Journal of Functional Analysis 273 (2017) 1049–1094

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Trace formulas for Wiener–Hopf operators with applications to entropies of free fermionic equilibrium states

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A R T I C L E   I N F O

Article history:
Received 2 June 2016
Accepted 3 April 2017
Available online 19 April 2017
Communicated by L. Gross

MSC:
primary 47G30, 35S05
secondary 45M05, 47B10, 47B35

Keywords:
Non-smooth functions of Wiener–Hopf operators
Asymptotic trace formulas
Entanglement entropy

A B S T R A C T

We consider non-smooth functions of (truncated) Wiener–Hopf type operators on the Hilbert space $\mathsf{L}^2(\mathbb{R}^d)$. Our main results are uniform estimates for trace norms ($d \geq 1$) and quasiclassical asymptotic formulas for traces of the resulting operators ($d = 1$). Here, we follow Harold Widom's seminal ideas, who proved such formulas for smooth functions decades ago. The extension to non-smooth functions and the uniformity of the estimates in various (physical) parameters rest on recent advances by one of the authors (AVS). We use our results to obtain the large-scale behaviour of the local entropy and the spatially bipartite entanglement entropy (EE) of thermal equilibrium states of non-interacting fermions in position space $\mathbb{R}^d$ ($d \geq 1$) at positive temperature, $T > 0$. In particular, our definition of the thermal EE leads to estimates that are simultaneously sharp for small T and large scaling parameter $\alpha > 0$ provided that the product $T\alpha$ remains bounded from below. Here α is the reciprocal quasiclassical parameter. For $d = 1$ we obtain for the thermal EE an asymptotic formula which is consistent with the large-scale

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http://dx.doi.org/10.1016/j.jfa.2017.04.005

behaviour of the ground-state EE (at $T = 0$), previously established by the authors for $d \geq 1$.

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1. Introduction

The present paper is devoted to the study of (bounded, self-adjoint) operators of the form

$$W_\alpha := W_\alpha(a; \Lambda) := \chi_\Lambda \operatorname{Op}_\alpha(a)\chi_\Lambda, \ \alpha > 0, \tag{1.1}$$

on $\mathsf{L}^2(\mathbb{R}^d)$, $d \geq 1$, where χ_Λ is the indicator function of a set $\Lambda \subset \mathbb{R}^d$. The parameter $1/\alpha$ can be interpreted as a quasiclassical parameter that tends to zero in our asymptotic results. The notation $\operatorname{Op}_\alpha(a)$ stands for the α-pseudo-differential operator with symbol $a = a(\boldsymbol{\xi})$, which acts on Schwartz functions u on $\mathbb{R}^d$ as

$$\big(\operatorname{Op}_\alpha(a)u\big)(\mathbf{x}) := \frac{\alpha^d}{(2\pi)^{\frac{d}{2}}} \iint e^{i\alpha\boldsymbol{\xi}\cdot(\mathbf{x}-\mathbf{y})} a(\boldsymbol{\xi})u(\mathbf{y})d\mathbf{y}d\boldsymbol{\xi}, \quad \mathbf{x} \in \mathbb{R}^d.$$

Integrals without indication of the integration domain always mean integration over $\mathbb{R}^d$ with the value of d which is clear from the context. More general symbols, depending on both variables $\mathbf{x}$ and $\boldsymbol{\xi}$, or operators with matrix-valued symbols can be also treated, but we limit our attention only to $\boldsymbol{\xi}$-dependent symbols. We call the operator (1.1) a Wiener–Hopf operator. A more precise term would be *truncated* Wiener–Hopf operator, but we always omit "truncated" for brevity. Our focus is on the operator difference

$$D_\alpha(a, \Lambda; f) := \chi_\Lambda f(W_\alpha(a;\Lambda))\chi_\Lambda - W_\alpha(f \circ a; \Lambda), \tag{1.2}$$

with some suitably chosen functions f. We are interested in the asymptotic properties of the trace $\operatorname{tr} D_\alpha(a, \Lambda; f)$ as $\alpha \to \infty$. If $f(0) = 0$, Λ is bounded and a decays sufficiently fast at infinity, then it is trivial to observe that the second operator on the right-hand side of (1.2) is trace-class and

$$\operatorname{tr} W_\alpha(f \circ a; \Lambda) = \frac{\alpha^d}{(2\pi)^d} |\Lambda| \int f\big(a(\boldsymbol{\xi})\big) d\boldsymbol{\xi}, \tag{1.3}$$

where $|\Lambda|$ is the d-dimensional Lebesgue measure of Λ. If $|\Lambda| = \infty$, then neither of the terms on the right-hand side of (1.2) is trace class (except in trivial cases), but their difference is trace class, under the conditions adopted in this paper. We must emphasise that it is essential to us to consider Λ in (1.2) also of infinite measure.

Asymptotic properties of $D_\alpha(a, \Lambda; f)$ have been extensively studied in the literature, with the majority of results obtained in the 1980's. All results obtained at that time pertained to the case of smooth functions f (or more precisely, smooth on the range of the symbol a) and bounded Λ. Under these assumptions, the case of a smooth symbol a was understood particularly well: the full asymptotic expansion of $\operatorname{tr} D_\alpha(a, \Lambda; f)$ in powers of α^{-1} was derived by A. Budylin–V. Buslaev [4] and H. Widom [30]. The paper [30] also provides a brief historical account of this problem. Out of all relevant bibliography we mention just one other paper by H. Widom, [29], whose ideas we exploit in some of our proofs.

Another important and challenging problem is to study the asymptotics of the trace of $D_\alpha(a, \Lambda; f)$ for discontinuous symbols, in particular, for symbols of the form $a = \chi_\Omega$ with a bounded region $\Omega \subset \mathbb{R}^d$. This problem was studied by H. Landau–H. Widom [12], H. Widom [28] (for $d = 1$) and by A.V. Sobolev [21,23] (for arbitrary $d \geq 1$). It was found that

$$\operatorname{tr} D_\alpha(a, \Lambda; f) = \mathfrak{W}_1\, \alpha^{d-1} \log(\alpha) + o(\alpha^{d-1} \log(\alpha)),\ \alpha \to \infty, \tag{1.4}$$

for a bounded domain $\Lambda \subset \mathbb{R}^d$ with an explicitly given coefficient $\mathfrak{W}_1 = \mathfrak{W}_1(\partial\Lambda, \partial\Omega, f)$. The discontinuity of the symbol a can be interpreted as the presence of one of the two *Fisher–Hartwig singularities* investigated in detail for truncated Toeplitz matrices, that is, for the discrete counterpart of Wiener–Hopf operators, see [6].

In recent years, new demands for the asymptotics of traces of Wiener–Hopf operators emerged, which have been triggered by applications to (quantum) statistical mechanics. Our interest originates from the large-scale behaviour of the spatially bipartite entanglement entropy (EE, also called mutual information) of free fermions in thermal equilibrium. Here one faces several mathematical challenges at the same time.

(1) **Non-smooth functions f.** One needs to consider the operator (1.2) with functions f that lack smoothness at finitely many points, or, which is the same in view of additivity, at one point. The functions of interest are the γ-*Rényi entropy functions* $\eta_\gamma, \gamma > 0$, that are defined in (10.1) and (10.2).
(2) **Unbounded Λ.** One needs to consider the operator (1.2) with unbounded domains Λ, in contrast to most of the previously known results.
(3) **Uniform estimates.** In quantum-mechanical applications, apart from the scaling parameter, it is natural to control the dependence of the symbol a on other parameters such as the temperature $T \geq 0$. Thus it is necessary to provide estimates and asymptotic remainder estimates that are uniform in the symbol a in some broad sense. For example, in the study of the entanglement entropy the symbol a in the operator (1.2) is given by the Fermi symbol $a_{T,\mu}$, see the definition (1.5), and one needs to control the T-dependence of the estimates. This requires substantial extra work since the results of [29,30] are not directly applicable.

A general approach to the study of operator differences of the form $Pf(PAP)P - Pf(A)P$ with a self-adjoint operator A, an orthogonal projection P and a non-smooth function f, was put forward in [25]. One application of the results in [25] is the extension of (1.4) to non-smooth functions f under the assumption that either Λ or its complement is bounded, thereby tackling challenges (1) and (2) above.

The special case $a(\boldsymbol{\xi}) = \chi_\Omega(\boldsymbol{\xi})$ for bounded $\Omega, \Lambda \subset \mathbb{R}^d$ was considered even earlier in [13]. In the quantum-mechanical context, formula (1.4), if used with $a = \chi_\Omega$ and the function $f = \eta_\gamma$, gives the large-scale asymptotics of the entanglement entropy at zero temperature with Fermi sea Ω, see also [7].

In the present paper we work exclusively with smooth symbols a with a fast decay at infinity. The function f is allowed to lack smoothness at one point, see Condition 2.1. A typical example of such a function is $f(t) = |t|^\gamma, \gamma > 0$. The region Λ is such that either Λ or $\mathbb{R}^d \setminus \Lambda$ is bounded, see Condition 3.1 for details.

The goal of this paper is two-fold, and it correspondingly splits in two parts.

Part 1: Sections 2–7. First we establish some explicit estimates for the (quasi-) norms of the operator (1.2) in the Schatten–von Neumann classes $\mathfrak{S}_q$, $q \in (0, 1]$. Later on we need only trace class norms, but the more general $\mathfrak{S}_q$-estimates are obtained at "no extra cost", and are provided for the sake of completeness. Here we rely on the results of [25] where this problem was studied in the abstract setting. We quote these results in Proposition 2.2. Indeed, the very fact that $D_\alpha(a, \Lambda; f) \in \mathfrak{S}_q$ is an almost direct consequence of Proposition 2.2, but this alone is insufficient for us since we need sharp explicit estimates, uniform in a. Thus we identify a class of symbols a that we call *multi-scale symbols*, and establish explicit estimates for $\|D_\alpha(a, \Lambda; f)\|_{\mathfrak{S}_q}$, which are uniform in some suitable sense, see Remark 3.3. They do turn out to be sharp in α and T when used for the symbol (1.5), which serves as our leading example. The main estimate is contained in Theorem 3.5. This takes care of issue (3) above.

Our next result is the asymptotic formula for $\operatorname{tr} D_\alpha(a, f; \Lambda)$ as $\alpha \to \infty$, for spatial dimension $d = 1$, see Section 4. Here we assume again that a is a multi-scale symbol, and the main objective is to have the explicit control of the remainder, see Theorems 4.4 and 4.7. As mentioned earlier, we follow the seminal ideas of H. Widom, who proved such asymptotic results for smooth functions f already in the 1980's, see [27,29–31]. The proofs of the main asymptotic results of Section 4 are presented in Sections 5 and 6. To accomplish this we use rather a standard methodology of quasiclassical analysis: first we prove the required asymptotics for smooth functions f, and then using the bounds from Theorem 3.5 we extend them to non-smooth ones. The starting point is the Helffer–Sjöstrand formula (see Appendix A) which rewrites the trace of $D_\alpha(a, \Lambda; f)$ for smooth f in terms of $D_\alpha(a, \Lambda; r_z)$ with the resolvent function $r_z(\lambda) := (\lambda - z)^{-1}$, $\lambda \in \mathbb{R}$, $z \in \mathbb{C}$.

Part 2: Sections 8–10. Here we apply the results obtained in Part 1 to the symbol

$$a(\boldsymbol{\xi}) := a_{T,\mu}(\boldsymbol{\xi}) := \frac{1}{1 + \exp\left(\frac{h(\boldsymbol{\xi}) - \mu}{T}\right)}, \quad \boldsymbol{\xi} \in \mathbb{R}^d, \tag{1.5}$$

which is nothing but the Fermi symbol of free fermions. The real-valued function $h = h(\boldsymbol{\xi})$ is the classical one-particle Hamiltonian of the free Fermi gas, $h(\boldsymbol{\xi}) \to \infty$ as $|\boldsymbol{\xi}| \to \infty$, the parameter $T > 0$ is the (absolute) temperature, and $\mu \in \mathbb{R}$ is the chemical potential. We always assume that μ is fixed and $T \in (0, T_0]$ for some $T_0 > 0$. We are interested in the behaviour of $D_\alpha(T) = D_\alpha(a_{T,\mu}, \Lambda; f)$ when $\alpha \to \infty$ and $T \downarrow 0$ simultaneously. The symbol $a_{T,\mu}$ fits in the formalism of multi-scale symbols, laid out in Section 3, and as a result we derive from Theorem 3.5 a sharp estimate for the trace norm of $D_\alpha(T)$ with explicit dependence on T and α, under the condition $\alpha T \geq 1$, see Theorem 8.3. For $d = 1$ the sharpness of this estimate is confirmed by the asymptotic formulas (8.24), (8.27) for the trace of $D_\alpha(T)$, that are derived from Theorem 4.7. The extension of this large-scale asymptotics to dimensions $d \geq 2$ is the content of a separate paper [26].

In Section 10 we specialise further to the function $f = \eta_\gamma, \gamma > 0$, which brings us to the main application of our results, that is, to the large scale asymptotic formulas for the entanglement entropy (EE) $\mathrm{H}_\gamma(T, \mu; \alpha\Lambda)$ of free fermions in thermal equilibrium associated with the bipartition $\mathbb{R}^d = \Lambda \cup \Lambda^c$ with a bounded $\Lambda \subset \mathbb{R}^d$, at temperature $T > 0$.

As pointed out earlier, by now the EE is well-understood at zero temperature (see [7,13]), which corresponds to the case when the Fermi symbol a is given by the indicator function χ_Ω of the Fermi sea $\Omega \subset \mathbb{R}^d$. In this case the EE exhibits a logarithmically enhanced area-law scaling of the form (1.4). The case $T > 0$ is somewhat trickier: the entropy of the total system on $\mathbb{R}^d$, that is, $\operatorname{tr} \eta_\gamma(W_\alpha(a_{T,\mu}; \mathbb{R}^d)) = \operatorname{tr} \operatorname{Op}_\alpha(a_{T,\mu})$, is infinite, and hence it is not clear in advance even how to define the EE in a meaningful way. Intuitively, the EE measures the difference between the sum of the entropies of the states localised to Λ and Λ^c and the entropy of the total system. Therefore, a physically and mathematically reasonable definition of the EE is given in (10.4) below. By that we not only ensure the finiteness of the EE, but we are also able to obtain sharp (in α and T) upper bounds in any spatial dimension $d \ge 1$. In Theorem 10.1 we show that[1] $|\mathrm{H}_\gamma(T,\mu;\alpha\Lambda)| \le C\alpha^{d-1}(|\log(T)|+1)$, if $\alpha \ge 1$, $\alpha T \ge 1$. This bound tallies well with the asymptotics (1.4) and thus supports the intuitive expectation that the scaling behaviour of the EE at $T > 0$ should resemble more and more the zero temperature behaviour, as $T \downarrow 0$. For $d = 1$ this expectation is further justified by the asymptotic formulas (10.9) and (10.11), derived from (8.24), see Section 9 for the low T-behaviour of the asymptotic coefficient. As a by-product, this leads to the two-term asymptotic expansion of the local thermal entropy of the free Fermi gas, which extends the hitherto known leading Weyl asymptotics (see [16,2]).

The paper [14] presents results on the EE for the one-dimensional and the multi-dimensional case without the underlying mathematical details. In combination with [26] and [15] the present paper provides then a full proof of these announcements.

Acknowledgements. Various parts of this paper were written during several visits of HL and AVS to the FernUniversität Hagen in 2014–2016, and during the stay of all authors at the International Newton Institute, Cambridge, in 2015. The authors are grateful to these institutions for hospitality and financial support.

AVS was supported by EPSRC grant EP/J016829/1.

2. Estimates

2.1. The Schatten–von Neumann ideals of compact operators

This paper relies on the results obtained in [25] for general quasi-normed ideals of compact operators. Here we limit our attention to the case of Schatten–von Neumann operator ideals $\mathfrak{S}_q$, $q > 0$. Detailed information on these ideals can be found e.g. in [3,8,18,20]. We shall point out only some basic facts. For a compact operator A on a separable Hilbert space $\mathcal{H}$ denote by $s_n(A)$, $n = 1, 2, \dots$ its singular values, that is, the eigenvalues of the operator $|A| := \sqrt{A^*A}$. We denote the identity operator on $\mathcal{H}$ by $\mathbb{1}$. The Schatten–von Neumann ideal $\mathfrak{S}_q$, $q > 0$ consists of all compact operators A, for which

$$\|A\|_{\mathfrak{S}_q} := \left[\sum_{k=1}^{\infty} s_k(A)^q\right]^{\frac{1}{q}} < \infty.$$

If $q \ge 1$, then the above functional defines a norm; if $0 < q < 1$, then it is a so-called quasi-norm. There is nevertheless a convenient analogue of the triangle inequality, which is called the *q-triangle inequality*:

[1] Here and everywhere below by C or c, with or without indices, we denote positive, finite constants, whose exact values are unimportant.

$$\|A_1 + A_2\|_{\mathfrak{S}_q}^q \le \|A_1\|_{\mathfrak{S}_q}^q + \|A_2\|_{\mathfrak{S}_q}^q, \quad A_1, A_2 \in \mathfrak{S}_q, \ 0 < g \le 1, \tag{2.1}$$

and the Hölder inequality,

$$\|A_1 A_2\|_{\mathfrak{S}_q} \le \|A_1\|_{\mathfrak{S}_{q_1}} \cdot \|A_2\|_{\mathfrak{S}_{q_2}}, \quad q^{-1} = q_1^{-1} + q_2^{-1}, \quad 0 < q_1, q_2 \le \infty, \tag{2.2}$$

see [19] and also [3]. In what follows we focus on the case $q \in (0, 1]$.

2.2. Non-smooth functions

We study non-smooth functions, satisfying the following condition:

Condition 2.1. *For some integer $n \ge 1$ the function $f \in \mathsf{C}^n(\mathbb{R} \setminus \{t_0\}) \cap \mathsf{C}(\mathbb{R})$ satisfies the bound*

$$\| f \|_n := \max_{0 \le k \le n} \sup_{t \ne t_0} |f^{(k)}(t)||t - t_0|^{-\gamma + k} < \infty \tag{2.3}$$

with some $\gamma > 0$, and is supported on the interval $(t_0 - R, t_0 + R)$ with some $R > 0$.

The case $R = \infty$ means no restriction on the support of the function f.

Below we denote by χ_R the indicator function of the interval $(-R, R)$, $R > 0$. For a function f satisfying the above condition the following bound holds for $t \ne t_0$:

$$|f^{(k)}(t)| \le \| f \|_n |t - t_0|^{\gamma - k} \chi_R(t - t_0), \ k = 0, 1, \dots, n. \tag{2.4}$$

If $n \ge 1$, then the above condition implies that with $\varkappa := \min\{1, \gamma\}$ the function f is $\varkappa$-Hölder continuous — we denote this set by $\mathsf{C}^{0,\varkappa}(\mathbb{R})$. In particular, one can show that for any $t_1, t_2 \in \mathbb{R}$,

$$|f(t_1) - f(t_2)| \le 2R^{\gamma - \varkappa} \| f \|_1 |t_1 - t_2|^{\varkappa}, \ \varkappa = \min\{1, \gamma\}. \tag{2.5}$$

The following Proposition was proved in [25]. For simplicity we state it only for bounded self-adjoint operators.

