When the rules of discourse change, but nobody tells you - the case of a class learning about negative numbers

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Abstract. In this article we introduce a research framework grounded in the assumption that thinking is a form of communication and that learning a school subject such as mathematics is modifying and extending one's discourse. This framework is then applied in the study devoted to the learning of negative numbers. The analysis of data is guided by questions about (a) the discourse on negative numbers as such, and the features that set it apart from the mathematical discourse with which the students have been familiar when the learning began; (b) students' and teacher's efforts toward the necessary transition to the new meta-discursive rules, and (c) effects of the learning teaching process, that is, the extent of discursive change resulting from these efforts. Our findings lead to the conclusion that discursive change, rather than being necessitated by an extra-discursive reality, is spurred by *communicational conflict*, that is, by the situation that arises whenever different interlocutors seem to be acting according to differing discursive rules. Another conclusion is that school learning requires an active lead of an experienced interlocutor and is fueled by a *realistic communicational agreement* between her and the learners.

Keywords: learning, negative numbers, mathematics, discourse, communication, discursive routine, communicational conflict, communicational agreement

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“How does it happen that there are people who do not understand mathematics?” wondered French mathematician Henri Poincaré in the beginning of the previous century. He had a good reason to puzzle: “If [mathematics] invokes only the rules of logic, those accepted by well-formed minds, how does it happen that there are so many people who are impervious to it?” Since then, much has been done to fathom the mechanism of mathematical success and failure, and today nobody seems to believe anymore that following rules of logic is all one needs to master mathematics. Nowadays, there is also a wide agreement that the three usual suspects – the curriculum, the teacher, and personal features of the students – are only a part of the story. The question of how these and other factors combine in the classroom to produce a given type of effect, however, is not easy to answer. As long as we decompose communicational processes into components to be investigated independently of each other, we are bound to miss something important. In the attempt to come to grips with this unyielding complexity, we adopted in our studies we have been employing theoretical lens that make us able to keep an eye on the evolving mathematics while also capturing some hitherto unnoticed aspects of the communicational events. In this article we present vignettes from one of these investigations, the study devoted to one 7th grade class learning about negative numbers.

Our wish to investigate students’ first encounters with negative numbers was motivated by the belief that this topic was somehow unique among mathematical subjects learned at school. We felt that for many students, negative numbers were particularly challenging, and not necessarily because of the intricacy of arithmetical techniques involved. Our conjecture was reinforced, among others, by the autobiographical account of the French writer Stendhal[2] who, in his memoirs, recalled his difficulty with the claim that “minus times minus is plus”. “That this difficulty was not explained to me was bad enough,” he said, “What was worse was that it was explained to me by means of reasons that were obviously unclear to those who employed them.” Since according to Stendhal himself, his teachers were certified “mathematical luminaries,” the claim that they were unsure of their reasons did not sound convincing. We conjectured instead that the teacher’s reasons were not Stendhal’s own: to be persuaded, Stendhal needed a different kind of justification. Our classroom investigation was driven, among others, by the wish to shed light on this puzzling, seemingly unbridgeable disparity between teachers’ offerings and student’s needs. We believed that the importance of what we would learn while unraveling this quandary would go beyond the particular case of negative numbers.

[2] Pseudonym of Marie-Henri Beyle, 1783 – 1842.

The study took place in a typical Israeli junior secondary school in a middle-class area. The class of 12-13 year olds was observed in the course of 30 one-hour meetings devoted to negative numbers. The teacher's expositions and whole-class discussions were videotaped and audio-recorded. In addition, during times when the students were working in small groups, a camera was directed at two designated pairs. These two pairs were also regularly interviewed before, during and after completing the learning sequence. The interactions, which were all held in Hebrew, were transcribed in their entirety and, for the sake of this article, partially translated into English. Since the aim of the study was to observe learning rather than to assess instruction, the teacher (the second author) was given a free hand in deciding about the manner to proceed. Her teaching turned out to be guided by the principle of always probing students' own thinking before presenting them with other people's ready-made ideas. This principle clearly manifests itself already in Episode 1, presented below, in which the teacher tries to elicit what knowledge about negative numbers the students might have already possessed at the time she started to teach the topic.

Episode 1: The first lesson on negative numbers

1	Teacher:	Have you heard about negative numbers? Like in temperatures, for instance?
2	Omri:	Minus!
3	Teacher:	What is minus?
4	Roi:	Below zero.
5	Teacher:	*Temperature* below zero?
6	Sophie:	Below zero… it can be minus five, minus seven… Any number.
7	Teacher:	Where else have you seen positive and negative numbers?
8	Omri:	In the bank.
9	Teacher:	And do you remember the subject “Altitude”? What is *sea level*?
10	Yaron:	Zero
11	Teacher:	And above sea level? More than zero?
12	Yaron:	From one meter up.

As already stated, the question that motivated our study regarded the course of the change that, so we believed, had to occur in students’ thinking before they could come to terms with the new type of numbers. Our goal was to fathom the ways in which the student and the teachers coped with invisible hurdles. We believed that answering this question would bring new insights about classroom processes. We also felt, however, that such insights are unlikely to come unless we operationalize our basic concepts. Thus, our preliminary question was “What does it mean to learn mathematics?” Our answer was that it may be useful to talk about all cognitive processes, and about learning in particular, in "communicational" terms.

1. What does it mean *to learn mathematics*? The communicational answer.

The first classroom conversation on negative numbers, as documented in Episode 1, has shown that the term *negative number* was not entirely unknown to the children. At the same time, the brief conversation, as well as additional data from earlier interviews with the children, indicated that they could not say much about it. What we saw can be summarized as showing that the children could identify the discourse on negative numbers when they heard it, and could associate the notion with some other expressions, such as *minus* or *below zero*, but they were much less likely to be proactive interlocutors. One can also say that the goal of the learning that was about to occur was for the children to become fluent participants in the discourse on negative numbers. Having said this, we can now define learning as a *process of changing one’s discursive ways in a certain well-defined manner*. We choose to speak about *changing* a discourse rather than about constructing a new one, because new discourses are never created from scratch; rather, they develop out of discourses in which the learners are already fluent. In particular, a person who learns about negative numbers alters and extends her discursive skills so as to become able to use this form of communication in solving mathematical problems.

The assumption that school learning can be seen as the activity of modifying and extending one's communicational practices is the basis of the conceptual framework which has been adopted in our study and which is called *communicational* (Sfard 2000a, b, c, 2001, 2002; Kieran, Forman, & Sfard, 2002; Sfard & Lavi 2004; Ben Yehuda et al., 2004). This framework is close in its basic tenets to *discursive psychology*, as described, for example, in Edwards & Potter (1992), Harré & Gillett (1995), Edwards (1997). This basic tenet comes hand in hand with the claim that *thinking is a form of communication*. Indeed, a person who thinks can be seen as communicating with herself. This is true whether the thinking is in words or in images and whether it is in spoken words or in writing. Our thinking is clearly a dialogical endeavor, where we inform ourselves, we argue, we ask questions, and we wait for our own responses. If so, becoming a participant in a mathematical discourse is tantamount to learning to *think* mathematically. The word *discourse* is used in this article in the broad sense of an act of communication which, let us stress it again, does not have to be verbal or public.

