3	3
TP9 Hiccup02 - Teacher's assumption that students 'notice' the brackets	2	6
TP9 Hiccup03 - Teacher's assumption that students 'see' the invisible multiplication sign	1	1
TP9 Hiccup04 - Students' task sequencing issue.	1	1
TP9 Hiccup05 - Teacher's assumption task sequencing will lead to counter-examples in stu	1	1
TP9 Hiccup06 - Technology fails as teacher goes to use live presenter in final plenary	1	1
TP9 Hiccup07 - Noticing that the worksheet uses letters other than x	1	5
TP9 Hiccup08 - Student doubts authority of MRT 'Why didn't it say false'	1	1
TP9 Hiccup09 - Insufficient specificity about labelling objects under discussion	1	3

Figure A9-37 [STP9] Activity hiccups

Several students made explicit their instrumentation difficulties when entering the expanded form of the quadratic expression. Common issues included: using the multiplication key  (which looked like the letter 'x'), instead of the alphanumeric key ; and pressing the x^2 key () without first pressing the alphanumeric 'x' key when entering x^2.

During Tim's introduction to the first activity, by reading out loud the expression (x+2)(x+5) as 'x plus two, x pus five' as he typed it into the function entry line he assumed the presence of the brackets [STP(Trans)]. This led to at least one student entering the function f1(x)=x+2x+5, which produced an unexpected linear function and, as he moved around the classroom several other students sought reassurance with respect to whether or not they should type the brackets. In the second lesson, the students continued to question Tim concerning his omission to mention the brackets when 'speaking out loud' as he entered the factorised expression (x+2)(x+9) into the MRT.

A similar hiccup occurred in the first lesson, which concerned Tim's hesitation with regard to whether the MRT would 'require' a multiplication sign between the two brackets. This again caused confusion as, a few minutes earlier, he had led a whole-class discourse concerning the 'invisible' multiplication sign.

A very subtle difficulty occurred for one pair of students resulting from the order in which they tried to match the functions. As Tim's introduction to the whole class had begun by 'wrongly', matching f1(x)=(x+2)(x+5) with f2(x)= x^2+7x+18, many students had these two functions entered on their handhelds. However, as they began to work independently, many chose to work on the next expression on the activity sheet, which was (x+3)(x+9). However, when they changed f1(x) to (x+3)(x+9), they were not prepared for this to result in a correct 'match' and were confused by the feedback from the MRT.

Tim initially thought that, by including many expanded quadratic expressions containing the term 9x, the students would be forced into cognitive conflict when they began to match the functions. However, as each pair of expressions matched uniquely by virtue of their integer terms, this situation did not arise.

A significant hiccup occurred when Tim looked closely at the format of the questions on the worksheet and noticed that the variable used in the questions, although initially represented by the letter 'x', changed to other letters.

There was an interesting moment when one student questioned the response from the MRT after Tim had entered a mathematical statement during the third lesson (Figure A9-38).

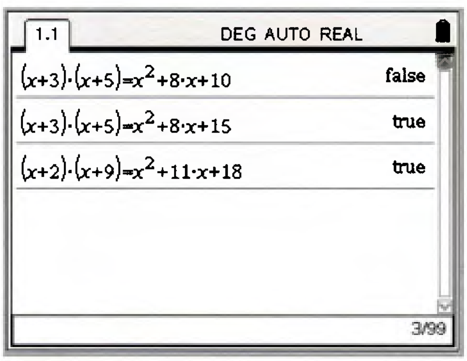


Figure A9-38 [STP9(tns-T)] Why didn't it say false?

Student (m): Why didn't it say false?

TP: Sorry?

Student (m): Why didn't it say false?

TP: Because we got it right. If I'd done, and I can copy this, Let's say I didn't put eighteen, let's say I put seventeen, instead of eighteen, that would have been wrong... 'cause we worked out it was eighteen yeah? If I put seventeen in instead, it says false. Alright? You get the idea?

Possible evidence of situated learning: Evidence from the various data sources led to a list of twelve actions by Tim that might provide evidence of his situated learning during this activity. These are shown in Figure A9-39.

Name	Sources	References
TP9 Action01 - Appreciated 'hiccup' over choice of expressions	1	2
TP9 Action02 - Appreciated that the MRT could have accepted any letter as variable	1	1
TP9 Action03 - Made explicit link between Medley's theory and grid multiplication.	1	1
TP9 Action04 - Noticed that students appreciated being sent the same file to work on	1	1
TP9 Action05 - Observed that students were highly motivated by MRT response 'true'	1	1
TP9 Action06 - Privileged 'press control z to undo'	1	1
TP9 Action07 - Responded to students' concerns about needing to type the brackets	2	2
TP9 Action08 - Reversed second section of task 2 to force inverse strategies	1	1
TP9 Action09 - Revised language wrt needing brackets	1	1
TP9 Action10 - Revised the task to account for the L1 hiccup	1	2
TP9 Action11 - Appreciated that student had noticed 'bold' graph	1	1
TP9 Action12 - Appreciated student's connection between 'Medley's rule' and grid multiplica	1	1

Figure A9-39 [STP9] Evidence of teacher's actions

Tim had clearly noticed the issue concerning the inadequacies of the generalisation that many of the students had made as a result of the first activity and the worksheet that he devised for the second lesson seemed to respond to these [STP9Hiccup5]. He also included a series of questions that would encourage the students to begin to consider the inverse relationship for the generality that they were beginning to conclude.

The number of students who questioned how they should enter the factorised expressions during the first lesson seemed to lead Tim to modify his language later in the lesson sequence [STP9Hiccup2]. For example, when speaking to one pair of students he said, 'The first one says x plus two close bracket open bracket, x plus nine. That's the first one...' [STP9(Trans)].

Following the third lesson, Tim and I had a brief conversation in which Tim commented that he felt that he had missed an opportunity for the students to appreciate that the choice of letters was not mathematically significant and that it would have been possible to set up the initial file such that all of the questions could have been checked in the form they had been written [STP9Hiccup7].

ACW: It was very motivating to have that true-false response... ... There are always issues between translating the specificities of the questions *when you take something off the shelf to [use alongside] the context of the technology.*

TP: yes, and that was an issue for me and I hadn't realised it. Yeah, it wouldn't have even taken very long to store a variable in every single letter of the alphabet, wouldn't take very long, but I didn't think of it... ...The better thing to do would have been to store a variable for all the letters of the alphabet and then it would work for anything. But there you go, you don't think of these things do you? [STP9(Int)]

STP10 Linear graphs

An activity offered during a single lesson with a year 12 group who were preparing to resit their GCSE examination. Only 10 students were present of which two were from another teacher's group due to a disrupted timetable on that day. In the previous lesson Tim had allowed the students to explore gradient properties of linear functions by 'making stars'. In this lesson he began by supporting the students to construct pairs of coordinate points and then join them with a geometric line (see Figure A9-40).

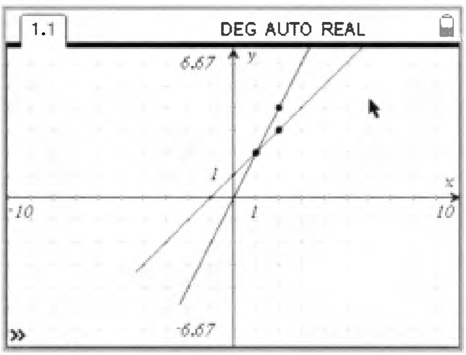


Figure A9-40 [STP10(tns-T)1] Comparing gradient properties of lines

Tim encouraged a whole class discussion about the steepness of each of the lines and used the dots on the grid to support the students to understand that the gradient of the (steeper) line was equal to two. He then showed them how to use MRT functionality to 'measure' the equation of the steeper line, which revealed an unanticipated outcome, causing Tim to express a sigh of despair (see Figure A9-41).

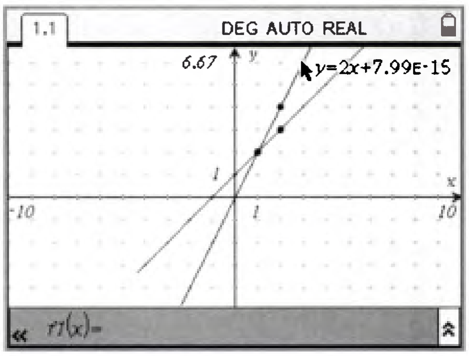


Figure A9-41 [STP10(tns-T)2] The revealed equation

Tim led a period of whole class discourse during which he discussed the unanticipated notation. He then instigated a quick poll, asking the students to send their response to the question 'what is the gradient of the other line?', a question which caused some confusion to those students who had not followed the whole class discussion closely enough to know what was meant by 'the other line' [STP10(Trans)]. The students' responses were discussed and, as most students had actually responded with an equation, rather than a numerical value for the gradient, Tim focused this discussion on the different syntactical formats that the students' responses had taken.

Tim then adapted the previous file to display a new line (Figure A9-42) and instructed the students, 'I want to know what the gradient of that line is... ...or even the equation - if you know what that is from yesterday' [STP10(Trans)].

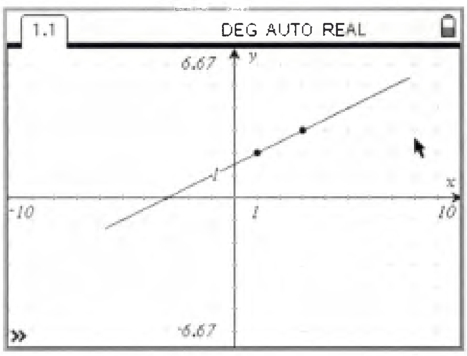


Figure A9-42 [STP10(tns-T)3] Describing the linear function

The students entered their responses onto a Notes page within the handhelds and Tim displayed these to the class using the screen capture view. Most of the students' responses were in the form of equations.

