

Noether symmetry approach in $f(T, B)$ teleparallel cosmology

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Abstract We consider the cosmology derived from $f(T, B)$ gravity where T is the torsion scalar and $B = \frac{2}{e} \partial_\mu (e T^\mu)$ a boundary term. In particular we discuss how it is possible to recover, under the same standard, the teleparallel $f(T)$ gravity, the curvature $f(R)$ gravity, and the teleparallel–curvature $f(R, T)$ gravity, which are particular cases of $f(T, B)$. We adopt the Noether Symmetry Approach to study the related dynamical systems and to find cosmological solutions.

1 Introduction

Nowadays, one of the most important problems in Physics is to understand the late-time accelerated expansion of the universe. Besides, large scale structure, ranging from galaxies to superclusters, presents the problem of missing matter, i.e. the luminous matter is not sufficient in order to guarantee the stability and the evolution of self-gravitating astrophysical systems. There are several candidates to explain these phenomena and the most popular ones are the dark energy and the dark matter, i.e. cosmic fluids gravitationally interacting and leading the evolution of the Hubble flow but without any electromagnetic counterpart.

In particular, the physics underlying the dark energy is still not understood since it behaves as a repulsive gravitational force in contrast to the usual gravitational field. In general, there exist two ways to study the dark energy problem: (1) one can retain the General Relativity (GR) and introduce a new kind of fluid which possess a negative pressure (e.g. by introducing a scalar field), or (2) one can think that GR

needs to be modified at high energy levels and hence the dark energy comes out from these modifications. One can change the left hand side of the Einstein field equations related with the theory of gravity (e.g. extending GR or considering alternatives [1]) or one can modify the right hand side of it by changing the matter content of the universe (i.e., the energy-momentum tensor).

Beside the issue to explain the energy-matter content of the universe, and then the source of accelerated expansion and structure aggregation, competing theories of gravity are posing several fundamental questions on the nature of gravitational field. In particular, if torsion has to be involved in the dynamics, if the equivalence principle is valid in any case, if geodesic structure and metric structure are related or not, if theories of gravity have to be formulated in metric, metric-affine or purely affine approaches [2,3]. In particular, the teleparallel formulation of gravity has recently acquired a lot of interest due to its applications at cosmological and fundamental level.

In this paper, we are interested in studying cosmology in teleparallel modified theories of gravity, which in contrast to GR, consider a curvatureless spacetime with a non-zero torsion. In this perspective, one needs to introduce the so-called Weitzenböck connection instead of the standard Levi-Civita connection [4]. By doing so, a spacetime endorsed with non-zero torsion and a vanishing curvature is achieved.

From the geometrical point of view, these spacetimes are different from the ones considered in GR. From the point of view of the field equations, the teleparallel equivalent of General Relativity (TEGR) is equivalent to GR (see [6,7] for further notions of TEGR). A natural extension of TEGR is, instead of considering only the trace of the torsion tensor T in the action, to introduce a function $f(T)$ in it (see the review paper of Ref. [5] for a discussion and references therein). $f(T)$ gravity remains a second order theory whereas

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the straightforward GR extension, $f(R)$ gravity is a fourth-order one in metric formalism. However, the price to pay in this approach is that $f(T)$ is not invariant under local Lorentz transformations, and hence different vierbeins could arise by different field equations [8, 9]. This theory has been used in cosmology to understand the accelerating cosmic expansion of the universe [10–12, 15], reconstruct cosmological models based on observational data [13, 14, 16, 17], among other models studied.

Recently, a new generalization of the standard $f(T)$ gravity was proposed in [18]. In this theory, the function $f(T)$ is extended to $f(T, B)$, where B is a boundary term related with the Ricci scalar via $R = -T + B$. By adding this dependency, one can recover $f(T)$ and $f(R)$ under suitable limits. Some cosmological features as reconstruction techniques and thermodynamics have been studied in [19] under the standard of this new theory. In [20], a non-minimally coupled scalar field with both the boundary term and the torsion scalar was presented in view to study cosmology by dynamical system techniques. There, it was shown that a dynamical crossing of the phantom barrier is possible and also without fine tuning, the system evolves to a late-time acceleration attractor solution. Some exact solutions and its thermodynamics properties were also discussed in [21]. In summary $f(T, B)$ gravity presents several interesting features by which it is possible to unify, under the same standard, issues coming from $f(T)$ and $f(R)$ gravity.

Here, we will explore cosmological solutions coming from $f(T, B)$ by the so-called *Noether symmetry approach*. This technique proved to be very useful for several reasons: (1) it allows one to fix physically interesting cosmological models related to the conserved quantities (i.e. in particular couplings and potentials) [22]; (2) the existence of Noether symmetries allows one to reduce dynamics and then to achieve exact solutions [23]; (3) symmetries act as a sort of selection rules to obtain viable models in quantum cosmology [24].

The plan of this paper is as follows: In Sect. 2, we briefly introduce TEGR and its extensions like $f(T)$ gravity and $f(T, B)$ gravity. $f(T, B)$ cosmology and particular cases that can be derived from it are introduced in Sect. 3. Section 4 is devoted to the study of Noether's symmetries for $f(T, B)$ gravity. In particular, we derive the Noether vector field and derive the Noether conditions for the function $f(T, B)$. In the related subsections, we study particular cases of $f(T, B)$ function discussing, in particular, how it reduces to $f(T)$, $f(R)$ and $f(R, T)$ gravities. The main point of this section is to demonstrate how several classes of modified gravity theories can be reduced to the $f(T, B)$ paradigm. Discussion and conclusions are drawn in Sect. 5. In our notation, Greek and Latin indices denote spacetime and tangent coordinates, respectively, and the signature $(+, -, -, -)$ is adopted for the metric.