The communicational approach can be traced back to Vygotsky's basic thesis according to which *patterned, collective forms of distinctly human forms of doing are developmentally prior to the activities of the individual*. Whereas more traditional schools of thought assumed that the individual development proceeds from personal acquisitions to the participation in collective activities, Vygotsky reversed the picture and claimed that people go from the participation in collectively implemented activities to similar forms of doing, but which they are now able to perform single-handedly. According to this vision, learning to speak, to solve mathematical problem or to cook means a gradual transition from being able to take a part in collective implementations of a given type of task to becoming capable of implementing such tasks in their entirety and on one's own accord. Eventually, a person can perform on her own and in her unique way entire sequences of steps which, so far, she would only execute with others. The tendency for individualization – for turning patterned collective doings into activities for an individual – seems to be one of the hallmarks of humanness, and it is made possible by our capacity for overtaking roles of others. Interpersonal communication is one of those collective activities that undergo individualization in the course of one's development. When it happens, the child becomes capable of uniquely human form of thinking.

The communicational conceptualization goes back also to Wittgenstein's critique of potentially harmful dichotomies implicit in the ways we talk about thinking (Wittgenstein 1953). These dichotomies, says Wittgenstein, interfere with our attempts to understand what thinking is all about. Among the principal targets of his criticism is the split between thought and its "expression," or between thinking and communicating. "Thought is not an incorporeal process which lends life and sense to speaking, and which it would be possible to detach from speaking" (§339), he says. True, speech may be said to be more or less *thoughtful* or *meaningful*, depending on how the speaker or listeners feel about it and how skillful they are in operating with, or on the grounds of, what is being said. However, one should not conclude from here that things said may contain greater or lesser proportion of the entity or process called "thought". We may simply experience diverse discursive acts differently. For example, we can communicate more or less flexibly, with greater or lesser facility, and with more or less self-assurance.

The dichotomy between thought and its content has been found equally problematic. This distinction seems unquestionable as long as the talk is about concrete material objects. And yet, as has been argued by many writers (Foucault 1972; Dreyfus & Rabinow 1982; Gottdiener 1995; Sfard 2000), the divide between thought and its referent loses ground when it comes to more abstract objects, such as numbers.[3] One can argue that these objects are, in fact, metaphors inspired by the discourse on material reality: They come into being when we replace discursive processes with nouns and then use these nouns within phrases modeled after the discourse on concrete objects. Think, for example, about the use of the noun *[number] five* as a substitute for counting up to 5 or about the term *function* x^2 which we use when trying to say something general about the operation of squaring numbers. In this context, think also about the phrases such as "Given a function…", "There are numbers such that…" – expressions that can be read as implying an existence of extra-discursive entities for which the nouns are but linguistic pointers. As a result of a prolonged use of such objectifying discursive forms, the putative entities often become "experientially real" to the user, who starts act up on them as of they were a part of a mind-independent reality.

Our focus in this paper is on one particular type of thinking, which we call *mathematical.* Mathematical discourse is made distinct by a number of unique features.

[3] Some writers go on to question even the ostensibly unproblematic case of the talk about concrete objects.

(1) *Word use.* A discourse counts as *mathematical* if it features mathematical words, such as those related to quantities and shapes. While becoming a participant of a mathematical discourse, the students may have to learn terms that they have never used before and that are unique to mathematics. Expressions such as *negative two* (or *minus two*) or *negative half* (*minus half*) are good examples. While number-related words may appear in non-specialized, colloquial discourses, mathematical discourses as practiced in schools or in academia dictate their own, more disciplined uses of these words. *Word use* is an all-important matter since, being tantamount to what others call "word meaning" ("The meaning of a word is its use in language," Wittgenstein 1967, p. 20, §43), it is responsible to a great extent for how the user sees the world.

(2) *Visual mediators* are means with which participants of mathematical discourses identify the object of their talk and coordinate their communication. While colloquial discourses are usually mediated by images of material things, that is, by concrete objects that are pointed to with the nouns or pronouns and that may be either actually seen or just imagined, mathematical discourses often involve symbolic artifacts, created specially for the sake of this particular form of communication. The most common examples include mathematical formulae, graphs, drawings, and diagrams. While communicating, we attend to the mediators in special ways. Think, for example, about the extended number line and the way you scan it with your eyes while trying to add two numbers. Contrary to what is implied by the common understanding of the role of tools, within the communicational framework one does not conceive of artifacts used in communication as mere auxiliary means for "conveying" or "giving expression to" pre-existing thought. Rather, one views them as a part and parcel of the act of communication, and thus of the cognitive processes themselves.

(3) *Narrative* is any text, spoken of written, which is framed as a description of objects, of relations between objects or activities with or by objects, and which is subject to *endorsement* or rejection, that is, to being labeled as *true* or *false*. Terms and criteria of endorsement may vary considerably from discourse to discourse, and more often than not, the issues of power relations between interlocutors may in fact play a considerable role. This is certainly true about social-sciences and humanistic narratives such as history or sociological theories. Mathematical discourse is conceived as one that should be impervious to any considerations other than purely deductive relations between the narrative's different elements. In the case of scholarly mathematical discourse, the consensually endorsed narratives are known as mathematical theories, and this includes such discursive constructs as definitions, proofs, and theorems. More generally, narrative is any text, spoken of written, framed as a description of objects, of relations between objects or activities with or by objects. Terms and criteria of endorsement may vary considerably from discourse to discourse, and more often than not, the issues of power relations between interlocutors may play a considerable role.

(4) *Routines* are well-defined repetitive patterns in interlocutors' actions, characteristic of a given discourse. Specifically mathematical regularities can be noticed whether one is watching the use of mathematical words and mediators or following the process of creating and substantiating narratives about number or geometrical shapes. In fact, such repetitive patterns can be seen in almost any aspect of mathematical discourses: in mathematical forms of categorizing, in mathematical modes of attending to the environment, and in ways of viewing situations as "the same" or different, which is crucial for the interlocutors' ability to apply mathematical discourse whenever appropriate. The list is still long. In the majority of discourses the participants are unaware of the fact that their actions disclose structural regularities, and they certainly cannot be said to "follow the rules" of the discourse in a conscious, intentional manner. The observed rules are termed *meta-discursive* because if formulated, they would take the form of propositions about the discourse. Some of these rules may be specific to the given mathematical topic. In this case, they would usually be stated explicitly. This is the case, for example, for the rules that regulate arithmetical operations on negative numbers. The more universal meta-rules, such as those that govern the endorsement of mathematical narratives (i.e., the rules of proving or defining), are rarely made explicit, and are usually learned from examples rather than from general verbal prescriptions. It must be emphasized that there is more than one type of communication that can count as mathematical, and that some mathematical routines that are acceptable in a school (e.g. school routines for endorsement of narratives) would be deemed inappropriate if applied in scholarly mathematical research.[4]

[4] Our use of the term *routine* is close to the usage that has been proposed by Schutz & Luckmann (1973) and applied in the context of mathematics learning by Voigt (1985). The notions *social norms* and *sociomathematical norms*, introduced by Cobb, Yackel, and their colleagues (see e.g. Yackel & Cobb, 1996), although not tantamount to the idea of *meta-discursive rule* (not every meta-rule is a norm; see Sfard 2000b), are clearly related to the same phenomena.

