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Deceleration versus acceleration universe in different frames of $F(R)$ gravity

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ABSTRACT

In this paper we study the occurrence of accelerating universe versus decelerating universe between the $F(R)$ gravity frame (Jordan frame) and non-minimally coupled scalar field theory frame, and the minimally coupled scalar field theory frame (Einstein frame) for various models. As we show, if acceleration is imposed in one frame, it will not necessarily correspond to an accelerating metric when transformed in another frame. As we will demonstrate, this issue is model and frame-dependent but it seems there is no general scheme which permits to classify such cases.

1. Introduction

One of the most profound questions in modified gravity is related with the choice of the physical frame. The frame corresponding to $F(R)$ gravity [1] is called Jordan frame, and by a conformal transformation it can be transformed to a minimally coupled scalar–tensor theory, with the corresponding frame being called Einstein frame. In addition to these, there are also frames in which the scalar field is non-minimally coupled to gravity, and these can be reached by a $F(R)$ gravity by also using a suitably chosen conformal transformation, or directly by the Einstein frame theory by conformally transforming the theory.

Generally speaking, one should confront the theoretical predictions of a specific gravitational theory with the observable Universe history supported by the current observational data. In this sense, each of the three mentioned frames, namely the $F(R)$ gravity, and the minimal and non-minimal scalar theories, may give a viable description of the observable Universe history. However, it is not sure that a viable description in one frame gives also viable and convenient description in the other frame. For instance, it may give a viable but physically inconvenient description. In other words, there appears the question which of these three frames is the most physical one (and in which sense) or, at least, which of these frames gives a convenient description of the Universe history. Eventually, the answer to this question depends very much from the confrontation with the observational data, from the specific choice of the theory and from the observer associated with specific frame. At the same time, the related question is about equivalent results in all three frames and/or about construction of the observable quantities which are invariant under conformal transformations between the three frames.

In the study of the inflationary epoch, when one is dealing with quasi-de Sitter space, it is expected that the spectral index of primordial curvature perturbations and the scalar-to-tensor ratio calculated in two frames ($F(R)$ and minimal scalar–tensor) are nearly the same. Indeed, the equivalence of two frames was explicitly demonstrated in Refs. [2] and also in [3]. However, this is surely not enough for number of reasons. For example, for the effect on neutron stars in $F(R)$ gravity, the Jordan and Einstein frame pictures are different, as was shown in Ref. [4].

In addition, finite-time singularities [5–9] between Jordan and Einstein frames belong to different types of singularity, see for example [8,10], because the conformal transformation does not work for singular points. In this research line, in this paper we shall investigate under which circumstances, an accelerating evolution in one frame may be transformed to a decelerating evolution in the other frame. We shall consider three types of frames, namely the

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$F(R)$ gravity frame (Jordan frame), the minimally coupled scalar–tensor theory frame (Einstein frame) and the non-minimally coupled scalar–tensor frame. By using several illustrative examples we shall demonstrate that an accelerating cosmology in one frame may correspond to a decelerating cosmology in the other frame, and thus the physical interpretation in the two frames may be different.

For a preliminary discussion along this research line, see [11]. Eventually, it depends on the model under investigation and on the specific choice of the conformal transformation. For simplicity, we do not add matter sector in this paper, since this may introduce extra complications due to the appearance of a non-minimal coupling in the matter sector for the Einstein frame.

This paper is organized as follows: In Sec. 2 we introduce some essential information about the correspondence between the Jordan and Einstein frame. In Sec. 3 we study the case where the scalar field is minimally coupled to the scalar curvature, so we start from the Einstein frame, and we study how an accelerating cosmology in the Einstein frame is transformed in the Jordan frame. Sec. 4 is devoted to generalize the latter idea by taking into account a theory non-minimally coupled between the scalar field and the Ricci scalar, so we study the correspondence between these frames. Finally the conclusions follow in Sec. 5.