Proposition 2.2. *Suppose that f satisfies Condition 2.1 with some $\gamma > 0$, $n \ge 2$ and some $t_0 \in \mathbb{R}$, $R \in (0, \infty)$. Let q be a number such that $(n - \sigma)^{-1} < q \le 1$ with some number $\sigma \in (0, 1]$, $\sigma < \gamma$. Let A be a bounded self-adjoint operator and let P be an orthogonal projection such that $PA(\mathbb{1} - P) \in \mathfrak{S}_{\sigma q}$. Then*

$$\|f(PAP)P - Pf(A)\|_{\mathfrak{S}_q} \le C \| f \|_n R^{\gamma - \sigma} \|PA(\mathbb{1} - P)\|_{\mathfrak{S}_{\sigma q}}^{\sigma}, \tag{2.6}$$

with a positive constant C independent of the operators A, P, the function f, and the parameters R, t_0.

Since the operator A is bounded, one does not have to assume that f is compactly supported. The function f can be always replaced by another function suitably localised to a bounded interval of size $2\|A\|$ around the origin. This observation allows us to obtain a bound of the correct degree of homogeneity. We state this fact as a corollary of Proposition 2.2.

Corollary 2.3. *Suppose that the conditions of Proposition 2.2 are satisfied with $R = \infty$. Assume in addition that $\|A\| \le 1$ and that $t_0 = 0$ in (2.3). Then for any $\lambda > 0$ we have*

$$\|f(\lambda PAP)P - Pf(\lambda A)\|_{\mathfrak{S}_q} \le C{\vert\kern-0.25ex\vert\kern-0.25ex\vert} f {\vert\kern-0.25ex\vert\kern-0.25ex\vert}_n \lambda^\gamma \|PA(\mathbb{1} - P)\|^\sigma_{\mathfrak{S}_{\sigma q}}, \tag{2.7}$$

with a positive constant C independent of the operators A, P, the function f and the parameter λ.

Proof. Let $f^{(\lambda)}(t) := \lambda^{-\gamma} f(\lambda t)$, so that ${\vert\kern-0.25ex\vert\kern-0.25ex\vert} f^{(\lambda)} {\vert\kern-0.25ex\vert\kern-0.25ex\vert}_n = {\vert\kern-0.25ex\vert\kern-0.25ex\vert} f {\vert\kern-0.25ex\vert\kern-0.25ex\vert}_n$. Since $\|A\| \le 1$, Proposition 2.2 with $R = 2$ leads to the bound

$$\|f^{(\lambda)}(PAP)P - Pf^{(\lambda)}(A)\|_{\mathfrak{S}_q} \le C{\vert\kern-0.25ex\vert\kern-0.25ex\vert} f {\vert\kern-0.25ex\vert\kern-0.25ex\vert}_n \|PA(\mathbb{1} - P)\|^\sigma_{\mathfrak{S}_{\sigma q}}.$$

Substituting the definition of $f^{(\lambda)}$ we get (2.7). □

As far as the λ-behaviour is concerned, the above estimate is sharp, since for $f(t) = |t|^\gamma$, $\gamma > 0$, both sides have the same homogeneity in λ. We include such estimates where an operator (or later, a symbol) is scaled by λ in this paper for completeness although the main application will appear only in [15].

We point out one special case of the non-homogeneous function η defined as

$$\eta(t) := -t \log|t|, t \in \mathbb{R}, \tag{2.8}$$

which nevertheless leads to a homogeneous estimate:

Corollary 2.4. *Let $q \in (0, 1]$, and let A be a bounded self-adjoint operator and let P be an orthogonal projection such that $\|A\| \le 1$ and $PA(\mathbb{1} - P) \in \mathfrak{S}_{\sigma q}$ for some $\sigma \in (0, 1)$. Then for any $\lambda > 0$,*

$$\|\eta(\lambda PAP) - P\eta(\lambda A)P\|_{\mathfrak{S}_q} \le C_\sigma \lambda \|PA(\mathbb{1} - P)\|^\sigma_{\mathfrak{S}_{\sigma q}}, \tag{2.9}$$

with a positive constant C_σ independent of the operators A, P and the parameter λ.

Proof. We write

$$\eta(\lambda PAP) - P\eta(\lambda A)P = \lambda\big(\eta(PAP)P - P\eta(A)\big)P.$$

The function η satisfies (2.3) with an arbitrary $\gamma \in (\sigma, 1)$, and arbitrarily large n, on any bounded interval centred at $t_0 = 0$. Now Proposition 2.2 leads to the claimed estimate. □

3. Estimates for multidimensional Wiener–Hopf operators

3.1. Definitions

Now we derive from Proposition 2.2 some estimates for Wiener–Hopf operators on $\mathsf{L}^2(\mathbb{R}^d)$. In this paper, under Wiener–Hopf operators we understand operators of the form (1.1) with a set $\Lambda \subset \mathbb{R}^d$ and symbol $a = a(\boldsymbol{\xi})$. Throughout the paper we assume that $a \in \mathsf{L}^\infty(\mathbb{R}^d)$ so that the operator $\operatorname{Op}_\alpha(a)$ is bounded with $\| \operatorname{Op}_\alpha(a)\| = \|a\|_{\mathsf{L}^\infty}$. Later we will assume that a satisfies some smoothness conditions. Our focus is on the operator difference (1.2) with suitable functions f. The right-hand side of (1.2) is well defined for a large class of functions f. We are mostly interested in functions f satisfying Condition 2.1. Our immediate objective is to obtain for the operator (1.2) estimates in the Schatten–von Neumann classes $\mathfrak{S}_q$, $q \in (0, 1]$. These will be derived from appropriate $\mathfrak{S}_q$-bounds for the operator

$$\chi_\Lambda \operatorname{Op}_\alpha(a)(\mathbb{1} - \chi_\Lambda). \tag{3.1}$$

Bounds of this type were proved in [22]. To state them properly we need to specify precise conditions on the set Λ and the symbol a.

We call a domain (an open, connected set) Lipschitz, if it can be described locally as a set above the graph of a Lipschitz function, see [22] for details. We call Λ a Lipschitz region if Λ is a union of finitely many Lipschitz domains such that their closures are pair-wise disjoint.

Condition 3.1. *For $d \geq 1$ the set $\Lambda \subset \mathbb{R}^d$ satisfies one of the following requirements:*

(1) *If $d = 1$, then Λ is a finite union of open intervals (bounded or unbounded) such that their closures are pair-wise disjoint.*
(2) *If $d \geq 2$, then Λ is a Lipschitz region, and either Λ or $\mathbb{R}^d \setminus \Lambda$ is bounded.*

We rely on the bounds for the operator (3.1) obtained in [22]. Apart from the explicit dependence on the parameter α, these bounds allow one to control the dependence on two scaling parameters: the momentum scaling $\tau > 0$ and the spatial scaling $\ell > 0$. The momentum scaling τ is introduced via the support condition for the symbol a:

$$\text{support of the symbol } a \text{ is contained in } B(\boldsymbol{\mu}, \tau) := \{\boldsymbol{\eta} \in \mathbb{R}^d : |\boldsymbol{\eta} - \boldsymbol{\mu}| < \tau\}, \tag{3.2}$$

with some $\boldsymbol{\mu} \in \mathbb{R}^d$, and via the family of norms

$$\mathsf{N}^{(m)}(a;\tau) := \max_{0\le r\le m} \sup_{\boldsymbol\xi\in\mathbb R^d} \tau^r |\nabla_{\boldsymbol\xi}^r a(\boldsymbol\xi)|, m = 0, 1, 2, \dots. \tag{3.3}$$

The spatial scaling parameter $\ell > 0$ is introduced by localising the operator (3.1) to the ball $B(\mathbf z, \ell)$ with an arbitrary $\mathbf z \in \mathbb R^d$. The constants in the estimates below are independent of the symbol a, the parameters α, τ, ℓ, and the points $\boldsymbol\mu$, $\mathbf z$.

Proposition 3.2. *See [22, Corollary 4.4] Let the region Λ satisfy Condition 3.1 and let the symbol $a = a(\boldsymbol\xi)$ satisfy the condition (3.2). Define for some $q \in (0, 1]$ the natural number m by*

$$m := \lceil (d + 1)q^{-1}\rceil + 1. \tag{3.4}$$

If the numbers τ and ℓ satisfy the condition $\alpha\tau\ell \ge \alpha_0 > 0$, then for any $q \in (0, 1]$,

$$\|\chi_\Lambda \chi_{B(\mathbf z,\ell)} \operatorname{Op}_\alpha(a)(\mathbb 1 - \chi_\Lambda)\|_{\mathfrak S_q} \le C_q(\alpha\tau\ell)^{\frac{d-1}{q}} \mathsf N^{(m)}(a;\tau).$$

In the next subsection we extend Proposition 3.2 to more general symbols a.

3.2. Multi-scale symbols, a

We consider $\mathsf C^\infty$-symbols $a = a(\boldsymbol\xi)$ for which there exist positive continuous functions $v = v(\boldsymbol\xi)$ and $\tau = \tau(\boldsymbol\xi)$ and constants C_k, $k = 0, 1, 2, \dots$ such that

$$|a(\boldsymbol\xi)| \le C_0 v(\boldsymbol\xi),\ |\nabla_{\boldsymbol\xi}^k a(\boldsymbol\xi)| \le C_k\tau(\boldsymbol\xi)^{-k} v(\boldsymbol\xi),\ k = 1, 2, \dots,\quad \boldsymbol\xi \in \mathbb R^d. \tag{3.5}$$

It is natural to call τ the *scale (function)* and v the *amplitude (function)*. We refer to symbols a satisfying (3.5) as *multi-scale* symbols. In fact, in what follows, only some finite smoothness of the symbol a is sufficient, but in most cases we impose the $\mathsf C^\infty$-smoothness in order to avoid cumbersome formulations. It is convenient to introduce the notation

$$V_{\sigma,\rho}(v, \tau) := \int \frac{v(\boldsymbol\xi)^\sigma}{\tau(\boldsymbol\xi)^\rho} d\boldsymbol\xi,\ \sigma > 0, \rho \in \mathbb R. \tag{3.6}$$

Apart from the continuity we often need some extra conditions on the scale and the amplitude. First we assume that τ is globally Lipschitz, that is,

$$|\tau(\boldsymbol\xi) - \tau(\boldsymbol\eta)| \le \nu|\boldsymbol\xi - \boldsymbol\eta|,\quad \boldsymbol\xi, \boldsymbol\eta \in \mathbb R^d, \tag{3.7}$$

with some $\nu > 0$. By adjusting the constants C_k in (3.5) we may assume that $\nu < 1$. It is straightforward to check that

$$(1 + \nu)^{-1} \le \frac{\tau(\boldsymbol\xi)}{\tau(\boldsymbol\eta)} \le (1 - \nu)^{-1},\quad \boldsymbol\eta \in B\big(\boldsymbol\xi, \tau(\boldsymbol\xi)\big). \tag{3.8}$$

Under this assumption on the scale τ, the amplitude v is assumed to satisfy the bounds

$$C_1 \le \frac{v(\boldsymbol{\eta})}{v(\boldsymbol{\xi})} \le C_2,\ \boldsymbol{\eta} \in B\big(\boldsymbol{\xi}, \tau(\boldsymbol{\xi})\big), \tag{3.9}$$

with some positive constants C_1, C_2 independent of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. The condition $\nu < 1$ guarantees that one can construct a covering of $\mathbb{R}^d$ by open balls centred at some points $\boldsymbol{\xi}_j$, $j = 1, 2, \dots$ of radii $\tau_j := \tau(\boldsymbol{\xi}_j)$, which satisfies the *finite intersection property,* that is, the number of intersecting balls is bounded from above by a constant depending only on the parameter ν, see [11, Chapter 1, Theorem 1.4.10]. We denote $B_j := B(\boldsymbol{\xi}_j, \tau_j)$. Moreover, there exists a partition of unity $\phi_j \in \mathsf{C}_0^\infty(\mathbb{R}^d)$ subordinate to the above covering such that

$$|\nabla_{\boldsymbol{\xi}}^k \phi_j(\boldsymbol{\xi})| \le C_k \tau_j^{-k},\ k = 0, 1, \dots,\ \boldsymbol{\xi} \in \mathbb{R}^d, \tag{3.10}$$

with some constants C_k independent of $j = 1, 2, \dots$.

It is useful to think of v and τ as (functional) parameters. They, in turn, can depend on other parameters, e.g. numerical parameters like α. In our leading example of the Fermi symbol (1.5), the function τ is naturally chosen to be dependent on the temperature $T > 0$, see (8.20).

Remark 3.3. Our aim is to derive various trace-norm estimates (resp. asymptotics) with explicit or implicit constants that are independent of the functions τ, v, a, but may depend on the constants in (3.5) and the domain Λ. If the functions τ, v are required to satisfy (3.7) and (3.9), then the constants in the trace-norm estimates (resp. asymptotics) may also depend on the constants ν and C_1, C_2 in (3.9). In all these cases we say that the estimates (resp. asymptotics) are uniform in τ, v and a.

In the example of the symbol (1.5), the above uniformity allows us to control explicitly the dependence of the obtained bounds on the temperature.

In what follows we always assume that

$$\tau_{\mathrm{inf}} := \inf_{\boldsymbol{\xi} \in \mathbb{R}^d} \tau(\boldsymbol{\xi}) > 0. \tag{3.11}$$

The constants in the obtained estimates will be independent of the parameters $\alpha, \tau_{\mathrm{inf}}, \ell$, satisfying the assumption

$$\alpha \tau_{\mathrm{inf}} \ge \alpha_0, \tag{3.12}$$

or

$$\alpha \ell \tau_{\mathrm{inf}} \ge \alpha_0, \tag{3.13}$$

with some $\alpha_0 > 0$, but may depend on α_0.

Lemma 3.4. *Suppose that the domain Λ satisfies Condition 3.1 and let the functions τ and v be as described above. Suppose that the symbol a satisfies (3.5) and that (3.13) holds. Then for any $q \in (0,1]$ we have*

$$\|\chi_\Lambda \chi_{B(\mathbf{z},\ell)} \operatorname{Op}_\alpha(a)(\mathbb{1} - \chi_\Lambda)\|_{\mathfrak{S}_q}^q \le C_q(\alpha\ell)^{d-1} V_{q,1}(v,\tau). \tag{3.14}$$

Suppose that (3.12) is satisfied. Then

$$\|\chi_\Lambda \operatorname{Op}_\alpha(a)(\mathbb{1} - \chi_\Lambda)\|_{\mathfrak{S}_q}^q \le C_q \alpha^{d-1} V_{q,1}(v,\tau). \tag{3.15}$$

The bound is uniform in τ, v and a in the sense specified in Remark 3.3.

Proof. Let m be as defined in (3.4). Denote $v_j := v(\boldsymbol{\xi}_j)$, $\tau_j := \tau(\boldsymbol{\xi}_j)$ and $B_j := B(\boldsymbol{\xi}_j, \tau_j)$, $j = 1, 2, \dots$. Due to (3.5) and (3.8), (3.9), the localised symbol $a_j = a\phi_j$ is supported in the ball $B(\boldsymbol{\xi}_j, \tau_j)$, and the bound holds:

$$|\nabla_{\boldsymbol{\xi}}^k a_j(\boldsymbol{\xi})| \le C_m \tau_j^{-k} v_j, \quad k = 0, 1, 2, \dots, m,$$

so that $\mathsf{N}^{(m)}(a_j; \tau_j) \le C v_j$, see (3.3). Since $\alpha\ell\tau_j \ge \alpha_0$, by Proposition 3.2, we have for any $q \in (0,1]$ that

$$\|\chi_\Lambda \chi_{B(\mathbf{z},\ell)} \operatorname{Op}_\alpha(a_j)(\mathbb{1} - \chi_\Lambda)\|_{\mathfrak{S}_q}^q \le C_q(\alpha\ell\tau_j)^{d-1} v_j^q, \ C_q = C_q(\alpha_0).$$

By the q-triangle inequality (2.1) we can write

$$\begin{aligned}\|\chi_\Lambda \chi_{B(\mathbf{z},\ell)} \operatorname{Op}_\alpha(a)(\mathbb{1} - \chi_\Lambda)\|_{\mathfrak{S}_q}^q &\le \sum_j \|\chi_\Lambda \chi_{B(\mathbf{z},\ell)} \operatorname{Op}_\alpha(a_j)(\mathbb{1} - \chi_\Lambda)\|_{\mathfrak{S}_q}^q \\ &\le C_q(\alpha\ell)^{d-1} \sum_j \tau_j^{d-1} v_j^q. \end{aligned} \tag{3.16}$$

In view of (3.8) and (3.9),

$$\tau_j^{d-1} v_j^q \le C \int\limits_{B_j} \tau(\boldsymbol{\xi})^{-1} v(\boldsymbol{\xi})^q d\boldsymbol{\xi},$$

and hence the sum on the right-hand side of (3.16) is bounded by

$$C \sum_j \int\limits_{B_j} \tau(\boldsymbol{\xi})^{-1} v(\boldsymbol{\xi})^q d\boldsymbol{\xi} \le \tilde{C} \int \tau(\boldsymbol{\xi})^{-1} v(\boldsymbol{\xi})^q d\boldsymbol{\xi}.$$

At the last step we used the finite intersection property of the covering $\{B_j\}$. This leads to (3.14).

The bound (3.15) immediately follows from (3.14) upon using a finite covering of Λ or $\mathbb{R}^d \setminus \Lambda$ by unit balls and an associated smooth partition of unity. $\square$

Lemma 3.4 leads to the following result.

Theorem 3.5. *Suppose that f satisfies Condition 2.1 with some $n \ge 2$ and $\gamma > 0$, and the domain Λ satisfies Condition 3.1. Let a be a real-valued symbol. Let the functions a and τ, v be as in Lemma 3.4, and let (3.12) be satisfied. Then for any $\sigma \in (0,1]$, $\sigma < \gamma$, and $q \in ((n-\sigma)^{-1}, 1]$ we have*

$$\|D_\alpha(a, \Lambda; f)\|_{\mathfrak{S}_q}^q \le C_q \alpha^{d-1} R^{q(\gamma-\sigma)} {|\!|\!|} f {|\!|\!|}_n^q \, V_{q\sigma,1}(v, \tau), \tag{3.17}$$

with a constant independent of t_0. Furthermore, if $t_0 = 0$ and $\|a\|_{\mathsf{L}^\infty} \le 1$, then for any $\lambda > 0$,

$$\|D_\alpha(\lambda a, \Lambda; f)\|_{\mathfrak{S}_q}^q \le C_q \alpha^{d-1} \lambda^{q\gamma} {|\!|\!|} f {|\!|\!|}_n^q \, V_{q\sigma,1}(v, \tau). \tag{3.18}$$

The above bounds are uniform in τ, v and a in the sense specified in Remark 3.3. Furthermore, the constants in (3.17) and (3.18) are independent of α, R, λ, but may depend on α_0 in (3.12).