2 Teleparallel equivalent of general relativity and its modifications

We briefly present the basis of the teleparallel equivalent of general relativity (TEGR) and its generalization, the so-called $f(T)$ gravity. In this theory, the vierbeins or tetrad fields e_μ^a are the dynamical variables which form an orthonormal basis for the tangent space at each point x^μ of the space-time manifold. Hence, the tetrads e_μ^a and their inverses E_m^μ obey the following orthogonality relations:

$$E_m^\mu e_\mu^n = \delta_m^n, \quad (1)$$

$$E_m^\nu e_\mu^m = \delta_\mu^\nu. \quad (2)$$

Using the tetrad fields, the metric tensor can be constructed as

$$g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab},$$

where η_{ab} denotes the Minkowski metric. The main idea of TEGR is to construct a theory with a geometry endorsed with torsion and having a globally flat curvature. To realize this program, we define the torsion tensor by considering the curvatureless Weitzenböck connection $W_\mu^a{}_\nu = \partial_\mu e_\nu^a$ such as

$$T^a{}_{\mu\nu} = W_\mu^a{}_\nu - W_\nu^a{}_\mu = \partial_\mu e_\nu^a - \partial_\nu e_\mu^a. \quad (3)$$

Additionally, it is convenient to define the contorsion tensor

$$2K_\mu{}^\lambda{}_\nu = T^\lambda{}_{\mu\nu} - T_{\nu\mu}{}^\lambda + T_\mu{}^\lambda{}_\nu, \quad (4)$$

and also the following tensor:

$$2S_\sigma{}^{\mu\nu} = K_\sigma{}^{\mu\nu} - \delta_\sigma^\mu T^\nu + \delta_\sigma^\nu T^\mu. \quad (5)$$

The combination $S_\sigma{}^{\mu\nu} T^\sigma{}_{\mu\nu}$ is denoted by T and it is usually called the torsion scalar. This quantity is a topological object and the TEGR is constructed by defining the action

$$S_{\text{TEGR}} = \frac{1}{\kappa} \int d^4x e T + S_m, \quad (6)$$

where S_m denotes the action of any matter field and $e = \det(e_\mu^a) = \sqrt{-g}$ is the volume element of the metric. The Ricci scalar R and the torsion scalar T differs by a boundary term via

$$R = -T + \frac{2}{e} \partial_\mu (e T^\mu) = -T + B. \quad (7)$$

Here, for simplicity we introduce $B = (2/e) \partial_\mu (e T^\mu) = \nabla_\mu T^\mu$. We can easily see that due to the above relation, the TEGR action reproduces the same field equations as GR (6) being equivalent to the Hilbert–Einstein action.

Now, we can straightforwardly generalize (6) by considering the following action:

$$S_{f(T)} = \frac{1}{\kappa} \int d^4x e f(T) + S_m, \tag{8}$$

where $f(T)$ is a smooth function of the torsion scalar. It is easy to see that, by setting $f(T) = T$, theTEGR action is recovered. In this theory is not possible to find the teleparallel equivalent of $f(R)$ gravity since now the boundary term in (7) contributes to the field equations. Since T itself is not invariant under local Lorentz transformations, this theory is also not invariant under Lorentz transformations. An important fact is that this theory is a second order one and hence, mathematically, it is easier than $f(R)$ gravity. The above action (8) can be immediately generalized by assuming that the function $f(T)$ depends also on the boundary term B . The action reads as follows [18]:

$$S_{f(T,B)} = \frac{1}{\kappa} \int d^4x e f(T, B) + S_m, \tag{9}$$

where f is a smooth function of two scalar fields, i.e. both the scalar torsion T and the boundary term B . The motivation of this action comes out from the fact that from $f(T)$ gravity, it is not possible to find an equivalent theory of its metric counterpart, the $f(R)$ gravity. From the above action, we can easily see that the $f(R)$ and $f(T)$ can be recovered by assuming $f(T, B) = f(-T + B) = f(R)$ and $f(T, B) = f(T)$, respectively.

By varying the above action with respect to the tetrad field, we get the field equations

$$\begin{aligned} &2eE_a^\lambda \square f_B - 2eE_a^\sigma \nabla^\lambda \nabla_\sigma f_B + eBE_a^\lambda f_B \\ &+ 4e[(\partial_\mu f_B) + (\partial_\mu f_T)] S_a^{\mu\lambda} + 4\partial_\mu(eS_a^{\mu\lambda}) f_T \\ &- 4ef_T T^\sigma{}_{\mu\alpha} S_\sigma{}^{\lambda\mu} - efE_a^\lambda = 16\pi e\Theta_a^\lambda, \end{aligned} \tag{10}$$

where $f_T = \partial f/\partial T$, $f_B = \partial f/\partial B$, ∇_σ is the covariant derivative with respect to the Levi-Civita connection and Θ_a^λ is the energy-momentum tensor. As said before, this theory can summarize features of $f(T)$, $f(R)$, and $f(R, T)$ gravities.

3 $f(T, B)$ cosmology

In this paper, we are interested in cosmological consequences of $f(T, B)$ gravity. In particular to find exact cosmological solutions by the Noether symmetry approach. Let us consider $f(T, B)$ cosmology in a flat Friedmann–Lemaître–Robertson–Walker (FLRW) universe. The spatially flat FLRW metric in Cartesian coordinates reads as follows:

$$ds^2 = dt^2 - a(t)^2 (dx^2 + dy^2 + dz^2), \tag{11}$$

where $a(t)$ is the scale factor of the universe. This metric can be constructed by the following tetrad field:

$$e_\mu^a = \text{diag}(1, a(t), a(t), a(t)). \tag{12}$$

Since $f(T, B)$ is not invariant under Lorentz transformations, one needs to be very careful with the choice of the tetrad. For instance, the unwanted condition $f_{TT} = 0$ appears when one considers a flat diagonal FLRW tetrad in spherical coordinates. The above vierbein is a “good tetrad” in the sense of Ref. [26] since it will not constrain our system.

By considering a standard perfect fluid as a content of the universe and using the above tetrad, we find that the modified Friedmann equations are given by

$$\begin{aligned} &-3H^2(3f_B + 2f_T) + 3H\dot{f}_B - 3\dot{H}f_B \\ &+ \frac{1}{2}f(T, B) = \kappa\rho(t), \end{aligned} \tag{13}$$

$$\begin{aligned} &-(3H^2 + \dot{H})(3f_B + 2f_T) - 2H\dot{f}_T + \ddot{f}_B \\ &+ \frac{1}{2}f(T, B) = -\kappa p(t). \end{aligned} \tag{14}$$

Here dots represent derivation with respect to the cosmic time and $H = \dot{a}/a$ is the Hubble parameter. In addition, $\rho(t)$ and $p(t)$ are the energy density and pressure of the cosmological fluid respectively. It is clear that by setting $f(T, B) = -f(T - B) = -f(-R)$ we recover the FLRW equations in $f(R)$ gravity with the standard notation (see for example [3,27]). Immediately we have

$$\begin{aligned} &-\frac{f(R)}{2} + 3(H^2 + \dot{H})f_R(R) \\ &- 18(4H^2\dot{H} + H\ddot{H})f_{RR}(R) = \kappa\rho(t), \end{aligned} \tag{15}$$

$$\begin{aligned} &\frac{f(R)}{2} - (H + 3H^2)f_R(R) \\ &+ 6(8H^2\dot{H} + 4\dot{H}^2 + 6H\ddot{H} + \ddot{H})f_{RR}(R) \\ &+ 36(4H\dot{H} + \ddot{H})^2 f_{RRR}(R) = \kappa p(t), \end{aligned} \tag{16}$$

where $f_R = df(R)/dR$. Moreover, we can choose $f(T, B) = f(T)$ to find the FLRW equations in $f(T)$ gravity given by

$$12H^2 f_T + f(T) = 2\kappa\rho(t), \tag{17}$$

$$48H^2 \dot{H} f_{TT} - (12H^2 + 4\dot{H})f_T - f(T) = 2\kappa p(t), \tag{18}$$

see also [30]. Note that the theory $f(R, T)$ can be viewed as a special case of $f(T, B)$ gravity since we can choose $f(T, B) = f(-T + B, T) = f(R, T)$. In this sense, it can be argued that $f(T, B)$ should be a more natural theory to consider than $f(R, T)$ as we will discuss below.