2. Correspondence between Jordan and Einstein frame: essential properties

Before we go into the main focus of the paper, we present in brief some essential information regarding the correspondence between the Jordan and Einstein frame. For details on these issues, we refer the reader to Ref. [1]. For simplicity, in this section we will assume that $\kappa^2 = 1$. Let us start from the Jordan frame $F(R)$ gravity action,

$$\mathcal{S} = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-\hat{g}}F(R)\,, \tag{1}$$

with $\hat{g}_{\mu\nu}$ being the metric tensor in the Jordan frame. By introducing an auxiliary field, which we denote as A, the action of Eq. (1) is written in the following way,

$$\mathcal{S} = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-\hat{g}}\left(F'(A)(R-A)+F(A)\right)\,. \tag{2}$$

Upon variation of the action (2), with respect to the auxiliary scalar degree of freedom, we obtain the solution $A = R$, and this actually proves that the actions of Eqs. (2) and (1) are mathematically equivalent. The Jordan and Einstein frames are connected via the following canonical transformation,

$$\varphi = \sqrt{\frac{3}{2}}\ln(F'(A)) \tag{3}$$

where φ will denote the canonical scalar field in the Einstein frame. By making the following conformal transformation,

$$g_{\mu\nu} = e^{-\varphi}\hat{g}_{\mu\nu} \tag{4}$$

with $g_{\mu\nu}$ denoting the Einstein frame metric, we finally obtain the following action,

$$\begin{aligned}\tilde{\mathcal{S}} &= \int \mathrm{d}^4x\sqrt{-g}\left(R-\frac{1}{2}\left(\frac{F''(A)}{F'(A)}\right)^2 g^{\mu\nu}\partial_\mu A\partial_\nu A\right.\\ &\left.-\left(\frac{A}{F'(A)}-\frac{F(A)}{F'(A)^2}\right)\right)\\ &= \int \mathrm{d}^4x\sqrt{-g}\left(R-\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right)\,,\end{aligned} \tag{5}$$

which is the Einstein frame action of the canonical scalar field φ. The scalar field potential $V(\varphi)$ appearing in Eq. (5), is equal to,

$$\begin{aligned}V(\varphi) &= \frac{1}{2}\left(\frac{A}{F'(A)}-\frac{F(A)}{F'(A)^2}\right)\\ &= \frac{1}{2}\left(e^{-\sqrt{2/3}\varphi}R\left(e^{\sqrt{2/3}\varphi}\right)\right.\\ &\left.-e^{-2\sqrt{2/3}\varphi}F\left[R\left(e^{\sqrt{2/3}\varphi}\right)\right]\right)\,.\end{aligned} \tag{6}$$

Therefore, for a specifically given $F(R)$ gravity, we can find the corresponding canonical scalar field potential by using the expression (6). The method can work in the converse way, by finding the function $R(\varphi)$. We can easily express the Ricci scalar as a function of the canonical scalar field, by solving Eq. (3) with respect to the auxiliary scalar A, bearing in mind that $A = R$. Then, for a specifically given potential, we can combine Eqs. (6) and (3), and by differentiating with respect to R, we obtain,

$$RF_R = 2\sqrt{\frac{3}{2}}\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\frac{V(\varphi)}{e^{-2(\sqrt{2/3})\varphi}}\right) \tag{7}$$

where $F_R = \frac{\mathrm{d}F(R)}{\mathrm{d}R}$. In this way, if the scalar potential is given, the $F(R)$ gravity easily follows by using Eq. (7).

3. Minimally curvature-coupled scalar–tensor theory

Let us start from the minimally coupled scalar–tensor theory action, which is the Einstein frame action,

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right\}. \tag{8}$$

By assuming a flat Friedmann–Robertson Walker (FRW) metric in the Einstein frame, with line element,

$$ds_E^2 = -dt^2 + a(t)^2 d\mathbf{x}^2, \tag{9}$$

the equations of motion corresponding to the action (8) are equal to,

$$3H^2 = \frac{1}{2}\dot{\phi}^2 + V\,, \tag{10}$$

$$3H^2 + 2\dot{H} = -\frac{1}{2}\dot{\phi}^2 + V\,, \tag{11}$$

where for simplicity we used a physical units system where $\kappa^2 = 1$.