Proof. Use Proposition 2.2 with $P = \chi_\Lambda$, $A = \mathrm{Op}_\alpha(a)$ to get

$$\begin{aligned}\|D_\alpha(a, \Lambda; f)\|_{\mathfrak{S}_q}^q &\le \big\| f(\chi_\Lambda \mathrm{Op}_\alpha(a)\chi_\Lambda)\chi_\Lambda - \chi_\Lambda \mathrm{Op}_\alpha(f \circ a)\big\|_{\mathfrak{S}_q}^q \\ &\le C_q {|\!|\!|} f {|\!|\!|}_n^q R^{q(\gamma-\sigma)} \big\|\chi_\Lambda \mathrm{Op}_\alpha(a)(\mathbb{1} - \chi_\Lambda)\big\|_{\mathfrak{S}_{q\sigma}}^{q\sigma}.\end{aligned}$$

To get (3.17) it remains to apply (3.15). The bound (3.18) follows from (3.15) and (2.7). □

We also state separately the estimate for the function (2.8):

Theorem 3.6. *Let the function η be as defined in (2.8). Suppose that the real-valued symbol a is as in Lemma 3.4 with $\|a\|_{\mathsf{L}^\infty} \le 1$ and that (3.12) is satisfied. Then for any $\lambda > 0$ and any $q \in (0,1]$, $\sigma \in (0,1)$ one has*

$$\|D_\alpha(\lambda a, \Lambda; \eta)\|_{\mathfrak{S}_q}^q \le C_{q,\sigma} \alpha^{d-1} \lambda^q \, V_{q\sigma,1}(v, \tau). \tag{3.19}$$

The bound is uniform in τ, v and a in the sense specified in Remark 3.3. Furthermore, the constant in (3.19) is independent of α, but may depend on α_0 in (3.12).

The proof is similar to that of (3.18), but instead of (2.7) one uses (2.9).

4. Asymptotic results for the one-dimensional case

4.1. Results for smooth functions

Now we focus on the asymptotic behaviour of the trace of $D_\alpha(a, \Lambda; f)$ as $\alpha \to \infty$ for dimension $d = 1$. In line with the general theme of the paper we put the emphasis on non-smooth functions f. Our starting point, however, is the asymptotic formula for smooth f. This type of asymptotics was studied in [29] and later in [17], and we use one result from [29] without proof. Conditions on the smoothness and decay of the symbol a imposed in [29] are quite mild, but we assume stronger restrictions that enable us to utilise the bounds derived in Section 3. More precisely, we impose the following condition.

Condition 4.1. *The symbol $a \in \mathsf{C}^m(\mathbb{R})$, $m \ge 1$, satisfies the bound (3.5) for all derivatives up to the order m, with some continuous positive functions τ and v satisfying (3.7) and (3.9) for all $\xi \in \mathbb{R}$, respectively.*

To state the result we first define asymptotic coefficients. For any function $g : \mathbb{C} \to \mathbb{C}$ and any $s_1, s_2 \in \mathbb{C}$ denote

$$U(s_1, s_2; g) := \int_0^1 \frac{g\big((1-t)s_1 + ts_2\big) - [(1-t)g(s_1) + tg(s_2)]}{t(1-t)} dt. \tag{4.1}$$

This integral is finite for functions $g \in \mathsf{C}^{0,\varkappa}(\mathbb{C})$, $\varkappa \in (0, 1]$. Note also that

$$U(s_1, s_1; g) = 0, \quad U(s_1, s_2; g) = U(s_2, s_1; g), \forall s_1, s_2 \in \mathbb{C}. \tag{4.2}$$

Note also that the integral equals zero if $g(t) = 1$ or $g(t) = t$. Now we define the asymptotic coefficient

$$\mathcal{B}(a; g) := \frac{1}{8\pi^2} \lim_{\varepsilon \downarrow 0} \iint_{|\xi_1 - \xi_2| > \varepsilon} \frac{U\big(a(\xi_1), a(\xi_2); g\big)}{|\xi_1 - \xi_2|^2} d\xi_1 d\xi_2. \tag{4.3}$$

Note that $\mathcal{B}$ is invariant under the change $a(\xi) \to a(\tau\xi)$ with an arbitrary $\tau > 0$. If g is such that $g'' \in \mathsf{L}^\infty(\mathbb{C})$, then the principal value integral can be replaced by the double integral, and the following bound holds:

$$|\mathcal{B}(a; g)| \le C\|g''\|_{\mathsf{L}^\infty} \iint \frac{|a(\xi_1) - a(\xi_2)|^2}{|\xi_1 - \xi_2|^2} d\xi_1 d\xi_2.$$

This estimate was first pointed out in [29, (17)]. As shown in [24], under Condition 4.1, one has

$$|\mathcal{B}(a; g)| \le C\|g''\|_{\mathsf{L}^\infty} V_{2,1}(v, \tau), \tag{4.4}$$

where the coefficient $V_{\sigma,m}$ for $\sigma > 0, m \in \mathbb{Z}$, is defined in (3.6).

Proposition 4.2. *See [29, Theorem 1(a)]. Suppose that Condition 4.1 is satisfied with $m \geq 2$ and that $V_{2,1}(v, \tau) < \infty$.*

(1) *Let g be analytic on a neighbourhood of the closed convex hull of the range of the function a. Then the operator $D_1(a; \mathbb{R}_\pm; g)$ is trace class and*

$$\operatorname{tr} D_1(a; \mathbb{R}_\pm; g) = \mathcal{B}(a; g). \tag{4.5}$$

(2) *If the symbol a is real-valued, then formula (4.5) holds under the condition $g \in \mathsf{C}_0^4(\mathbb{R})$.*

Formula (4.5) was obtained in [29] under weaker conditions on the symbol a. Moreover, for real-valued symbols a the smoothness conditions on g in [29] are less restrictive than in the above proposition. Note also that for real-valued a the paper [17] allows further relaxation on the functions a and g but we omit the details.

By rescaling $x \to \alpha x$ one immediately concludes that the left-hand side of (4.5) coincides with $\operatorname{tr} D_\alpha(a; \mathbb{R}_\pm; g)$. It is worth pointing out that, formally speaking, the estimates in Section 3 do not ensure that the trace on the left-hand side of (4.5) is finite, since neither $\mathbb{R}_\pm$ itself nor its complement is bounded. However, those estimates in combination with Proposition 6.1 below do guarantee that $D_1(a; \mathbb{R}_\pm; g)$ is trace-class.

We apply Proposition 4.2 to the case of a real-valued symbol a and the function $g : \mathbb{R} \mapsto \mathbb{C}$ defined as

$$g(\lambda) := r_z(\lambda) := \frac{1}{\lambda - z}, \ \operatorname{Im} z \neq 0.$$

Now our immediate objective is to derive from (4.5) a similar asymptotic formula for the operator $D_\alpha(a; \Lambda; g)$ with a set Λ satisfying Condition 3.1(1). For $d = 1$, instead of Λ we use the notation I. According to Condition 3.1(1),

$$I = I_0 \cup I_{K+1} \bigcup_{k=1}^{K} I_k \tag{4.6}$$

where $\{I_k\}$, $k = 1, 2, \dots, K$ is a finite collection of bounded open intervals such that their closures are disjoint, the set I_0 (resp. I_{K+1}) is either empty or $(-\infty, x_0)$ (resp. (x_0, ∞)) with some $x_0 \in \mathbb{R}$, and its closure is also disjoint from the other intervals. Below we use the following notation for the number of endpoints of I, namely

$$\omega := \begin{cases} 2K \text{ if } I_0 = I_{K+1} = \varnothing, \\ 2K + 1 \text{ if only one of } I_0, I_{K+1} \text{ is non-empty}, \\ 2K + 2 \text{ if both } I_0, I_{K+1} \text{ are non-empty}. \end{cases} \tag{4.7}$$

By writing $K = K(I)$ and $\omega = \omega(I)$ we emphasise the dependence on the set I. We observe that

$$\omega(I) = \omega(I^c), \text{ with } I^c = \mathbb{R} \setminus I. \tag{4.8}$$

For arbitrary symbols a, b we introduce the notation

$$M^{(m)}(a,b) := \|\partial_\xi^m a\|_{\mathsf{L}^1}\|b\|_{\mathsf{L}^\infty} + \|a\|_{\mathsf{L}^\infty}\|\partial_\xi^m b\|_{\mathsf{L}^1},\ m = 1, 2, \dots, \tag{4.9}$$

and denote

$$\delta(z,a) := \operatorname{dist}(z, [-\|a\|_{\mathsf{L}^\infty}, \|a\|_{\mathsf{L}^\infty}]) > 0. \tag{4.10}$$

Theorem 4.3. *Let I and ω be as described in (4.6) and (4.7). Assume that*

$$\inf_{k,j:k\neq j}\{|I_k|, \operatorname{dist}(I_k, I_j)\} \geq 1, \quad k, j = 0, 1, 2, \dots, K+1. \tag{4.11}$$

Suppose that $a \in \mathsf{C}^m(\mathbb{R})$, $m \geq 3$, is real-valued. Then for any $\alpha > 0$ we have

$$\begin{aligned} &\big|\operatorname{tr} D_\alpha(a, I; r_z) - \omega\mathcal{B}(a; r_z)\big| \\ &\qquad \leq C_m \alpha^{-m+1}\frac{1}{\delta(z,a)}\left(\frac{|z| + \|a\|_{\mathsf{L}^\infty}}{\delta(z,a)}\right)^3 M^{(m)}(a_z, a_z^{-1}), \end{aligned} \tag{4.12}$$

with a constant C_m independent of a, z, $\delta(z,a)$, and α, and the intervals I_k, $k = 0, 1, \dots, K+1$.

Clearly, by scaling we may replace the 1 on the right-hand side in condition (4.11) by any (strictly) positive real number.

In the case of one bounded interval, the convergence of the left-hand side of (4.12) to zero as $\alpha \to \infty$ was proved in [29, Theorem 2], see also [17, Theorem 9]. Note that for an infinitely smooth a the right-hand side of (4.12) decays as $\alpha^{-\infty}, \alpha \to \infty$. For one bounded interval, this effect was pointed out in [4, formula (1.5)]. These conclusions of [4,17,29] are not sufficient for us, as our aim is to have a more explicit control of the remainder as a function of the symbol a as in Theorem 4.3. In particular, when considering symbols $a = a_{T,\mu}$ defined in (1.5), Theorem 4.3 allows us to obtain estimates that depend explicitly on the temperature T, and possibly, on the chemical potential μ. The proof of Theorem 4.3 draws on the ideas of [29] and it is postponed until Section 6.

We extend the above bound to arbitrary functions of finite smoothness satisfying some decay conditions. Precisely, for $g \in \mathsf{C}^n(\mathbb{R})$, $n \in \mathbb{N}_0$ and a constant $r > 0$ we define

$$N_n(g) := N_n(g; r) := \sum_{k=0}^n \int |g^{(k)}(t)|\langle t\rangle_r^{k-2} dt,\ \langle t\rangle_r := \sqrt{t^2 + r^2}. \tag{4.13}$$

Theorem 4.4. *Let I and ω be as in the previous theorem. Suppose that $a \in \mathsf{C}^m(\mathbb{R})$, $m \geq 3$, is real-valued and satisfies the bound (3.5) with some continuous positive functions τ, v. Suppose further that $f \in \mathsf{C}_0^n(\mathbb{R})$ with $n \geq m+6$. Then for any $r \geq \|v\|_{\mathsf{L}^\infty}$ and any $\alpha > 0$ we have*

$$\big|\mathrm{tr}\, D_\alpha(a, I; f) - \omega \mathcal{B}(a; f)\big| \le C_{m,n} N_n(f; r) \alpha^{-m+1} V_{1,m}(v, \tau). \tag{4.14}$$

This bound is uniform in τ, v and a in the sense specified in Remark 3.3. The constant $C_{m,n}$ in (4.14) is independent of the parameters α, r and the function f.

4.2. Results for non-smooth functions

Now we assume that f satisfies Condition 2.1 with some $\gamma > 0$. In this case, if $\gamma > 0$ is small, it is not immediately clear why and under which conditions on the symbol a the coefficient $\mathcal{B}(a; f)$ is finite. This issue was investigated in [24]. We quote the appropriate bound, adjusted for the use in the forthcoming calculations. We use the integral $V_{\sigma,\rho}(v, \tau)$ defined in (3.6) and the notation $\varkappa := \min\{1, \gamma\}$.

Proposition 4.5. *See [24, Theorem 6.1]. Suppose that f satisfies Condition 2.1 with $n = 2$, $\gamma > 0$ and some $R > 0$. Let $a \in \mathsf{C}^\infty(\mathbb{R})$ satisfy Condition 4.1. Then for any $\sigma \in (0, \varkappa]$ we have*

$$|\mathcal{B}(a; f)| \le C_\sigma {\vert\!\vert\!\vert} f {\vert\!\vert\!\vert}_2 R^{\gamma-\sigma} V_{\sigma,1}(v, \tau), \tag{4.15}$$

with a constant C_σ independent of f, uniformly in the functions τ, v, and the symbol a in the sense specified in Remark 3.3.

We note another useful result from [24]. It describes the contribution of "close" points ξ_1 and ξ_2 in the coefficient (4.3). Suppose that $\tau_{\mathrm{inf}} := \inf_{\xi\in\mathbb{R}} \tau(\xi) > 0$, then we define

$$\mathcal{B}^{(1)}(a; f) := \frac{1}{8\pi^2} \lim_{\varepsilon\downarrow 0} \iint\limits_{\varepsilon<|\xi_1-\xi_2|<\frac{\tau_{\mathrm{inf}}}{2}} \frac{U\big(a(\xi_1), a(\xi_2); f\big)}{|\xi_1 - \xi_2|^2} d\xi_1 d\xi_2. \tag{4.16}$$

This quantity is estimated in the following proposition.

Proposition 4.6. *Suppose that f satisfies Condition 2.1 with $n = 2$ and $\gamma > 0$. Let $a \in \mathsf{C}^\infty(\mathbb{R})$ satisfy Condition 4.1. Suppose also that $\tau_{\mathrm{inf}} > 0$. Then for any $\delta \in [0, \varkappa)$, the following bound holds:*

$$|\mathcal{B}^{(1)}(a; f)| \le C_\delta {\vert\!\vert\!\vert} f {\vert\!\vert\!\vert}_2 \tau_{\mathrm{inf}}^\delta V_{\varkappa,1+\delta}(v, \tau), \tag{4.17}$$

uniformly in the functions τ, v, and the symbol a in the sense specified in Remark 3.3.

This bound follows from [24, Corollary 6.5].

The bound (4.15) plays a central role in the proof of the following theorems. From now on we assume that $\tau_{\mathrm{inf}} > 0$ and that τ_{inf} and α satisfy (3.12). The convergence in the next theorems is uniform in the functions τ, v, and the symbol a in the sense specified in Remark 3.3, but no uniformity is claimed in the parameter α_0 in (3.12).

Theorem 4.7. *Let I and ω be as described in (4.6) and (4.7). Suppose that f satisfies Condition 2.1 with some $\gamma > 0$, $n = 2$ and some $t_0 \in \mathbb{R}$. Let $a \in \mathsf{C}^\infty(\mathbb{R})$ be a real-valued symbol satisfying Condition 4.1, and let $\alpha\tau_{\inf} \ge \alpha_0$. Suppose that $\|v\|_{\mathsf{L}^\infty} \le 1$ and $V_{\sigma,1}(v,\tau) < \infty$ for some $\sigma \in (0,1]$, $\sigma < \gamma$, and*

$$\lim_{\alpha\to\infty} \alpha^{-m+1} \frac{V_{1,m}(v,\tau)}{V_{\sigma,1}(v,\tau)} = 0, \tag{4.18}$$

uniformly in v, τ (see Remark 3.3), for some $m \ge 3$. Then

$$\lim_{\alpha\to\infty} \frac{1}{V_{\sigma,1}(v,\tau)} \big(\operatorname{tr} D_\alpha(a, I; f) - \omega\mathcal{B}(a; f)\big) = 0, \tag{4.19}$$

and the convergence is uniform in v, τ and a.

In order to avoid possible confusion we recall that v, τ are thought of as functional parameters of the problem, and they may depend on the numerical parameter α. Thus the equality (4.18) is a genuine, non-vacuous assumption.

For the next theorem recall that the function η is defined in (2.8).

Theorem 4.8. *Let I and ω be as in the previous theorem. Suppose that f satisfies Condition 2.1 with some $\gamma > 0$, $t_0 = 0$, and all n. Let $a \in \mathsf{C}^\infty(\mathbb{R})$ be a real-valued symbol satisfying Condition 4.1, and let $\alpha\tau_{\inf} \ge \alpha_0$. Suppose that $\|v\|_{\mathsf{L}^\infty} \le 1$ and $V_{\sigma,1}(v,\tau) < \infty$ for some $\sigma \in (0,1]$, $\sigma < \gamma$, and that (4.18) is satisfied. Then for any real $\lambda > 0$, one has*

$$\lim_{\alpha\to\infty} \frac{1}{\lambda^\gamma V_{\sigma,1}(v,\tau)} \big(\operatorname{tr} D_\alpha(\lambda a, I; f) - \omega\mathcal{B}(\lambda a; f)\big) = 0. \tag{4.20}$$

In addition, if $V_{\sigma,1}(v,\tau) < \infty$ with some $\sigma < 1$, then

$$\lim_{\alpha\to\infty} \frac{1}{\lambda V_{\sigma,1}(v,\tau)} \big(\operatorname{tr} D_\alpha(\lambda a, I; \eta) - \omega\mathcal{B}(\lambda a; \eta)\big) = 0. \tag{4.21}$$

The convergence in (4.20) and (4.21) is uniform in the functions τ, v, and the symbol a in the sense specified in Remark 3.3, and in (4.21) it is also uniform in the parameter $\lambda \in (0, \lambda_0]$ for any $\lambda_0 < \infty$.

We point out that the smoothness conditions on the function f in Theorem 4.8 are much more restrictive than those in Theorem 4.7. This difference will be briefly explained after the proof of Theorem 4.8.

The main difficulty lies in the proof of Theorem 4.3, whereas the remaining theorems are derived from it via relatively standard methods. In the next section we concentrate on this derivation. The proof of Theorem 4.3 is deferred until Section 6.