The special feature of communicational research is that it considers the entire discourse – in the present case, the discourse on negative numbers – as the unit of analysis. Here, asking what the participants of a study have yet to learn becomes equivalent to inquiring how students' ways of communicating must change if they are to become skilful participants of a given mathematical discourse. In our project, we followed the discursive development of the class by identifying changes occurring in each of the four discursive characteristics: the use of words characteristic of the discourse on negatives, the use of mediators, endorsed narratives, and routines. The students were observed becoming increasingly proactive and linguistically accurate in conversations featuring such new terms as "minus two" or "minus three and a half". We were particularly attentive to the question whether the children's use of these terms was becoming objectified. We also watched the learners operating on specially designed visual mediators - extended number line, arrow model of negative numbers and "magic cubes model". While doing so, we tried to discern the slowly evolving mediating routines. Here, our main question was whether the children used different mediators interchangeably. Finally, we documented the growing repertoire of narratives endorsed by the students, as well as the transformations that occurred – or failed to occur - in the children's discursive meta-rules. Full results of our study are yet to be reported. In this article, we focus on the change that could be held responsible for Stendhal's complaints, the one that has been long known as a major challenge to many students. To identify the nature of this change and to see what happened when actors in the classroom drama tried to come to grips with the difficulty, we will now analyze the communicational process by focusing on: (a) the discourse on negative numbers as such and, in particular, on features that set it apart from the mathematical discourse with which the students have been familiar when the learning began; (b) students' and teacher's efforts toward the necessary transition to new meta-discursive rules, and (c) effects of the learning teaching process, that is, the extent of discursive change that resulted from these efforts.

2. Focus on the discourse (mathematics): What is to change in the endorsement routines when negative numbers are introduced?

Stendhal's story made us aware of a certain uniqueness of the discourse on negative numbers among other mathematical discourses learned in school. If Stendhal found the rule "minus times minus is plus" insurmountably challenging, it was probably because it was not clear to him where this claim had come from and why it had been endorsed; and if the substantiation offered by the teachers did not help, it was probably because their argument was not of the type that young Stendhal would find convincing.

One possible way in which one may substantiate the rule in question is presented in Figure 1. The argument originates in the principle that the extended discourse must preserve some critical features of the original numerical discourse. Basic rules of addition and multiplication – associativity, commutativity, distributivity, etc. – had been identified by mathematicians of the past as the ones that epitomized the nature of numbers. These were, therefore, the properties that were chosen to be retained. In Figure 1, the rule "minus times minus is plus" is derived as a necessary implication of this requirement.

Taking as a point of departure the request that the basic laws of numbers, as have been known so far, should not be violated, and assuming that the rules $[a \cdot (-b) = -ab]$ and $-(-a) = a$ have already been derived from these laws we may now argue that for any two positive numbers, a and b, the following must hold:

On the one hand,

(1) $$0 = 0 \cdot (-b) = [a + (-a)] (-b)$$

and on the other hand, because of the distributive law which is supposed to hold,

(2) $$[a + (-a)] (-b) = a(-b) + (-a)(-b)$$

Since it was already agreed that $a(-b) = - ab$, we get from (1) and (2):

$$- ab + (-a)(-b) = 0$$

From here, and from the law $-(-x) = x$, one now gets:

$$(-a)(-b) = -(-ab) = ab$$

Fig. 1: Deriving the rule for multiplication of two negative numbers from the basic rules of the discourse on numbers

The speculation that the substantiation given to Stendhal by his teachers, whatever its actual form, followed a similar path is highly plausible simply because no other argument seems available. In particular, there is no concrete model from which this rule could be deduced.[5] If so, it is quite clear why Stendhal was so hesitant to accept the explanations and, more generally, why other learners are likely to go through a similar experience: For those who knew only unsigned numbers so far, a concrete model had always been the stepping stone – and the ultimate reason – for mathematical claims. Indeed, before the appearance of negative numbers, mathematical and colloquial discourses were unified in their endorsement routines: In both cases, the narratives were verified by confronting propositions in question with extra-discursive reality. Consequently, decisions about the endorsability of mathematical statements were perceived by the participants of mathematical discourse as imposed by the world itself. This impression was fortified by the fact that the mathematical discourse was fully objectified, with all the traces of human agency removed from the stories told by its participants. The substantiation routine of the new discourse, instead of pointing to mind-independent, extra-discursive reasons, rests on the exclusive attention to the inner coherence of the discourse itself. This intra-discursive argument, so far removed from anything that counts as convincing in colloquial discourses, is a rather dramatic change in the rules of a mathematical game – and a major challenge to the learners.

[5] On the face of it, this claim may be contested since many ideas have been proposed to model negative numbers (e.g. there is the model of movement where time, velocity and distance can be measured in negative as well as positive numbers; numbers may be represented as vectors, etc.). And yet, at the closer look, all of these explanations turn out to be derivatives of the same basic decisions about preserving certain former rules of numbers while giving up some others; these fundamental choices are exactly the same as the ones that find their expression in the acceptance of axioms of numerical field as a basis for any further decision, and they must be (possibly in a tacit way) accepted prior to any justification (see also Sfard 2000b).

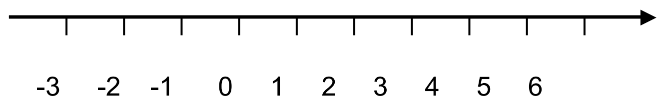
An additional difficulty stems from the fact that in the process of extending the numerical discourse, preserving some former discursive features goes hand in hand with compromising some others. Among the numerical properties that mathematicians agreed to give up in the transition to the signed numbers were those that involved inequalities. For example, in the extended set of numbers the claim “If $a > b$ then $a/b > 1$ for every a and $b \neq 0$” is no longer true. Mathematicians’ tacit criteria for deciding what to preserve and what to give up cannot possibly be clear to children. A cursory look at the history of negative numbers suffices to see that for a long time, these criteria were far from obvious to the mathematicians themselves. The fact that the negatives lacked some of the properties which, so far, appeared as the defining characteristics of numbers led Chuquet, Stifel, and Cardan to claim that the negatives were “absurd”, “false”, “imaginary”, “empty symbols”. Nearly two centuries later Descartes stated that these numbers were “false, because they represent numbers smaller than nothing”, whereas Pascal declared: “I know people who don’t understand that if we subtract 4 from zero, nothing will be left” (Kline, 1980). Back then in the 17th century the real, albeit unspoken, question was about the rules of mathematical game: Who is the one to decide what counts as mathematically acceptable - the reality itself or the participant of the mathematical discourse? Hundreds of years passed before this dilemma was finally resolved. Our study was to show how contemporary students and teachers come to terms with this uneasy problem.

3. Focus on learning interaction: What do the teacher and the students do to make the change happen?

3.1 Teaching: Helping children out of the inherent circularity of discourse development

Although no major difficulty was expected until the class arrived at the two-minuses multiplication rule, a certain inherent circularity of the discursive development was likely to obstruct students’ learning from the very beginning. To illustrate, let us look at the introduction to the topic taken from a typical textbook (Figure 2). The crux of this definition is in the interesting conceptual twist: points on the number line are marked with decimal numerals preceded by dash and, subsequently, these marked points are called *negative numbers*. One may wonder about the reasons for this verbal acrobatics: giving new names to points on a line and saying these are numbers. Whereas it is virtually impossible to introduce a new discourse without actually naming its objects from the very beginning, it is also very difficult to use the new names without anchoring them in something familiar. Alas, negative number are not anything that could be associated with easily identifiable “referents”. Unlike in other discourses, where one can indicate a new object by referring the students to some familiar perceptual experience (think e.g. about teaching velocity or exotic animal species), in the discourse on negative numbers the initial remarks on the new “mathematical objects” have almost no concrete instantiations to build on. Points on the extended number line, although far from sufficient, are probably the best visual mediator one can think of in this very first phase of learning.

Let's choose a point on a straight line and name it "zero." Let's choose a segment and call it "the unit of length." Let's place the unit head-to-tail repeatedly on the line to the right of the point "zero." *The points made this way will be denoted by 1, 2, 3 and so on …*



To the left of the point "zero," we put the unit segment head-to-tail again and *denote the points obtained in this way with numbers -1, -2, -3,... The set of numbers created in this way is called the set of negative numbers*.