Now we conformally transform the metric $g_{\mu\nu} = e^{\sqrt{\frac{2}{3}}\phi}\tilde{g}_{\mu\nu}$ to switch the Einstein frame to the Jordan frame, in which the action of Eq. (8) takes the form

$$S = \int d^4x\sqrt{-\tilde{g}}\left\{\frac{1}{2}e^{\sqrt{\frac{2}{3}}\phi}\tilde{R} - e^{2\sqrt{\frac{2}{3}}\phi}V(\phi)\right\}. \tag{12}$$

Hence the line element of the Jordan frame metric reads,

$$ds_J^2 = e^{\sqrt{\frac{2}{3}}\phi}\left(-dt^2 + a(t)^2 d\mathbf{x}^2\right), \tag{13}$$

or by introducing a new time coordinate $\tilde{t}$, which is defined as follows,

$$d\tilde{t} = e^{\frac{1}{2}\sqrt{\frac{2}{3}}\phi}dt, \tag{14}$$

the metric of Eq. (13) may be rewritten as follows,

$$ds_J^2 = -d\tilde{t}^2 + \tilde{a}(\tilde{t})^2 d\mathbf{x}^2, \tag{15}$$

and the scale factors of the Jordan and Einstein frames are related as follows,

$$\tilde{a}(\tilde{t}(t)) = e^{\frac{1}{2}\sqrt{\frac{2}{3}}\phi} a(t). \tag{16}$$

For an arbitrary function b wich depends on time, we will use the following notation: $\dot{b} \equiv \frac{db}{dt}$ and $b' \equiv \frac{db}{d\tilde{t}}$. Now, if we calculate first derivative of expression (16):

$$\tilde{a}' = \frac{1}{2}\sqrt{\frac{2}{3}}\dot{\phi}a + \dot{a}, \tag{17}$$

and also the second derivative reads,

$$\tilde{a}'' = \left(\frac{1}{2}\sqrt{\frac{2}{3}}\ddot{\phi}a + \frac{1}{2}\sqrt{\frac{2}{3}}\dot{\phi}\dot{a} + \ddot{a}\right) e^{-\frac{1}{2}\sqrt{\frac{2}{3}}\phi}. \tag{18}$$

Now the conditions that must be satisfied in order for an accelerated expansion in one frame corresponds to decelerated expansion in another one, are $\ddot{a} > 0$ and simultaneously $\tilde{a}'' < 0$. In addition the conditions $\dot{a} >$ and $\tilde{a}' > 0$ must hold true in order to have accelerating expansion in the both frames. It is clear from expression (18) that in order for the above constraints to be satisfied, it suffices if the following conditions hold true,

$$A \equiv \frac{1}{2}\sqrt{\frac{2}{3}}\ddot{\phi}a + \frac{1}{2}\sqrt{\frac{2}{3}}\dot{\phi}\dot{a} + \ddot{a} > 0, \tag{19}$$

$$\ddot{a} < 0, \tag{20}$$

and $\dot{a} >$ and $\tilde{a}' > 0$ as well.

Consider the following cosmological evolution in the Einstein frame,

$$H(t) \equiv \frac{\dot{a}}{a} = f_0 (t - t_s)^{\alpha}, \tag{21}$$

where for simplicity we will assume that $t_s = 0$. By integrating Eq. (21) we find,

$$a(t) = a_0 e^{\frac{f_0}{\alpha+1} t^{\alpha+1}}. \tag{22}$$

We can see that expressions for $\tilde{a}'$ and $\tilde{a}''$ contain only $a(t)$, $\phi(t)$ and its derivatives with respect to the cosmic time t. In order to find $\phi(t)$ let us subtract equations (10)–(11), and we find,

$$-2\dot{H} = \dot{\phi}^2, \tag{23}$$

which is true for any type of potential $V(\phi)$. Therefore, by using Eq. (21) we obtain,

$$\dot{\phi}(t) = \sqrt{-2 f_0 \alpha} t^{\frac{\alpha-1}{2}}, \tag{24}$$

and in addition we get,

$$\ddot{\phi}(t) = \frac{\alpha - 1}{2}\sqrt{-2 f_0 \alpha} t^{\frac{\alpha-3}{2}}. \tag{25}$$