5. Proofs of Theorems 4.4, 4.7 & 4.8

We use the almost analytic extension constructed in Lemma A.1 with some $r \ge \|v\|_{\mathsf{L}^\infty}$, where v is the amplitude of the symbol a as in (3.5). Let $\tilde{f}$ be the almost analytic extension of f constructed in Lemma A.1. It follows from (A.1) that

$$\begin{aligned}\operatorname{tr} D_\alpha(a, I; f) &- \omega \mathcal{B}(a; f) \\ &= \frac{1}{\pi} \iint \frac{\partial}{\partial \overline{z}} \tilde{f}(x, y; r) \big(\operatorname{tr} D_\alpha(a, I; r_z) - \omega \mathcal{B}(a; r_z)\big) dx dy.\end{aligned}$$

Thus by Theorem 4.3 and by (A.3) we have

$$\begin{aligned}&|\operatorname{tr} D_\alpha(a, I; f) - \omega \mathcal{B}(a; f)| \\ &\quad \le C\alpha^{-m+1} \iint \Big|\frac{\partial}{\partial \overline{z}} \tilde{f}(x, y; r)\Big| \big(|x| + |y| + \|a\|_{\mathsf{L}^\infty}\big)^3 |y|^{-4} M^{(m)}(a_z, a_z^{-1}) dx dy \\ &\quad \le C\alpha^{-m+1} \int \int_{|y| < \langle x \rangle_r} F(x; r) \langle x \rangle_r^3 |y|^{n-5} M^{(m)}(a_z, a_z^{-1}) dy dx, \end{aligned} \tag{5.1}$$

for any $r \ge \|v\|_{\mathsf{L}^\infty}$. Let us now estimate $M^{(m)}(a_z, a_z^{-1})$.

Lemma 5.1. *Suppose that $a \in \mathsf{C}^m(\mathbb{R})$ satisfies (3.5) with some $m \ge 1$. Then*

$$M^{(m)}(a_z, a_z^{-1}) \le C_m \frac{\|v\|_{\mathsf{L}^\infty} + |z|}{|\operatorname{Im} z|^2} \left(1 + \frac{\|v\|_{\mathsf{L}^\infty}^{m-1}}{|\operatorname{Im} z|^{m-1}}\right) V_{1,m}(v, \tau). \tag{5.2}$$

Moreover, for any $r \ge \|v\|_{\mathsf{L}^\infty}$, and all y with $|y| < \langle x \rangle_r$, we have

$$M^{(m)}(a_z, a_z^{-1}) \le C_m \frac{\langle x \rangle_r}{|y|^2} \left(1 + \frac{\langle x \rangle_r^{m-1}}{|y|^{m-1}}\right) V_{1,m}(v, \tau), \tag{5.3}$$

with a constant C_m independent of r.

Proof. By definition (4.9),

$$M^{(m)}(a_z, a_z^{-1}) = \|\partial_\xi^m a\|_{\mathsf{L}^1} \|a_z^{-1}\|_{\mathsf{L}^\infty} + \|\partial_\xi^m a_z^{-1}\|_{\mathsf{L}^1} \|a_z\|_{\mathsf{L}^\infty}. \tag{5.4}$$

In view of the bound (3.5) the first summand in the above formula is bounded by

$$C_m \frac{1}{|\operatorname{Im} z|} \int \tau(\xi)^{-m} v(\xi) d\xi.$$

To estimate the second term on the right-hand side of (5.4) we use the Leibniz formula and (3.5) to obtain

$$|\partial_\xi^m a_z^{-1}| \le C_m\left(\frac{v}{|\operatorname{Im} z|^2} + \frac{v^m}{|\operatorname{Im} z|^{m+1}}\right)\tau^{-m}.$$

Thus the second summand in (5.4) does not exceed

$$C_m(\|v\|_{\mathsf{L}^\infty} + |z|)\left(\frac{1}{|\operatorname{Im} z|^2} + \frac{\|v\|_{\mathsf{L}^\infty}^{m-1}}{|\operatorname{Im} z|^{m+1}}\right)\int \tau(\xi)^{-m} v(\xi)d\xi.$$

This leads to the claimed bound (5.2).

For $r \ge \|v\|_{\mathsf{L}^\infty}$ and $|y| < \langle x\rangle_r$, the bound (5.3) immediately follows from (5.2). $\square$

Proof of Theorem 4.4. By (5.3), the integral on the right-hand side of (5.1) is estimated by

$$V_{1,m}(v,\tau)\int \int_{|y|<\langle x\rangle_r} F(x;r)\langle x\rangle_r^4 |y|^{n-7}\left(1 + \frac{\langle x\rangle_r^{m-1}}{|y|^{m-1}}\right)dydx.$$

Since $n \ge m+6$, this integral is finite and it is bounded by

$$CV_{1,m}(v,\tau)\int F(x;r)\langle x\rangle_r^{n-2}dx = CV_{1,m}(v,\tau)N_n(f;r),$$

where we have used the definition (4.13). Therefore (5.1) yields the bound

$$|\operatorname{tr} D_\alpha(a,I;f) - \omega\mathcal{B}(a;f)| \le CN_n(f;r)\alpha^{-m+1}V_{1,m}(v,\tau),$$

as claimed. $\square$

Proof of Theorem 4.7. For brevity we denote $D_\alpha(f) := D_\alpha(a,I;f)$ and $\mathcal{B}(f) := \mathcal{B}(a;f)$.

Step 1. *Proof of formula (4.19) for $f \in \mathsf{C}^2(\mathbb{R})$.* Without loss of generality we may assume that $\|a\|_{\mathsf{L}^\infty} \le 1/2$ and that the function f is supported on the interval $[-1,1]$. By the Weierstrass Theorem, for any $\varepsilon > 0$ one can find a real polynomial f_ε such that the function $g_\varepsilon := f - f_\varepsilon$ satisfies the bound

$$\max_{0\le k\le 2}\max_{|t|\le 1}|g_\varepsilon^{(k)}(t)| < \varepsilon. \tag{5.5}$$

Clearly,

$$D_\alpha(f) = D_\alpha(f_\varepsilon) + D_\alpha(g_\varepsilon).$$

In order to estimate $D_\alpha(g_\varepsilon)$ we extend the function g_ε to the interval $[-2,2]$ as a C_0^2-function in such a way that $\|g_\varepsilon\|_{\mathsf{C}^2} \le C\varepsilon$ with some universal constant $C > 0$. Observe now that such g_ε satisfies Condition 2.1 with $t_0 = -3$, $R = 5$, $n = 2$ and arbitrary $\gamma > 0$. Furthermore, $\lvert\!\lvert\!\lvert g_\varepsilon \rvert\!\rvert\!\rvert_2 < C\varepsilon$. To be definite we take $\gamma = 2$. Since the condition (3.12) is satisfied, we may use Theorem 3.5 with $q = 1$ and arbitrary $\sigma \in (0,1)$, so that

$$\|D_\alpha(g_\varepsilon)\|_{\mathfrak{S}_1} \le C\varepsilon V_{\sigma,1}(v,\tau). \tag{5.6}$$

Moreover, by (4.4),

$$|\mathcal{B}(g_\varepsilon)| \le C\varepsilon V_{2,1}(v,\tau) \le C\varepsilon V_{\sigma,1}(v,\tau). \tag{5.7}$$

In order to handle the trace of $D_\alpha(f_\varepsilon)$, extend the polynomial f_ε as a C_0^∞-function on the interval $[-2,2]$. Thus by Theorem 4.4 with $r = 1 \ge \|v\|_{\mathsf{L}^\infty}$,

$$|\operatorname{tr} D_\alpha(f_\varepsilon) - \omega\mathcal{B}(f_\varepsilon)| \le CN_n(f_\varepsilon;1)\alpha^{-m+1}V_{1,m}(v,\tau),$$

with arbitrary $m \ge 3$. In view of the condition (4.18) and by virtue of (5.6) and (5.7), we have

$$\begin{aligned}
&\limsup \frac{1}{V_{\sigma,1}(v,\tau)}\big|\operatorname{tr} D_\alpha(f) - \omega\mathcal{B}(f)\big| \\
&\quad\le \limsup \frac{1}{V_{\sigma,1}(v,\tau)}\big|\operatorname{tr} D_\alpha(f_\varepsilon) - \omega\mathcal{B}(f_\varepsilon)\big| \\
&\quad + \limsup \frac{1}{V_{\sigma,1}(v,\tau)}\|D_\alpha(g_\varepsilon)\|_{\mathfrak{S}_1} + \omega \limsup \frac{1}{V_{\sigma,1}(v,\tau)}|\mathcal{B}(g_\varepsilon)| \\
&\quad\le N_n(f_\varepsilon;1)\ \limsup\ \alpha^{-m+1}\frac{V_{1,m}(v,\tau)}{V_{\sigma,1}(v,\tau)} + C\varepsilon = C\varepsilon.
\end{aligned}$$

Here the lim sup is taken as $\alpha \to \infty$, $\alpha\tau_{\mathrm{inf}} \ge \alpha_0$, and it is uniform in v, τ and a. Since $\varepsilon > 0$ is arbitrary, this leads to (4.19) for arbitrary C^2-functions f.

Step 2. *Completion of the proof.* Let f be a function as specified in the theorem. Let $\zeta \in \mathsf{C}_0^\infty(\mathbb{R})$ be a real-valued function satisfying (A.2). We represent

$$\begin{aligned}
f &= f_R^{(1)} + f_R^{(2)}, 0 < R \le 1, \\
f_R^{(1)}(t) &:= f(t)\zeta\big((t-t_0)R^{-1}\big), \\
f_R^{(2)}(t) &:= f(t) - f_R^{(1)}(t).
\end{aligned}$$

For $f_R^{(1)}$ we use Theorem 3.5 with $q = 1$, $n = 2$, and a $\sigma \in (0,1], \sigma < \gamma$, such that $V_{\sigma,1} < \infty$:

$$\|D_\alpha(f_R^{(1)})\|_{\mathfrak{S}_1} \le CR^{\gamma-\sigma} {\left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert f_R^{(1)} \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert}_2 V_{\sigma,1}(v,\tau).$$

By (4.15), the coefficient $\mathcal{B}(f_R^{(1)})$ satisfies the same bound. Note also that ${\left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert f_R^{(1)} \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert}_2 \le C {\left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert f \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert}_2$, so that

$$\frac{1}{V_{\sigma,1}(v,\tau)}\big|D_\alpha(f_R^{(1)}) - \omega\mathcal{B}(f_R^{(1)})\big| \le C {\left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert f \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert}_2 R^{\gamma-\sigma}. \tag{5.8}$$

Now, it is clear that $f_R^{(2)} \in \mathsf{C}^2(\mathbb{R})$, so one can use formula (4.19) established in Part 1 of the proof:

$$\lim_{\alpha\to\infty,\ \alpha\tau_{\mathrm{inf}}\ge\alpha_0} \frac{1}{V_{\sigma,1}(v,\tau)} |\operatorname{tr} D_\alpha(f_R^{(2)}) - \omega\mathcal{B}(f_R^{(2)})| = 0.$$

Together with (5.8), this equality gives

$$\begin{aligned} &\limsup \frac{1}{V_{\sigma,1}(v,\tau)} |\operatorname{tr} D_\alpha(f) - \omega\mathcal{B}(f)| \\ &\quad\le \lim \frac{1}{V_{\sigma,1}(v,\tau)} |\operatorname{tr} D_\alpha(f_R^{(2)}) - \omega\mathcal{B}(f_R^{(2)})| + C{\left\vert\kern-0.25ex\left\vert f \right\vert\kern-0.25ex\right\vert}_2 R^{\gamma-\sigma} \\ &\quad\le C{\left\vert\kern-0.25ex\left\vert f \right\vert\kern-0.25ex\right\vert}_2 R^{\gamma-\sigma}. \end{aligned}$$

Again, the limits above are taken as $\alpha \to \infty$, $\alpha\tau_{\mathrm{inf}} \ge \alpha_0$. Since $\sigma < \gamma$, and $R > 0$ is arbitrary, the required asymptotics follow. □

Proof of Theorem 4.8. Instead of f we introduce for $\lambda > 0$ the function

$$f^{(\lambda)}(t) := \lambda^{-\gamma} f(\lambda t), t \in \mathbb{R}.$$

It is clear that ${\left\vert\kern-0.25ex\left\vert f \right\vert\kern-0.25ex\right\vert}_n = {\left\vert\kern-0.25ex\left\vert f^{(\lambda)} \right\vert\kern-0.25ex\right\vert}_n$ for all n. As in the previous proof, without loss of generality we may assume that $\|a\|_{\mathsf{L}^\infty} \le 1/2$, so that the function $f^{(\lambda)}$ may be assumed to be supported on the interval $[-1, 1]$. Note that

$$D_\alpha(\lambda a, I; f) = \lambda^\gamma D_\alpha(a, I; f^{(\lambda)}), \quad \mathcal{B}(\lambda a; f) = \lambda^\gamma \mathcal{B}(a; f^{(\lambda)}).$$

Thus the asymptotic formula (4.20) is equivalent to the following relation:

$$\lim_{\alpha\to\infty,\alpha\tau_{\mathrm{inf}}\ge\alpha_0} \frac{1}{V_{\sigma,1}(v,\tau)} \big(\operatorname{tr} D_\alpha(a, I; f^{(\lambda)}) - \omega\mathcal{B}(a; f^{(\lambda)})\big) = 0. \tag{5.9}$$

The further proof now follows essentially Step 2 of the proof of Theorem 4.7. As before, for brevity we use the notation $D_\alpha(f) := D_\alpha(a, I; f)$, $\mathcal{B}(f) := \mathcal{B}(a; f)$.

Let $\zeta \in \mathsf{C}_0^\infty(\mathbb{R})$ be a real-valued function, satisfying (A.2). Represent

$$\begin{aligned} f^{(\lambda)} &= g_R^{(1)} + g_R^{(2)}, 0 < R \le 1, \\ g_R^{(1)}(t) &:= f^{(\lambda)}(t)\zeta\big(tR^{-1}\big), \\ g_R^{(2)}(t) &:= f^{(\lambda)}(t) - g_R^{(1)}(t). \end{aligned}$$

Since the condition (3.12) is satisfied, for $g_R^{(1)}$ we may use Theorem 3.5 with $q = 1$, $n = 2$, and a $\sigma \in (0, 1]$, $\sigma < \gamma$, such that $V_{\sigma,1} < \infty$:

$$\|D_\alpha(g_R^{(1)})\|_{\mathfrak{S}_1} \le CR^{\gamma-\sigma} \bm{|} g_R^{(1)} \bm{|}_2 V_{\sigma,1}(v,\tau) \le CR^{\gamma-\sigma} \bm{|} f \bm{|}_2 V_{\sigma,1}(v,\tau).$$

By (4.15) the coefficient $\mathcal{B}(g_R^{(1)})$ satisfies the same bound. It is clear that $g_R^{(2)} \in \mathsf{C}^\infty(\mathbb{R})$, and by definition (4.13),

$$N_n(g_R^{(2)}; r) \le C_{n,R,r} \bm{|} f^{(\lambda)} \bm{|}_n \le \tilde{C}_{n,R,r} \bm{|} f \bm{|}_n, n = 1, 2, \dots,$$

for any $r > 0$. Thus by (4.14),

$$|\operatorname{tr} D_\alpha(g_R^{(2)}) - \omega\mathcal{B}(g_R^{(2)})| \le CN_n(g_R^{(2)}; 1 + \lambda_0)\alpha^{-m+1} V_{1,m}(v,\tau),$$

with arbitrary $m \ge 3$. Therefore, using (4.18) and arguing as in the proof of Theorem 4.7, we obtain

$$\begin{aligned} &\limsup \frac{1}{V_{\sigma,1}(v,\tau)} \, |\operatorname{tr} D_\alpha(f^{(\lambda)}) - \omega\mathcal{B}(f^{(\lambda)})| \\ &\le \lim \frac{1}{V_\sigma(v,\tau)} |\operatorname{tr} D_\alpha(g_R^{(2)}) - \omega\mathcal{B}(g_R^{(2)})| + C \bm{|} f \bm{|}_2 R^{\gamma-\sigma} \\ &\le C \bm{|} f \bm{|}_2 R^{\gamma-\sigma}. \end{aligned}$$

The limits above are taken as $\alpha \to \infty$, $\alpha\tau_{\inf} \ge \alpha_0$. Since $R > 0$ is arbitrary, the required asymptotics (5.9) follow. As explained previously, this implies (4.20).

Proof of (4.21): We write

$$D_\alpha(\lambda a, I; \eta) = \lambda D_\alpha(a, I; \eta), \ \mathcal{B}(\lambda a; \eta) = \lambda \mathcal{B}(a; \eta).$$

Since the function η satisfies Condition 2.1 for any $\gamma \in (\sigma, 1)$, the proclaimed asymptotic formula is a direct consequence of the formula (4.19) for the operator $D_\alpha(a, I; \eta)$. □

Observe that the proof of Theorem 4.8 has only one step, in contrast to that of Theorem 4.7. Namely, in the former we do not prove that the sought asymptotics holds for arbitrary $f \in \mathsf{C}^2(\mathbb{R})$ since this would require approximating $f^{(\lambda)}$ with polynomials whose dependence on λ would have to be explicitly controlled. We do not go into these difficulties.

6. Proof of Theorem 4.3: the case of a single interval

We recall the notation (1.1) for the Wiener–Hopf operator: $W_\alpha(a; I) = \chi_I \operatorname{Op}_\alpha(a) \chi_I$ with the notation Λ replaced by a subset $I \subset \mathbb{R}$. A central role in our argument plays the operator

$$H_\alpha(a, b; I) := W_\alpha(ab; I) - W_\alpha(a; I) W_\alpha(b; I) = \chi_I \operatorname{Op}_\alpha(a) \big(\mathbb{1} - \chi_I\big) \operatorname{Op}_\alpha(b) \chi_I, \quad (6.1)$$

with C^m-symbols $a = a(\xi)$ and $b = b(\xi)$. At the first step of the proof we assume that the set I is just a bounded interval (x_0, y_0) with $y_0 - x_0 \ge 1$.

6.1. Preliminary bounds

For any $z \in \mathbb{R}$ denote

$$\mathbb{R}_z^{(+)} := (z, \infty), \mathbb{R}_z^{(-)} := (-\infty, z), \quad \chi_z^{(+)} := \chi_{(z,\infty)}, \chi_z^{(-)} := \chi_{(-\infty,z)}.$$

We define

$$Z_\alpha(a, b; I) := H_\alpha(a, b; I) - H_\alpha(a, b; \mathbb{R}_{y_0}^{(-)}) - H_\alpha(a, b; \mathbb{R}_{x_0}^{(+)}).$$

Most of the estimates for the introduced operators will follow from the next proposition, which is a consequence of [22, Theorem 2.7].