Interpretation of variance and invariance: By choosing to construct two line segments and compare their gradients using the dotty grid as a reference, Tim was expecting that the students would conclude that, the calculated value of the gradient would equal the vertical translation for every unit horizontal translation.

The variance was provided by the different positioning of the two coordinate points which defined the lines and the invariant property was the measured value of its gradient. In this respect, the MRT was acting the classroom authority. It appeared that Tim hoped that by contrasting the two initial lines and, later in the lesson introducing a third line, his students would begin to generalise about the relationship between the positioning of the dots and the value of the gradient.

Implied instrument utilisation scheme: In this lesson Tim began by co-constructing the TI-Nspire file with the students, however, as there were several students in the lesson that day who struggled to follow his instructions he also sent the resulting file directly to their handhelds[3].

The initial representation was one of geometric points that were positioned on a Cartesian plane with the dotty grid displayed. These points were used to construct geometric line segments. The equation of the line segments were measured and the respective equations were displayed in y=mx+c format. Figure A9-43 shows this IUS diagrammatically.

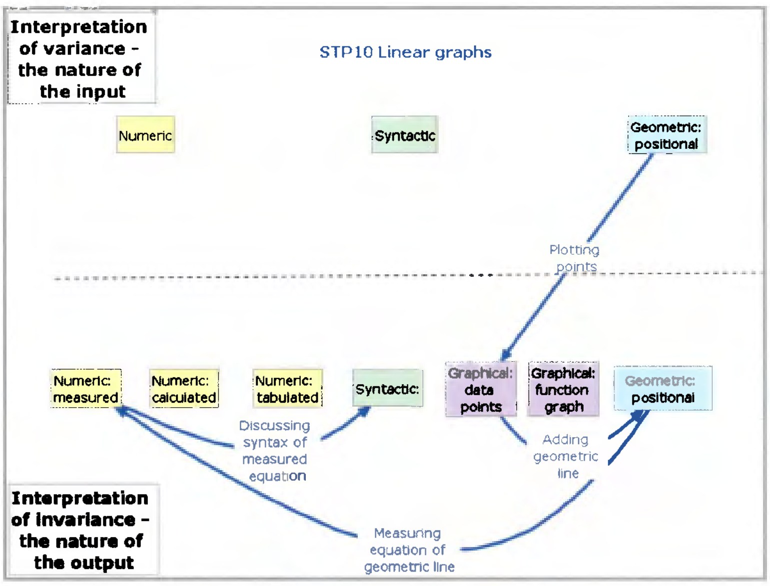


Figure A9-43 IUS [STP10] Linear functions

Hiccups identified from the lesson data: Twenty-four hiccups were observed in this activity and they were grouped into the thirteen categories as shown in Figure A9-44.

[3] Due to a staff absence and a subsequent regrouping of the students, three of the students in the class that day were normally taught by another teacher and had no previous experience with the MRT.

Name	Sources	Reference
TP10 Hiccup01 - Network communication errors	1	4
TP10 Hiccup02 - Students's instrumentation difficulty locating document in folder	1	1
TP10 Hiccup03 - Students' instrumentation difficulties plotting free coordinate points	1	3
TP10 Hiccup04 - Students's instrumentation difficulty locating the equals sign on the key pad	1	1
TP10 Hiccup05 - Teacher's laptop hibernates, so has to give instrumentation instructions verbally	1	1
TP10 Hiccup06 - Teacher's instrumentation issues - moved from handheld to computer software	1	2
TP10 Hiccup07 - Teacher's instrumentation difficulty - could not display quick poll and graph screen	1	1
TP10 Hiccup08 - Measured equation is in standard form	1	3
TP10 Hiccup10 Students interpretation of gradient is not what was expected	1	3
TP10 Hiccup11 - Student uneasy about having a different response	1	1
TP10 Hiccup12 - Realised that initial example was not a helpful one	1	1
TP10 Hiccup13 - Hampered by 'gap between students' existing understanding and lesson intention	2	3

Figure A9-44 [STP10] Activity hiccups

There were many hiccups caused by technical/instrumentation problems in this lesson and these were experienced by both Tim himself and individual students (Hiccups 1 to 7). Tim's technical issues included a failure of the display technology at a crucial point during the activity introduction when his laptop began to hibernate. As a result he resorted to providing students with the necessary instrumentation instructions orally, without the benefit of the whole class display. This may have partly accounted for the later instrumentation issues experienced by the students. A second technical issue concerned the inability to switch between the graph about which a quick poll question had been asked, and the question itself. A number of these hiccups resulted from instrumentation issues experienced by students who had not previously used the TI-Nspire handhelds as they were normally taught by another teacher. As Tim had could not have anticipated this problem when he planned the lesson, there was little that he could have done to alleviate this.

The most significant hiccup in the lesson, which was clearly unanticipated by Tim, concerned the display of the measured equation as y=2x+7.99E-15 at a point in the lesson where he really wanted to highlight the importance of the two in this equation [STP10Hiccup8].

Other hiccups in this lesson related to the inconsistent naming (or labelling) of the objects under scrutiny, which made it difficult for both Tim and his students to be specific when trying to describe the different properties ([STP10Hiccup10] and [STP10Hicup11]). Initially they were described as the first line and the second line in the order that they had appeared. However, minutes later this 'lesson history' seemed to have been lost. In addition, some students had constructed their own versions of the activity, which meant that their objects differed from those constructed by Tim and it became problematic for them to follow the whole-class discourse.

Possible evidence of situated learning: Evidence from the various data sources led to a list of twelve actions by Tim that might provide evidence of his situated learning during this activity. These are shown in Figure A9-45.

Name	Sources	References
TP10 Action1 - Appreciated student's ability to connect maths - 'are we drawing triangles'	1	1
TP10 Action2 - Appreciated student's awareness of previous lesson 'are we making a star sir'	1	2
TP10 Action3 - Appreciated student's recollection of previous lesson's work 'uphill or downhill'	1	1
TP10 Action4 - Appreciated student's sense of being 'more' mathematically correct	1	1
TP10 Action5 - Recognised student's need to know if he was correct	1	1
TP10 Action6 - Reflected deeply on the notion of an example	1	1
TP10 Action7 - Responded to instrumentation issues by sending students his file	1	2
TP10 Action8 - Responded to students' instrumentation hiccups by sending students his file	1	1
TP10 Action9 - Tried to find something generalisable from the students' responses	2	2

Figure A9-45 [STP10] Evidence of teacher's actions

Tim demonstrated a number of pedagogic strategies to respond to the instrumentation issues experienced by the students during the activity. He began the lesson by talking through the steps needed to create the initial TI-Nspire file, with the students being expected to follow these on their own handhelds. However, partly as a result of his laptop hibernating and the screen display being temporarily blank, he told the students that he would send them his file, rather than expect them to follow the steps to create it for themselves. This had the advantage of enabling the students to focus on his screen (when it returned to view) and make sense of the construction that he was building.

The hiccup that resulted in the display of standard form notation that undoubtedly contributed to the students' confusion over the definition of gradient [STP10Hiccup] required Tim to respond by looking to identify commonalities in the students' responses to the quick poll.

The students' oral responses and their interactions with the technology provided Tim with substantial information about their prior understanding of the topic, as well as their recollections of previous lesson activities that they related to it.

The diversity of the students' responses to the activity presented Tim with dilemmas concerning how he responded to those students' who questioned the correctness of their answers. For example, one student wanted reassurance that his response (y=x+1) was more mathematically correct than that of another student (y=1x+1).

APPENDIX 10 SUMMARY OF ELEANOR'S PHASE TWO DATA

CEL6 Transforming graphs

Data	Description
[Journ]	Email from EL with outline for lesson Lesson observation notes Suggestions for revised approach (Responded to concerns: -lack of common focus –resulting in not very meaningful class discussions; Opportunities to capture and record observations – too much variation.)
[CEL6(Struct)]	Eleanor's Smart Notebook lesson structure
[CEL6(Activity)]	Eleanor's activity sheet for the students
[CEL6(Trans)]	Lesson transcript
[CEL6(ScreenCapt)]	screen capture of students' work
[CEL6(Quest2)]	Eleanor's written lesson evaluation
[CEL6(Activityv2)]	Revised activity sheet for the students (ACW and EL)

CEL7 Generating circles

Data	Description
[Journ]	Notes from pre-lesson discussion. Lesson observation notes.
[CEL7(tns-T)]	Eleanor's tns file sent to the students during the lesson
[CEL7(tns-S)]	Students tns files
[CEL7(Quest2)]	Eleanor's written lesson reflection
[CEL7(Int-T)]	Post lesson discussion with Eleanor

CEL8 Triangles and squares

Data	Description
[CEL8(Plan)]	Eleanor's hand written lesson plan.
[Journ]	Notes from pre-lesson discussion. Lesson observation notes.
[CEL8(Plan)]	Eleanor's handwritten lesson plan
[CEL8(Trans)]	Lesson transcript

Data	Description
[CEL8(tns-T)]	Eleanor's tns file distributed to the students at the beginning of the lesson.
[CEL8(ScreenCapt)]	Images of students' handhelds as a screen capture
[CEL8(tns-S)]	Students' tns files
[CEL8(StudWork)]	Class set of students' handwritten response to Activity 3-1.
[CEL8(Quest2)]	Eleanor's detailed lesson reflection.