First of all, note that we need a positive parameter f_0 for an expanding Universe in the Einstein frame, and in addition, the parameter α should be negative in order to have real values of the scalar field. The first derivative $\dot{a}$ reads,

$$\dot{a} = a f_0 t^{\alpha}, \tag{26}$$

and it is always positive for the parameters chosen as we discussed above. By taking into account Eq. (17), the first derivative $\tilde{a}'$ reads,

$$\tilde{a}' = a f_0 t^{\alpha}\left(1 + \sqrt{\frac{-\alpha}{3 f_0}} t^{\frac{-\alpha-1}{2}}\right), \tag{27}$$

and we can see that for any set of parameters, it is impossible to have expansion in one frame and contraction in the other frame. Now let us calculate the second derivatives of the scale factors. For $\ddot{a}$ we have,

$$\ddot{a} = \dot{a} f_0 t^{\alpha} + a f_0 \alpha t^{\alpha-1} = a f_0 t^{\alpha-1}\left(\alpha + f_0 t^{\alpha+1}\right), \tag{28}$$

and we can see that depending on time, this function may have different sign for the parameters chosen as above. By calculating the expression (19), we get,

$$A = a t^{\frac{\alpha-3}{2}}\left(\frac{\alpha-1}{2}\sqrt{\frac{-f_0\alpha}{3}} + \sqrt{\frac{-f_0\alpha}{3}} f_0 t^{\alpha+1} + f_0^2 t^{\frac{3\alpha+3}{2}} + f_0 \alpha t^{\frac{\alpha+1}{2}}\right). \tag{29}$$

Thus, according to our previous considerations, we are interested in the case that $\ddot{a} < 0$ and $A > 0$ or equivalently,

$$\alpha + f_0 t^{\alpha+1} < 0, \tag{30}$$

and in addition,

$$\frac{\alpha-1}{2}\sqrt{\frac{-f_0\alpha}{3}} + \sqrt{\frac{-f_0\alpha}{3}} f_0 t^{\alpha+1} + f_0^2 t^{\frac{3\alpha+3}{2}} + f_0 \alpha t^{\frac{\alpha+1}{2}} > 0. \tag{31}$$

From a general point of view we have the next situation: two inequalities (30)–(31) with three parameters α, f_0 and t. So it is quite natural to expect that both these inequalities may be satisfied at least for some time instance. But actually the real picture is more complicated, because there are additional restrictions for the parameters, namely, $\alpha < 0$, $f_0 > 0$, which must be satisfied as well. The inequality of Eq. (30) indicates,

$$t^{\alpha+1} < \frac{-\alpha}{f_0}. \tag{32}$$

Let us suppose that (32) may hold true at the time instance t_*, thus we may put,

$$t_*^{\alpha+1} = m\frac{-\alpha}{f_0}, \tag{33}$$

where m is some numerical parameter restricted according to (32) as

$$0 < m < 1. \tag{34}$$

Substituting expression (33) in (31), we find,

$$\frac{\alpha-1}{2}\sqrt{\frac{-f_0\alpha}{3}} - \alpha m\sqrt{\frac{-f_0\alpha}{3}} + m^{\frac{3}{2}}|\alpha|\sqrt{-f_0\alpha} + m^{\frac{1}{2}}\alpha\sqrt{-f_0\alpha} > 0. \tag{35}$$

Note at this point, that in expression (31) the first and fourth terms are negative, whereas the second and the third are positive. Also note that in Eq. (35) we have $\alpha < 0$. The above expression can be simplified as follows,

$$B = \frac{\alpha-1}{2\sqrt{3}} - \frac{\alpha m}{\sqrt{3}} + m^{\frac{3}{2}}|\alpha| + m^{\frac{1}{2}}\alpha > 0. \tag{36}$$

We rewrite the above expression as follows,

$$A_1 + A_2 > 0, \tag{37}$$

where A_1 is equal to,

$$A_1 = \frac{\alpha-1}{2\sqrt{3}} - \frac{\alpha m}{\sqrt{3}}, \tag{38}$$

while A_2 is,

$$A_2 = m^{\frac{3}{2}}|\alpha| + m^{\frac{1}{2}}\alpha. \tag{39}$$

We can see that A_2 is always negative in the range ($\alpha < 0$, $0 < m < 1$), whereas A_1 may be positive due to the last term. Thus it is clear Eq. (36) will hold true if $A_1 > 0$. A detailed analysis of this inequality, imposes additional restrictions for parameters, namely, $\alpha < -1$ and $\frac{1}{2} < m < 1$. By taking into account the negative values of α, it is possible to rewrite (36) as follows,

$$|\alpha| > \frac{1}{2\sqrt{3}m^{\frac{3}{2}} + 2m - 2\sqrt{3}m^{\frac{1}{2}} - 1}. \tag{40}$$

The expression in the denominator is monotonically increasing function of m in the range $\frac{1}{2} < m < 1$, which crosses zero near the point $m \approx 0.8042$. This means, that all interesting for us values of m lie in the narrow interval $0.8042 \lesssim m < 1$. Let us take for instance $m = 0.9$, then expression (40) indicates that $|\alpha| \gtrsim 2.12$, so let us assume $\alpha = -3$. By substituting these values into (36) we find $B \backsimeq 0.119$. In addition, note that the result is true for any positive values of f_0, whereas the time t may be calculated by using (33). Thus we explicitly demonstrated how to obtain an accelerating cosmology in one frame, which corresponds to a decelerating one in a conformal frame, for the Jordan and Einstein frames. In the next section we consider the non-minimally coupled scalar field frame.