Fig. 2: From a school textbook (Mashler, M., 1976, *Algebra for 7th grade*; Translated from Hebrew by AS).

This means that the process of introduction to the new discourse is inherently circular: Although the learning sequence that begins with giving a new name to an old thing seems somehow implausible, it can hardly be avoided simply because even the first step in a new discourse is, by definition, already the act of participation in this discourse. This, however, faces the learner with a dilemma: On the one hand, in order to objectify the new number words and see them as numbers, not just labels, the student needs to use these words “the numerical way,” that is, has to speak about adding them, multiplying etc.; on the other hand, how can a person talk “the numerical way” about something that is not yet seen as a number? The learner’s dilemma becomes the teacher’s challenge: Her task is to help the children out of the circularity. The teacher has to find a way to break the vicious circle and make the students actually talk about negative numbers even if the young interlocutors do not yet have a full sense of the new entities’ number-like nature.

In our study, the teacher's solution was to provide the students with additional tools to think with. She introduced familiar perceptual mediators about which the children would be able talk without much explanation, and which would generate a discourse very similar, perhaps even identical with, the talk on negative numbers. The choice of the mediators was to be made carefully, so as to ensure they would not be treatable in terms of the "old" (unsigned) numbers more easily than in terms of the new ones (as is the case with the majority of "real life" situations supposedly supporting the use of negative numbers; for example, questions about changes in temperatures do not, in fact, necessitate manipulating negative numbers). Two such mediators were introduced: the *arrow model* that featured positive and negative numbers as arrows pointing to the right and to the left, respectively; and the *magic cubes model* said to consist of cold and hot cubes that, when added to water, lowered or heightened its temperature by 1 degree. The teacher's assumption was that the children, once provided with these two generators of the relevant talk, would be able to make much progress on their own. She hoped that they would arrive at the generally accepted rules for *adding* and *subtracting* signed numbers, and at the rule for *multiplying a negative number by a positive*. She anticipated no serious hurdle until the appearance of the "minus-times-minus" question, regarding the only rule which could not be deduced from the models. Classroom events, however, took an unexpected course.

3.2 Learning: Breaking out of the discursive circularity by recycling old routines

The two models did help the students in making their first steps in the new discourse. For better or worse, the children seemed to know what to expect from something that has been labeled as "number": When asked to perform a number-like operation on "positive" or "negative" arrows or cubes, they summoned discursive routines associated with the 'old' numbers. This reliance on the former discursive habits could be seen in the following conversation that took place between two students, Sophia and Adva, when they had already been well acquainted with both mediators and knew how to add and subtract signed numbers. They were now trying to figure out how to multiply a positive by negative:

Episode 2: The children try to find the value of $(+2)\cdot(-5)$

1121 Sophia: Plus-two times minus five…

1122 Adva: Two times minus five

1123 Sophia: Aha, hold on, hold on, plus-two... it is as if you said minus five multiplied two times [*looks at the written expression:* $(+2)\cdot(-5)$]. So, minus five two times it is minus ten…

1124 Adva: How about plus-two? How about the two?

1125 Sophia: [*Looks at the written text*] Minus five… one, two, three, four, five [*counts notches on the number axis left to the zero and eventually marks the fifth of them with '–5'*] Times two. You know that plus-two is two, you can take the plus away, right? So it is like two times minus-five, two times minus-five, so it is minus-five and one more [add[6]] minus-five [*turns to Adva*]. It gives minus-ten.

1126 Adva: I don't know…two, the plus – maybe it does mean something.

1127 Sophia: Ok, you can take this plus away.

1128 Adva: So, it is like I can take this minus away

[6] The Hebrew expression "ve-od," which literally means "and [one] more," is used in school mathematics in the sense of "add" or "plus" (the word *plus* itself may also be used).

1129 Sophia: No, not the minus, because this means two times minus-five.

So far, so good. In concert with the teacher's prediction, expectations evoked by the word *number* helped the students find their way into the new discourse. Although not without some telling hesitation, Sophia and Adva were able to arrive at the formula that the teacher had in mind while posing the question. The girls discovered this rule by projecting in a metaphorical manner from their former discursive experience into the new, unfamiliar context: in the realm of unsigned numbers, multiplication of a number by 2 meant adding that number to itself, and they used the same interpretation now, when the doubled number was negative. However, during the whole-class discussion that followed the work in pairs, not everybody shared this opinion.

Episode 3: In response to the question, "What $(+2)\cdot(-5)$ could be equal to?"

1226	Roi:	Minus-ten.
1227	Teacher:	Why?
1228	Roi:	We simply did… two times minus-five is minus-ten because five is the bigger number, and thus… uhmm… It's like two times five is ten, but [it's] minus-ten because it's minus-five.
……	………	……………………………………………………
1248	Noah:	And if it was the positive seven instead of positive two?
1249	Yoash:	Then it would be positive thirty five
1250	Sophie:	Why?
1251	Yoash:	Because the plus [the positive] is bigger.

At the first sight, Roi's idea might have sounded surprising. On the closer look, it was grounded in the principle of continuity with the former discourse, similar to the one that had guided Sophie. As presented schematically in Figure 3, Sophie substituted the new numbers for old numbers: In the familiar multiplication procedure for unsigned numbers, the negatives had slid into the slot of the second multiplier, occupied so far exclusively by unsigned numbers. In Roi's case, the new task evoked the formerly developed routine for the *addition* of signed numbers and the students substituted operation for operation: The *multiplication* of signed numbers was obtained from the multiplication of unsigned numbers in the way in which the *addition* of signed numbers had been obtained from the subtraction of the unsigned, more or less. To sum up, Roi, just like Sophie before him, drew on previously developed discursive routines, except that his choice did not fit with the historical decision made by the mathematical community.

Successful try: substitution into the discursive template

$$2 \cdot b = b + b$$

Unsuccessful try: substitution into the discursive template

$$(+a) + (-b) = \begin{cases} |a - b| & \text{if } a > b \\ -\ |a - \mathrm{b}| & \text{if } a \le b \end{cases}$$

in which *a* and *b* are "unsigned" and both '+' and '–' are substituted with '·'

Fig. 3: Recycling old discursive templates in the new context

In an attempt to account for the difference in Sophie's and Roi's choice of the rule to preserve, let us take a closer look at the two children's discourse on negatives. Roi's ways of talking were not unlike those of Adva in Episode 2 or of Yoash in Episode 3. Nothing indicated that any of these children have objectified negatives, that is, could speak about them the way they spoke about more familiar numbers: as self-sustained entities remaining in a numerical relation one to another and to the other numbers. On the contrary, evidence abounded that the signs + and - had not yet turned for any of them into an integral part of the names of number-like entities. In utterances 1126-1129, the children discussed taking the sign "away". This, by itself, might not have been the sufficient evidence for the lack of objectification. And yet, the question "Maybe [the plus] does mean something?" (1126) asked by Adva when she tried to decide whether to delete the sign from +2 showed that, for her, only 2 deserved being called *number*, whereas the sign was somehow tacked on and not necessarily relevant for the course of numerical conversation. In other places, children who shared Roi's idea about multiplication could be heard using phrases such as "the plus [the positive] is bigger" (1251) with respect to the pair of numbers +7 and –5. Clearly, the announcement about the "bigger" [number] resulted from the comparison between "numbers without the signs". All this indicated that in expressions such as +7 or –5, only the numeral part counted as a number, whereas the sign was something that did not affect this numerical identity anymore than, say, the change in the name or in the external appearance affects one's identity as a human being.