CEL9 Crossing linear functions

Data	Description
[Journ]	Notes from pre-lesson discussion. Lesson observation notes.
[CEL9(Int-T)]	Planning discussions
[CEL9(Plan)]	Eleanor's lesson plan
[CEL9(Struct)]	Eleanor's Smart Notebook lesson structure
[CEL9(tns-T)]	Eleanor's tns file distributed to the students at the beginning of the lesson.
[CEL9(Image)]	Image of the quick poll responses
[CEL9(Trans)]	Lesson transcripts
[CEL9(tns-S)]	Students' tns files
[CEL9(ScreenCapt)]	Images of students' handhelds as a screen capture

APPENDIX 11: DETAILED DESCRIPTIONS AND ANALYSES OF ELEANOR'S PHASE TWO ACTIVITIES

CEL6 Transforming graphs

A single lesson taught in May 2009 with a year 11 class of 29 students working at National Curriculum levels 7-8 from the GCSE higher tier examination syllabus. Eleanor's lesson objective was for students to develop 'An understanding of standard transformations of graphs' [CEL6(Plan)] and she expanded on this by saying 'I wanted the students to explore the effects of different transformations of linear and quadratic functions to enable them to make generalisations for themselves' [CEL6(Int)]. In the lesson the students were given a worksheet devised by Eleanor that included six 'sets' of linear, quadratic, and cubic functions laid out as three pairs. Each pair was intended to encourage students to compare particular transformations, for example the first set compared the effects of y =f(x)±a with y=f(x±a). There were thirty-nine different functions in total and the activity sheet did not label the sets of functions in any way (Figure A11-1).

$y = f(x)$	$y = f(x) \pm a$	$y = f(x)$	$y = f(x \pm a)$
$y = x^2$	$y = x^2 + 2$	$y = x^2$	$y = (x + 2)^2$
$y = x^2$	$y = x^2 - 2$		$y = (x + 2)^2$
$y = x^3$	$y = x^3 + 1$	$y = x^3$	$y = (x + 1)^3$
$y = x^3$	$y = x^3 - 1$		$y = (x + 1)^3$
$y = 1/x$	$y = 1/x + 2$	$y = x$	$y = x + 4$
$y = 1/x$	$y = 1/x - 2$		$y = x - 4$

$y = f(x)$	$y = -f(x)$	$y = f(x)$	$y = f(-x)$
$y = x$	$y = -x$	$y = x^2$	$y = -x^2$
$y = x^2$	$y = -x^2$	$y = x^2 + 3$	$y = (-x)^2 + 3$
$y = x^2 - 1$	$y = -(x^2 - 1)$	$y = x^3 - 1$	$y = (-x)^3 - 1$
$y = x^3 + 1$	$y = -(x^3 + 1)$	$y = 3x + 4$	$y = -3x + 4$

$y = f(x)$	$y = kf(x)$	$y = f(x)$	$y = f(kx)$
$y = x^2$	$y = 3x^2$	$y = x^2$	$y = (3x)^2$
$y = x^2 - 3$	$y = 2(x^2 - 3)$	$y = x^2 - 3$	$y = (2x)^2 - 3$
$y = x^3$	$y = 2x^3$	$y = x^3$	$y = (2x)^3$

Figure A11-1 [CEL6(Activity)]

The students were asked to enter the functions syntactically into Graphing application on their handhelds and to describe the transformations they observed within each set of functions. Eleanor questioned the students about different types of transformations (reflection, translation, rotation, and enlargement) and encouraged them to use these words when describing their observations. They were not instructed as to how they should communicate their observations, however it seemed to be an expected part of the classroom ethos that they would discuss their outcomes with their neighbours. The Smart Notebook file that Eleanor developed to present the activity to the students included the suggestion that the students should 'use 2 graphs per page' [CEL6(Struct)]. A typical student's response to the first stage of the activity is shown in Figure A11-2.

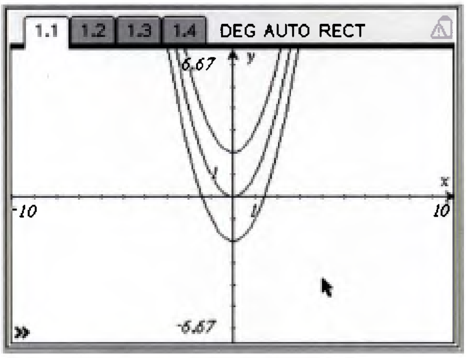


Figure A11-2 [CEL6(tns-S)]

During the lesson Eleanor moved around the classroom and responded to questions initiated by the students. These were mainly related to instrumentation issues concerning entering the functions, 'where is the squared key?' and 'how do I insert a new page' [CEL6(Trans)]. Ten minutes prior to the end of the lesson, Eleanor instigated one episode of whole class discourse in which she asked the students to open 'your page where you've explored this set' whilst gesturing to the set of functions shown in Figure A11-3.

$y = f(x)$	$y = f(x \pm a)$
$y = x^2$	$y = (x + 2)^2$
	$y = (x + 2)^2$
$y = x^3$	$y = (x + 1)^3$
	$y = (x + 1)^3$
$y = x$	$y = x + 4$

Figure A11-3 [CEL6] Function set selected for whole class display

The resulting screen capture view was on public display in classroom in Figure A11-4.

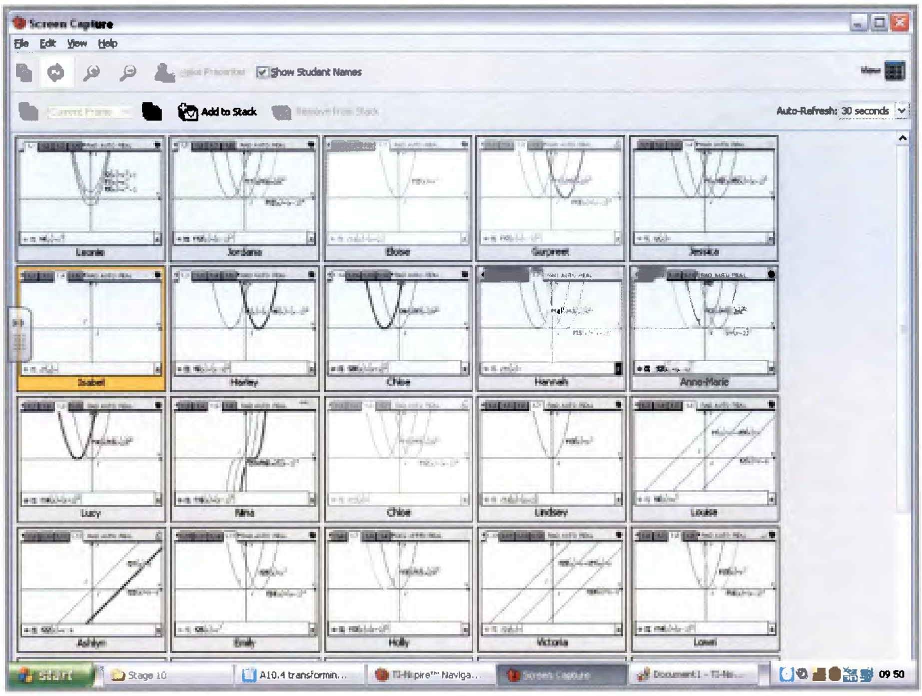


Figure A11-4 [CEL6(ScreenCapt)]

Eleanor attempted to elicit from the students the key generalisation for this transformation, i.e. that it resulted in a 'sideways shift' of ±a. No other mathematical representations were used during this discussion to justify or explore why this was true.

Interpretation of variance and invariance: Eleanor's interpretation of variance and invariance concerned the four different generalisations that can be concluded as a function f(x) is transformed according to the following conventions.

Function transformation	Geometric transformation
f(x)+a	vertical translation by a units
f(x+a)	horizontal translation by −a units
-f(x)	reflection in the x-axis
f(-x)	reflection in the y-axis
af(x)	stretch, parallel to the y-axis, scale factor a
f(ax)	stretch, parallel to the x-axis, scale factor 1/a

The invariant properties were the generalisations that could be made by varying different input functions according to the above conventions. As a consequence there were two variant properties being explored in this activity which were the variance resulting from the chosen function transformation and the variance in the syntactic inputs and geometric and outputs of the individual functions.

Implied instrument utilisation scheme: The activity used a scheme which was resonant with IUS5 in that Eleanor used a syntactic input to generate a graphical output, the geometric features of which were the focus for the classroom discourse (Figure A11-5).

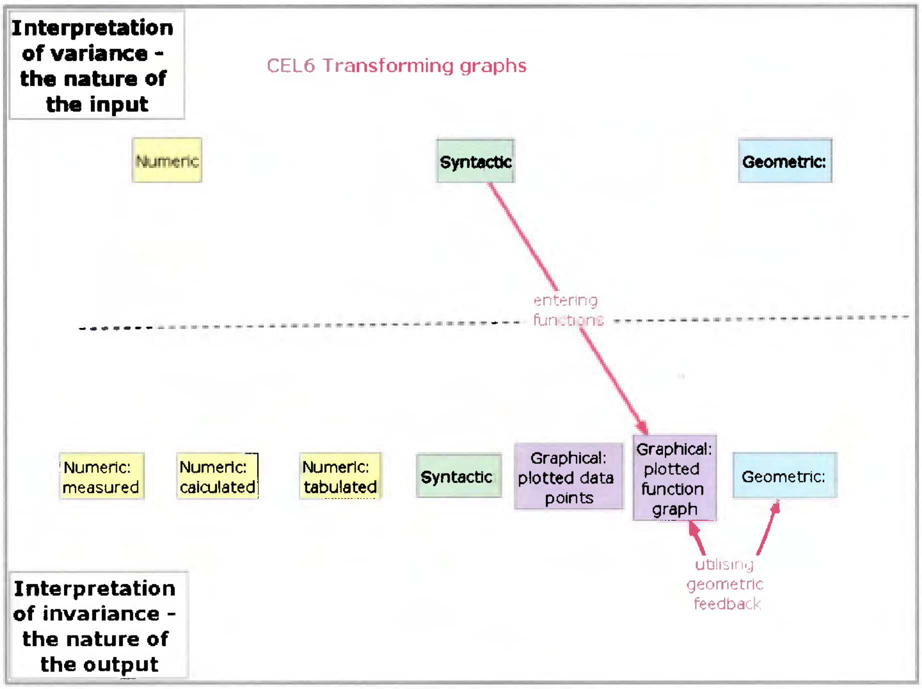


Figure A11-5 [CEL6] Instrument utilisation scheme

Hiccups identified from the lesson data: Nine hiccups were observed in this activity and they were grouped into the six categories as shown in Figure A11-6.