Proposition 6.1. *Let a be a symbol such that $\partial_\xi^m a \in \mathsf{L}^1(\mathbb{R})$ with some $m \ge 3$. Let z, t be numbers such that $z - t = \ell > 0$. Then for any $\alpha > 0$, we have*

$$\|\chi_t^{(-)} \operatorname{Op}_\alpha(a) \chi_z^{(+)}\|_{\mathfrak{S}_1} + \|\chi_z^{(+)} \operatorname{Op}_\alpha(a) \chi_t^{(-)}\|_{\mathfrak{S}_1} \le C_m (\alpha\ell)^{-m+1} \|\partial_\xi^m a\|_{\mathsf{L}^1}.$$

Proof. The operator $\chi_t^{(-)} \operatorname{Op}_\alpha(a) \chi_z^{(+)}$ is trivially unitarily equivalent to $\chi_0^{(-)} \operatorname{Op}_1(b) \chi_1^{(+)}$ with the symbol b defined as $b(\xi) := a\big((\alpha\ell)^{-1}\xi\big)$. By [22, Theorem 2.7],

$$\|\chi_0^{(-)} \operatorname{Op}_1(b) \chi_1^{(+)}\|_{\mathfrak{S}_1} \le C_m \|\partial^m b\|_{\mathsf{L}^1} \le C_m (\alpha\ell)^{1-m} \|\partial^m a\|_{\mathsf{L}^1},$$

for any $m \ge 3$. This is the required estimate. □

Remark. Theorem 2.7 in [22] contains two misprints: the number n should be defined by the formula $n := \lceil 2q^{-1} \rceil + 1$, and the main estimate of the Theorem should have the factor $r^{2q^{-1}-m}$ instead of $r^{q^{-1}-m}$.

Now we proceed to estimating trace norms of the operators H_α, Z_α introduced above. Recall that $M^{(m)}(a, b)$ is defined in (4.9).

Lemma 6.2. *Let $I = (x_0, y_0)$ with $y_0 - x_0 \ge 1$. Then for $m \ge 3$ and any $\alpha > 0$ we have*

$$\|Z_\alpha(a, b; I)\|_{\mathfrak{S}_1} \le C_m \alpha^{-m+1} M^{(m)}(a, b), \tag{6.2}$$

and

$$\|H_\alpha(a, b; \mathbb{R}_{y_0}^{(-)})\, H_\alpha(a, b; \mathbb{R}_{x_0}^{(+)})\|_{\mathfrak{S}_1} + \|H_\alpha(a, b; \mathbb{R}_{x_0}^{(+)}) H_\alpha(a, b; \mathbb{R}_{y_0}^{(-)})\|_{\mathfrak{S}_1} \\ \le C_m \alpha^{-m+1} \|a\|_{\mathsf{L}^\infty} \|b\|_{\mathsf{L}^\infty} M^{(m)}(a, b). \tag{6.3}$$

Proof. We denote $A := \operatorname{Op}_\alpha(a), B := \operatorname{Op}_\alpha(b)$. Clearly, the operator $Z := Z_\alpha(a, b; I)$ splits into the sum

$$Z = Z^{(1)} + Z^{(2)},$$
$$Z^{(1)} := \chi_I A \chi_{y_0}^{(+)} B \chi_I - \chi_{y_0}^{(-)} A \chi_{y_0}^{(+)} B \chi_{y_0}^{(-)},$$
$$Z^{(2)} := \chi_I A \chi_{x_0}^{(-)} B \chi_I - \chi_{x_0}^{(+)} A \chi_{x_0}^{(-)} B \chi_{x_0}^{(+)}.$$

Let us rewrite

$$Z^{(1)} = -\chi_{x_0}^{(-)} A \chi_{y_0}^{(+)} B \chi_I - \chi_{y_0}^{(-)} A \chi_{y_0}^{(+)} B \chi_{x_0}^{(-)},$$
$$Z^{(2)} = -\chi_{y_0}^{(+)} A \chi_{x_0}^{(-)} B \chi_I - \chi_{x_0}^{(+)} A \chi_{x_0}^{(-)} B \chi_{y_0}^{(+)}.$$

Then, by Proposition 6.1,

$$\begin{aligned}\|Z^{(1)}\|_{\mathfrak{S}_1} &\le \|\chi_{x_0}^{(-)} A \chi_{y_0}^{(+)}\|_{\mathfrak{S}_1} \|B\| + \|A\| \; \|\chi_{y_0}^{(+)} B \chi_{x_0}^{(-)}\|_{\mathfrak{S}_1} \\ &\le C_m \alpha^{-m+1} \big(\|\partial_\xi^m a\|_{\mathsf{L}^1} \|b\|_{\mathsf{L}^\infty} + \|a\|_{\mathsf{L}^\infty} \|\partial_\xi^m b\|_{\mathsf{L}^1} \big) = C_m \alpha^{-m+1} M^{(m)}(a,b).\end{aligned}$$

Adding it up with the same bound for the operator $Z^{(2)}$ completes the proof of (6.2) for $Z(a,b;I)$.

Proof of (6.3): Let $z_0 := (x_0 + y_0)/2$, so that the trace norm of the operator

$$H_\alpha(a,b;\mathbb{R}_{y_0}^{(-)}) H_\alpha(a,b;\mathbb{R}_{x_0}^{(+)}) = \chi_{y_0}^{(-)} A \chi_{y_0}^{(+)} B \chi_I A \chi_{x_0}^{(-)} B \chi_{x_0}^{(+)}$$

can be estimated by

$$\|A\|^2 \; \|B\| \; \|\chi_{y_0}^{(+)} B \chi_{(x_0,z_0)}\|_{\mathfrak{S}_1} + \|A\| \; \|B\|^2 \; \|\chi_{(z_0,y_0)} A \chi_{x_0}^{(-)}\|_{\mathfrak{S}_1}.$$

Now Proposition 6.1 leads to the bound (6.3) for the first term on the left-hand side of (6.3). In the same way one proves the same bound for the second term on the left-hand side. □

Lemma 6.3. *Let the conditions of Lemma 6.2 be satisfied and let $g \in \mathsf{C}^m(\mathbb{R})$ be another symbol. Then*

$$\begin{aligned}&\big\| \big[W_\alpha(a;I) - W_\alpha(a;\mathbb{R}_{x_0}^{(+)}) \big] H_\alpha(b,g;\mathbb{R}_{x_0}^{(+)}) \big\|_{\mathfrak{S}_1} \\ &\qquad + \big\| \big[W_\alpha(a;I) - W_\alpha(a;\mathbb{R}_{y_0}^{(-)}) \big] H_\alpha(b,g;\mathbb{R}_{y_0}^{(-)}) \big\|_{\mathfrak{S}_1} \\ &\qquad\qquad \le C_m \alpha^{-m+1} \|g\|_{\mathsf{L}^\infty} M^{(m)}(a,b). \qquad (6.4)\end{aligned}$$

Proof. With $A := \mathrm{Op}_\alpha(a), B := \mathrm{Op}_\alpha(b)$ we write

$$W_\alpha(a;\mathbb{R}_{x_0}^{(+)}) - W_\alpha(a;I) = \chi_I A \chi_{y_0}^{(+)} + \chi_{y_0}^{(+)} A \chi_{x_0}^{(+)} \qquad (6.5)$$

and estimate

$$\|\chi_I A\chi_{y_0}^{(+)} H_\alpha(b, g; \mathbb{R}_{x_0}^{(+)})\|_{\mathfrak{S}_1} \le \|a\|_{\mathsf{L}^\infty}\|g\|_{\mathsf{L}^\infty}\|\chi_{y_0}^{(+)} B\chi_{x_0}^{(-)}\|_{\mathfrak{S}_1}.$$

For the second term on the right-hand side of (6.5) let $z_0 := (x_0 + y_0)/2$. Then

$$\begin{aligned}&\|\chi_{y_0}^{(+)} A\chi_{x_0}^{(+)}\, H_\alpha(b, g; \mathbb{R}_{x_0}^{(+)})\|_{\mathfrak{S}_1}\\ &\qquad \le \|\chi_{y_0}^{(+)} A\chi_{z_0}^{(-)}\|_{\mathfrak{S}_1}\|b\|_{\mathsf{L}^\infty}\|g\|_{\mathsf{L}^\infty} + \|a\|_{\mathsf{L}^\infty}\|g\|_{\mathsf{L}^\infty}\|\chi_{z_0}^{(+)} B\chi_{x_0}^{(-)}\|_{\mathfrak{S}_1}.\end{aligned}$$

Now Proposition 6.1 leads to inequality (6.4) for the first term on the left-hand side of (6.4). The remaining inequality is derived in the same way. □

6.2. *Estimates for $D_\alpha(a, I; r_z)$: one-dimensional case*

We apply definition (6.1) to the symbols $a_z := a - z$ and a_z^{-1}. Now we assume that a is a real-valued symbol and that $\delta(z, a) > 0$, see definition (4.10). Thus we obtain (replace Λ by I)

$$\chi_I - W_\alpha(a_z; I)W_\alpha\big(a_z^{-1}; I\big) = H_\alpha(a_z, a_z^{-1}; I).$$

Clearly, both operators $W_\alpha(a_z; I)$ and $W_\alpha(a_z^{-1}; I)$ are invertible on $\mathsf{L}^2(I)$ and

$$\big\|\,W_\alpha(a_z; I)|_I^{-1}\big\| \le \frac{1}{\delta(z, a)},\ \ \big\|\,W_\alpha(a_z^{-1}; I)\big|_I^{-1}\big\| \le |z| + \|a\|_{\mathsf{L}^\infty}.$$

Thus $\mathbb{1} - H_\alpha(a_z, a_z^{-1}; I)$ is invertible on $\mathsf{L}^2(\mathbb{R})$ and

$$\big(\mathbb{1} - H_\alpha(a_z, a_z^{-1}; I)\big)^{-1}\chi_I = W_\alpha(a_z^{-1}; I)\big|_I^{-1}\, W_\alpha(a_z; I)|_I^{-1},$$

with the bound

$$\|\big(\mathbb{1} - H_\alpha(a_z, a_z^{-1}; I)\big)^{-1}\| \le \frac{|z| + \|a\|_{\mathsf{L}^\infty}}{\delta(z, a)}. \tag{6.6}$$

As a consequence,

$$\begin{aligned}&\big(W_\alpha(a; I) - z\big)^{-1}\chi_I - W_\alpha\big(a_z^{-1}; I\big)\\ &\qquad = W_\alpha\big(a_z^{-1}; I\big)\Big[W_\alpha(a_z^{-1}; I)\big|_I^{-1}\, W_\alpha(a_z; I)|_I^{-1} - \chi_I\Big]\\ &\qquad = W_\alpha\big(a_z^{-1}; I\big)H_\alpha(a_z, a_z^{-1}; I)\big[\mathbb{1} - H_\alpha(a_z, a_z^{-1}; I)\big]^{-1}.\end{aligned} \tag{6.7}$$

Let us analyse the part of the right-hand side which contains H_a.

Lemma 6.4. *Let $I = (x_0, y_0)$, and let $y_0 - x_0 \ge 1$. Denote*

$$H_\alpha := H_\alpha(a_z, a_z^{-1}; I),\ H_\alpha^{(1)} := H_\alpha(a_z, a_z^{-1}; \mathbb{R}_{x_0}^{(+)}),\ H_\alpha^{(2)} := H_\alpha(a_z, a_z^{-1}; \mathbb{R}_{y_0}^{(-)}).$$

Then for any $\alpha > 0$ *and any* $m \ge 3$,

$$\big\|H_\alpha(\mathbb{1} - H_\alpha)^{-1} - H_\alpha^{(1)}(\mathbb{1} - H_\alpha^{(1)})^{-1} - H_\alpha^{(2)}(\mathbb{1} - H_\alpha^{(2)})^{-1}\big\|_{\mathfrak{S}_1} \le C_m \alpha^{-m+1}\left(\frac{|z| + \|a\|_{\mathsf{L}^\infty}}{\delta(z,a)}\right)^3 M^{(m)}(a_z, a_z^{-1}).$$

Proof. We use the representation

$$H_\alpha = H_\alpha^{(1)} + H_\alpha^{(2)} + Z_\alpha,\ \ Z_\alpha := Z_\alpha(a_z, a_z^{-1}; I).$$

The required bound for $Z_\alpha(1 - H_\alpha)^{-1}$ follows from (6.2) and (6.6). Now, by the resolvent identity,

$$\big\|H_\alpha^{(1)}(\mathbb{1} - H_\alpha)^{-1} - H_\alpha^{(1)}(\mathbb{1} - H_\alpha^{(1)})^{-1}\big\|_{\mathfrak{S}_1} \le \|(\mathbb{1} - H_\alpha^{(1)})^{-1}\| \times \Big[\|H_\alpha^{(1)} H_\alpha^{(2)}\|_{\mathfrak{S}_1} + \|H_\alpha^{(1)}\|\ \|Z_\alpha\|_{\mathfrak{S}_1}\Big]\|(\mathbb{1} - H_\alpha)^{-1}\|.$$

The required bound for this operator follows from (6.6), (6.2) and (6.3). $\square$

Lemma 6.5. *For* $z \in \mathbb{C}$ *let* g *be the function defined as* $g(\lambda) := r_z(\lambda) := (\lambda - z)^{-1}$ *for* $\lambda \in \mathbb{R}$. *Also, with the symbol* a *as above, let* $a_z := a - z$. *Then for any* $\alpha > 0$,

$$\big\|D_\alpha(a, I; r_z) - D_\alpha(a, \mathbb{R}_{y_0}^{(-)}; r_z) - D_\alpha(a, \mathbb{R}_{x_0}^{(+)}; r_z)\big\|_{\mathfrak{S}_1} \le C_m \alpha^{-m+1}\frac{1}{\delta(z,a)}\left(\frac{|z| + \|a\|_{\mathsf{L}^\infty}}{\delta(z,a)}\right)^3 M^{(m)}(a_z, a_z^{-1}).$$

Proof. We use the notation $H_\alpha, H_\alpha^{(1)}, H_\alpha^{(2)}$ from Lemma 6.4, and

$$W_\alpha := W_\alpha(a_z^{-1}; I),\ \ W_\alpha^{(1)} := W_\alpha(a_z^{-1}; \mathbb{R}_{x_0}^{(+)}),\ \ W_\alpha^{(2)} := W_\alpha(a_z^{-1}; \mathbb{R}_{y_0}^{(-)}).$$

By Lemma 6.3 and the bound (6.6),

$$\|(W_\alpha - W_\alpha^{(k)})H_\alpha^{(k)}\big(\mathbb{1} - H_\alpha^{(k)}\big)^{-1}\|_{\mathfrak{S}_1} \le C\alpha^{-m+1}\frac{|z| + \|a\|_{\mathsf{L}^\infty}}{\delta(z,a)^2} M^{(m)}(a_z, a_z^{-1}),$$

for $k = 1, 2$. Together with Lemma 6.4 this gives

$$\|W_\alpha H_\alpha(\mathbb{1} - H_\alpha)^{-1} - \sum_{k=1}^{2} W_\alpha^{(k)} H_\alpha^{(k)}(\mathbb{1} - H_\alpha^{(k)})^{-1}\|_{\mathfrak{S}_1} \le C\alpha^{-m+1}\frac{1}{\delta(z,a)}\left(\frac{|z| + \|a\|_{\mathsf{L}^\infty}}{\delta(z,a)}\right)^3 M^{(m)}(a_z, a_z^{-1}), j = 1, 2.$$

Now formula (6.7) leads to the proclaimed estimate. $\square$

Proof of Theorem 4.3 for the case $I = (x_0, y_0)$. Lemma 6.5 shows that for the function r_z defined as $r_z(\lambda) := (\lambda - z)^{-1}$ the trace of $D_\alpha(a, I; r_z)$ coincides with the sum

$$\operatorname{tr} D_\alpha(a, \mathbb{R}^{(-)}_{y_0}; r_z) + \operatorname{tr} D_\alpha(a, \mathbb{R}^{(+)}_{x_0}; r_z) \tag{6.8}$$

up to the remainder specified in the lemma. As we have observed earlier, due to the translation and reflection invariance, each of the intervals $\mathbb{R}^{(+)}_{x_0}, \mathbb{R}^{(-)}_{y_0}$ in the above trace sum can be replaced by $(0, \infty)$. When calculating the traces in (6.8), by making the change of variables $x \to \alpha x$ we can take $\alpha = 1$. Now Theorem 4.3 follows from Proposition 4.2(1). □

7. Proof of Theorem 4.3: the case of multiple intervals

In this section we consider general sets I of the form (4.6), and assume that (4.11) is satisfied. Throughout this section we assume that $a \in \mathsf{C}^m(\mathbb{R})$ with some $m \geq 3$ and that a is real-valued. The parameter α is allowed to take any positive value and the constants in all estimates obtained are independent of the function a or the parameters z with $\delta(z, a) > 0$ and α. Our strategy is to reduce the case of general I's either to the case of one bounded interval, considered in the previous section, or to the case of the half-line, covered by Proposition 4.2. More precisely, our objective is to prove the following result:

Theorem 7.1. *For all $\alpha > 0$ we have*

$$\begin{aligned} \|D_\alpha(a, I; r_z) - \sum_k D_\alpha(a, I_k; r_z)\|_{\mathfrak{S}_1} \\ \leq C\alpha^{-m+1} \frac{1}{\delta(z, a)} \left(\frac{|z| + \|a\|_{\mathsf{L}^\infty}}{\delta(z, a)} \right)^3 M^{(m)}(a_z, a_z^{-1}), \end{aligned} \tag{7.1}$$

where $M^{(m)}(a, b)$ is defined in (4.9).