Similar analysis of the way Sophie used numerical words has shown that in contrast to her classmates, the girl had already made a significant step toward objectification of the talk on negatives. This was particularly salient in the episode that follows, taken from the whole-class discussion:

Episode 4: Sophie's response to the question, "What (-2) · (+6) could be equal to?"

1364 Sophie: I say that if you have one minus and one plus, then you go with the plus, that is, if you have here minus-two times plus-six *[(-2) · (+6)]*, then you do six times minus-two... *[6 ·(-2)]*

1365 Teacher: You mean, I need to reverse their order?

1366 Sophie: The order here doesn't matter.

This brief conversation brings into even stronger relief what could be seen already in Episode 2: Sophie could treat expressions such as “negative five” (or “minus-five”) as integrated wholes and was capable of incorporating them into the numerical discourse simply by putting them into slots reserved for numbers. Moreover, the fact that she had little difficulty extending the endorsed narrative “*Two times a number* means adding this latter number to itself” to the negatives adds plausibility to the conjecture that for her, these new entities were as “addible” as the numbers she knew before. This claim finds its further reinforcement in Sophie’s naturally adopted assumption about the commutativity of the extended multiplication (see her utterance 1357: “Order doesn’t matter”).

To sum up, Sophie’s ability to view such symbols as –2 or –7 as representing numbers expressed itself in her tendency to “plug” these symbols into numerical slots of formerly endorsed numerical narratives and then to examine the resulting narratives for their endorsability, possibly with modifications. In contrast Roi, for whom –2 and –7 did not yet represent integrated entities that deserved being called *numbers*, was more arbitrary in his projections from the old numerical discourse to the new one. The rule he chose, according to which the numeral part of the numerical symbol and the sign attached to the numeral were to be treated separately, reflected his “split vision” of the negatives as “numbers with signs attached”.

3.3 Teaching: Transition to the “telling” mode

Inspired by differing ideas on what it might mean to multiply a positive number by a negative one, the children ended up with two competing routines for multiplication. They had now to decide which of the resulting incompatible narratives should be endorsed and which had to be disqualified as inappropriate. However, the lesson ended soon after the introduction of the two proposal and before the class had an opportunity to reach a resolution. The teacher, being convinced that the children will soon find out their way out of the momentary confusion regretted to have lost the opportunity to watch the process in its entirety. Later that day she wrote in her journal:

> The lesson ended and I had to let the children go. I am afraid that they will check it at home and I will lose the opportunity to listen to their further thinking. But I have no choice. I don’t give them any homework and hope to resume our conversation in two days, exactly from the point where it ended today.

The teacher's fears did not materialize, though. The next lesson began with the whole class debate and it was clear that the disagreement about the "plus times minus" persisted. The teacher hoped, however, that the explicit confrontation between the two alternatives would soon lead the class to the unequivocal decision about the preferrability of Sophie’s proposal. The following excerpt from this conversation aptly instantiates the general spirit of the lengthy debate that followed:

Episode 5: Trying to decide between the two proposals for “plus times minus”

1341	Teacher:	Come on, let's take the expression… minus-two times 6 [*writes* '(-2)·(+6)=' *alongside the expression* '(+2)·(-5)=' *which already appeared on the blackboard*]. What is the answer and why?
1342	Naor:	Plus-12 because 6 is bigger than 2.
1343	Teacher:	Plus-12 [*writes on the blackboard:* '(-2)·(+6)=12?']. I added the question mark because we don't know yet.
1344	Student:	When will you tell us?
1345	Teacher:	I will tell you today, but… in fact, what is your opinion? What do you say, Vladis?
1346	Vladis:	Me too: Plus-12 because 6 is bigger.
1347	Teacher:	What do you say, Sophie?
1348	Sophie:	I say *minus*-12.
1349	Teacher:	[*Writes*: '(-2)·(+6)=-12'] Why?
1350	Sophie:	Because you can take the plus of the six away and then you get 6 times minus-2.
1351	Roi:	But you can do the opposite.
1352	Teacher:	What do you mean by "the opposite"?
1353	Roi:	You can do 2 times plus-6. Why do we have to do 6 times minus-2?
1354	Teacher:	Because 2 has the minus.
1355	Roi:	So what?
1356	Teacher:	Are you saying that I should ignore also these brackets? [*points to the brackets around* – 2]
1357	Sophie:	What does it mean "minus-two times"? This is what you are saying [*she addresses Roi*]. You ignore the minus…
1358	Roi:	Ok, you have to make both of them plus or both of them minus.
1359	Teacher:	Do we have to "make them both plus or minus"…
1360	Roi:	Yes, somehow.
1361	Teacher:	... or should we decide whether the result is plus or minus?

1362 Roi: In these two exercises [*points to the two expressions on the blackboard:* '(-2)·(+6)=' *and* '(+2)·(-5)='] we decide according to the bigger.

This exchange is remarkable for at least two reasons. First, it shows that contrary to the teacher's expectations, the children did not converge on Sophie's proposal. Surprisingly, it was Roi's version of the multiplication law that was winning the broader following. The second thought-provoking fact is the teacher's restrain and her persistent refusal to step in with decisive judgment.

The classroom debate went on for another full period and an even greater majority of students decided to give support to Roi, who continued to claim that the sign of the product should be like that of the multiplier with the bigger absolute value. Recurring demonstrations with arrows and magic cubes did not help. At a certain point, some of the children began showing signs of impatience: They were asking for the teacher's authoritative intervention (1344). The teacher could no longer persist in her refusal to act as an arbitrator. Although initially reluctant (1345), she finally stepped in with the explicit ruling:

Episode 6: The teacher tells children how to multiply numbers with different signs

[1372] Teacher: I want to explain what Sophie said. What she said is true, and this is the rule that guides us. Sophie did not manage to convince all of you, but I believe that some of you did get convinced that to multiply is to add time and again…for example, here [*points to* '(-2)·6 =' *written on the blackboard*] you add the number –2 six times [*she marks arcs that symbolize –2 on the number line, from point 0 to the left*] … and I reach –12 and this is the right answer.

As if against herself, the teacher resolved the problem by imposing her vision of “who is right.” True, there was an attempt at substantiating this decision by pointing to the repeated addition procedure. And yet, there was also something defeatist in the way the explanation was presented. The very fact that the teacher repeated the argument that had already been tried and had not worked for the children made her sound resigned and unconvincing. The teacher’s disappointment found its expression in the note she made for herself after the lesson:

> In the beginning of the lesson I said to myself: “Fortunately, the children were not too interested in the topic .. They are back without the answer….” [Now, after the lesson] I can see that even my repeated emphasis on the correct proposal did not help – the only thing that counts is the kids’ wish to be like the leaders of the class.

The disillusionment as to the prospects of children’s independent “reconstruction” of the numerical laws led to a change in the teacher’s strategy. The last formula the class had yet to learn was *minus times minus is plus.* Although the teacher did not give up the idea of letting the students probe their own creative ideas before being exposed to other people’s discursive constructs, she did try to prevent a lengthy discussion. This time the children were not expected to be able to reinvent the rule by themselves, anyway.

As can be seen in Episode 6, the students found the task of figuring out the product of –2 and –3 quite confusing. This was true even of Sophie, one of the few children who had little difficulty 'reinventing' mathematicians' way of multiplying positive numbers by negative.

Episode 7: *Sophie and Adva are trying to figure out what (-3)·(-2) might be*

1400 Sophie [*reads from the worksheet*] “How, in your opinion, can we perform each of the following operations and why…” And this is exactly what she said, minus three times minus two… OK.

1401 Adva: Minus six, because they are both minus.. No, I don’t understand... don’t know what we are supposed to do.

1402 Sophie: Two minus, see, do you remember how we did plus-four times minus two? You can delete the plus, [so we have] four times minus two, you do four times minus two, this is minus eight.. but now she gave us this worksheet so that we do operations with both [umbers] with minus. So, what do we do when both are minus?