Name	Sources	Reference
EL6 Hiccup01 - Students' reluctance to focus on the outcomes related to their inputs	2	2
EL6 Hiccup02 - Students' struggling to see 'sets' of transformations	2	2
EL6 Hiccup03 - Instrumentation (S) - 'How do you draw them'	1	1
EL6 Hiccup04 - Instrumentation (S) - Entering x^3	1	2
EL6 Hiccup05 - Instrumentation(S) - all pages change the same	1	1
EL6 Hiccup06 - Diverse student responses make generalisations difficult	1	1

Figure A11-6 [CEL6] Activity hiccups

Several students' instrumentation issues were observed at the beginning of the activity to which Eleanor responded.

The omission of any labelling of the 'sets' of functions as they were laid out on the worksheet (or related teacher explanation) seemed to trigger the following hiccups during the lesson:

- Difficulties experienced by the students in making 'global' sense of the activity and 'notice' the invariant properties as Eleanor had intended through her activity design.

- Whilst the students were competent with entering the functions into the MRT, they did this in different combinations on different pages.
- The large number of different functions that the students were being asked to plot focussed the students' activity on entering as many of them as they could, rather than looking closely at any individual set and discussing or making written notes in relation to the outcomes. Some students had worked very diligently to input all thirty-nine functions into the MRT, but had failed to appreciate the 'sets' as Eleanor had envisaged.

As a consequence, Eleanor experienced difficulties in identifying any specific generalities on which to focus the whole-class discourse.

Possible evidence of situated learning: Evidence from the various data sources led to a list of seven actions by Eleanor that might provide evidence of her situated learning. These are shown in Figure A11-7.

Name	Sources	Reference
EL6 Action01 - Loaded TE to show how to input functions	1	1
EL6 Action02 - Led discussion about types of transformation	1	1
EL6 Action03 - Noticed students' expression of the generality	1	1
EL6 Action04 - Appreciated that the comparisons the students could make related to their existing knowledge of y=	1	1
EL6 Action05 - Attempts to be specific - but a lack of common 'labelling' led to issues....	2	2
EL6 Action06- Realised that the students don't all need to do so much -	1	1
EL6 Action07 - Suggested revisions to the activity design wrt her own questioning strategies	1	1

Figure A11-7 [CEL6] Evidence of teacher's actions

Eleanor was confident in her responses to students' instrumentation difficulties, giving quick tips such as 'control escape to undo' and 'press escape' and loading the teacher edition software to demonstrate how to input functions.

However, the hiccups experienced by Eleanor in this lesson led her to reflect deeply on aspects that she felt she would change in our post-lesson discussion.

Reflecting on her activity design, Eleanor commented,

> *I did not need all of the students to work through many similar problems - it was actually much more memorable to look at screens that appeared different, but, because of an underlying mathematical concept had something similar about them. This meant that I could have let the students choose their own functions to transform in particular ways - something that I will try next time. [CEL6(Eval-T)]*

Eleanor and I agreed that the underlying approach for the lesson was sound. However we discussed a redesigned format for the lesson, which responded to

Eleanor's comment that she could allow the students to explore their own functions. We also incorporated an element of the lesson which I had felt was a constituent part in developing the students' understanding of the outcomes of each of the transformations. To exemplify this, when the function y=f(x) is compared with y=f(x±a), the visible horizontal 'shift' in the graph is linked with the apparent 'shift' in corresponding values of x within the table of values for the functions when viewed side-by-side. This is shown for the function $y=x^2$ and $y=(x+2)^2$ in Figure A11-8.

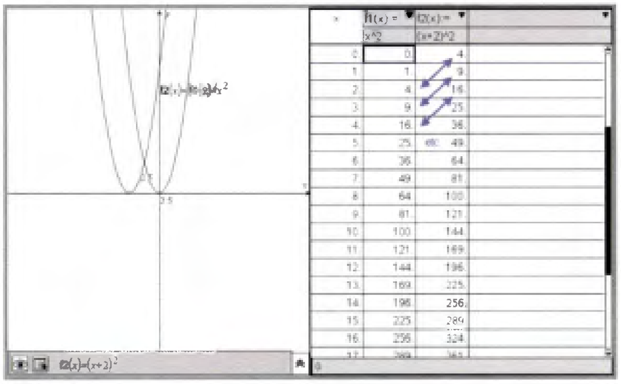


Figure A11-8 Using the MRT to explore the function table

In our discussion, when I showed this to Eleanor, she commented that she had never thought about this connection before, partly because she had learned the various transformations herself by rote. We jointly developed a recording sheet for the students to use in the subsequent lesson (see Figure A11-9). This required them to choose their own functions, record selected values from the function table and make their own written notes with respect to what they had noticed.

My chosen function	f(x) =

Exploration 1

x	f(x)	f(x) ± a

Exploration 2

x	f(x)	f(x ±a)

Notes

Figure A11-9 [CEL6(Activityv2)] The student recording sheet

Eleanor suggested that she might ask pairs of students to focus on particular transformation types with a view to them being able to summarise and justify the outcomes of their explorations to other members of the class.

CEL7 Generating circles

A single lesson taught in May 2009 with a year 11 class of twenty-nine students working at National Curriculum levels 7-8 from the GCSE higher tier examination syllabus. Eleanor wrote that her lesson objective was 'To establish the equation $x^2 + y^2 = r^2$' [CEL7(Quest2)].

Eleanor began the lesson with a quick poll in which she asked the students to give two numbers such that, when they were individually squared and summed they equalled twenty five. Eleanor shared the responses publicly, which initially included mainly the values 3 and 4. Eleanor resent the poll, asking the students to think again and provide a different pair of numbers and this time the responses included zeros. Some students' perturbation at the inclusion of zero led to a period of classroom discourse concerning the range and type of permissible variables that they were 'allowed' to include. Eleanor opened a TI-Nspire file that she had prepared before the lesson and moved to the first page, which included a spreadsheet into which she typed a pair of 'correct' numbers. The third column automatically summed these numbers as a 'check'. She then sent the file to the students' handhelds, and asked the students to continue to complete the spreadsheet with as many pairs of numbers as they could.

Examples of two students' responses during this activity are shown in Figure A11-10 and Figure A11-11.

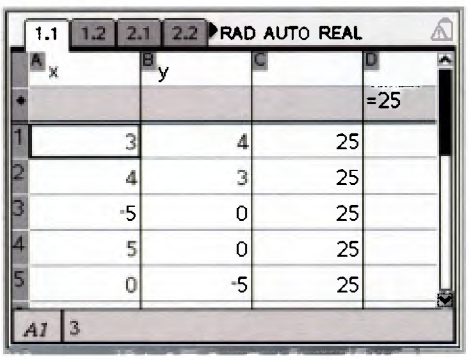


Figure A11-10 [CEL7(tns-S1)page1]

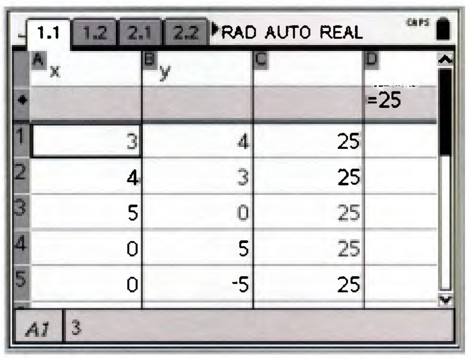


Figure A11-11 [CEL7(tns-S2)page1]

Having given the students about ten minutes to enter as many values into the spreadsheet as they were able, Eleanor asked them to move to the next page of the file, which was a graphing page onto which the data was (unknowingly to the students) being plotted automatically (Figure A11-12 and Figure A11-13).

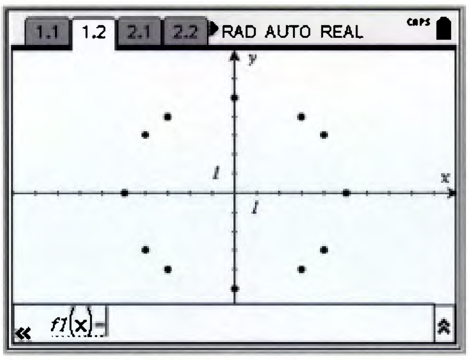


Figure A11-12 [CEL7(tns-S1)page2]

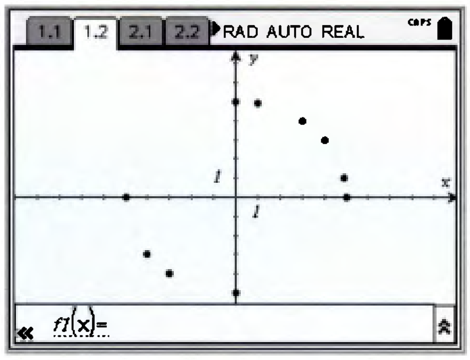


Figure A11-13 [CEL7(tns-S2)page2]

At this point Eleanor displayed the screen capture view of all of the students' handhelds to the class, prompting a period of purposeful classroom discourse concerning: the range of permissible values (some students had used the square root function), the symmetrical pictures that some students had produced; the nature of the shape that was emerging and how to form a 'more perfect circle' [CEL7(Trans)].