In the specific case we are dealing with, the cosmological equations can be derived both from the field Eq. (10) or deduced by a point-like canonical Lagrangian $\mathcal{L}(a, \dot{a}, T, \dot{T}, B, \dot{B})$ related to the action (9), where dots represent derivation with respect to the cosmic time t . Here, $\mathbb{Q} \equiv \{a, T, B\}$ is the configuration space from which it is possible to derive $\mathbb{TQ} \equiv \{a, \dot{a}, T, \dot{T}, B, \dot{B}\}$, the corresponding tangent space on which \mathcal{L} is defined as an application. The variables $a(t)$, $T(t)$, and $B(t)$ are, respectively, the scale factor, the torsion scalar and the boundary term defined in the FLRW metric. The Euler–Lagrange equations are given by

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{a}} = \frac{\partial \mathcal{L}}{\partial a}, \quad \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{T}} = \frac{\partial \mathcal{L}}{\partial T}, \quad \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{B}} = \frac{\partial \mathcal{L}}{\partial B}, \quad (19)$$

with the energy condition

$$E_{\mathcal{L}} = \frac{\partial \mathcal{L}}{\partial \dot{a}} \dot{a} + \frac{\partial \mathcal{L}}{\partial \dot{T}} \dot{T} + \frac{\partial \mathcal{L}}{\partial \dot{B}} \dot{B} - \mathcal{L} = 0. \quad (20)$$

As a consequence, the infinite number of degrees of freedom of the original field theory are reduced to a finite number as in mechanical systems.

Let us consider the canonical variables a, T, B in order to derive the $f(T, B)$ action as follows:

$$S_{f(T,B)} = \int \mathcal{L}(a, \dot{a}, T, \dot{T}, B, \dot{B}) dt.$$

In a flat FLRW metric, it is

$$T = -6 \left[\frac{\dot{a}(t)}{a(t)} \right]^2, \quad (21)$$

$$B = -6 \left[\frac{\ddot{a}(t)}{a(t)} + 2 \left(\frac{\dot{a}(t)}{a(t)} \right)^2 \right]. \quad (22)$$

Therefore, the Ricci scalar is

$$R = -T + B = -6 \left[\left(\frac{\dot{a}(t)}{a(t)} \right)^2 + \frac{\ddot{a}(t)}{a(t)} \right]. \quad (23)$$

By using (21) and (22), we can rewrite the action (9) into its point-like representation using the Lagrange multipliers λ_1 and λ_2 as

$$S_{f(T,B)} = 2\pi^2 \int dt \left\{ (f(T, B))a^3 - \lambda_1 \left[T + 6 \left(\frac{\dot{a}}{a} \right)^2 \right] - \lambda_2 \left(B + 6 \left[\frac{\ddot{a}}{a} + 2 \left(\frac{\dot{a}}{a} \right)^2 \right] \right) \right\}. \quad (24)$$

By varying this action with respect to T and B , we find

$$(a^3 f_T - \lambda_1) \delta T = 0 \rightarrow \lambda_1 = a^3 f_T, \quad (25)$$

$$(a^3 f_B - \lambda_2) \delta B = 0 \rightarrow \lambda_2 = a^3 f_B. \quad (26)$$

Thus, the action (24) becomes

$$S_{f(T,B)} = 2\pi^2 \int dt \left\{ (f(T, B))a^3 - a^3 f_T \left(T + 6 \left(\frac{\dot{a}}{a} \right)^2 \right) - a^3 f_B \left(B + 6 \left[\frac{\ddot{a}}{a} + 2 \left(\frac{\dot{a}}{a} \right)^2 \right] \right) \right\}, \quad (27)$$

and the point-like Lagrangian is

$$\mathcal{L}_{f(T,B)} = a^3 [f(T, B) - T f_T - B f_B] - 6a\dot{a}^2 f_T + 6a^2 \dot{a} (f_{BT} \dot{T} + f_{BB} \dot{B}), \quad (28)$$

where we have integrated by parts. This Lagrangian is canonical and depends on the three time-dependent fields a, T , and B . If we choose $f(T, B) = f(T)$, we recover the teleparallel $f(T)$ cosmology with the Lagrangian [30]

$$\mathcal{L}_{f(T)} = a^3 [f(T) - T f_T] - 6a\dot{a}^2 f_T. \quad (29)$$

In addition, if we choose $f(T, B) = f(-T + B) = f(R)$ we obtain the point-like Lagrangian action of $f(R)$ gravity [31]

$$\mathcal{L}_{f(R)} = a^3 [f(R) - R f_R] + 6a\dot{a}^2 f_R + 6a^2 \dot{a} \dot{R} f_{RR}. \quad (30)$$

Moreover, we can recover the teleparallel–curvature gravity assuming $f(T, B) = f(-T + B, T) = f(R, T)$ and starting from the following considerations. In this case, we need to be careful in adopting the suitable variables. Assuming $x_1 = -T + B = R$ and $x_2 = T$, we have

$$f_T = \frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial T} + \frac{\partial f}{\partial x_2} \frac{\partial x_2}{\partial T} = -f_{x_1} + f_{x_2} = -f_R + f_T, \quad (31)$$

$$f_B = \frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial B} + \frac{\partial f}{\partial x_2} \frac{\partial x_2}{\partial B} = f_{x_1} = f_R. \quad (32)$$

Using the derivative chain rule, the second and third derivatives of T and B are given by

$$f_{TT} = f_{RR} + f_{TT} - 2f_{RT}, \quad (33)$$

$$f_{TB} = -f_{RR} + f_{RT} \quad (34)$$

$$f_{BB} = f_{RR}. \quad (35)$$

The $f(R, T)$ point-like Lagrangian is given by

$$\mathcal{L}_{f(R,T)} = a^3 [f(R, T) - T f_T - R f_R] - 6a\dot{a}^2 (f_T - f_R) + 6a^2 \dot{a} (f_{RT} \dot{T} + f_{RR} \dot{R}). \quad (36)$$

see also [33] for a discussion. With these considerations in mind, let us search for cosmological solutions for the above models by the Noether symmetry approach.