1403 Adva Aah [……..]

1404 Sophie You can do minus three times minus two, but what is “minus tree times?”

1405 Adva: Three times minus two.

1406 Sophie: But you have to consider the minus!

1407 Adva: In this case there will be minus in the end.

1408 Sophie: What? Do you think that you can erase the minus and do three times minus two?

1409 Adva: But this will be minus in the end in any case [….].

1410 Sophie: But I am not so sure about it. Look, you can perhaps do something like that: you can delete the minus [*points to the minus of the number –3*] and you get three times minus two, and this gives minus six – you think you can do this?

1411 Adva: I don’t know.

1412 Sophie: I am not sure about this. Can you can delete the minus when both are minus? This would mean that the result would be minus, and that you can erase the minus of the first or of the second…

This conversation did not seem to lead to anywhere. The girls were grappling in the dark, never sure of what they were saying. The teacher, anxious to spare her students additional frustration (or perhaps afraid that, as before, some of the children would develop an attachment to unwanted formulas!), decided to present her own answer. Always respectful toward students but unable to advance their own thinking any further, she opted for the second best: Rather than parachuting the new law on the class, she derived this rule from what the children already knew.

Episode 8: *Teacher's intra-discursive substantiation of the laws of multiplication*

[1556] Teacher: Well, I wish to explain it [2 · (-3)=-6] now in a different way. [*Writes on the blackboard the following column of equations, stopping at each line and asking the children about the result before actually writing it down and stressing that the decrease of 1 in the multiplied number decreases the result by 2; this rule, she says, must be preserved when the right multiplier becomes negative*

$$
\begin{aligned}
2\cdot 3 &= 6\\
2\cdot 2 &= 4\\
2\cdot 1 &= 2\\
2\cdot 0 &= 0\\
2\cdot(-1) &= -2\\
2\cdot(-2) &= -4\\
2\cdot(-3) &= -6]
\end{aligned}
$$

Let us now compute (-2) (-3) = in a similar way [*as before, writes on the blackboard the following column of equations, stopping at each line and asking the children about the result before actually writing it down and noting that the decrease of 1 in the multiplied number increases the result by 3; this rule, she says, must be preserved when the left multiplier becomes negative*:

$$3\cdot(-3) = -9$$
$$2\cdot(-3) = -6$$
$$1\cdot(-3) = -3$$
$$0\cdot (-3) = 0$$
$$(-1)\cdot(-3) = 3$$
$$(-2)\cdot(-3) = 6$$]
[7]

[7] Note that the teacher's argument is a school version of the one that has been presented formally in Figure 1: In both cases the laws of multiplying signed numbers are derived from the laws that hold for unsigned numbers according to the principle of preserving certain basic features of numerical operations.

Summing up, one may say that in spite of the gradual evolution of the teacher's instructional strategies, two salient features of her way of teaching endured. First, she was deeply convinced that the students should play an active role in the advancing of mathematical discourse. This principle remained in force even when she had to compromise her initial intention to build on children's own inventions. In this latter case, while presenting to the students other people's discursive constructions, she was careful to justify these ideas in such a way as to make sure that nobody accepts what she was saying merely because of her privileged position as a teacher. Second, at no point did she attend directly to the meta-discursive rules for endorsement of narratives that influenced her decisions from behind the scenes and that, unnoticed, underwent a substantial change in the span of a few lessons. These rules were left hidden even when the law of multiplying two negatives was discussed.

4. Focus on effects: Did the expected change in endorsement routine occur?

A number of questions have to be asked now. How effective was the teacher's attempt to introduce the new endorsement routines simply by implementing them? Can the children satisfy themselves with the inner consistency of mathematical discourse as the sole criterion for the endorsement of narratives about numbers? Students' reactions to the teacher's derivation of the formula for the multiplication of two negatives demonstrate that this was not the case:

Episode 9: Children's reactions to teacher's derivation of the laws of multiplication

1557	Shai:	I don't understand why we need all this mess. Is there no simpler rule?

1559 Sophie: And if they ask you, for example, how much is (-25)·(-3), will you start from zero, do 0·(-3), and then keep going till you reach (-25)·(-3)?

Evidently, the children did not even recognize the function of the teacher's argument. Rather than viewing it as an attempt at mathematical *substantiation*, they interpreted the exposition as a demonstration of the routine *for producing* endorsed narratives such as $(-2) \cdot (-3) = 6$ or $(-25) \cdot (-3) = 75$, and a very ineffective one, at that. Unable to tell substantiation of narratives from their production, they had still a long way to go until their endorsement routines undergo the necessary transformation.

This conclusion is reinforced by certain utterances made by the children in response to the teacher's recurrent queries about their reasons for choosing Roi's rule for "plus times minus". Here is a representative sample from the conversation that followed one such query:

Episode 10: The teacher tries to understand why the children opted for Roi's formula

1334	Teacher:	You repeat time and again what Roi said last time. I need to understand *why* you think this is how things work?
1335	Yoash:	Because this is what Roi said.
1336	Teacher:	But Roi did not explain why it is so – why it is according to the bigger …
1337	Roi:	Because there must be a law, one rule or another
1338	Teacher:	Ok, there must be some rule. Does it mean that we should do it according to the magnitude?

1339 Leah: Yeah… The bigger is the one that decides.

Roi's exclamation "there must be a law, one rule or another" (1337) showed that the children fully accepted at least one basic rule of the numerical discourse: They agreed that whenever one dealt with entities called *numbers*, there had to be formulas that would tell one what to do. On the other hand, the conversation made it equally clear that the students did not yet develp routines for producing and substantiating such formulas. When faced with the request to look for laws of multiplication on their own, the students grappled in the dark. Leah's appeal to the "universal rules of the world" ("The bigger is the one that decides," 1339) was a reminder that the student's previous experience with numbers made her think about them the way she thought about concrete objects: As entities that existed in the world and were subject to extra-discursive laws of nature (and for many people, the latter type of laws includes rules that govern human societies.)

A different message, the message about the role of human factor in shaping mathematical discourse, was conveyed by one child's remark about "those who invented mathematics" (1426):

Episode 11: *Vladis's proposal for the value of (-2)·(-3)*

1458	Vladis	It is plus-six
1459	Teacher:	Why?
1460	Vladis:	According to the rule.
1461	Teacher:	According to which rule?

1462 Vladis: According to the rule of those who invented the mathematics.

Later we found out that Vladis learned the new rules of multiplication from his mathematically versed father. For all we know, this was also how he came across the claim about these law's human origins. Although probably far from truly convinced about mathematics being a human invention, some of the children admitted that social considerations played a role in their decision-making. Suffices to recall Yoash's frank assertion that his preference for the unconventional formula for multiplication was motivated by its being proposed by his friend Roi (see 1335 in Episode 10), on whom he relied. Roi's own explanation for the fact the class voted for his proposal indicates children's awareness of their sensitivity to social circumstances:

Episode 12: Why choose one template rather than another?

1374	Teacher:	Six times minus two is minus twelve – is this too complicated?
1375	Roi:	But I am more charismatic… I managed to influence them all.

However we interpret one classroom utterance or another, it is clear that the tacit upheaval in the rules of the mathematical discourse bewildered the children. Moreover, the meta-questions asked by the teacher ("Why should the numbers be multiplied this way?", "Why did you choose this rule?") were hardly the children's questions, ones that they would be likely to tackle on their own. So far, the students did not need to bother themselves with meta-quandaries to be successful with numbers. Once such questions were asked, the children lost confidence. Unsure of the rules of the game anymore, they were now prepared to follow the lead of anybody who appeared to have a sense of direction and could show a measure of self-assurance.