During the final lesson plenary Eleanor supported the students to make connections between the original problem and the algebraic function that would 'join' the points, leading to the establishment of the equation $x^2 + y^2 = 25$.

Eleanor reported that, in the next lesson (which I did not observe), she had supported the students to rearrange this equation to give $y = \sqrt{(25-x^2)}$, which they plotted on the graphing page and established that it fitted the plotted points.

Eleanor concluded that this lesson had been a very successful one for the following reasons:

- The numerical starting point for the lesson was highly accessible to all of the students, with everyone able to contribute a response.
- Having populated the spreadsheet with some initial values that satisfied the given condition, the 'reveal' of the graph promoted a high level of intrigue amongst the students, and provided a visible challenge for them to try find many more points.
- The sharing of responses prompted a high level of discussion, initially concerning the use of negative values and later on, the use of the square root symbol within the spreadsheet to generate many more numerical solutions.
- The students were quick to notice the 'symmetry' within the numbers for the possible solutions, i.e. if they knew (3, 4) was a solution then (4, 3), (-3, 4),

(3, -4), (-3, -4), (-4, -3), (-4, 3) and (4, -3) would also be solutions. They made strong links between the symmetry of the numbers within the coordinates with the geometric symmetry of their position on the graph.

Interpretation of variance and invariance: The starting point for this activity was finding values of x and y that satisfied $x^2 + y^2=25$, which Eleanor was interpreting as being in the real number domain only. However, as Eleanor left the definition of the domain open as a point of discussion throughout the lesson, most students began by considering positive integers and then expanded this domain to include zero and negative integers. Some students extended this domain to include positive and negative square roots of non-square numbers. The variant property was the range of permissible values for x and y, whereas the invariant property was the given initial condition.

Although Eleanor had designed the activity to include similar explorations of $x^2 + y^2=16$ and $x^2 + y^2=36$, and had included a page in the TI-Nspire file which encouraged the students to begin to generalise (Figure A11-14), this part of the activity was not observed during the lesson.

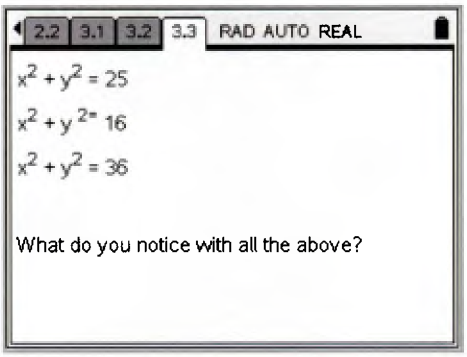


Figure A11-14 [CEL7(tns-S2)page3]

Implied instrument utilisation scheme: The instrument utilisation scheme developed by Eleanor for this lesson, whilst resonating with the type IUS5, clearly illustrates the range of representational forms that she utilised and the associated flow of the students' attentions and actions (Figure A11-15).

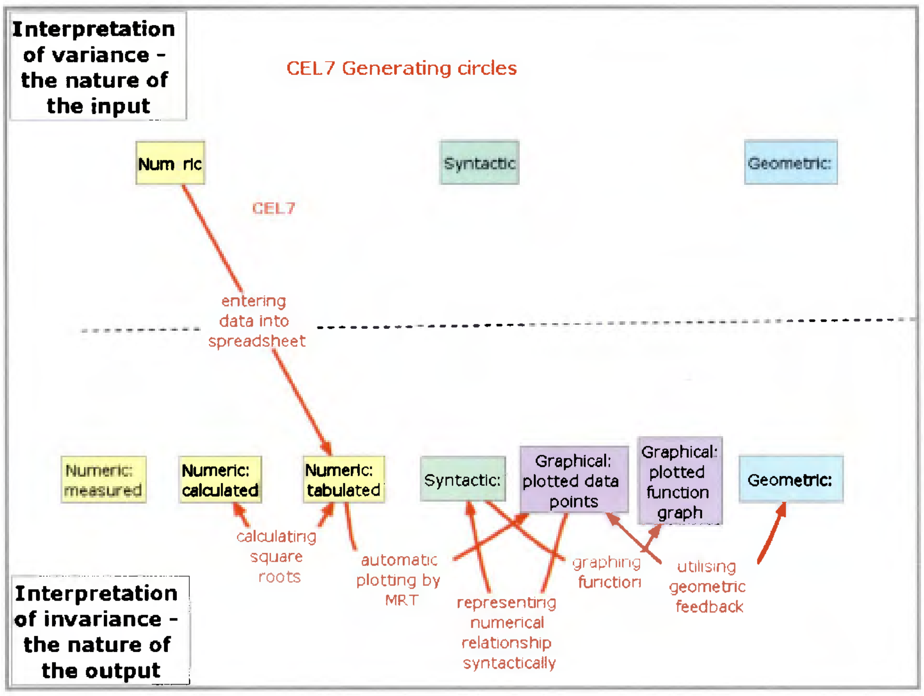


Figure A11-15 [CEL7] Instrument utilisation scheme

Hiccups identified from the lesson data: Four hiccups were observed in this activity and they were grouped into the categories as shown in Figure A11-16.

Name	Sources	Reference
EL7 Hiccup01 - Students question the range of possible variation	1	1
EL7 Hiccup02 - Instrumentation(S) - over writing formula in spreadsheet	1	1
EL7 Hiccup03 - Students are perturbed by the range of different pictures	1	1
EL7 Hiccup04 - Instrumentation (S) - 'Forgot' how to use the spreadsheet	1	1

Figure A11-16 [CEL7] Activity hiccups

Apart from a few early instrumentation issues experienced by the students, which were quickly responded to by Eleanor, most of the hiccups in the lesson related to the perturbations that the students were experience with respect to the domain of possible variation of the problem.

Possible evidence of situated learning: Evidence from the various data sources led to a list of eleven actions by Eleanor that might provide evidence of her situated learning. These are shown in Figure A11-17.

Name	Sources	Referenc
EL7 Action01 - Gave litle guidance as to how to respond to Quick Poll	1	1
EL7 Action02 - Using MRT to offer another representation	1	1
EL7 Action03 - Led a discussion about the domain of possible values for the variables	1	1
EL7 Action04 - Used students' own strategies to expand domain of possible answers	1	1
EL7 Action05 - Realised that the approach developed methods and concepts	1	2
EL7 Action06 - Commented that the increased speed and accuracy was motivating for the students.	1	1
EL7 Action07 - Commented that by connecting the different representations, the students' learning was conceptually deeper.	1	1
EL7 Action08 - Appreciated that files could be prepared in advance to speed up lesson and overcome students' instrumentatio	1	2
EL7 Action09 - Realised that, by having her own handheld she could be part of the class view and partipate if necessary	1	2
EL7 Action10 - Appreciated that the students were in control of the mathematical progression	1	1
EL7 Action11 - Learned about the students' domains of thinking about the variables	1	1
EL7 Action12 - Developed activity to model function syntactically in next lesson	1	1

Figure A11-17 [CEL7] Evidence of teacher's actions

This seemed to relate to Eleanor's:

- Adoption of an open-minded attitude to eliciting students' existing knowledge – and allowing the students' own responses to become the focus for the resulting responses from the quick poll and public screen capture display.
- Acknowledgement of and responses to the students' perturbations regarding the domain for the invariant properties.
- Development of a new strategy to overcome students' instrumentation difficulties (i.e. send a pre-written TI-Nspire file).
- Willingness to involve multiple representations to provide multiple perspectives for the variant and invariant properties.

Interestingly, Eleanor's perception of her own use of the MRT in the lesson was 'It's not very high tech' [CEL7(Int)].

CEL8 Triangles and squares

A single lesson taught in July 2009 with a group of thirty 13-14 year old girls working at National Curriculum levels 6-7 who were soon to commence the higher tier GCSE course. This is Eleanor's adaptation of the lesson developed and taught by Tim [STP6], which I had observed a week previously and described to Eleanor. As it was an appropriate topic on her scheme of work for this class we met and adapted the activity into a form with which she felt comfortable. Eleanor reported that she planned the lesson as 'a very initial look into Pythagoras' theorem in a completely different way that could only be thought about using the handheld technology' [CEL(Journ-T)].

As this was the first time this class would have used dynamic geometry, in common with Tim, she began the lesson with a activity which introduced the students to the instrumentation process required to 'grab and drag' a geometric point. Whereas Tim's original activity included three squares that were constructed with common vertices free to move in the 2-D plane (Figure A11-18), Eleanor adapted this so that the three squares were constructed on a common horizontal base (Figure A11-19). Eleanor also included a Calculator page on the right hand side of the handheld screen and hid selected vertices of the squares to focus the students' attention on just four dynamic aspects.