4 Noether symmetry approach for $f(T, B)$ cosmology

The Noether symmetry approach has been widely used in the literature to find cosmological solutions in modified gravity (see [22] for a comprehensive review). The main idea is to find symmetries in a given model and then to use them to reduce related dynamical systems and find exact solutions. As a byproduct, the existence of the symmetries selects the functions inside the models (e.g. couplings and self-interaction potentials) that, in most cases, have a physical meaning. In this sense, the existence of a Noether symmetry is a sort of selection rule. Essentially, the technique consists in deriving constants of motions. Any constant of motion is related to a conserved quantity that allows one to reduce the dynamical system and then to obtain exact solutions. If the number of constants is equal to the number of degrees of freedom, the system is completely integrable.

In general, a Noether symmetry for a given Lagrangian exists if the condition

$$L_X \mathcal{L} = 0 \rightarrow X \mathcal{L} = 0 \tag{37}$$

is satisfied. X is the Noether vector field and L_X is the Lie derivative. For generalized coordinates q_i , we can construct the Noether vector field X . We have

$$X = \alpha^i(q) \frac{\partial}{\partial q^i} + \frac{d\alpha^i(q)}{dt} \frac{\partial}{\partial \dot{q}^i}, \tag{38}$$

where α^i are functions defined in a given configuration space \mathbb{Q} that assign the Noether vector. In our case, a symmetry generator X in the space $\mathbb{Q} \equiv \{a, T, B\}$ is

$$X = \alpha \partial_a + \beta \partial_T + \gamma \partial_B + \dot{\alpha} \partial_{\dot{a}} + \dot{\beta} \partial_{\dot{T}} + \dot{\gamma} \partial_{\dot{B}}, \tag{39}$$

where α, β, γ depend on a, T , and B . Therefore we have

$$\dot{\alpha} = \left(\frac{\partial \alpha}{\partial a}\right) \dot{a} + \left(\frac{\partial \alpha}{\partial T}\right) \dot{T} + \left(\frac{\partial \alpha}{\partial B}\right) \dot{B}, \tag{40}$$

$$\dot{\beta} = \left(\frac{\partial \beta}{\partial a}\right) \dot{a} + \left(\frac{\partial \beta}{\partial T}\right) \dot{T} + \left(\frac{\partial \beta}{\partial B}\right) \dot{B}, \tag{41}$$

$$\dot{\gamma} = \left(\frac{\partial \gamma}{\partial a}\right) \dot{a} + \left(\frac{\partial \gamma}{\partial T}\right) \dot{T} + \left(\frac{\partial \gamma}{\partial B}\right) \dot{B}. \tag{42}$$

A Noether symmetry exists if at least one of the functions α, β , and γ is different from zero. Their analytic forms can be found by making explicit Eq. (37), which corresponds to a set of partial differential equations given by equating to zero the terms in $\dot{a}^2, \dot{a}\dot{T}, \dot{a}\dot{B}, \dot{T}^2, \dot{B}^2, \dot{B}\dot{T}$ and so on. For a n dimensional configuration space, we have $1 + n(n + 1)/2$ equations derived from Eq. (37). In our case, the configuration space is three dimensional, so we have seven partial differential equations. Explicitly, from (37), we find the fol-

lowing system of partial differential equations:

$$f_T \left(2a \frac{\partial \alpha}{\partial a} + \alpha \right) + f_{TB} \left(a\gamma - a^2 \frac{\partial \beta}{\partial a} \right) + af_{TT}\beta - a^2 f_{BB} \frac{\partial \gamma}{\partial a} = 0, \tag{43}$$

$$f_{TB} \left(a \frac{\partial \alpha}{\partial a} + a \frac{\partial \beta}{\partial T} + 2\alpha \right) - 2f_T \frac{\partial \alpha}{\partial T} + af_{BB} \frac{\partial \gamma}{\partial T} + a(f_{TTB}\beta + f_{TBB}\gamma) = 0, \tag{44}$$

$$f_{BB} \left(a \frac{\partial \alpha}{\partial a} + a \frac{\partial \gamma}{\partial B} + 2\alpha \right) + af_{TB} \frac{\partial \beta}{\partial B} + a(\beta f_{TBB} + \gamma f_{BBB}) - 2f_T \frac{\partial \alpha}{\partial B} = 0, \tag{45}$$

$$f_{TB} \frac{\partial \alpha}{\partial T} = 0, \tag{46}$$

$$f_{BB} \frac{\partial \alpha}{\partial B} = 0, \tag{47}$$

$$f_{TB} \frac{\partial \alpha}{\partial B} + f_{BB} \frac{\partial \alpha}{\partial T} = 0 \tag{48}$$

$$3(f - Bf_B - Tf_T)\alpha - a(Bf_{TB} + Tf_{TT})\beta - a(Bf_{BB} + Tf_{TB})\gamma = 0. \tag{49}$$

where the unknown variables are α, β, γ , and the function $f(T, B)$. There are two different strategies to solve it and to find symmetries: (1) one can directly solve the system (43)–(49) and then find the unknown functions; (2) one can impose specific forms of $f(T, B)$ and search for the related symmetries [25]. From a physical viewpoint, the second approach is better because it allows one to study reliable models. By the first strategy, solutions can be achieved but, in most cases, they are implicit functions that do not allow a physical analysis [22]. We will adopt the second one to discuss the $f(T, B)$ cosmology.

4.1 Case 1: $f(T, B) = b_0 B^k + t_0 T^m$

For a power-law like function given by $f(T, B) = b_0 B^k + t_0 T^m$, where b_0, t_0, k and m are constants, we find that the unique solution of (43)–(49) is for $k = 1$. This is trivial because it gives $f(T, B) = b_0 B + t_0 T^m$, which is the same as a power-law $f(T)$ function. This comes from the fact that B is a boundary term so that a linear form of the function in B does not introduce any change in the field equations. Hence, this kind of function gives the same results as reported in [30].