The proof consists of several steps:

Lemma 7.2. *Under the above conditions*

$$\|W_\alpha(a_z^{-1}; I) - \sum_k W_\alpha(a_z^{-1}; I_k)\|_{\mathfrak{S}_1} \leq C\alpha^{-m+1} \frac{1}{\delta(z, a)} M^{(m)}(a_z, a_z^{-1}), \tag{7.2}$$

and with $H_\alpha(a, b; I)$ defined in (6.1),

$$\|H_\alpha(a_z, a_z^{-1}; I) - \sum_k H_\alpha(a_z, a_z^{-1}; I_k)\|_{\mathfrak{S}_1} \leq C\alpha^{-m+1} M^{(m)}(a_z, a_z^{-1}). \tag{7.3}$$

Proof. In order to prove (7.2) we write

$$W_\alpha(a_z^{-1}; I) - \sum_k W_\alpha(a_z^{-1}; I_k) = \sum_{j,k: j\neq k} \chi_{I_k} \operatorname{Op}_\alpha(a_z^{-1})\chi_{I_j}.$$

Due to the condition (4.11), by Proposition 6.1, the trace norm of the right-hand side does not exceed

$$\alpha^{-m+1}\|\partial_\xi^m a_z^{-1}\|_{\mathsf{L}^1} \le \alpha^{-m+1}\frac{1}{\delta(z,a)} M^{(m)}(a_z, a_z^{-1}),$$

as required. For the proof of (7.3) we write

$$H_\alpha(a_z, a_z^{-1}; I) - \sum_k H_\alpha(a_z, a_z^{-1}; I_k) = -\sum \chi_{I_k} \operatorname{Op}_\alpha(a_z)\chi_{I_j} \operatorname{Op}_\alpha(a_z^{-1})\chi_{I_s},$$

where the sum is taken over the indices such that either $j \neq k$ or $j \neq s$. By Proposition 6.1, the trace norm of the right-hand side does not exceed

$$\alpha^{m+1}\big(\|a_z\|_{\mathsf{L}^\infty}\|\partial_\xi^m a_z^{-1}\|_{\mathsf{L}^1} + \|a_z^{-1}\|_{\mathsf{L}^\infty}\|\partial_\xi^m a_z\|_{\mathsf{L}^1}\big) = \alpha^{-m+1}M^{(m)}(a_z, a_z^{-1}),$$

as required. □

Lemma 7.3. *Under the above conditions*

$$\big\|\big[\mathbb{1} - H_\alpha(a_z, a_z^{-1}; I)\big]^{-1} H_\alpha(a_z, a_z^{-1}; I) - \sum_k \big[\mathbb{1} - H_\alpha(a_z, a_z^{-1}; I_k)\big]^{-1} H_\alpha(a_z, a_z^{-1}; I_k)\big\|_{\mathfrak{S}_1}$$
$$\le C\alpha^{-m+1}\bigg(\frac{|z| + \|a\|_{\mathsf{L}^\infty}}{\delta(z,a)}\bigg)^3 M^{(m)}(a_z, a_z^{-1}).$$

Proof. For brevity we denote $H_\alpha := H_\alpha(a_z, a_z^{-1}; I)$, $H_\alpha^{(k)} := H_\alpha(a_z, a_z^{-1}; I_k)$. Due to (6.6) and (7.3), in the first term we can replace H_α with $\sum_k H_\alpha^{(k)}$. Now we estimate, using the resolvent identity:

$$\begin{aligned}
&\big\|(\mathbb{1} - H_\alpha)^{-1}H_\alpha^{(k)} - (\mathbb{1} - H_\alpha^{(k)})^{-1}H_\alpha^{(k)}\big\|_{\mathfrak{S}_1} \\
&\qquad \le \|(\mathbb{1} - H_\alpha)^{-1}\|\ \|(\mathbb{1} - H_\alpha^{(k)})^{-1}\|\ \|(H_\alpha - H_\alpha^{(k)})H_\alpha^{(k)}\|_{\mathfrak{S}_1} \\
&\qquad \le \bigg(\frac{|z| + \|a\|_{\mathsf{L}^\infty}}{\delta(z,a)}\bigg)^2 \bigg[\|H_\alpha - \sum_j H_\alpha^{(j)}\|_{\mathfrak{S}_1}\|H_\alpha^{(k)}\| + \sum_{j\neq k}\|H_\alpha^{(j)}H_\alpha^{(k)}\|_{\mathfrak{S}_1}\bigg],
\end{aligned}$$

where we have used (6.6) again. As $j \neq k$ in the last term in the square brackets, this term equals zero. Now the required bound follows from (7.3) and the bound

$$\|H_\alpha^{(k)}\| \le \frac{|z| + \|a\|_{\mathsf{L}^\infty}}{\delta(z,a)}. \qquad \square \tag{7.4}$$

Proof of Theorem 7.1. As in the previous proof we use the notation $H_\alpha, H_\alpha^{(k)}$. Also, we denote $W_\alpha := W_\alpha(a_z^{-1}; I)$, $W_\alpha^{(k)} := W_\alpha(a_z^{-1}; I_k)$. In view of (6.7),

$$\begin{aligned}\|D_\alpha(a, I; r_z) - \sum_k D_\alpha(a, I_k; r_z)\|_{\mathfrak{S}_1} \\ &\le \big\|W_\alpha - \sum_k W_\alpha^{(k)}\big\|_{\mathfrak{S}_1} \|(\mathbb{1} - H_\alpha)^{-1} H_\alpha\| \\ &\quad + \sum_k \|W_\alpha^{(k)}\| \, \big\|(\mathbb{1} - H_\alpha)^{-1} H_\alpha - \sum_k (\mathbb{1} - H_\alpha^{(k)})^{-1} H_\alpha^{(k)}\big\|_{\mathfrak{S}_1}.\end{aligned}$$

The first term on the right-hand side satisfies (7.1) by (7.2) and (6.6), (7.4). The second term satisfies (7.1) by Lemma 7.3 and due to the bound $\|W_\alpha^{(k)}\| \le \delta(z, a)^{-1}$. □

Proof of Theorem 4.3. By (7.1), it remains to use the results for individual operators $D_\alpha(a, I_k; r_z)$. For $k = 1, 2, \dots, K$, that is, when I_k is a bounded interval, we use the bound (4.12) proved previously. If $k = 0$ or $k = K + 1$, that is, when I_k is a half-line, we use the identity (4.5). This leads to the bound (4.12), and completes the proof of Theorem 4.3. □

8. Estimates for $D_\alpha(a, \Lambda; f)$ with Fermi symbol $a = a_{T,\mu}$: multi-dimensional case

As explained in the Introduction, the asymptotic analysis in this paper was partly motivated by the study of the entanglement entropy of free fermions. Thus in this section we apply the results obtained above to the special choice of the symbol a featuring in definition (1.2). We choose the symbol a to be the Fermi symbol $a_{T,\mu}$ defined in (1.5). The choice of the (non-smooth) function f remains arbitrary for the time being. Further on, in Section 10, we specialise to the Rényi entropy function $f = \eta_\gamma$, $\gamma > 0$.

The physical context of the various quantities is as follows. We assume that the energy of a single particle in position space $\mathbb{R}^d$ consists only of kinetic energy in the absence of external forces and is determined by a Hamiltonian $h = h(\boldsymbol{\xi})$ and that, for simplicity, particles do not have a spin-degree of freedom. The free Fermi gas is then a collection of infinitely many such particles obeying the (Pauli–)Fermi–Dirac statistics. An equilibrium state of this free Fermi gas is uniquely determined by specifying the temperature $T > 0$, the chemical potential $\mu \in \mathbb{R}$, and the Fermi symbol (1.5). We will assume that μ is fixed and $T \in (0, T_0]$, and, in particular, T is allowed to become small, that is, $T \downarrow 0$. Our aim is to find estimates with explicit dependence on T and α.

In what follows it will be convenient to use the following notation. For any two non-negative functions x and y depending on all or some of the variables/parameters α, T, $\boldsymbol{\xi}$, we write $x \asymp y$ if there exist two constants C, c independent of α, T, $\boldsymbol{\xi}$ such that $cy \le x \le Cy$.

The assumptions on the function $h = h(\boldsymbol{\xi})$ are as follows:

Condition 8.1.

(1) *The function* $h \in \mathsf{C}^\infty(\mathbb{R}^d)$ *is real-valued, and for sufficiently large* $\boldsymbol{\xi}$ *and with some constants* $\beta_1 > 0$ *and* $c > 0$ *we have*

$$h(\boldsymbol{\xi}) \ge c|\boldsymbol{\xi}|^{\beta_1}. \tag{8.1}$$

Moreover, for some $\beta_2 \ge 0$

$$|\nabla^n h(\boldsymbol{\xi})| \le C_n(1+|\boldsymbol{\xi}|)^{\beta_2},\ n = 1, 2, \dots, \quad \boldsymbol{\xi} \in \mathbb{R}^d. \tag{8.2}$$

(2) *On the set* $S := \{\boldsymbol{\xi} \in \mathbb{R}^d : h(\boldsymbol{\xi}) = \mu\}$ *the condition*

$$\nabla h(\boldsymbol{\xi}) \ne 0,\ \boldsymbol{\xi} \in S \tag{8.3}$$

is satisfied.

(3) *The Fermi sea* $\Omega := \{\boldsymbol{\xi} \in \mathbb{R}^d : h(\boldsymbol{\xi}) < \mu\}$ *has finitely many connected components.*

Let us record some useful inequalities for the symbol $a = a_{T,\mu}$ from (1.5).

Lemma 8.2. *Suppose that* $0 < T \le T_0$. *Then*

$$|\nabla^n a(\boldsymbol{\xi})| \le C_n a(\boldsymbol{\xi})\big(1 - a(\boldsymbol{\xi})\big)(1+|\boldsymbol{\xi}|)^{n\beta_2} T^{-n}, n = 1, 2, \dots,$$

with constants C_n *depending on* T_0, μ, *and the constants in* (8.2).

The proof is elementary and thus omitted.

A straightforward calculation leads to the bounds

$$|a(\boldsymbol{\xi}) - \chi_\Omega(\boldsymbol{\xi})| \le \exp\left(-\frac{|h(\boldsymbol{\xi}) - \mu|}{T}\right),\ \boldsymbol{\xi} \in \mathbb{R}^d, \tag{8.4}$$

and

$$a(\boldsymbol{\xi})\big(1 - a(\boldsymbol{\xi})\big) \le \exp\left(-\frac{|h(\boldsymbol{\xi}) - \mu|}{T}\right),\ \boldsymbol{\xi} \in \mathbb{R}^d. \tag{8.5}$$

Our objective is to obtain the following estimate.

Theorem 8.3. *Suppose that the function* f *satisfies* Condition 2.1 *with some* $n \ge 2$ *and* $\gamma > 0$. *Suppose also that the region* Λ *and the function* h *satisfy* Conditions 3.1 and 8.1 *respectively. Let* $\alpha T \ge \alpha_0 > 0$, $0 < T \le T_0$ *for some* α_0 *and* T_0. *Then for any* $\sigma \in (0, \gamma)$, $\sigma \le 1$, *we have*

$$\big\|D_\alpha(a_{T,\mu}, \Lambda; f)\big\|_{\mathfrak{S}_1} \le C R^{\gamma-\sigma}\alpha^{d-1}(|\log(T)| + 1) {\big| f \big|}_n, \tag{8.6}$$

with a constant C independent of T, R, t_0, α, and the function f, but depending on α_0, T_0, μ.

Until the end of this section we always assume that the region Λ and the function h satisfy Conditions 3.1 and 8.1 respectively.

Because of (8.1) the set Ω is bounded, so that $\Omega \subset B(\mathbf{0}, R_0)$ with some $R_0 > 0$. Assume first that $d \ge 2$. Due to condition (8.3), the set S is locally a C^∞-surface, which is called the *Fermi surface*. More precisely, for any $\boldsymbol{\xi}_0 \in S$ there is a radius $r > 0$ such that $|\partial_{\xi_d} h(\boldsymbol{\xi})| \ge c$ for all $\boldsymbol{\xi} \in B(\boldsymbol{\xi}_0, 2r)$ with a suitable choice of coordinates $\boldsymbol{\xi} = (\hat{\boldsymbol{\xi}}, \xi_d)$, and hence there exists a function $\Psi \in \mathsf{C}^\infty(\mathbb{R}^{d-1})$ such that

$$S \cap B(\boldsymbol{\xi}_0, 2r) = \{\boldsymbol{\xi} \in \mathbb{R}^d : \xi_d = \Psi(\hat{\boldsymbol{\xi}})\} \cap B(\boldsymbol{\xi}_0, 2r). \tag{8.7}$$

For definiteness we assume that $B(\boldsymbol{\xi}_0, 2r) \subset B(\mathbf{0}, R_0)$. We may also assume that

$$\Omega \cap B(\boldsymbol{\xi}_0, 2r) = \{\boldsymbol{\xi} \in \mathbb{R}^d : \xi_d > \Psi(\hat{\boldsymbol{\xi}})\} \cap B(\boldsymbol{\xi}_0, 2r). \tag{8.8}$$

This can be achieved by replacing ξ_d and $\Psi(\hat{\boldsymbol{\xi}})$ with $-\xi_d$ and $-\Psi(\hat{\boldsymbol{\xi}})$ and by taking a smaller r, if necessary. Without loss of generality we may assume that $\|\nabla\Psi\|_{\mathsf{L}^\infty} \le M$ with some constant $M > 0$. By choosing a sufficiently small $r > 0$, due to the condition $|\partial_{\xi_d} h| \ge c$ one can also guarantee that

$$|\xi_d - \Psi(\hat{\boldsymbol{\xi}})| \asymp |h(\boldsymbol{\xi}) - \mu|, \ \boldsymbol{\xi} \in B(\boldsymbol{\xi}_0, 2r), \tag{8.9}$$

with some $C \ge 1$. It is clear that $|\xi_d - \Psi(\hat{\boldsymbol{\xi}})| \ge \operatorname{dist}(\boldsymbol{\xi}, S)$. On the other hand, $|\boldsymbol{\xi} - \boldsymbol{\eta}| \ge (1 + M^2)^{-1/2}|\xi_d - \Psi(\hat{\boldsymbol{\xi}})|$, for any $\boldsymbol{\xi} \in B(\boldsymbol{\xi}_0, 2r)$ and any $\boldsymbol{\eta} \in S \cap B(\boldsymbol{\xi}_0, 2r)$. Consequently,

$$|\xi_d - \Psi(\hat{\boldsymbol{\xi}})| \asymp \operatorname{dist}(\boldsymbol{\xi}, S), \ \forall \boldsymbol{\xi} \in B(\boldsymbol{\xi}_0, r). \tag{8.10}$$

Since the set Ω is in fact a C^∞-region, we can cover its boundary S with finitely many open balls $\{D_j(r)\}$ of radius r centred at some $\boldsymbol{\xi}_j \in S$, such that in each $D_j(2r)$ one can find an appropriate function $\Psi = \Psi_j$ that satisfies the properties (8.7)–(8.9) after an appropriate choice of coordinates in every ball $D_j(2r)$. From now on for brevity we denote $D_j = D_j(r)$.

Let $\tilde{D} \subset \mathbb{R}^d$ be a region such that $\tilde{D} \cap S = \varnothing$, and

$$\mathbb{R}^d = (\cup_j D_j) \cup \tilde{D}, \quad \tilde{D} = (\cup_j \tilde{D}_j) \cup \{\boldsymbol{\xi} \in \mathbb{R}^d : |\boldsymbol{\xi}| > R_0\}. \tag{8.11}$$

If $d = 1$, then we modify the definitions of $\{D_j\}$ and $\tilde{D}$ in an obvious way. For example, each D_j is now an interval such that with an appropriate choice of the coordinate ξ the open set $D_j \cap \Omega$ is simply $D_j \cap \{\xi \in \mathbb{R} : \xi > 0\}$. Thus the covering (8.11) holds for $d = 1$ as well.

The idea of the proof of Theorem 8.3 is to observe that the symbol (1.5) satisfies (3.5) on each element of the covering (8.11) with some functions τ and v defined individually on each of the domains D_j and $\tilde{D}$. After that Theorem 3.5 produces Theorem 8.3.

Let us first describe the construction of the scaling function τ and amplitude function v on D_j and $\tilde{D}$. We do this for the case $d \ge 2$, as for $d = 1$ only obvious modifications are required.

Let $\Psi = \Psi^{(j)} \in \mathsf{C}^\infty(\mathbb{R}^{d-1})$ be a function describing the surface S inside D_j, see (8.7). Recall that we always assume that $\|\nabla\Psi\|_{\mathsf{L}^\infty} \le C$. We introduce the functions $\ell^{(j)}$ and $w^{(j)}$ defined on $\mathbb{R}^d$ as

$$\ell^{(j)}(\boldsymbol{\xi}) := |\xi_d - \Psi^{(j)}(\hat{\boldsymbol{\xi}})| + T,\ w^{(j)}(\boldsymbol{\xi}) := \exp\left(-c_1\frac{\ell^{(j)}(\boldsymbol{\xi})}{T}\right). \tag{8.12}$$

Due to (8.4), (8.5) and (8.9), the constant c_1 can be chosen to guarantee that

$$|a(\boldsymbol{\xi}) - \chi_\Omega(\boldsymbol{\xi})| \le w^{(j)}(\boldsymbol{\xi}),\ \boldsymbol{\xi} \in D_j, \tag{8.13}$$

and

$$a(\boldsymbol{\xi})\big(1 - a(\boldsymbol{\xi})\big) \le w^{(j)}(\boldsymbol{\xi}),\ \boldsymbol{\xi} \in D_j. \tag{8.14}$$

Since $D_j \subset B(\mathbf{0}, R_0)$, we get from Lemma 8.2 that for $\boldsymbol{\xi} \in D_j$

$$|\nabla^n a(\boldsymbol{\xi})| \le C_n T^{-n} a(\boldsymbol{\xi})(1 - a(\boldsymbol{\xi})) \le \tilde{C}_n T^{-n} w^{(j)}(\boldsymbol{\xi}), C_n = C_n(R_0).$$

Using the fact that $\sup_{t>0} t^n e^{-t}$ is finite for all $n = 0, 1, \dots$, we can estimate the right-hand side by $C_n \ell^{(j)}(\boldsymbol{\xi})^{-n}$. Therefore

$$|\nabla^n a(\boldsymbol{\xi})| \le C_n \ell^{(j)}(\boldsymbol{\xi})^{-n},\ n = 0, 1, 2, \dots,\ \forall \boldsymbol{\xi} \in D_j. \tag{8.15}$$

This shows that on D_j the symbol a satisfies (3.5) with $\tau(\boldsymbol{\xi}) := \ell^{(j)}(\boldsymbol{\xi})$ and $v(\boldsymbol{\xi}) := v^{(j)}(\boldsymbol{\xi}) = 1$.

On the domain $\tilde{D}$ the construction is different. Define the function $\widetilde{w}$ as

$$\widetilde{w}(\boldsymbol{\xi}) := \exp\left(-c_1\frac{(1 + |\boldsymbol{\xi}|)^{\beta_1}}{T}\right),\ \boldsymbol{\xi} \in \mathbb{R}^d. \tag{8.16}$$

Since h satisfies (8.1), and $|h(\boldsymbol{\xi}) - \mu| \ge c$ for $\boldsymbol{\xi} \in \tilde{D}$, one can find a constant $c_1 > 0$ such that

$$\exp\left(-\frac{|h(\boldsymbol{\xi}) - \mu|}{T}\right) \le \widetilde{w}(\boldsymbol{\xi})^2,\ \boldsymbol{\xi} \in \tilde{D}.$$

Hence, by (8.4) and (8.5),

$$|a(\boldsymbol{\xi}) - \chi_\Omega(\boldsymbol{\xi})| \le \widetilde{w}(\boldsymbol{\xi})^2,\ \boldsymbol{\xi} \in \tilde{D}, \tag{8.17}$$

and

$$a(\boldsymbol{\xi})\big(1-a(\boldsymbol{\xi})\big) \le \widetilde{w}(\boldsymbol{\xi})^2,\ \boldsymbol{\xi}\in\tilde{D}. \tag{8.18}$$

Consequently by Lemma 8.2,

$$|\nabla^n a(\boldsymbol{\xi})| \le C_n T^{-n}(1+|\boldsymbol{\xi}|)^{\beta_2 n}\widetilde{w}(\boldsymbol{\xi})^2, n=1,2,\dots,$$

for all $\boldsymbol{\xi}\in\tilde{D}$. Using the fact that $\sup_{t\ge 1}(t^{\beta_2}T^{-1})^n e^{-c_1 t^{\beta_1}T^{-1}} \le C(n,T_0)$ for all $T\in(0,T_0]$ and $n=1,2,\dots$, we conclude that

$$|\nabla^n a(\boldsymbol{\xi})| \le \tilde{C}_n\widetilde{w}(\boldsymbol{\xi}),\ n=1,2,\dots,\ \boldsymbol{\xi}\in\tilde{D}.$$

This implies that with a suitable constant $c_2=c_2(T_0,h)$,

$$|\nabla^n a(\boldsymbol{\xi})| \le C_n e^{-c_2|\boldsymbol{\xi}|^{\beta_1}},\ n=0,1,\dots,\ \boldsymbol{\xi}\in\mathbb{R}^d.$$

It is more convenient to replace the exponential by a power-like function $\tilde{v}(\boldsymbol{\xi}) := \tilde{v}_\sigma(\boldsymbol{\xi}) := (1+|\boldsymbol{\xi}|)^{-(d+1)\sigma^{-1}}$ with some $\sigma\in(0,1]$, so that for all $\boldsymbol{\xi}\in\tilde{D}$

$$|\nabla^n a(\boldsymbol{\xi})| \le C_n(1+|\boldsymbol{\xi}|)^{-(d+1)\sigma^{-1}},\ n=0,1,\dots. \tag{8.19}$$

Thus a satisfies (3.5) with $\tilde{\tau}=1$ and $v=\tilde{v}_\sigma$. The choice of the value of σ will be made later.