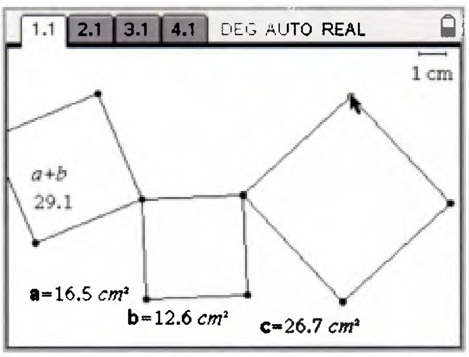


Figure A11-18 [STP6(tns-T)page1]

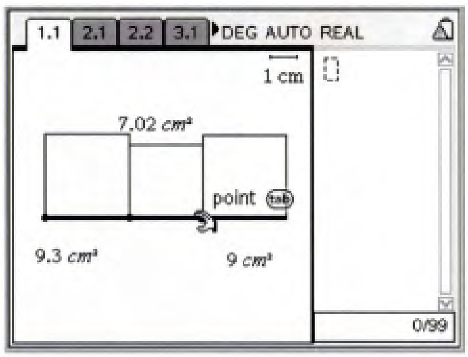


Figure A11-19 [CEL8(tns-T)page1]

In a similar initial activity to that devised by Tim, Eleanor asked the students to drag the vertices to try to make the squares have equal areas. She then supported the students to remember how to work out the side length of a square, if its area was known, using the adjacent *Calculator* page for this purpose.

The student then moved to the second activity which required the students to explore a dynamic geometric construction that related the areas of squares constructed on the edges of a triangle. They were asked to try to find a condition whereby the areas of the two smaller squares summed to give an area that was close to that of the largest triangle (Figure A11-20).

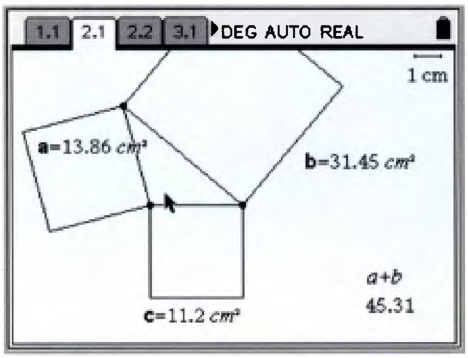


Figure A11-20 [CEL8(tns-T)page2]

Following this, Eleanor displayed the class view of the students' handhelds publicly and invited the students to offer their own conjectures in response to the many different cases that were visible (Figure A11-21).

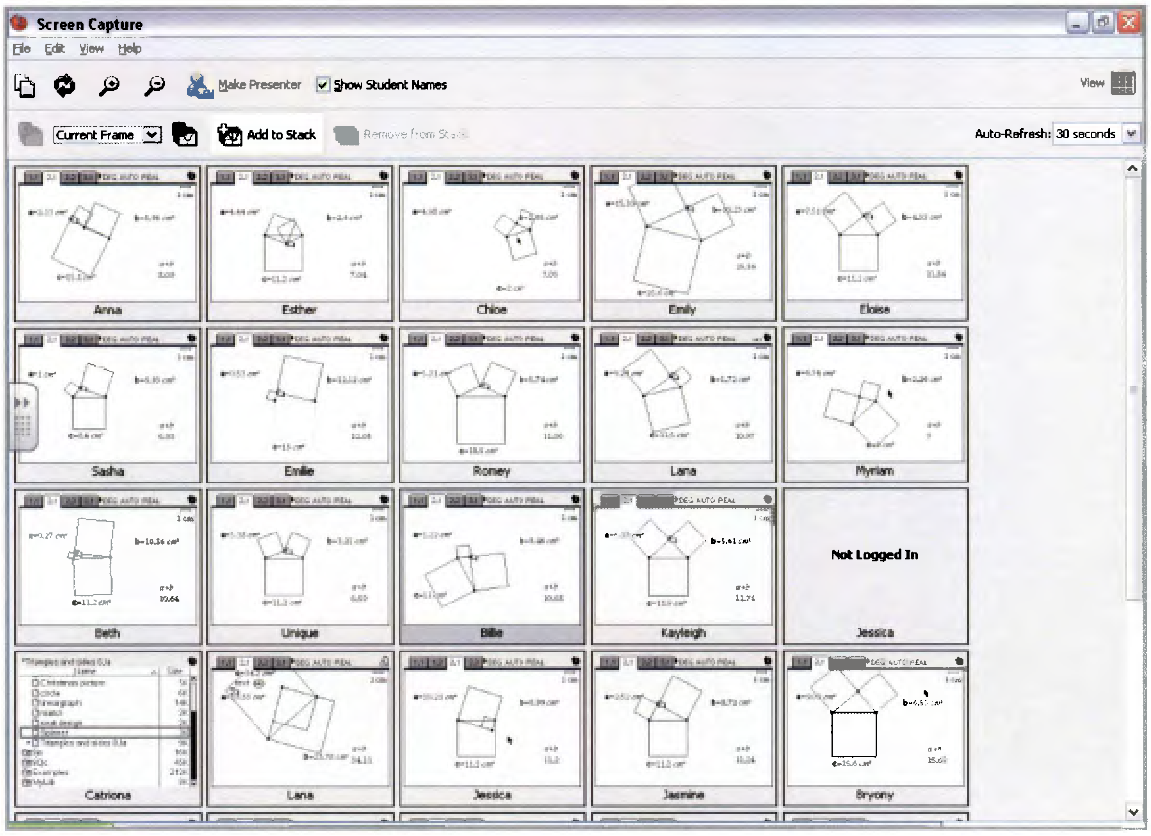


Figure A11-21 [CEL8(ScreenCapt)]

Eleanor then invited the students to respond to a quick poll question concerning their observations (Figure A11-22).

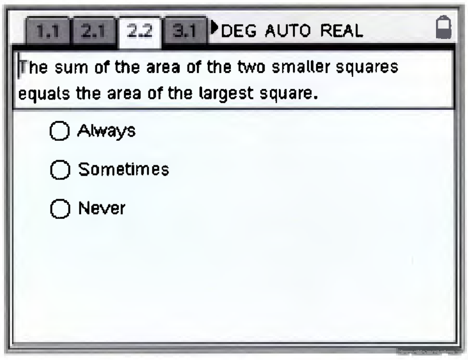


Figure A11-22 [CEL8(tns-T)page2.2]

During last few minutes of the lesson Eleanor asked the students to drag points within a traditional Pythagoras construction to create their own numerical problem (Figure A11-23). They were then asked to try to find as many of the missing measurements on their diagram as they could.

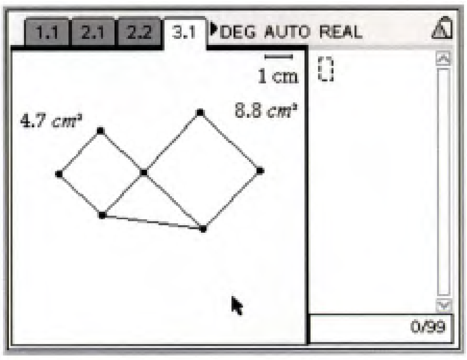


Figure A11-23 [CEL8(tns-T)page3.1]

Interpretation of variance and invariance: In the main activity, the invariant property concerned the condition for the sum of the two smaller squares equalling the area of the larger square when the contained triangle appeared to be right-angled. The invariant property was the positioning of the vertices of the triangle, which dictated the values of the resulting areas and area calculation.

Implied instrument utilisation scheme: In common with Tim's original lesson, the instrument utilisation scheme developed by Eleanor for this lesson was resonant with the type IUS3 as shown in Figure A11-24.

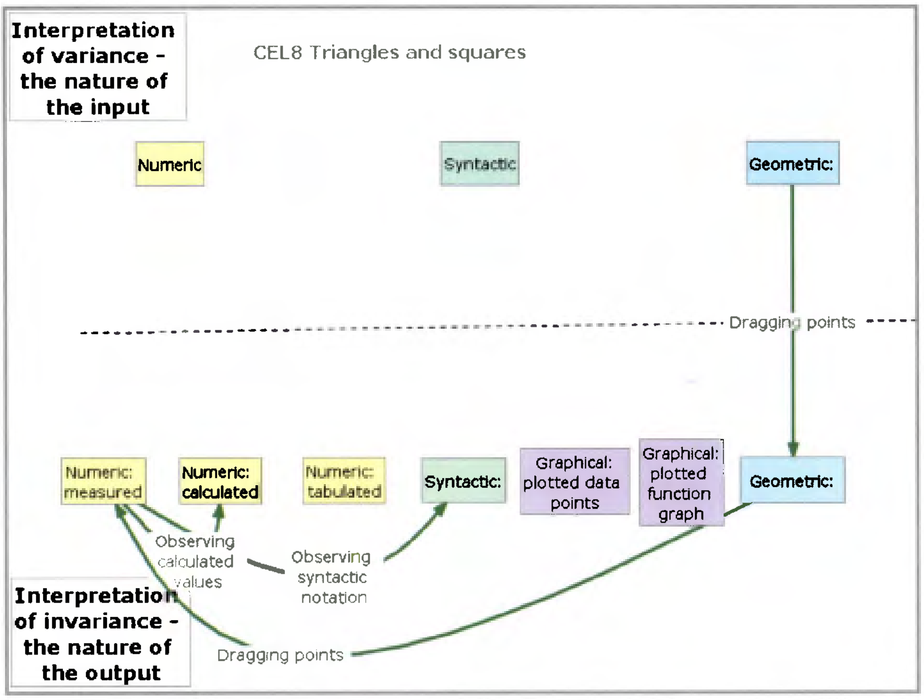


Figure A11-24 [CEL8] Instrument utilisation scheme

Hiccups identified from the lesson data: Nine hiccups were observed in this activity and they were grouped into the nine categories as shown in Figure A11-25.