4. Non-minimally curvature-coupled scalar theory

Let us now consider the case where the Ricci scalar is non-minimally coupled to the scalar field. In this case, the gravitational action takes the following form,

$$S = \int d^4x\sqrt{-g}\left[(1 + f(\phi))\frac{R}{\kappa^2} - \frac{1}{2}\omega(\phi)\partial_\mu\phi\partial^\mu\phi - V(\phi)\right]. \tag{41}$$

We shall refer to this non-minimally coupled frame as Jordan frame too. In Ref. [12], it was shown that by performing the following conformal transformation,

$$g_{\mu\nu} = [1 + f(\phi)]^{-1}\tilde{g}_{\mu\nu}\,, \tag{42}$$

we recover the Einstein frame minimally-coupled scalar–tensor theory given by the action,

$$S = \int dx^4\sqrt{-\tilde{g}}\left(\frac{\tilde{R}}{\kappa^2} - \frac{1}{2}W(\phi)\,\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - U(\phi)\right), \tag{43}$$

where the functions $W(\phi)$ and $U(\phi)$ are defined as follows,

$$W(\phi) = \frac{\omega(\phi)}{1 + f(\phi)} + \frac{3}{\kappa^2(1 + f(\phi))^2}\left(\frac{df(\phi)}{d\phi}\right)^2, \tag{44}$$

$$U(\phi) = \frac{V(\phi)}{[1 + f(\phi)]^2}. \tag{45}$$

For the action of Eq. (43) and for a flat FRW metric, the cosmological equations can be written as follows (taking $\kappa^2 = 1$),

$$\tilde{H}^2 = \frac{\kappa^2}{6}\rho_\phi\,, \tag{46}$$

$$\tilde{H}' = -\frac{\kappa^2}{4}\left(\rho_\phi + p_\phi\right), \tag{47}$$

$$2W(\phi)\left[\phi'' + 3\tilde{H}\phi'\right] + \left[W_\phi\left(\phi'\right)^2 + 2U_\phi\right] = 0\,, \tag{48}$$

where ρ_ϕ stands for,

$$\rho_\phi = \frac{1}{2}W(\phi)(\phi')^2 + U(\phi)\,, \tag{49}$$

$$p_\phi = \frac{1}{2}W(\phi)(\phi')^2 - U(\phi)\,, \tag{50}$$

and the "prime" denotes differentiation with respect to $\tilde{t}$, whereas W_ϕ denotes partial differentiation with respect to ϕ. Therefore, we have that

$$W(\phi)(\phi')^2 = -4\tilde{H}'\,, \tag{51}$$

$$U(\phi) = 6\tilde{H}^2 + 2\tilde{H}'. \tag{52}$$

In the rest of this section, we will study how the acceleration might change from one frame and another.

4.1. How does acceleration change from one frame to another?

Let us now find the conditions which when are satisfied, we can have acceleration in one frame and deceleration in the other. The scale factors and time-coordinates are related by,

$$a(t) = [1 + f(\phi)]^{-1/2}\,\tilde{a}(\tilde{t})\,, \quad \frac{d\tilde{t}}{dt} = [1 + f(\phi)]^{1/2}\,. \tag{53}$$

Now, by differentiating with respect to the time t, we find,

$$\frac{da(t)}{dt} \equiv \dot{a} = \tilde{a}' - \frac{1}{2}[1 + f(\phi)]^{-1}f_\phi\phi'\tilde{a}, \tag{54}$$

and the second derivative is,

$$\ddot{a} = \frac{1}{2}[1 + f(\phi)]^{-\frac{1}{2}}\left\{2[1 + f(\phi)]\tilde{a}'' + [1 + f(\phi)]^{-1}f_\phi^2(\phi')^2\tilde{a} - f_{\phi\phi}(\phi')^2\tilde{a} - f_\phi\phi''\tilde{a} - f_\phi\phi'\tilde{a}'\right\}. \tag{55}$$