Now we can put together the definitions of $\ell^{(j)}, v^{(j)}$ and $\tilde{\tau},\tilde{v}$ to define the scaling function and amplitude on the entire space. Let $\{\phi_j\},\tilde{\phi}$ be a partition of unity subordinate to the covering (8.11). Then we define for $\boldsymbol{\xi}\in\mathbb{R}^d$

$$\begin{cases} \tau(\boldsymbol{\xi}) := \theta\big(\sum_j \ell^{(j)}(\boldsymbol{\xi})\phi_j(\boldsymbol{\xi}) + \tilde{\phi}(\boldsymbol{\xi})\big), \\ v(\boldsymbol{\xi}) := \sum_j \phi_j(\boldsymbol{\xi}) + (1+|\boldsymbol{\xi}|)^{-\frac{d+1}{\sigma}}\tilde{\phi}(\boldsymbol{\xi}). \end{cases} \tag{8.20}$$

The constant $\theta>0$ is chosen to guarantee the bound $\|\nabla\tau\|_{\mathsf{L}^\infty}\le\nu$ with some $\nu\in(0,1)$. It is straightforward to check that v satisfies (3.9) and that $\tau\asymp 1$ on $\tilde{D}$. Moreover, by virtue of (8.10), $\tau\asymp\ell^{(j)}$ on D_j. Consequently, the symbol a satisfies (3.5) with the functions τ and v defined above.

Let us establish some bounds for $V_{\sigma,\rho}(v,\tau)$, see (3.6).

Lemma 8.4. *Let $T\in(0,T_0]$. Let τ be defined as in (8.20) with the same $\sigma\in(0,1]$ as in (8.20). Then*

$$V_{\sigma,1}(v,\tau) \asymp \Big|\log(T)\Big| + 1, \tag{8.21}$$

$$V_{\sigma,\rho}(v,\tau) \le C_{\sigma,\rho}T^{-\rho+1},\ \rho>1, \tag{8.22}$$

with a constant independent of $T\in(0,T_0]$.

Proof. We estimate integrals of the type (3.6) over the domains that form the covering (8.11). Denote

$$V^{(j)}_{\sigma,\rho}(v,\tau) := \int \phi_j(\boldsymbol{\xi}) \frac{v(\boldsymbol{\xi})^\sigma}{\tau(\boldsymbol{\xi})^\rho} d\boldsymbol{\xi}, \quad \tilde{V}_{\sigma,\rho}(v,\tau) := \int \tilde{\phi}(\boldsymbol{\xi}) \frac{v(\boldsymbol{\xi})^\sigma}{\tau(\boldsymbol{\xi})^\rho} d\boldsymbol{\xi}.$$

As we have observed previously, $\tau \asymp \ell^{(j)}$ on D_j, so that

$$V^{(j)}_{\sigma,\rho}(v,\tau) \asymp \int_{D_j} (|\xi_d - \Psi^{(j)}(\hat{\boldsymbol{\xi}})| + T)^{-\rho} d\boldsymbol{\xi}$$
$$\asymp r^{d-1} \int_{-2r}^{2r} (|t| + T)^{-\rho} dt.$$

This leads to (8.21) and (8.22) for $V^{(j)}_{\sigma,\rho}$. Furthermore, $\tau(\boldsymbol{\xi}) \asymp 1$ for all $\boldsymbol{\xi} \in \tilde{D}$. Therefore

$$\tilde{V}_{\sigma,\rho}(v,\tau) \asymp \int (1 + |\boldsymbol{\xi}|)^{-d-1} d\boldsymbol{\xi} \le C,$$

for any $\rho \in \mathbb{R}$. The obtained bounds together prove (8.21) and (8.22). □

Proof of Theorem 8.3. We use (3.17) with $q = 1$ and any $\sigma \in (0, \gamma)$, $\sigma \le 1$:

$$\|D_\alpha(a, \Lambda; f)\|_{\mathfrak{S}_1} \le C_{\sigma,\gamma} \alpha^{d-1} {|\!|\!|} f {|\!|\!|}_n R^{\gamma-\sigma} V_{\sigma,1}(v,\tau).$$

Now Lemma 8.4 leads to (8.6). □

The case of a homogeneous function h deserves special attention since in this case one can explicitly control the dependence on the chemical potential μ. We illustrate this with the example of the function $h(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^2$. The parameter μ can be "scaled out" with the help of the following formula:

$$\mathrm{Op}_\alpha(a_{T,\mu}) = \mathrm{Op}_\nu(a_{T',1}),\ T' := T\mu^{-1},\ \nu := \alpha\sqrt{\mu},$$

so that $D_\alpha(a_{T,\mu}, \Lambda; f) = D_\nu(a_{T',1}, \Lambda; f)$. Thus Theorem 8.3 leads to the following result.

Theorem 8.5. *Suppose that f satisfies Condition 2.1 with some $n \ge 2$ and $\gamma > 0$, and that the region Λ satisfies Condition 3.1. Let $a = a_{T,\mu}$ be given by (1.5) with $h(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^2$ and let $\alpha T \mu^{-1/2} \ge \alpha_0, 0 < T\mu^{-1} \le T_0$. Then for any $\sigma \in (0, \gamma)$, $\sigma \le 1$,*

$$\big\|D_\alpha(a_{T,\mu}, \Lambda; f)\big\|_{\mathfrak{S}_1} \le C_\sigma R^{\gamma-\sigma} {|\!|\!|} f {|\!|\!|}_n (\alpha\sqrt{\mu})^{d-1}(|\log(T\mu^{-1})| + 1), \tag{8.23}$$

with a constant C_σ independent of R, α, μ, and the function f, but depending on α_0, T_0 and γ, σ.

The final result in this section is specific to dimension one.

Theorem 8.6. *Let I and ω be defined as in (4.6) and (4.7) respectively, and let the constituent intervals I_j satisfy (4.11). Suppose h satisfies Condition 8.1 and that f satisfies Condition 2.1 with some $\gamma > 0$, $t_0 \in \mathbb{R}$ and $n = 2$. Furthermore, suppose that $T \in (0, 1/2]$ and $\alpha T \geq \alpha_0 > 0$. Then*

$$\lim_{\alpha\to\infty} \frac{1}{|\log(T)|}\big(\operatorname{tr} D_\alpha(a_{T,\mu}, I; f) - \omega \mathcal{B}(a_{T,\mu}; f)\big) = 0, \tag{8.24}$$

uniformly in $t_0 \in \mathbb{R}$. Moreover, for any $T \in (0, 1/2]$,

$$|\mathcal{B}(a_{T,\mu}; f)| \leq C_{\gamma,\sigma} \big\| f \big\|_2 \, |\log(T)|, \tag{8.25}$$

uniformly in $t_0 \in \mathbb{R}$.

Proof. Define the scale and the amplitude as in (8.20). Then the log-bound (8.21), together with (4.15) implies (8.25).

In order to prove (8.24) we use the asymptotics (4.19). First we check the condition (4.18). By (8.21) and (8.22), the left-hand side of (4.18) is estimated by

$$\alpha^{-m+1} \frac{V_{\sigma,m}(v,\tau)}{V_{\sigma,1}(v,\tau)} \leq C(\alpha T)^{-m+1} \frac{1}{|\log(T)|},$$

and hence it tends to zero under the conditions $\alpha T \geq \alpha_0$ and $\alpha \to \infty$. As a result, the condition (4.18) is satisfied, and therefore one can use (4.19), which leads to (8.24), as required. □

The above formulas hold for arbitrary T satisfying the condition $\alpha T \geq \alpha_0$. If we assume additionally that $T \downarrow 0$, then the asymptotics (8.24) can be written in a more explicit form, thanks to the asymptotic formula for $\mathcal{B}(a_{T,\mu}; f)$, $T \downarrow 0$, obtained in Theorem 9.1, which, incidentally, confirms the sharpness of the estimate (8.25). Recall that according to Condition 8.1, for $d = 1$ the set Ω is represented as

$$\Omega = \bigcup_{j=1}^{N} J_j, \ \ N < \infty, \tag{8.26}$$

where $\{J_j\}$ are bounded open intervals such that their closures are pairwise disjoint.

Corollary 8.7. *Let the set I, number ω and the functions h, f be as in Theorem 8.6. Suppose that $T \downarrow 0$ and $\alpha T \geq \alpha_0 > 0$. Then*

$$\operatorname{tr} D_\alpha(a_{T,\mu}, I; f) = |\log(T)| \bigg(\frac{\omega N}{2\pi^2} U(1, 0; f) + o(1) \bigg), \tag{8.27}$$

uniformly in $t_0 \in \mathbb{R}$, where N is as in (8.26).

Proof. The claimed asymptotics follows immediately from Theorems 8.6 and 9.1. □

9. Asymptotics of $\mathcal{B}(a_{T,\mu}; f)$ as $T \downarrow 0$

Here we study the behaviour of $\mathcal{B}(a_{T,\mu}; f)$ with the Fermi symbol $a_{T,\mu}$ defined in (1.5) as $T \downarrow 0$. The number N below is as in the representation (8.26).

Theorem 9.1. *Let $a_{T,\mu}$ be as in (1.5), and let h satisfy Condition 8.1. Suppose that f satisfies Condition 2.1 with some $t_0 \in \mathbb{R}$, $\gamma > 0$ and some $R \le 1$. Then, as $T \downarrow 0$*

$$\mathcal{B}(a_{T,\mu}; f) = \frac{N}{2\pi^2}\, U(1, 0; f)\, |\log(T)| + O(1), \tag{9.1}$$

with $U(1, 0; f)$ defined in (4.1).

Let τ and v be as defined in (8.20), so that $\tau_{\inf} = \theta T$. We study separately the integral $\mathcal{B}^{(1)}$ defined in (4.16) and

$$\mathcal{B}^{(2)}(a; f) := \frac{1}{8\pi^2} \iint\limits_{|\xi_1 - \xi_2| > \frac{\theta T}{2}} \frac{U(a(\xi_1), a(\xi_2); f)}{|\xi_1 - \xi_2|^2} d\xi_1 d\xi_2, \ \ a = a_{T,\mu}. \tag{9.2}$$

Using (4.17) and (8.22), we conclude that for all $T \in (0, T_0]$,

$$|\mathcal{B}^{(1)}(a; f)| \le C. \tag{9.3}$$

To study $\mathcal{B}^{(2)}$ we intend to replace a with the indicator function χ_Ω. To this end we note the following properties of the function f, and as a result, of the integral (4.1). The bound (2.5) says that the function f is Hölder continuous:

$$|f(t_1) - f(t_2)| \le 2 {|\!|\!|} f {|\!|\!|}_1 |t_1 - t_2|^\varkappa, \ \ \varkappa := \min\{1, \gamma\}.$$

An elementary calculation shows that for any $\mu \in (0, 1)$ and for any real s_1, r_1, s_2, r_2 (see [24, Formulas (2.4) and (3.8)]) we have

$$\begin{aligned} |U(s_1, s_2; f) - U(r_1, r_2; f)| \le\ & C {|\!|\!|} f {|\!|\!|}_1 |\log(\mu)| \big(|s_1 - r_1|^\varkappa + |s_2 - r_2|^\varkappa\big) \\ & + C {|\!|\!|} f {|\!|\!|}_1 \mu^\varkappa \big(|s_1 - s_2|^\varkappa + |r_1 - r_2|^\varkappa\big). \end{aligned} \tag{9.4}$$

This leads to the following result.

Lemma 9.2. *Let f be as above and suppose that $|s_1 - s_2| + |r_1 - r_2| \le C$. Then for any $\delta \in [0, \varkappa)$ we have*

$$|U(s_1, s_2; f) - U(r_1, r_2; f)| \le C_\delta {|\!|\!|} f {|\!|\!|}_1 \big(|s_1 - r_1|^\delta + |s_2 - r_2|^\delta\big). \tag{9.5}$$

Proof. It suffices to assume that $|s_1 - r_1| + |s_2 - r_2| < 1/2$. Now (9.5) follows from (9.4) if one sets $\mu = |s_1 - r_1| + |s_2 - r_2|$. $\square$

Lemma 9.3. *Let the condition of Theorem 9.1 be satisfied. Then*

$$|\mathcal{B}^{(2)}(a; f) - \mathcal{B}^{(2)}(\chi_\Omega; f)| \leq C\,{|\!|\!|} f {|\!|\!|}_1, \tag{9.6}$$

uniformly in $T \in (0, T_0]$.

Proof. In view of (9.5),

$$\begin{aligned} |\mathcal{B}^{(2)}(a; f) - \mathcal{B}^{(2)}(\chi_\Omega; f)| &\leq C_\delta {|\!|\!|} f {|\!|\!|}_1 \iint\limits_{\frac{\theta T}{2} < |\xi_1 - \xi_2|} \frac{|a(\xi_1) - \chi_\Omega(\xi_1)|^\delta}{|\xi_1 - \xi_2|^2} d\xi_1 d\xi_2 \\ &\leq C_\delta T^{-1} {|\!|\!|} f {|\!|\!|}_1 \int |a(\xi) - \chi_\Omega(\xi)|^\delta d\xi, \end{aligned} \tag{9.7}$$

for any $\delta \in [0, \varkappa)$. To estimate this integral we use a partition of unity subordinate to the covering (8.11), as in Section 8. Thus, in view of (8.4) and (8.9), for each D_j we obtain

$$\int\limits_{D_j} |a(\xi) - \chi_\Omega(\xi)|^\delta d\xi_1 \leq C \int e^{-c\delta \frac{|\xi|}{T}} d\xi \leq C_\delta T.$$

Also, since the set $\tilde{D}$ is separated from Ω, we have

$$|a(\xi_1) - \chi_\Omega(\xi_1)| \leq C \exp\Big(-\frac{(1 + |\xi_1|)^{\beta_1}}{T}\Big), \ \xi \in \tilde{D},$$

and hence

$$\int\limits_{\tilde{D}} |a(\xi_1) - \chi_\Omega(\xi_1)|^\delta d\xi_1 \leq C \int e^{-c\delta \frac{|\xi_1 + 1|^{\beta_1}}{T}} d\xi_1 \leq C_\delta e^{-\frac{c}{T}}.$$

Together with (9.7), the above estimates lead to (9.6). $\square$

It remains to calculate $\mathcal{B}^{(2)}(\chi_\Omega; f)$. Since $U(1, 1; f) = U(0, 0; f) = 0$ and $U(1, 0; f) = U(0, 1; f)$, this coefficient reduces to

$$\mathcal{B}^{(2)}(\chi_\Omega; f) = \frac{U(1, 0; f)}{4\pi^2} \int\limits_{\xi_1 \notin \Omega} \int\limits_{\xi_2 \in \Omega : \frac{\theta T}{2} < |\xi_1 - \xi_2|} \frac{1}{|\xi_1 - \xi_2|^2} d\xi_2 d\xi_1.$$

The next lemma seems to be useful in its own right, where we claim a certain uniformity in the size of the intervals J_k, $k = 1, 2, \ldots, N$, although Theorem 9.1 does not need this.

Lemma 9.4. *Let $J_k = (s_k, t_k) \subset \mathbb{R}$, $k = 1, 2, \dots, N$ be a finite collection of bounded open intervals, such that their closures are pairwise disjoint, and let $J = \cup_k J_k$. Suppose that $T \in (0, T_0]$ and $|J_k| \le d_1$, $k = 1, 2, \dots, N$, with some $d_1 > 0$. Then*

$$\sum_{k=1}^{N} \int_{t \notin J} dt \int_{|t-s| \ge T, s \in J_k} \frac{ds}{|t-s|^2} \le CN\,|\log(T)|, \tag{9.8}$$

with a constant C depending only on d_1.

Assume in addition that

$$|J_k| \ge d_0,\ k = 1, 2, \dots, N, \quad \min_{j \ne k} \operatorname{dist}\{J_k, J_j\} \ge d_0,$$

with some $d_0 \in (0, d_1]$. Let $\varphi \in \mathsf{C}(\mathbb{R}) \cap \mathsf{L}^\infty(\mathbb{R})$ be a function. Then, as $T \downarrow 0$,

$$\sum_{k=1}^{N} \int_{t \notin J} \varphi(t) dt \int_{|t-s| \ge T, s \in J_k} \frac{ds}{|t-s|^2}$$
$$= |\log(T)| \sum_{k=1}^{N} \big(\varphi(s_k) + \varphi(t_k)\big) + N\|\varphi\|_{\mathsf{L}^\infty} O(1), \tag{9.9}$$

where $O(1)$ depends only on d_0 and d_1.