Name	Sources	Referenc
EL8 Hiccup01 - MRT responses prompts students' question 'why is that a polygon'	1	1
EL8 Hiccup02 - Student's instrumentation issues - dragging	1	1
EL8 Hiccup03 - Need to define 'exactly' the same	2	3
EL8 Hiccup04 - Student's instrumentation issue	1	2
EL8 Hiccup05 - Students add areas to find length of line	1	1
EL8 Hiccup06 - Instrumentation issues - Ctrl G to function remove entry line	1	1
EL8 Hiccup08 - Students responses show squares in unexpected orientation	1	1
EL8 Hiccup09 - Students understanding of always, sometimes never	1	1
EL8 Hiccup10 - Student questions the missing square	1	1

Figure A11-25 [CEL8] Activity hiccups

It was noticeable that only one student experienced problems grabbing and dragging, which was unusual as this was the first time the students had used the dynamic geometry features of the handheld.

There were three incidents whereby the students questioned what was meant by 'equal', prompted by their concerns over what was an acceptable degree of accuracy whereby they had met the required condition.

The most significant hiccup in the lesson concerned the students' responses to the second activity in which they had satisfied the required condition, however, the orientation of their triangles what not what Eleanor had expected to see. She hesitated and asked me, 'Does Isla's still work?' (Figure A11-26) and later, to the class, 'Does Lana's work' (Figure A11-27) [CEL8(trans)].

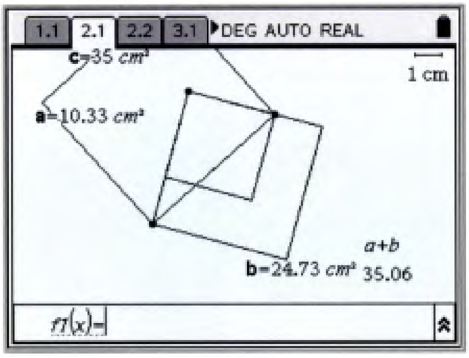


Figure A11-26 [CEL8(tns-S)] 'Does Isla's still work?

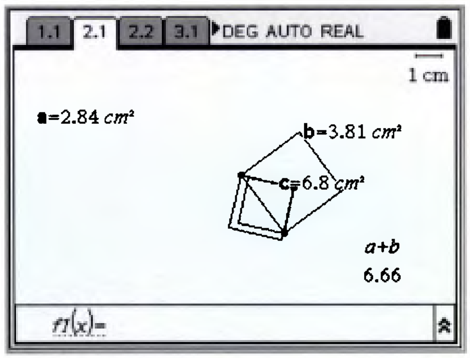


Figure A11-27 [CEL8(tns-S)] 'Does Lana's work?'

Possible evidence of situated learning: Evidence from the various data sources led to a list of twenty actions by Eleanor that might provide evidence of her situated learning. These are shown in Figure A11-28.

Name	Sources	Referenc
EL8 Action01 - Clear instrumentation instructions to students	1	3
EL8 Action02 - Responds to student's question 'why is that a polygon'	1	1
EL8 Action03 - Supports individual student's instrumentation issue	1	3
EL8 Action04 - Draws students' attentions to other MRT application to use later in activity	1	1
EL8 Action05 - Gives instrumentation tips - Esc, Ctrl z	1	1
EL8 Action06 - Explicitly uses term 'area'	1	1
EL8 Action07 - Sets 'bridging task to find lengths of sides of square using MRT	1	1
EL8 Action08 - Encourages students to share their responses with each other	1	1
EL8 Action09 - Leads discussion around student's work on handheld	1	1
EL8 Action10 - Directs students to look at what's changing	1	1
EL8 Action11 - Directs students to make sense of the variables	1	1
EL8 Action12 - Noted that the students' interpretation of the generality was not as expected	1	1
EL8 Action13 - Encourages students to think about the missing square	1	1
EL8 Action14 - Appreciated change in didactic contract	1	1
EL8 Action15 - Praises Myriam's screen	1	1
EL8 Action16 - Gives clear instrumentation to students	1	1
EL8 Action17 - Noted that accuracy isses - conflict raised	1	1
EL8 Action18 - Directs focus from student's own screen to the class few	1	1
EL8 Action19 - Acknowledged that the activity needed more time	2	3
EL8 Action20 - Suggests changes to the task	1	1

Figure A11-28 [CEL8] Evidence of teacher's actions

The number of actions taken by Eleanor in the lesson was extensive and those which imply that she was actively developing strategies that may have developed as a result of her growing classroom experiences were:

- The very explicit directions that she was giving to the students to focus on the specific aspects of each representation and, in particular the features that concerned the generality that she was supporting them to notice.
- The 'scripts' that she had developed to support students to learn the basic instrumentation skills to enable them to interact with the technology.
- The recognition that some bridging activities were sometimes needed to support the students to make connections between the various aspects of the MRT activity.

CEL9 Crossing linear graphs

A detailed description of this lesson is provided in the main thesis (Section 6.3.2.1)

Interpretation of variance and invariance: In the initial activity, the invariant property was the co-ordinate point and the variant property was the infinite number of linear functions that might pass through it. This led to the generalisation that, if two lines were to cross, they would share a common coordinate.

Implied instrument utilisation scheme: The implied instrument utilisation scheme developed for this lesson was resonant with the type IUS3 as shown in Figure A11-29.

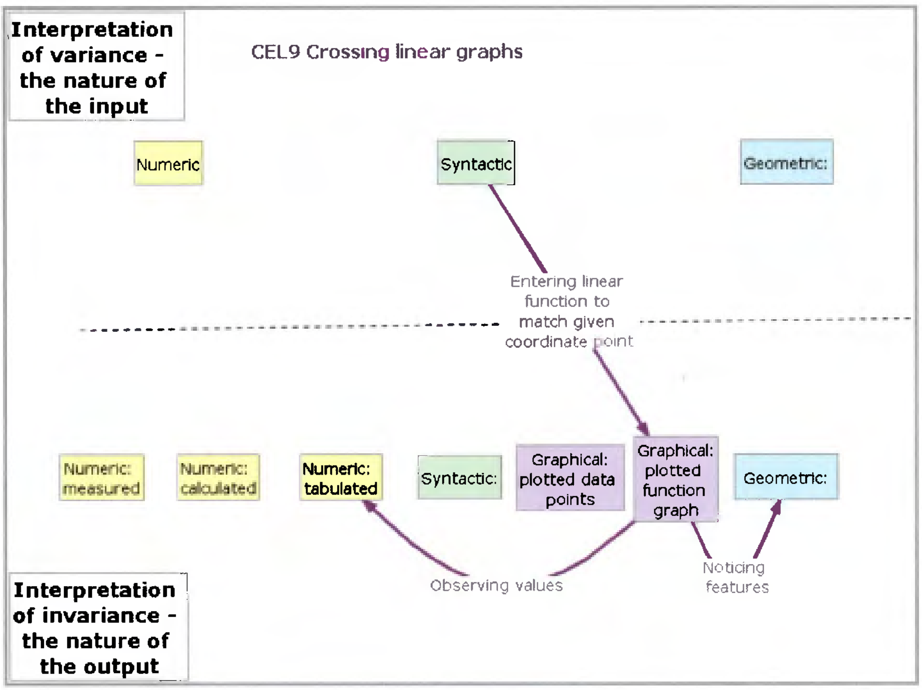


Figure A11-29 [CEL9] Instrument utilisation scheme

Hiccups identified from the lesson data: Nine hiccups were observed in this activity and they were grouped into the nine categories as shown in Figure A11-30.

Name	Sources	Referen
EL9 Hiccup01 - Students give diverse responses - quick poll - used positiviely by teacher	2	4
EL9 Hiccup02 - Instrumentation (S) - 'where's equals'	1	1
EL9 Hiccup03 - Instrumentation (S) 'how do you get down' - to send quick poll response	1	1
EL9 Hiccup04- 'Emily how did you do it'	1	5
EL9 Hiccup05 - 'How do you get a graph that goes straight up'	1	1
EL9 Hiccup06 - Instrumentation (S) 'How do you get onto it' - referring to full function entry line	1	1
EL9 Hiccup07 - Instrumentation(T) struggles to display table of values	1	1
EL9 Hiccup08 - Instrumentation (S) - 'How do you get that' - referring to the table	1	1

Figure A11-30 [CEL9] Activity hiccups

A hiccup within the lesson 'Crossing linear functions' [CEL9] challenged Eleanor's understanding of the meaning of 'different' further when we discussed the screens produced by Maria in Figure A11-31 and

Figure A11-32 in response to the same activity.

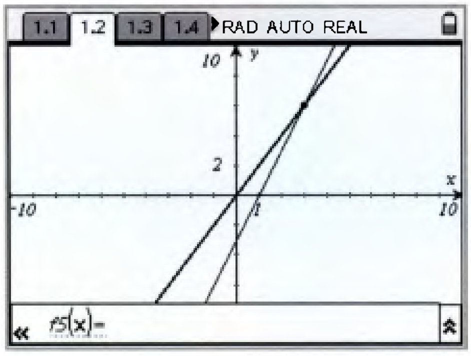


Figure A11-31 [CEL9(tns-S)] Maria's graphs

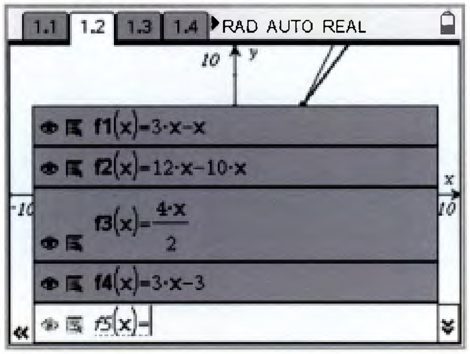


Figure A11-32 [CEL9(tns-S)] Maria's functions

Eleanor did not have a chance to make sense of Maria's problem in order to respond to it during the lesson. However, it did become the topic of one of our post-lesson discussions.