Now, let us assume that in the Jordan (Einstein) frame the Universe is decelerating (accelerating). To materialize such a scenario, the following inequalities need to hold true,

$$2[1 + f(\phi)]\tilde{a}'' + [1 + f(\phi)]^{-1}f_\phi^2(\phi')^2\tilde{a} - f_{\phi\phi}(\phi')^2\tilde{a} - f_\phi\phi''\tilde{a} - f_\phi\phi'\tilde{a}' < 0\,, \tag{56}$$

$$\tilde{a}'' > 0\,. \tag{57}$$

Additionally, we need to impose that in each frame the Universe is expanding, hence

$$\tilde{a}' - \frac{1}{2}[1 + f(\phi)]^{-1}f_\phi\phi'\tilde{a} > 0\,, \tag{58}$$

$$\tilde{a}' > 0\,. \tag{59}$$

4.1.1. Example I

As a first example, let us consider a de-Sitter expansion, $\tilde{a}(\tilde{t}) = \tilde{a}_0 e^{\tilde{H}_0\tilde{t}}$ and we assume that the function coupling is chosen as follows,

$$f(\phi) = \frac{1 - \alpha\phi}{\alpha\phi}\,, \tag{60}$$

where α is a constant. As a first task, we need to demonstrate that such kind of solution really exists. Equations (46)–(47) indicate that (recall that we assumed $\kappa^2 = 1$),

$$\rho_\phi = 6\tilde{H}_0^2\,, \quad p_\phi = -6\tilde{H}_0^2\,, \tag{61}$$

so by using definitions (49)–(50) we easily find,

$$(\phi')^2W(\phi) = 0\,, \quad U(\phi) = 6\tilde{H}_0^2\,. \tag{62}$$

Thus, we have three possibilities to satisfy these relations: either $\phi' = 0$ together with $W = 0$ or $W \neq 0$ and finally if $\phi' \neq 0$

whereas $W = 0$. Let us focus on the last possibility. In this case from Eq. (44) we have,

$$W(\phi) = 0 \rightarrow \quad \omega(\phi) = \frac{-3}{\alpha\phi^3} \tag{63}$$

and the potential may be found from Eq. (45) and it reads,

$$V(\phi) = \frac{6\tilde{H}_0^2}{\alpha^2\phi^2}\,. \tag{64}$$

In Ref. [12] it was explicitly demonstrated that the solution in this case is,

$$\phi = \tilde{t} = \alpha^{\frac{-1}{3}}\left(\frac{3}{2}t\right)^{\frac{2}{3}}\,. \tag{65}$$

With regard to Eq. (59) we have $\tilde{a}' = \tilde{H}_0\tilde{a} > 0$. The expression (58) reads $\tilde{a}\left[\tilde{H}_0 + (2\tilde{t})^{-1}\right] > 0$ and it is also true for any time $\tilde{t}$. With regard to (57) we have $\tilde{a}'' = \tilde{H}_0^2\tilde{a} > 0$. Finally with regard to the expression (56) we have,

$$\frac{\tilde{a}}{\alpha\tilde{t}^3}\left(2\tilde{H}_0^2\tilde{t}^2 + \tilde{H}_0\tilde{t} - 1\right) < 0\,, \tag{66}$$

and by also taking into account that $\tilde{t} > 0$, we find that the expression (56) is satisfied for $0 < \tilde{t} < 1/(2\tilde{H}_0)$.

It is interesting to note, that case (60) may be easy generalized to

$$f(\phi) = \frac{1 - \alpha\phi^n}{\alpha\phi^n}\,. \tag{67}$$

In this case, the general solution $\phi(\tilde{t}) = \tilde{t}$ is valid, but the time t-dependence, will be completely different. The condition (66) is transformed as follows,

$$\frac{\tilde{a}}{\alpha\tilde{t}^{n+2}}\left(2\tilde{H}_0^2\tilde{t}^2 + n\tilde{H}_0\tilde{t} - n\right) < 0\,, \tag{68}$$

and it is satisfied when the following conditions hold true, $0 < \tilde{t} < \left[-\frac{n}{4} + \frac{n}{4}\sqrt{1 + \frac{8}{n}}\right]/\tilde{H}_0$.