4.2 Case 2: $f(T, B) = f_0 B^k T^m$

Let us now study the case where the function takes the form

$$f(T, B) = f_0 B^k T^m, \tag{50}$$

where f_0, k and m are constants. From (46)–(48), it is $\alpha = \alpha(a)$. If we replace the function (50) into (43)–(49), we find the following Noether vector:

$$X = \frac{\alpha_0}{a^2} \partial_a - \frac{6\alpha_0 T}{a^3} \partial_T - \frac{3\alpha_0 B}{a^3} \partial_B, \tag{51}$$

and also the constraint $k = 1 - m$, which gives us $f(T, B) = f_0 B^k T^{\frac{1-k}{2}}$. Here, α_0 is an integration constant that can be set equal to 1 without loss of generality [22]. Let us now find cosmological solutions for this type of function. The point-like Lagrangian (28) takes the following form:

$$\mathcal{L} = \frac{1}{2} f_0 (k - 1) a(t) B^{k-2} T^{-\frac{1}{2}(k+1)} \left[6B^2 \dot{a}(t)^2 - 6ka(t) \dot{a}(t) (B\dot{T} - 2\dot{B}T) - B^2 T a(t)^2 \right]. \tag{52}$$

It is easy to see that the trivial case $k = 1$, which produces $f = f_0 B$, gives the expected result where the field equations are identically zero. The Euler–Lagrange equation for the scale factor $a(t)$ gives, for $k \neq 1$ and $f_0 \neq 0$,

$$\begin{aligned} a(t)^2 \left(4(k - 2)kT^2 \dot{B}^2 + 4kBT (T\ddot{B} - (k - 1)\dot{B}\dot{T}) \right. \\ \left. + kB^2 \left((k + 1)\dot{T}^2 - 2T\ddot{T} \right) + B^3 T^2 \right) \\ + 2B^3 T \dot{a}(t)^2 + 2a(t)B^2 (2BT\ddot{a}(t) \\ + \dot{a}(t) (2kT\dot{B} - (k + 1)B\dot{T})) = 0. \end{aligned} \tag{53}$$

Additionally, the energy equation becomes

$$-6B^2 \dot{a}(t)^2 + 6ka(t) \dot{a}(t) (B\dot{T} - 2\dot{B}T) + B^2 T a(t)^2 = 0. \tag{54}$$

If we replace T and B given by (21) and (22) we find that Eqs. (53) and (54) become

$$\begin{aligned} (k - 1)a(t)^4 \ddot{a}(t)^4 + 4(k - 2)\dot{a}(t)^8 - 4(k - 4)a(t)^2 \\ \times \ddot{a}(t)\dot{a}(t)^5 - 8(k - 1)a(t)\dot{a}(t)^6 \ddot{a}(t) \\ + 4(k - 2)a(t)^3 \ddot{a}(t)\dot{a}(t)^3 \ddot{a}(t) + 2a(t)^2 \dot{a}(t)^4 \left(a(t)\ddot{a}(t) \right. \\ \left. + 2(2k - 5)\ddot{a}(t)^2 \right) + a(t)^3 \dot{a}(t)^2 \\ \times \left((k - 2)a(t)\ddot{a}(t)^2 + (10 - 4k)\ddot{a}(t)^3 + a(t)\ddot{a}(t)\ddot{a}(t) \right) \\ - 2(k - 1)a(t)^4 \ddot{a}(t)\dot{a}(t)\ddot{a}(t)^2 = 0, \end{aligned} \tag{55}$$

$$\begin{aligned} -(k - 1)a(t)^2 \ddot{a}(t)^2 - 2(k - 2)\dot{a}(t)^4 + ka(t)^2 \ddot{a}(t)\dot{a}(t) \\ + 2(k + 2)a(t)\dot{a}(t)^2 \ddot{a}(t) = 0. \end{aligned} \tag{56}$$

These equations admit power law solutions for the scale factor being

$$a(t) = a_0 t^s, \quad s = \frac{1 + k}{3}. \tag{57}$$

The torsion scalar and the boundary term are $T = -6s^2/t^2$ and $B = 6s(1 - 3s)/t^2$, respectively. Immediately we see that

several cosmologically interesting cases can be recovered. A radiation solution is for

$$a(t) = a_0 t^{1/2}, \quad \text{with } k = \frac{1}{2}. \tag{58}$$

A dust solution is for

$$a(t) = a_0 t^{2/3}, \quad \text{with } k = 1. \tag{59}$$

A stiff matter one is for

$$a(t) = a_0 t^{1/3}, \quad \text{with } k = 0. \tag{60}$$

Power-law inflation is recovered for $s \geq 1$ and $k \geq 2$.

4.3 Case 3: $f(T, B) = -T + F(B)$

The case $f(T, B) = -T + F(B)$ is a deviation ofTEGR up to a function which depends on the boundary term. The Noether condition gives

$$2a \frac{\partial \alpha}{\partial a} + \alpha + a^2 F_{BB} \frac{\partial \gamma}{\partial a} = 0, \tag{61}$$

$$2 \frac{\partial \alpha}{\partial T} + a F_{BB} \frac{\partial \gamma}{\partial T} = 0, \tag{62}$$

$$F_{BB} \left(a \frac{\partial \alpha}{\partial a} + a \frac{\partial \gamma}{\partial B} + 2\alpha \right) + a\gamma F_{BBB} + 2 \frac{\partial \alpha}{\partial B} = 0, \tag{63}$$

$$F_{BB} \frac{\partial \alpha}{\partial B} = 0, \tag{64}$$

$$F_{BB} \frac{\partial \alpha}{\partial T} = 0, \tag{65}$$

$$3\alpha (f(B) - BF_B) - aBF_{BB}\gamma = 0. \tag{66}$$

Discarding the trivial case $F(B) = B$, which gives standardTEGR, from (64) and (65) we obtain again $\alpha = \alpha(a)$. Using this condition in (62), we find that $\gamma = \gamma(B, a)$ and the equations become

$$2a \frac{d\alpha}{da} + \alpha - a^2 F_{BB} \frac{\partial \gamma}{\partial a} = 0, \tag{67}$$

$$F_{BB} \left(a \frac{d\alpha}{da} + a \frac{\partial \gamma}{\partial B} + 2\alpha \right) + a\gamma F_{BBB} = 0, \tag{68}$$

$$3\alpha (F(B) - BF_B) - aBF_{BB}\gamma = 0. \tag{69}$$

We can rewrite (68) as

$$\partial_B (\gamma F_{BB}) = -F_{BB} \left(\frac{d\alpha}{da} + 2 \frac{\alpha}{a} \right), \tag{70}$$

which can be solved for γ , yielding

$$\gamma = - \left(\frac{d\alpha}{da} + 2 \frac{\alpha}{a} \right) \frac{F_B}{F_{BB}} + \frac{g(a)}{F_{BB}}, \tag{71}$$

where $g(a)$ is an arbitrary function of the scale factor. Therefore, from (67) one finds that

$$F_B \left(2\alpha - 2a \frac{d\alpha}{da} - a^2 \frac{d^2\alpha}{da^2} \right) + \alpha + 2a \frac{d\alpha}{da} + a^2 \frac{dg}{da} = 0, \tag{72}$$

which has the following solution:

$$\alpha(a) = c_1 a + \frac{C_2}{a^2}, \quad g(a) = c_3 - \frac{C_2}{a^3} - 3c_1 \log a. \tag{73}$$