Possible evidence of situated learning: Evidence from the various data sources led to a list of nineteen actions by Eleanor that might provide evidence of her situated learning. These are shown in Figure A11-33.

Name	Sources	Referenc
EL9 Action01 - Acknowledges MRT dictates alternative teaching approach for her	1	1
EL9 Action02 - Acknowledges that the MRT necessitates a different approach	1	1
EL9 Action02 - Presents a 'big picture' problem to the students	1	1
EL9 Action03 - Appreciates that students have interpreted the initial problem differently	2	2
EL9 Action04 - Makes herself live presenter for the first time	1	1
EL9 Action05 - Asks question that focuses students on the representation	2	5
EL9 Action06 - Asked a question that requires students to consider the 'big picture'	2	3
EL9 Action07 - Sends quick poll qustion to identify students' prior knowledge	1	1
EL9 Action08 - Leads discussion to develop knowledge	1	2
EL9 Action09 - Resends same poll to expand responses	1	1
EL9 Action10 - Gives clear instrumentation steps	1	4
EL9 Action11 - Instructs Emily 'to avoid it at the moment'	1	3
EL9 Action12 - Asks a question to extend the students' repertoires of responses	1	1
EL9 Action13 - Notices that there are enough correct responses to begin a class discussion	1	1
EL9 Action14 -Uses one student's response to stimulate discourse	2	3
EL9 Action15 - Asks students to identify different responses that satisfy the required constraint	1	1
EL9 Action16 - Appreciates that her use of the shared display has developed	1	1
EL9 Action17 - Suggests plans for subsequent lesson that build on lesson outcomes	1	2
EL9 Action18 - Recognises significance of student's response	0	0
EL9 Action19 - Bridges paper and pencil approaches with MRT approach	1	1

Figure A11-33 [CEL9] Evidence of teacher's actions

APPENDIX 12: ELEANOR'S LESSON PLAN AND LESSON STRUCTURE FOR 'CROSSING LINEAR GRAPHS' [CEL9]

Lesson plan [CEL9(LessPlan)]

Purpose and context of lesson Lesson 1/3	To explore the intersection of linear equations graphically as a means of an introduction to simultaneous equations and to explore how the Nspire technology can enable us to do this. To revise the linear equation y=mx + c and connections between linear graphs drawn.		
Learning Objective(s) PLTS	To find linear equations that have one common solution and to explore the mathematics behind these common factors. reflective learners, creative thinkers		
Literacy/numeracy			
Cross Curricular Opportunities			
Approx time	**Activity**	**Resources**	**AFL opportunities**
Introduction Development (Focus/step) (Focus/step) Plenary	Choose own working pair. Nspires and Nspire paper. Starbucks, Seattle – cappuccino & latte prices. Could we work this out, discuss. Our intention is to be able to eventually be able to solve this and problems like this. Send tns file, on first page point (3,6) has been plotted. How many straight lines go through this point? Quick poll 'Write a rule that is true for (3,6). 30 secs only to come up with an idea, save to portfolio, re-submit. Discussion to lead to x,y notation and axis and linear equation. Add a function that goes through the point. NB y=f(x). Now add a 2^{nd}. How do we know that they cross? Now add a results table and look carefully at values. Come up with 10 different linear equations that all go through (3,6). Record thoughts, findings. Pull together ideas.	Nspires, A3 paper Open Smart, TE, navigator	
Personal Study Task	N/A		
PLTS* - independent enquirers, creative thinkers, reflective learners, team workers, self managers, effective participators			

Lesson structure [CEL9(LessStruct)]

THIS IMAGE HAS BEEN REDACTED DUE TO THIRD PARTY RIGHTS OR OTHER LEGAL ISSUES

Appendix 13: Tim's repertoire of responses to his classroom hiccups

Description of hiccup	No immediate response repertoire	Developing response repertoire	Well-rehearsed response repertoire
TP6 Hiccup1 - Difficulties over identification of dynamic objects			
TP6 Hiccup2 - Students' mis-interpretations of activity - 'different way around'			
TP6 Hiccup3 - Instrumentation (T) 'your c has gone off the screen'			
TP6 Hiccup4 - Instrumentation (S) - grabbing and dragging			
TP6 Hiccup5 - Jump from MRT activity to trad paper and pencil problem			
TP7 Hiccup1 - Instrumentation (T) - Lines crossing circle only once			
TP7 Hiccup2 - Student not believing the MRT response to sqrt(-11)			
TP7 Hiccup3 - Student questioned the robustness of the generality			
TP7 Hiccup4 - Student's dissatisfaction about line and circle only touching once			
TP7 Hiccup5 - Students struggled to see the purpose for the generalisation			
TP7 Hiccup6 - Unprompted student question - responded with MRT (moving centre of circle)			
TP7 Hiccup7 - Students own IUS is in conflict with the activity design			
TP8 Hiccup1 - Instrumentation (T) - Transfer of file resulted in lost functionality			
TP8 Hiccup2 - Students appear to enter 'crazy' functions			
TP8 Hiccup3 - Instrumentation (S) - Not understanding how MRT file is working			
TP9 Hiccup01 - Student's instrumentation issue entering X^2			
TP9 Hiccup02 - Teacher's assumption that students 'notice' the brackets			

Description of hiccup	No immediate response repertoire	Developing response repertoire	Well-rehearsed response repertoire
TP9 Hiccup03 - Teacher's assumption that students 'see' the invisible multiplication sign		■	
TP9 Hiccup04 - Students' activity sequencing issue.	■		
TP9 Hiccup05 - Teacher's assumption activity sequencing will lead to counter-examples in students' theories		■	
TP9 Hiccup06 - Technology fails as teacher goes to use live presenter in final plenary		■	
TP9 Hiccup07 - Noticing that the worksheet uses letters other than x		■	
TP9 Hiccup08 - Student doubts authority of MRT 'Why didn't it say false'			■
TP9 Hiccup09 - Insufficient specificity about labelling objects under discussion		■	
TP10 Hiccup01 - Network communication errors		■	
TP10 Hiccup02 – Student's instrumentation difficulty locating document in folder			■
TP10 Hiccup03 - Students' instrumentation difficulties plotting free coordinate points			■
TP10 Hiccup04 - Student's instrumentation difficulty locating the equals sign on the key pad			■
TP10 Hiccup05 - Teacher's laptop hibernates, so has to give instrumentation instructions verbally		■	
TP10 Hiccup06 - Teacher's instrumentation issues - moved from handheld to computer software			■
TP10 Hiccup07 - Teacher's instrumentation difficulty - could not display quick poll/graph screen simultaneously		■	
TP10 Hiccup08 - Measured equation is in standard form		■	
TP10 Hiccup10 - Students interpretation of gradient is not what was expected		■	
TP10 Hiccup11 - Students uneasy about having a different response		■	
TP10 Hiccup12 - Realised that initial example was not a helpful one		■	
TP10 Hiccup13 - Students' confusion over gradient		■	

APPENDIX 14: ELEANOR'S REPERTOIRE OF RESPONSES TO HER CLASSROOM HICCUPS

Description of hiccup	No immediate response repertoire	Developing response repertoire	Well-rehearsed response repertoire
EL6 Hiccup01 - Students' reluctance to focus on the outcomes related to their inputs			
EL6 Hiccup02 - Students' struggling to see 'sets' of transformations			
EL6 Hiccup03 - Instrumentation (S) - 'How do you draw them'			
EL6 Hiccup04 - Instrumentation (S) - Entering x^3			
EL6 Hiccup05 - Instrumentation(S) - all pages change the same			
EL6 Hiccup06 - Diverse student responses make generalisations difficult			
EL7 Hiccup01 - Students question the range of possible variation			
EL7 Hiccup02 - Instrumentation(S) - over writing formula in spreadsheet			
EL7 Hiccup03 - Students are perturbed by the range of different pictures			
EL7 Hiccup04 - Instrumentation (S) - 'Forgot' how to use the spreadsheet			
EL8 Hiccup01 - MRT responses prompts students' question 'why is that a polygon'			
EL8 Hiccup02 - Student's instrumentation issues - dragging			
EL8 Hiccup03 - Need to define 'exactly' the same			
EL8 Hiccup04 - Instrumentation (S) - Dragging			
EL8 Hiccup05 - Students add areas to find length of line			
EL8 Hiccup06 - Instrumentation issues - Ctrl G to function remove entry line			

Description of hiccup	No immediate response repertoire	Developing response repertoire	Well-rehearsed response repertoire
EL8 Hiccup07 - Students responses show squares in unexpected orientation			
EL8 Hiccup08 - Students understanding of always, sometimes never			
EL8 Hiccup09 - Student questions the missing square			
EL9 Hiccup01 - Students give diverse responses to quick poll			
EL9 Hiccup02 - Instrumentation (S) - 'where's equals?'			
EL9 Hiccup03 - Instrumentation (S) 'how do you get down?' - to send quick poll response			
EL9 Hiccup04- 'Emily how did you do it?'			
EL9 Hiccup05 - 'How do you get a graph that goes straight up?'			
EL9 Hiccup06 - Instrumentation (S) 'How do you get onto it?' - referring to full function entry line			
EL9 Hiccup07 – Instrumentation (T) struggles to display table of values			
EL9 Hiccup08 - Instrumentation (S) - 'How do you get that?' - referring to the table			