6. Discussion and conclusions: On communicational conflict and realistic communicational agreement as the necessary conditions for learning

On the face of it, the story of the class learning negative numbers has a disappointing ending: The students did not manage to make the expected transition to a new, qualitatively different, set of meta-discursive rules. The teacher was visibly displeased with the course of events in her classrooms, and she repeatedly expressed her discontent not just with the students' progress but also with herself. The question arose of what could have been done differently so as to assure a more satisfactory learning outcome. In an attempt to answer this query, let us harness communicational tools in analyzing the learning process and its demands.

Perhaps the most dramatic difference between the more traditional vision of mathematical thinking and the one discussed in this article is in their respective messages about the origins of mathematical learning. Whereas most traditional perspectives view learning as resulting from the learners' direct efforts to arrive at a coherent vision of the world, the present framework conceptualizes learning as arising mainly from one's attempt to make sense of other people's vision of this world (the term "vision of the world" refers to all the narratives about the world the person endorses or is likely to endorse). The former perspectives imply that learning, at least in theory, could take place without participation of other people. In contrast, the idea of mathematics as a form of discourse entails that individual learning originates in communication with others and is driven by the need to adjust one's discursive ways to those of other people. In other words, the most powerful opportunities for learning arise when the learner stumbles upon a difference between her own and her interlocutors' discursive ways. We say that these are situations of *communicational conflict*.[8] This type of conflict emerges whenever interlocutors differ in their uses of words, in the manner of looking at visual mediators or in the ways they match discursive procedures with problems and situations. More often than not, these differences find their explicit, most salient expression in the fact that the different participants endorse differing, possibly contradicting, narratives.[9]

[8] Let me remark that communicational conflict is often involved also in mathematical invention (or any other scientific invention, for that matter). In this case, the conflict is likely to occur *within* a person: in the transition from a familiar discourse to a new one the mathematician may find himself endorsing conflicting narratives. One of a well-known cases of such inner conflict is that of George Cantor, the inventor of set theory, who in his letters to another mathematician, Richard Dedekind complained on his inability to overcome the contradiction between and the well-known "truth" that a part is smaller than the whole and the conclusion he reached on the grounds of his new theory, according to which a subset of an infinite set may be "as big as" the whole set (Cavaill`es, 1962).

[9] Of course, some cases of conflicting narratives may stem from differing opinions rather than from discursive conflict. Discursive conflict should be suspected only in those cases when the conflicting narratives are factual and the possibility of an error in their construction has been eliminated. A narrative counts as factual if there are well-defined meta-discursive rules for their substantiation.

The notion of *communicational conflict* brings to mind the idea of *cognitive conflict*, central to the well-known, well developed theory of conceptual change (Vosniadou, 1994; Schnotz, Vosniadou, & Carretero, 1999). Cognitive conflict is defined as resulting from one's holding two contradicting beliefs about the world, with the contradiction stemming from incompatibility of one of these beliefs with the real state of affairs. In one's attempt to resolve the conflict, the person will thus try to employ the world itself as an ultimate arbitrator. The idea of communicational conflict, on the other hand, rests on the assumption that learning is a change of discourse resulting from interactions with others. According to this latter approach, most opportunities for learning arise not from discrepancies between one's endorsed narratives and certain external evidence, but rather from differences in interlocutors' ways of communicating.[10] The communicational framework, therefore, questions the traditional relation between the world and the discourse: Rather than assuming that what we say (think) about the world is determined by what we find in the world, we claim a reflexive relation between what we are able to say and what we are able to perceive and endorse. Our discourses remain fully consistent with our experience of reality until a discursive change opens our eyes to new possibilities and to new visions. We thus often need a change in how we talk (think) before we can experience a change in what we see.

Yet another difference between the two types of conflict is in their role in learning: Whereas creating cognitive conflict is considered as an optional pedagogical move, particularly useful when the students display "misconceptions," the communicational conflict is indispensable for mathematical learning. Our study made this necessity quite clear: Without other people's example, the children who were supposed to learn about negative numbers would have no incentive for changing their discursive ways. The discourse they practiced when the learning began did not seem to have any particular weaknesses as a tool for making sense of the world around them. The differences between the concepts of cognitive and communicational conflict are summarized in Table 1.

Table 1: Comparison of concepts: communicational conflict versus cognitive conflict

	Cognitive conflict	**Communicational conflict**
Ontology The conflict is between:	the interlocutor and the world	two interlocutors' discursive ways (in endorsed narratives, use of words, the "when" of routines)
Role in learning	is an optional way for removing misconceptions	is indispensable for a substantial change in discourse (learning)

[10] Only too often, communicational conflicts are mistaken for factual disagreements. Most well known incompatibilities between scientific theories may, in fact, be resulting from communicational conflicts rather than from correct versus incorrect factual beliefs. Thus, for example, what appears as a straightforward contradiction between Aristotle and Newton – the former thinker's claim that a constant force applied to a body results in the body's constant movement, versus Newton's second law of dynamics asserting that constant force results in a constant acceleration – may, in fact be the outcome of the two men's differing uses of the word *force*.

How is it resolved?	by student's rational effort	by student's acceptance and rationalization (individualization) of the discursive ways of expert interlocutor

Although indispensable for substantial learning, communicational conflict is also potentially dangerous. While usually invisible and easily mistaken for a factual disagreement, it may hinder any further communication. The nest question to ask, therefore, regards the conditions that turn communicational conflict from an obstacle into a true learning event. The process of overcoming the conflict is complicated by a certain inherent circularity. To be able to implement the complex change in the unwritten rules of the discursive game, the children must have a very good reason. The most powerful driving force would be the awareness of the necessity of the required change or at least of its prospective gains (here, *necessary* means imposed from outside, by an extra-discursive reality). And yet in our study, neither a necessity nor the expected usefulness seemed to be likely incentives for children's learning. As stressed many times along these pages, the new meta-discursive rules introduced by the teacher were not dictated by a logical necessity. Although mathematicians had found the resulting discourse useful, the young learners had no means of envisioning and appreciating the value of this discourse before actually gaining some experience in applying it in problem solving.

This somehow paradoxical, indeed, "impossible" nature of the discursive change has many pedagogical implications, one of them being that we cannot expect this difficult transition to happen rapidly, in a single decisive step. In those special cases when learning requires a change in meta-rules, time seems to be an all-important factor. If so, even if in our study we were not able to see the desired results, some learning might, in fact, have been taking place, except that the period of observations was too short to show this complex process coming to fruition.

All this said, the passage of time and the awareness of the conflict do not yet seem sufficient to ensure that the students take advantage of the learning opportunity such conflict affords. We now wish to argue that one of the central factors that make the difference between instructionally effective communicational conflicts and the ones that remain an insurmountable hurdle to communication is a *realistic communicational agreement* – a set of unwritten understandings between the participants of the communicational process about those aspects of this process that are essential to its success. This kind of agreement occurs when the participants are unanimous, if only tacitly, about the three basic aspects of the communicational process: the leading discourse, their own respective roles, and the nature of the expected change. Let me elaborate on each of these requirements.[11]

[11] The notion of *reasonable communicational agreement* can be seen as a communicational counterpart and elaboration of Brousseau's idea of *didactic contract*, that is, of "the system of [students' and teachers'] reciprocal obligations" (Brusseau, 1997, p. 31). We do not claim that communicational agreement is sufficient for the success in overcoming the communicational conflict – we only say it is crucially important for learning. This is a theoretical assertion, analytically derived from basic tenets of our approach, but our findings in the present study seem to corroborate it.