Here, c_1 , C_2 and c_3 are integration constants. Now, by using (69) and (71) we find that

$$a^3(3Bc_1 \log(a) - Bc_3 + 3c_1 F) + C_2(B + 3F - 3BF_B) = 0. \tag{74}$$

Since $F = F(B)$, we see that the first term is zero, so that $c_1 = c_3 = 0$, yielding

$$B + 3F - 3BF_B = 0, \tag{75}$$

which can be solved obtaining

$$F(B) = f_0 B + \frac{1}{3} B \log(B). \tag{76}$$

Therefore, we find the following symmetry solutions:

$$X = \frac{C_2}{a^2} \partial_a + \beta \partial_T - \frac{C_2}{a^3 F_{BB}} \partial_B, \tag{77}$$

$$f(T, B) = T + f_0 B + \frac{1}{3} B \log(B). \tag{78}$$

Let us now search for cosmological solutions for this model. Considering (77), it is convenient to introduce the following coordinates:

$$u = \frac{1}{3C_2} a^3, \quad v = \frac{1}{3C_2} [F_B + \log(a)], \tag{79}$$

which transform the Noether vector as

$$X = \partial_u + \beta \partial_T. \tag{80}$$

Lagrangian (28) reads as follows:

$$\mathcal{L} = \frac{2C_2}{\ddot{u}(t)} \left[\ddot{u}(t)^2 + \ddot{u}(t)\dot{u}(t) \right], \tag{81}$$

and hence, the Euler–Lagrange equation for $u(t)$ is

$$\ddot{\ddot{u}}(t) - \frac{\ddot{u}(t)^2}{\ddot{u}(t)} = 0. \tag{82}$$

Hence, it is easily to find the following solution:

$$u(t) = \frac{u_3}{u_1^2} e^{u_1 t} + u_2 t + u_0, \tag{83}$$

where u_0 , u_1 , u_2 , and u_3 are integration constants. Additionally, since $\mathcal{L} = E - 2V$, with E being the Hamiltonian (the energy) of the system and $V(t) = 2C_2 u_3 e^{u_1 t}$ can be understood as an energy potential, we find the following constraint:

$$2C_2 u_1 v_1 = E. \tag{84}$$

Finally, using (79) we can express this cosmological solution in terms of the scale factor as follows:

$$a(t) = \left[\frac{3C_2 u_3 e^{u_1 t}}{u_1^2} + 3C_2 (tu_2 + u_0) \right]^{1/3}. \tag{85}$$

It is easy to see that this solution gives a de Sitter universe for the specific choice $u_2 = u_0 = 0$. This de Sitter solution is reported also in [19] where a cosmological reconstruction technique is adopted.

In the next subsections, remarkable theories that can be recovered from $f(T, B)$ gravity are discussed. We will see that all symmetries found in earlier studies for $f(T)$, $f(R)$, and $f(R, T)$ can be achieved starting from the Noether symmetry equations of Eqs. (43)–(49) derived for $f(T, B)$ cosmology.

4.4 Case 4: $f(T, B) = f(T)$

The first remarkable example is $f(T)$ gravity. The cases studied in [28,29] are straightforwardly obtained. Equations (46)–(48) are identically satisfied since $f_{TB} = f_{BB} = 0$. The other equations become

$$f_T \left(2a \frac{\partial \alpha}{\partial a} + \alpha \right) + a f_{TT} \beta = 0, \tag{86}$$

$$f_T \frac{\partial \alpha}{\partial T} = 0, \tag{87}$$

$$f_T \frac{\partial \alpha}{\partial B} = 0, \tag{88}$$

$$3(f - T f_T) \alpha - a \beta T f_{TT} = 0. \tag{89}$$

By discarding the TEGR case ($f(T) = -T$) we have $f_T \neq 0$ and hence, from Eqs. (87) and (88), we find again $\alpha = \alpha(a)$. From Eq. (89), we find that

$$\alpha(a) = \frac{a f_{TT} T}{3(f - T f_T)} \beta(a, T, B) \tag{90}$$

By replacing this expression in (86) we get the following differential equation for β :

$$\frac{\partial \beta}{\partial a} = -\frac{3f}{2a f_T T} \beta(a, T, B). \tag{91}$$

To solve this equation, let us assume that β can be separated as $\beta(a, T, B) = \beta_1(a)\beta_2(T)\beta_3(B)$. We obtain

$$\frac{2a}{\beta_1} \frac{d\beta_1}{da} = -\frac{3f}{f_T T} = -\frac{3}{C}. \tag{92}$$

Here we have used the fact that the l.h.s. of the equation only depends on a and the r.h.s. only on T , so that C is a constant. Thus, it is easy to solve the above equation yielding

$$f(T) = f_0 T^C, \tag{93}$$

where f_0 is an integration constant. Moreover, it is straightforward to find that the Noether symmetry vector becomes

$$X = -\frac{1}{3}\beta_0 a^{1-\frac{3}{2C}} \partial_a + \frac{\beta_0 T a^{-\frac{3}{2C}}}{C} \partial_T + \gamma \partial_B, \tag{94}$$

where β_0 is an integration constant. As shown in [28,29], using this symmetry, one finds that $f(T)$ gravity admits power-law cosmological solutions of the form of $a(t) \propto t^{-2C/C_3}$. A more general study of power-law $f(T)$ cosmology is in [30].