4.1.2. Example II

Now let us try to modify the previous example in such a way so that it becomes more realistic. Firstly, notice that the solution $\phi = \tilde{t}$ is the general solution, as it was also demonstrated in previous works [12]. So by adding by hand some specific cosmological behavior $\tilde{a}(\tilde{t})$ we may find the functions ω and V which generate such a cosmological evolution. In this sense, this method is analog of the reconstruction procedure [12]. In the previous example, the cosmological evolution was that of an exact de Sitter solution, but it is well known that inflation is actually a quasi-de Sitter solution. Thus consider the following quasi-de Sitter cosmological evolution,

$$\tilde{H}(\tilde{t}) = H_0 - h(\tilde{t}), \tag{69}$$

where H_0 is some positive constant and h is a slow-varying function of the time variable $\tilde{t}$. This function will determine the exit from inflation, and also it crucially affects the spectrum of cosmological perturbation, but for the moment we leave this undetermined. Note also that the expression in Eq. (56) (which was derived by using general considerations) contains only the functions $\tilde{a}$, $\tilde{a}'$, $\tilde{a}''$ and also the function f and its derivatives. Let us assume that the function f has the form (69), in effect we have,

$$\tilde{a}' = \tilde{a}H_0 - \tilde{a}h\,, \tag{70}$$

$$\tilde{a}'' = \tilde{a}H_0^2 - 2\tilde{a}hH_0 - \tilde{a}h^2 - \tilde{a}h'. \tag{71}$$

By substituting these expressions in Eq. (56) and by rearranging the terms we get,

$$\frac{\tilde{a}}{\alpha\tilde{t}^{n+2}}\left[2\tilde{t}^2H_0^2 + nH_0\tilde{t} - n - \left(2\tilde{t}^2h' + 2\tilde{t}^2h^2 + 4H_0h\tilde{t}^2 + nh\tilde{t}\right)\right] < 0. \tag{72}$$

This is an algebraic[1] expression of time variable $\tilde{t}$, provides us an explicit example where acceleration (inflation) in one frame corresponds to deceleration in another. It is quite clear that the expression (72) may be valid at least for some time instance, but let us demonstrate this fact explicitly by using a concrete example. We assume that $h(\tilde{t}) = k\ln(1 + \tilde{t})$. For small values of time (early universe) we may decompose h in Taylor series $h \approx k\tilde{t}$, $h' \approx k - k\tilde{t}$ and the expression in square brackets (it is clear that the coefficient of it, is always positive) becomes,

$$2\tilde{t}^2H_0^2 + nH_0\tilde{t} - n - 2\tilde{t}^2k + 2\tilde{t}^3k - 2\tilde{t}^4k^2 - 4H_0k\tilde{t}^3 - nk\tilde{t}^2 < 0. \tag{73}$$

By keeping only leading order terms, from the inequality (73), we find that (56) in this case will be true for,

$$0 < \tilde{t} \lesssim \frac{1}{H_0}. \tag{74}$$

Hence we showed explicitly that accelerating expansion in one frame corresponds to decelerating expansion in the other frame.

5. Discussion

The question which of the different mathematical frames is the physical one is very profound, and in this paper we investigated how it is possible to have accelerating expansion of the Universe in one frame and decelerating expansion in the other. The frames which we studied are the Jordan frame of an $F(R)$ gravity, the Einstein frame which corresponds to a minimally coupled scalar–tensor theory, and finally the frame which corresponds to a non-minimally coupled scalar field. As we showed, if certain conditions are satisfied, it is possible to have acceleration in the Jordan frame, and when the metric is conformally transformed to the Einstein frame, the transformed metric can describe a decelerating Universe. The same situation can occur when one considers a non-minimally coupled scalar field that is conformally transformed to the Einstein frame. We illustrated our arguments by using several characteristic examples. According to our findings, the various mathematical frames are physically equivalent when conformal invariant quantities are considered, like for example quantities related to the comoving curvature. However, in principle, the physical interpretation in the various frames can be different if non-conformal invariant quantities are used.

Finally, let us note that the question of physical equivalence of the different frames becomes much more involved when quantum effects are taken into account, see for example Ref. [13] for minimal and non-minimal frames, and also consult [14].

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[1] We suppose that $h(\tilde{t})$ is some known function.

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