Agreement on the leading discourse. In the case of communicational conflict, interlocutors are facing two differing discourses. It is clear that the conflict will not be resolved if each of the participants goes on acting according to his or her own discursive rules. Agreement on a more or less uniform set of discursive routines is the condition for effective communication. Although this agreed set of rules will be negotiated by the participants and will end up being probably somehow different from each of those with which the students and the teacher entered the interaction, the process of change may be ineffective if the interlocutors do not agree about which of these initial discourses should be given the lead, that is, which of them should be regarded as setting the standards. In traditional classrooms, the discourse of teachers and textbooks counted, by default, as the leading form of communication. This leadership may be not so obvious in schools that follow reform curricula. However, as argued below, this shift of authority does not mean the disappearance of the need for a well-defined, explicitly present model discourse.

Agreement on interlocutors' roles. This next ingredient of the communicational agreement ensues from the former: For the learning to occur it is not enough to agree about whose discourse should be the example to follow. Once the choice of the model-discourse is made, those who are given the lead must be willing to play the role of teachers, whereas those whose discourses require adaptation must agree to act as learners. The acceptance of roles is not a formal act. Rather than expressing itself in any explicit declaration, this role-taking means a genuine commitment to the communicational rapprochement. Such agreement implies that those who agreed to be teachers feel responsible for the change in students' discourse and those who agreed to learn show confidence in the leader's guidance and are genuinely willing to follow in the expert participants' discursive footsteps (as documented in research literature, cases of student's resistance are not infrequent these days; see e.g. Forman & Ansell, 2002). It is important to stress that this acceptance of another person's leadership does not mean readiness for mindless imitation. Rather, it means a genuine interest in the new discourse and a strong will to explore its inner logic.

Agreement on the necessary course of the discursive change. Agreeing about the discourse to follow and the readiness to shape one's own discourse in its image are important factors in learning, but it is not yet clear how the children can possibly "bootstrap themselves" out of the circularities inherent in communicational conflicts. At a closer look, it seems that they have no other option than to engage in the leading discourse even before having a clear sense of its inner logic and of its advantages. As was repeatedly emphasized, awareness of the gains can only be acquired through participation. At this initial stage, the children's participation is possible only if initiated and heavily scaffolded by expert participants. For some time to come the child cannot be expected to be a proactive user of the new discourse: At this point in time, the discourse may only be practiced by the learner as a *discourse-for-others*, that is, a discourse which is used solely for the sake of communication with those to whom it makes sense, and in spite of the fact that if it does not yet fully make sense to the child herself. The goal of further learning will be to turn this discourse into a *discourse-for-oneself*, that is, the type of communication in which the person is likely to engage on her own accord, while trying to solve her own problems[12].

To sum up, students' persistent participation in mathematical talk when this kind of communication is for them but a discourse-for-others seems to be an inevitable stage in learning mathematics. If learning is to succeed, all the participants, the students and the teachers, have to have a realistic vision of what can be expected to happen in the classroom. In particular, all the parties to the learning process must agree to live with the fact that the new discourse will initially be seen by the participating students as somehow foreign, and that it will be practiced only because of its being a discourse that others use and appreciate. Let me stress that the exhortation to involve the student in other people's discourses is not an attempt to capitalize on the students' well-known, and commonly disparaged, wish "to please the teacher". Entering foreign forms of talk (and thought) requires a genuine interest and a measure of creativity. To turn the discourse-for-others into a discourse-for-oneself, the student must actively explore other people's reasons for engaging in this discourse. This process of thoughtful imitation seems to be the most natural, indeed, the only imaginable way to enter new discourses.[13] Without the urge to participate in discourses of others and without a strong determination to communicate – so strong that it may motivate acceptance of rules enacted by other interlocutors – we might never even be able to learn our first language.[14]

[12] The term *discourse-for-oneself* is close to Vygotsky's idea of *speech-for-oneself*, introduced to denote a stage in the development of children's language (see e.g. Vygotsky 1987, p.71). These ideas also brings to mind the Bakhtinian distinction between *authoritative discourse*, a discourse that "binds us, quite independently of any power it might have to persuade us internally"; and *internally persuasive* discourse, one that is "tightly woven with 'one's own world.' (Bakhtin, 1981, pp. 110-111.)

[13] As Vygotsky (1978) reminds us, a sociocultural vision of learning (and, in particular, his own notion *zone of proximal development*) must result in "reevaluation of the role of imitation in learning" (p. 87)

[14] Let me add that we often insist on children's own inventions not only because of the learning opportunities they create, but also because we believe that in this way we show more respect to the learner and, while doing so, help her or him to be a better person. However, belief in the possibility of children's unmediated learning from the world and the wish to sustain the democratic sprit of the classroom discourse are not an indissoluble whole, and abandoning the former does not necessitate compromising the latter. It seems that realistic communicational agreement can be cultivated in schools without any harm to the democratic spirit of classrooms interactions. Indeed, making sense of other people's discourse is not any less creative or demanding than "reading the codes" of nature.

Armed with the idea of cognitive conflict and of the realistic communicational agreement let us return to the 7th graders grappling with the negative numbers and to reflect on possible reasons for the apparent ineffectiveness of their learning. In hindsight, the teacher's didactic decisions can be interpreted as an attempt to spare the children the experience of discourse-for-others altogether. This was probably why she decided to withhold her discursive initiatives and to request from the students to "discover" the rules of the new form of communication for themselves. In this manner, the teacher inadvertently violated not only the third component of the communicational agreement, but also the first. Her refusal to demonstrate her own discursive ways left the children without a clue about what the leading discourse might have been. In the thus created leadership void, the children chose to follow the discourse of the person who was known as a leader of many other discourses. The teacher's reticence had an unhelpful impact also on the children's further learning. Leadership once renounced cannot easily be regained. When the teacher decided to be more explicit about the new meta-rules, the children greeted her attempts with disbelief. They were no longer taking the superiority of teacher's discursive ways for granted.

We are now ready to address the question of what the teacher could have done differently so as to ensure more effective learning. The first thing to note is that not everything was in teacher's hands. Although communicational conflict and learning agreement are necessary to make mathematical learning possible, much more than this is needed for substantial learning to occur. In order to succeed in implementing the difficult discursive transformation, the learner must be strongly motivated. Motivation, in turn, is not just a matter of what happens in school. When it comes to wishes, needs and desires, cultural factors that come from outside mathematical classroom may be of principal importance.

This, however, does not mean that there is nothing a mathematics teacher can do to support students in turning the discourse-for-others into discourse-for-themselves. First, a proper message about the sources of mathematical discourse may be of help. We conjecture that much can be attained by putting human agency back into the talk about "mathematical objects." In other words, it is important to make it clear that mathematics is a matter of human decisions rather than of externally-imposed necessity. Aware of the social origins of the objects of the discourse, the student will have a better sense of where to turn to while searching for answers to certain questions. Another potentially helpful, even if not an easy, didactic move would be an explicit conversation about meta-discursive rules and the change they are supposed to undergo.

To conclude, although at the first sight the results of the teaching-learning effort observed in our study was unsatisfactory, the upshot of our analysis is that there is no reason to despair. As we tried to make clear, the difficulties revealed on these pages, rather than being an unintended result of a particular instructional approach, are part and parcel of the process of learning. These difficulties are to the change in discourse what friction is to the change in movement: the necessary condition for such change to occur. The more knowledgeable teachers are about the hidden mechanisms of learning, the more realistic they become in their expectations, and their help to the learners becomes more effective.

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