4.5 Case 5: $f(T, B) = f(-T + B) = f(R)$

We can recover $f(R)$ gravity by assuming $f(T, B) = f(-T + B) = f(R)$. Hence, $f_R(R) = f'(-T + B) = -f_T = f_B$ and the system of differential equations (43)–(49) related to the Noether symmetry in $f(R)$ gravity becomes

$$f_R \left(2a \frac{\partial \alpha}{\partial a} + \alpha \right) - a f_{RR} \left(\beta + a \frac{\partial \beta}{\partial a} - \gamma - a \frac{\partial \gamma}{\partial a} \right) = 0, \tag{95}$$

$$-f_{RR} \left(a \frac{\partial \alpha}{\partial a} + a \frac{\partial \beta}{\partial T} + 2\alpha - a \frac{\partial \gamma}{\partial T} \right) + 2f_R \frac{\partial \alpha}{\partial T} + a f_{RRR}(\beta - \gamma) = 0, \tag{96}$$

$$f_{RR} \left(a \frac{\partial \alpha}{\partial a} + a \frac{\partial \gamma}{\partial B} + 2\alpha - a \frac{\partial \beta}{\partial B} \right) + a f_{RRR}(\gamma - \beta) + 2f_R \frac{\partial \alpha}{\partial B} = 0, \tag{97}$$

$$-f_{RR} \frac{\partial \alpha}{\partial T} = 0, \tag{98}$$

$$f_{RR} \frac{\partial \alpha}{\partial B} = 0, \tag{99}$$

$$-f_{RR} \left(\frac{\partial \alpha}{\partial B} - \frac{\partial \alpha}{\partial T} \right) = 0 \tag{100}$$

$$3\alpha (f - R f_R) + a R f_{RR}(\beta - \gamma) = 0. \tag{101}$$

In addition, we require that $\beta = -\gamma$ to obtain the same generators as in $f(R)$ gravity. In doing this, Eqs. (96) and (97) are identical and hence the Noether equations become

$$f_R \left(2a \frac{\partial \alpha}{\partial a} + \alpha \right) + 2a f_{RR} \left(\gamma + a \frac{\partial \gamma}{\partial a} \right) = 0, \tag{102}$$

$$f_{RR} \frac{\partial \alpha}{\partial B} = 0. \tag{103}$$

$$f_{RR} \left(a \frac{\partial \alpha}{\partial a} + 2\alpha + 2a \frac{\partial \gamma}{\partial R} \right) + 2f_R \frac{\partial \alpha}{\partial R} + 2a \gamma f_{RRR} = 0 \tag{104}$$

$$3\alpha (f - R f_R) - 2a \gamma R f_{RR} = 0. \tag{105}$$

It is worth noticing that, in order to recover the same Noether symmetry equations as in [31], we require that $\gamma = \frac{1}{2}\tilde{\beta}$. This issue appears in the computation of the Lie derivative since the generator and some terms related with the generator of T and B are summed twice. Therefore, by changing $\gamma = \frac{1}{2}\tilde{\beta}$ we find the same equations as in [31], that is,

$$f_R \left(2a \frac{\partial \alpha}{\partial a} + \alpha \right) + a f_{RR} \left(\tilde{\beta} + a \frac{\partial \tilde{\beta}}{\partial a} \right) = 0, \tag{106}$$

$$f_{RR} \left(a \frac{\partial \alpha}{\partial a} + 2\alpha + a \frac{\partial \tilde{\beta}}{\partial R} \right) + 2f_R \frac{\partial \alpha}{\partial R} + a \tilde{\beta} f_{RRR} = 0 \tag{107}$$

$$f_{RR} \frac{\partial \alpha}{\partial R} = 0, \tag{108}$$

$$3\alpha (f - R f_R) - a R f_{RR} \tilde{\beta} = 0. \tag{109}$$

Since we are not interested in the GR case, $f_{RR} \neq 0$ and, from (108), we directly find that $\alpha = \alpha(a)$. Hence, Eq. (107) can be rewritten as

$$\partial_R(\beta f_{RR}) = -f_{RR} \left(\frac{d\alpha}{da} + \frac{2\alpha}{a} \right) \tag{110}$$

and solved yielding

$$\beta(a, R) = \frac{g(a)}{f_{RR}(R)} - \frac{(a\alpha'(a) + 2\alpha(a)) f_R(R)}{a f_{RR}(R)}, \tag{111}$$

where $g(a)$ is an arbitrary function depending on a . Note that the latter solution is very similar to the one found in (71) for the case $f(T, B) = -T + F(B)$. Now if we replace this solution into (106), we obtain

$$f_R(R) [\alpha(a) - a (a\alpha''(a) + \alpha'(a))] + a [ag'(a) + g(a)] = 0, \tag{112}$$

which is satisfied only if each bracket is zero. We have

$$\alpha(a) = \frac{(a^2 + 1)\alpha_0}{2a} - \frac{(a^2 - 1)\alpha_1}{2a}, \tag{113}$$

$$g(a) = \frac{c}{a}, \tag{114}$$

where c , α_0 and α_1 are integration constants. It is important to mention that this result is more general than that in [31] where some terms in $\alpha(a)$ are not present; however, the final result does not changes since the symmetry vectors are similar. By placing the above expression into (109), we find

$$\frac{(\alpha_0 + \alpha_1)(3f(R) - 2Rf_R(R))}{2a} + \frac{3}{2}a(\alpha_0 - \alpha_1) f(R) - cR = 0, \tag{115}$$

which is valid only if $c = 0$ and $\alpha_0 = \alpha_1$. This gives the result

$$f(R) = f_0R^{3/2}, \tag{116}$$

where f_0 is an integration constant. By considerations similar to those in Sect. 4.3, it is possible to show that $f(R)$ gravity admits power-law solution of the form

$$a(t) \propto t^{1/2}, \text{ and } a(t) = a_0[c_4t^4 + c_3t^3 + c_2t^2 + c_1t + c_0]^{1/2}. \tag{117}$$

For a discussion on the physical meaning of such solutions, see [32].

4.6 Case 6: $f(T, B) = f(-T + B, T) = f(R, T)$

Let us now discuss the case where $f(T, B) = f(-T + B) = f(R, T)$. First of all, in order to have the same generator as in $f(R, T)$, we require to change the function $\gamma(T, B, a) \rightarrow \gamma(T, B, a) + \beta(T, B, a)$. Additionally, for the derivative terms, we need to use Eqs. (31)–(35) and hence the transformation $\partial/\partial T \rightarrow \partial/\partial T - \partial/\partial R$. After these replacements, the Noether conditions become

$$\alpha(f_R - f_T) + a\gamma(f_{RR} - f_{RT}) + a\beta(f_{RT} - f_{TT}) + 2a\frac{\partial\alpha}{\partial a}(f_R - f_T) + a^2f_{RR}\frac{\partial\gamma}{\partial a} + a^2f_{RT}\frac{\partial\beta}{\partial a} = 0, \tag{118}$$