Proof. Proof of (9.9): Without loss of generality assume that $\|\varphi\|_{\mathsf{L}^\infty} = 1$. It is immediate to see that

$$\sum_{k=1}^{N} \int_{t \notin J} \varphi(t) dt \int_{|t-s| \ge T, s \in J_k} \frac{ds}{|t-s|^2} = \sum_{k=1}^{N} \int_{t \notin J_k} \varphi(t) dt \int_{|t-s| \ge T, s \in J_k} \frac{ds}{|t-s|^2} + NO(1)$$

so that (9.9) reduces to showing that

$$\sum_{k=1}^{N} \int_{t \notin J_k} \varphi(t) dt \int_{|t-s| \ge T, s \in J_k} \frac{ds}{|t-s|^2} = |\log(T)| \sum_{k=1}^{N} \big(\varphi(s_k) + \varphi(t_k)\big) + NO(1),\ T \downarrow 0.$$

Hence it suffices to prove (9.9) for one integral only, that is, that

$$\int_{t \notin J} \varphi(t) dt \int_{|t-s| \ge T, s \in J} \frac{ds}{|t-s|^2} = \big(\varphi(s_0) + \varphi(t_0)\big)\,|\log(T)| + O(1),\ T \downarrow 0, \tag{9.10}$$

for a bounded interval $J = (s_0, t_0)$ with $|s_0 - t_0| \ge d_0$. Without loss of generality assume that $J = (0, 1)$. Split the sought integral into the sum $X_1 + X_2 + X_3 + X_4$, with

$$X_1 := \int_{1+T}^{\infty} \varphi(t)dt \int_0^1 \frac{ds}{(t-s)^2},$$

$$X_2 := \int_{-\infty}^{-T} \varphi(t)dt \int_0^1 \frac{ds}{(t-s)^2},$$

$$X_3 := \int_1^{1+T} \varphi(t)dt \int_0^{1-T} \frac{ds}{(t-s)^2} + \int_1^{1+T} \varphi(t)dt \int_{|s-t|>T, 1-T<s<1} \frac{ds}{(t-s)^2},$$

$$X_4 := \int_{-T}^{0} \varphi(t)dt \int_T^1 \frac{ds}{(t-s)^2} + \int_{-T}^{0} \varphi(t)dt \int_{|s-t|>T, 0<s<T} \frac{ds}{(t-s)^2}.$$

Direct calculations show that $X_3 + X_4 \le C$ uniformly in $T \in (0, T_0]$. The integral X_1 differs from

$$X_1' = \varphi(1) \int_{1+T}^{2} dt \int_0^1 \frac{ds}{(t-s)^2}$$

at most by a constant independent of T. An elementary calculation shows that

$$X_1' = \varphi(1)|\log(T)| + O(1).$$

Thus X_1 satisfies the same formula. In the same way one proves the appropriate formula for X_2. This leads to (9.10), and hence to (9.9).

The bound (9.8) is proved in a similar way by estimating integrals of the same type as in the first part of the proof. We omit the details. □

Proof of Theorem 9.1. Writing

$$\mathcal{B}(a; f) = \mathcal{B}^{(1)}(a; f) + \big(\mathcal{B}^{(2)}(a; f) - \mathcal{B}^{(2)}(\chi_\Omega; f)\big) + \mathcal{B}^{(2)}(\chi_\Omega; f),$$

and combining (9.3), (9.6) and formula (9.9) with $\varphi = 1$, we obtain the claimed asymptotics (9.1). □

10. Entanglement entropy and local entropy

In this section we keep using the Fermi symbol $a = a_{T,\mu}$ as in (1.5) and investigate the special case of the function f given by the γ-*Rényi entropy function* $\eta_\gamma : \mathbb{R} \mapsto [0, \log(2)]$ defined for all $\gamma > 0$ as follows. If $\gamma \ne 1$, then

$$\eta_\gamma(t) := \begin{cases} \frac{1}{1-\gamma} \log\left[t^\gamma + (1-t)^\gamma\right] & \text{for } t \in (0,1), \\ 0 & \text{for } t \notin (0,1), \end{cases} \tag{10.1}$$

and for $\gamma = 1$ (the von Neumann case) it is defined as the limit

$$\eta_1(t) := \lim_{\gamma\to 1} \eta_\gamma(t) = \begin{cases} -t\log(t) - (1-t)\log(1-t) & \text{for } t \in (0,1), \\ 0 & \text{for } t \notin (0,1). \end{cases} \tag{10.2}$$

From now one we assume that the region Λ and the Hamiltonian h satisfy Conditions 3.1 and 8.1 respectively. The operator $D_\alpha(\,\cdot\,)$ is as defined in (1.2) and the notation Ω is used for the Fermi sea, see Condition 8.1.

If Λ is bounded, then *the local (thermal) γ-Rényi entropy* of the equilibrium state at temperature $T > 0$ and chemical potential $\mu \in \mathbb{R}$ is defined as

$$\mathrm{S}_\gamma(T,\mu;\Lambda) := \operatorname{tr}\big[\eta_\gamma(W_1(a_{T,\mu};\Lambda))\big], \tag{10.3}$$

see for example [10]. If one lifts the condition of boundedness, then the above quantity may be infinite, but *the γ-Rényi entanglement entropy (EE)* with respect to the bipartition $\mathbb{R}^d = \Lambda \cup (\mathbb{R}^d \setminus \Lambda)$, defined as

$$\mathrm{H}_\gamma(T,\mu;\Lambda) := \operatorname{tr} D_1(a_{T,\mu},\Lambda;\eta_\gamma) + \operatorname{tr} D_1(a_{T,\mu},\mathbb{R}^d\setminus\Lambda;\eta_\gamma), \tag{10.4}$$

is finite, as the next theorem shows. Note that these definitions also make sense for $T = 0$, if one adopts the notation $a_{0,\mu} := \lim_{T\downarrow 0} a_{T,\mu} = \chi_\Omega$. A somewhat surprising fact is that for bounded Λ,

$$\mathrm{H}_\gamma(0,\mu;\Lambda) = 2\,\mathrm{S}_\gamma(0,\mu;\Lambda). \tag{10.5}$$

As explained in [13], this is a consequence of the following two identities: $\operatorname{tr}\eta_\gamma(\chi_\Lambda P_\Omega \chi_\Lambda) = \operatorname{tr}\eta_\gamma(P_\Omega\chi_\Lambda P_\Omega)$, where $P_\Omega = \mathrm{Op}_1(\chi_\Omega)$, and $\eta_\gamma(P_\Omega\chi_\Lambda P_\Omega) = \eta_\gamma(P_\Omega\chi_{\Lambda^c}P_\Omega)$. The first identity holds since the non-zero spectra of $\chi_\Lambda P_\Omega\chi_\Lambda$ and $P_\Omega\chi_\Lambda P_\Omega$ coincide. The second one follows from the symmetry of η_γ, that is, from the equality $\eta_\gamma(t) = \eta_\gamma(1-t),\ t\in[0,1]$.

We are interested in the behaviour of the above quantities when Λ is replaced with $\alpha\Lambda$, with a large scaling parameter α. While the case $T = 0$ was investigated in detail in [13], in the current paper we concentrate on the case $T > 0$ and the limit $T \downarrow 0$. The next theorem shows that the entropies (10.3) and (10.4) are both finite, and establishes sharp bounds when α and T both vary within certain limits.

Theorem 10.1. *Let $d \ge 1$. Suppose that $\alpha T \ge \alpha_0$ and $T \in (0, T_0]$ with some $\alpha_0 > 0$, $T_0 > 0$. Then the γ-Rényi entanglement entropy satisfies*

$$|\mathrm{H}_\gamma(T,\mu;\alpha\Lambda)| \le C\alpha^{d-1}\big(|\log(T)| + 1\big). \tag{10.6}$$

If Λ is bounded, then the local γ-Rényi entropy satisfies

$$\big|\mathrm{S}_\gamma(T,\mu;\alpha\Lambda) - \alpha^d s_\gamma(T,\mu)|\Lambda|\big| \le C\alpha^{d-1}(|\log(T)| + 1), \tag{10.7}$$

where

$$s_\gamma(T,\mu) := \frac{1}{(2\pi)^d}\int \eta_\gamma(a_{T,\mu}(\boldsymbol{\xi}))d\boldsymbol{\xi}.$$

The constants in (10.6) and (10.7) are independent of α and T, but may depend on the parameters α_0, T_0, μ, the function h and the region Λ.

The coefficient $s_\gamma(T,\mu)$ is called the *γ-Rényi entropy density* (cf. [14]). It can be expressed in the form:

$$s_\gamma(T,\mu) = \begin{cases} \dfrac{\gamma}{(\gamma-1)T}\big(p(T,\mu) - p(T/\gamma,\mu)\big), & \text{if } \gamma \neq 1, \\ \dfrac{\partial p}{\partial T}(T,\mu), & \text{if } \gamma = 1, \end{cases} \tag{10.8}$$

in terms of the *pressure*

$$p(T,\mu) := \int \frac{\mathcal{N}(E)}{1+e^{(E-\mu)/T}}\, dE,$$

and the *integrated density of states*

$$\mathcal{N}(E) := \frac{1}{(2\pi)^d}\int \chi_{[0,\infty)}(E - h(\boldsymbol{\xi}))\, d\boldsymbol{\xi},\ E\in\mathbb{R},$$

of the free Fermi gas. The relation (10.8) for $\gamma = 1$ is a standard thermodynamic relation, see for instance [1].

For $d = 1$, apart from the bounds, we can also determine the asymptotic behaviour of the local (or thermal) entropy and of the EE.

Theorem 10.2. *Let $d = 1$ and let $I \subset \mathbb{R}$ be given by (4.6). Then the EE satisfies*

$$\mathrm{H}_\gamma(T,\mu;\alpha I) = 2\omega\mathcal{B}(a_{T,\mu},\eta_\gamma) + o(|\log(T)| + 1), \tag{10.9}$$

and if $I_0 = I_{K+1} = \varnothing$, then with s_γ from (10.8) the local entropy satisfies

$$\mathrm{S}_\gamma(T,\mu;\alpha I) = \alpha s_\gamma(T,\mu)|I| + 2K\mathcal{B}(a_{T,\mu};\eta_\gamma) + o(|\log(T)| + 1), \tag{10.10}$$

as $\alpha T \geq \alpha_0$, $\alpha\to\infty$.

A proof of the leading large-scale behaviour of the local entropy $\mathrm{S}_\gamma(T,\mu;\alpha\Lambda)$ at fixed $T > 0$ appeared (among other things) first in [16,2] (for $\gamma = 1$). The sub-leading correction in dimension $d = 1$ in (10.10) is new. The extension to dimension $d \geq 2$ is subject of [26, Section 3].

If in Theorem 10.2 we also assume that $T \downarrow 0$, then the asymptotic formulas take a more explicit form. To state this result recall that due to Condition 8.1, the Fermi sea has the form (8.26) with a finite $N \in \mathbb{N}$.

Corollary 10.3. *Let $d = 1$, I be as in (4.6), and let N be the number of connected components of the Fermi sea, see (8.26). Let $\alpha T \geq \alpha_0$ and $T \downarrow 0$. Then the EE satisfies*

$$\mathrm{H}_\gamma(T, \mu; \alpha I) = \omega N \,\frac{1+\gamma}{6\gamma} |\log(T)| + o(|\log(T)| + 1). \tag{10.11}$$

It is worth pointing out that the coefficient in front of $|\log(T)|$ agrees with the asymptotic coefficient found in [7,13] for the zero temperature case. Indeed, with the notation that we presently use, the theorem in [13] implies that for a bounded I, that is, with $\omega = 2K$, we have (see (10.5))

$$\begin{aligned}\mathrm{H}_\gamma(0, \mu; \alpha I) &= 2\,\mathrm{S}_\gamma(0, \mu; \alpha I)\\ &= \omega N \,\frac{1+\gamma}{6\gamma}\, \log(\alpha) + o(\log(\alpha)),\ \ \alpha \to \infty.\end{aligned}$$

Clearly, the coefficient in this formula is the same as in Corollary 10.3. Therefore, if we identify α inside the logarithm with $1/T$ we recover the above asymptotic expansion in Corollary 10.3.

Proof of Theorem 10.1. It is easy to see that

$$\mathrm{H}_\gamma(T, \mu; \alpha\Lambda) = \operatorname{tr} D_\alpha(a_{T,\mu}, \Lambda; \eta_\gamma) + \operatorname{tr} D_\alpha(a_{T,\mu}, \mathbb{R}^d \setminus \Lambda; \eta_\gamma). \tag{10.12}$$

Now, let $\phi \in \mathsf{C}^\infty(\mathbb{R})$ be such that $0 \leq \phi \leq 1$ and

$$\phi(t) = \begin{cases} 1 & \text{for } t \leq 1/4,\\ 0 & \text{for } t \geq 3/4.\end{cases}$$

If $\gamma \neq 1$, then $\eta_\gamma\phi$ and $\eta_\gamma(1-\phi)$ satisfy Condition 2.1 with $t_0 = 0$ and $t_0 = 1$, respectively, and with $\varkappa = \min\{1, \gamma\}$. The functions $\eta_1\phi$ and $\eta_1(1-\phi)$ satisfy Condition 2.1 with arbitrary $\gamma < 1$. Since the mapping $f \mapsto D_\alpha(a, \Lambda; f)$ is linear, we have

$$D_\alpha(a, \Lambda; \eta_\gamma) = D_\alpha(a, \Lambda; \eta_\gamma\phi) + D_\alpha(a, \Lambda; \eta_\gamma(1-\phi)). \tag{10.13}$$

Applying Theorem 8.3 with $R = 1$ to each term on the right-hand side, we conclude for $\alpha T \geq \alpha_0$ and $0 < T \leq T_0$, that

$$\|D_\alpha(a_{T,\mu}, \Lambda; \eta_\gamma)\|_{\mathfrak{S}_1} + \|D_\alpha(a_{T,\mu}, \mathbb{R}^d \setminus \Lambda; \eta_\gamma)\|_{\mathfrak{S}_1} \leq C\alpha^{d-1}\big(|\log(T)| + 1\big). \tag{10.14}$$

In view of (10.12), this leads to (10.6).

In order to prove (10.7), we rewrite (10.3):

$$\mathrm{S}_\gamma(T,\mu;\Lambda) = \mathrm{tr}[\chi_\Lambda \eta_\gamma(\mathrm{Op}_\alpha(a_{T,\mu}))\chi_\Lambda] + \mathrm{tr}\, D_\alpha(a_{T,\mu},\Lambda;\eta_\gamma). \tag{10.15}$$

For the first trace we have the simple identity

$$\mathrm{tr}[\chi_\Lambda \eta_\gamma(\mathrm{Op}_\alpha(a_{T,\mu}))\chi_\Lambda] = \mathrm{tr}[\chi_\Lambda \mathrm{Op}_\alpha(\eta_\gamma(a_{T,\mu}))\chi_\Lambda] = \alpha^d s_\gamma(T,\mu)\,|\Lambda|.$$

Together with the bound (10.14) for the second trace, this yields (10.7). □

Proof of Theorem 10.2. Applying Theorem 8.6 to each term on the right-hand side of (10.13), and using (4.8), we obtain as $\alpha T \ge \alpha_0$, $\alpha \to \infty$, that

$$\mathrm{tr}\, D_\alpha(a_{T,\mu}, I; \eta_\gamma\phi) = \omega \mathcal{B}(a_{T,\mu};\eta_\gamma\phi) + o(|\log(T)| + 1),$$

and

$$\mathrm{tr}\, D_\alpha(a_{T,\mu}, I^c; \eta_\gamma\phi) = \omega \mathcal{B}(a_{T,\mu};\eta_\gamma\phi) + o(|\log(T)| + 1),$$

where $\omega = \omega(I) = \omega(I^c)$. Similar formulas can be written with the function $(1-\phi)\eta_\gamma$ as well. Thus, remembering the linearity of the map $f \mapsto \mathcal{B}(a;f)$ (see definition (4.3)), we obtain (10.9).

The asymptotics in (10.10) is obtained in the same way using (10.15) and (1.3). □

Proof of Corollary 10.3. The claimed formula immediately follows from (10.9) and the asymptotic relation (9.1) after observing that (cf. [13])

$$U(1,0;\eta_\gamma) = \int_0^1 \frac{\eta_\gamma(t)}{t(1-t)}dt = \pi^2\,\frac{1+\gamma}{6\gamma}. \qquad \square$$

One should note that in the same way one could replace the coefficient $\mathcal{B}(a_{T,\mu},\eta_\gamma)$ by its asymptotics (9.1) in formula (10.10) as well. However, the specific entropy density $s_\gamma(T,\mu)$ in the leading term would also need to be expanded in $T \downarrow 0$, and the precise place of the $\mathcal{B}(\,\cdot\,)$-term in the resulting expansion of S_γ will depend on the relationship between αT and T. We do not go into these details.

Appendix A. The Helffer–Sjöstrand formula

When studying functions of self-adjoint operators we rely on the Helffer–Sjöstrand formula which holds for arbitrary operators $A = A^*$ and arbitrary smooth functions $f \in \mathsf{C}_0^n(\mathbb{R})$, $n \ge 2$ ($z := x + iy$, $\frac{\partial}{\partial \bar z} := \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$):

$$f(A) = \frac{1}{\pi}\iint \frac{\partial}{\partial \bar z}\tilde f(x,y)\,(A - x - iy)^{-1} dx dy, \tag{A.1}$$

where $\tilde{f} = \tilde{f}(x, y)$ is an almost analytic extension of the function f, see [9, Proposition 7.2] and [5, Chapter 2]. An almost analytic extension of $f \in \mathsf{C}^n(\mathbb{R})$ is a $\mathsf{C}^1(\mathbb{R}^2)$-function $\tilde{f}$, such that $f(x) = \tilde{f}(x, 0)$ and $\left|\frac{\partial}{\partial \bar{z}} \tilde{f}(x, y)\right| \le C|y|$. For the sake of brevity we use the representation (A.1) for compactly supported functions only, so that the integral (A.1) is norm-convergent.

Let us describe a convenient almost analytic extension of a function $f \in \mathsf{C}_0^n(\mathbb{R})$. For an arbitrary $r > 0$ introduce the function

$$U_r(x, y) := \begin{cases} 1, & |y| < \langle x \rangle_r, \\ 0, & |y| \ge \langle x \rangle_r, \end{cases} \quad \langle x \rangle_r := \sqrt{x^2 + r^2}.$$

Later, we need a function $\zeta \in \mathsf{C}_0^\infty(\mathbb{R})$ to be a function such that

$$\zeta(t) = 1 \quad \text{for} \quad |t| \le 1/2, \quad \text{and} \quad \zeta(t) = 0 \quad \text{for} \quad |t| \ge 1. \tag{A.2}$$

Lemma A.1. *Let $f \in \mathsf{C}^n(\mathbb{R})$, $n \ge 2$. Then for any $r > 0$ the function f has an almost analytic extension $\tilde{f} = \tilde{f}(\,\cdot\,, \,\cdot\,; r) \in \mathsf{C}^1(\mathbb{R}^2)$ such that $\tilde{f}(x, y; r) = 0$ if $|y| > \langle x \rangle_r$. Moreover, the derivative, $\frac{\partial}{\partial \bar{z}} \tilde{f}(x, y; r)$, satisfies the bound*

$$\left|\frac{\partial}{\partial \bar{z}} \tilde{f}(x, y; r)\right| \le C_n F(x; r) |y|^{n-1} U_r(x, y), \tag{A.3}$$

where

$$F(x; r) := \sum_{l=0}^{n} |f^{(l)}(x)| \langle x \rangle_r^{-n+l}.$$

The constant C_n does not depend on f or the constant r.

The proof of this lemma is a marginal modification of the proof contained in [5, Chapter 2] and is thus omitted.

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