$$\begin{aligned} &(f_{RT} - f_{RR})\left(a\frac{\partial\alpha}{\partial a} + a\frac{\partial\beta}{\partial T} - a\frac{\partial\beta}{\partial R} + 2\alpha\right) \\ &- 2(f_T - f_R)\left(\frac{\partial\alpha}{\partial T} - \frac{\partial\alpha}{\partial R}\right) \\ &+ af_{RR}\left(\frac{\partial(\gamma + \beta)}{\partial T} - \frac{\partial(\gamma + \beta)}{\partial R}\right) \\ &+ a((f_{TRR} - f_{RRR})(\beta + \gamma) \\ &+ (f_{RRR} + f_{TTR} - 2f_{TRR})\beta) = 0, \tag{119} \end{aligned}$$

$$\begin{aligned} &f_{RR}\left(a\frac{\partial\alpha}{\partial a} + a\frac{\partial(\beta + \gamma)}{\partial R} + 2\alpha\right) + a(f_{RT} - f_{RR})\frac{\partial\beta}{\partial R} \\ &+ a(\beta(f_{TRR} - f_{RRR}) + (\gamma + \beta)f_{RRR}) \\ &- 2(f_T - f_R)\frac{\partial\alpha}{\partial R} = 0, \tag{120} \end{aligned}$$

$$(f_{RT} - f_{RR})\left(\frac{\partial\alpha}{\partial T} - \frac{\partial\alpha}{\partial R}\right) = 0, \tag{121}$$

$$f_{RR}\frac{\partial\alpha}{\partial R} = 0, \tag{122}$$

$$(f_{RT} - f_{RR})\frac{\partial\alpha}{\partial R} + f_{RR}\left(\frac{\partial\alpha}{\partial T} - \frac{\partial\alpha}{\partial R}\right) = 0, \tag{123}$$

$$3\alpha(f - Rf_R - Tf_T) - a\gamma(Tf_{RT} + Rf_{RR}) - a\beta(Tf_{TT} + Rf_{RT}) = 0. \tag{124}$$

By adding (119) with (120), we get

$$\begin{aligned} &2\alpha f_{RT} + a\gamma f_{TRR} + a\beta f_{TTR} + af_{RT}\frac{\partial\alpha}{\partial a} + 2f_R\frac{\partial\alpha}{\partial T} \\ &- 2f_T\frac{\partial\alpha}{\partial T} + af_{RR}\frac{\partial\gamma}{\partial T} + af_{RT}\frac{\partial\beta}{\partial T} = 0. \tag{125} \end{aligned}$$

In addition, by subtracting (121) with (123) and using (122) and then adding (121) with (123), we get $f_{RT}\frac{\partial\alpha}{\partial T} = 0$ and $f_{RT}\frac{\partial\alpha}{\partial R} = 0$. Therefore, the Noether symmetry equations can be rewritten as follows:

$$\begin{aligned} &\alpha(f_R - f_T) + a\gamma(f_{RR} - f_{RT}) + a\beta(f_{RT} - f_{TT}) \\ &+ 2a\frac{\partial\alpha}{\partial a}(f_R - f_T) + a^2f_{RR}\frac{\partial\gamma}{\partial a} + a^2f_{RT}\frac{\partial\beta}{\partial a} = 0, \tag{126} \end{aligned}$$

$$\begin{aligned} &2\alpha f_{RT} + a\gamma f_{TRR} + a\beta f_{TTR} + af_{RT}\frac{\partial\alpha}{\partial a} + 2f_R\frac{\partial\alpha}{\partial T} \\ &- 2f_T\frac{\partial\alpha}{\partial T} + af_{RR}\frac{\partial\gamma}{\partial T} + af_{RT}\frac{\partial\beta}{\partial T} = 0, \tag{127} \end{aligned}$$

$$\begin{aligned} &2\alpha f_{RR} + af_{RRR}\gamma + af_{TRR}\beta + af_{RR}\frac{\partial\alpha}{\partial a} + 2f_R\frac{\partial\alpha}{\partial R} \\ &- 2f_T\frac{\partial\alpha}{\partial R} + af_{RR}\frac{\partial\gamma}{\partial R} + af_{RT}\frac{\partial\beta}{\partial R} = 0, \tag{128} \end{aligned}$$

$$f_{RT}\frac{\partial\alpha}{\partial T} = 0, \tag{129}$$

$$f_{RR}\frac{\partial\alpha}{\partial R} = 0, \tag{130}$$

$$f_{RT}\frac{\partial\alpha}{\partial R} = 0,$$

$$3\alpha(f - Rf_R - Tf_T) - a\gamma(Tf_{RT} + Rf_{RR}) - a\beta(Tf_{TT} + Rf_{RT}) = 0. \tag{131}$$

It is clear that by changing $\beta \rightarrow \gamma$ and $\gamma \rightarrow \beta$, the system of differential equations (126)–(131) for the Noether symmetry of $f(R, T)$ result the same as those studied in [33] with the same physical implications.

5 Discussion

In this paper, we discussed an extension of modified teleparallel gravity including functions of the torsion scalar T and its related boundary term $B = \frac{2}{e}\partial_\mu(eT^\mu)$. In such a way, a gravitational theory with two fields, i.e. T and B , can be taken into account. If not assumed in a trivial way, that is linear in

B , interesting features come out from the combinations of T and B , in particular, the possibility to relate $f(T)$ and $f(R)$ gravity under the same standard. This means that not only GR and TEGR (respectively, theories linear in the Ricci scalar R and the torsion scalar T in their actions) result in the “same” effective theory but also their extensions, also if conceptually very different, can show analogies and similitudes.

Here we consider the Noether symmetry approach in order to investigate the related cosmologies. The main result is that the Noether vector fields emerging from $f(T, B)$ gravity are a general standard to find solutions both in the starting theory and in the particular cases like $f(T)$, $f(R)$, and $f(R, T)$. In this last case, the Noether technique allows one to deal with curvature R and torsion T scalars as two scalar fields.

The related cosmological solutions are of physical interest and, essentially, all the main cosmological behaviors can be recovered. However, this is only a preliminary study where no effective comparison with observations has been made and only toy models have been analyzed in order to test the technique.

In forthcoming studies, we will adopt an approach for $f(T, B)$ gravity as in [30], where the condition (37) is extended to the possibility of discussing singular Lagrangians. Furthermore, a similar approach can be used for teleparallel modified Gauss–Bonnet gravity $f(T, B, T_G, B_G)$ as studied in [34]. Under this standard, other interesting models can naturally arise by taking into account some specific functions of T and B as $f = f(-T + B, -T_G + B_G) = f(R, G)$ (modified Gauss–Bonnet) or $f = f(T, T_G)$ gravity (modified teleparallel Gauss–Bonnet). The final issue is to define a *mother theory* by which all extensions and modifications of GR can be generated.

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