THE BENEFITS OF AN ALTERNATIVE APPROACH TO ANALYTIC NUMBER THEORY

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Abstract. In our talk we will survey the alternative (or "pretentious") approach to analytic number theory, which has the potential to be more flexible and broad reaching than traditional zeta-function methods, highlighting some spectacular recent developments. In this "sampler" we give some indication of the change in perspective that motivates this shift in technique.

1. Riemann's memoir

In a nine-page memoir written in 1859, Riemann outlined an extraordinary plan to attack the elementary question of counting prime numbers using deep ideas from the theory of complex functions. His approach begins with what we now call the *Riemann zeta-function*:

$$\zeta(s) := \sum_{n\geq 1} \frac{1}{n^s}.$$

To make sense of an infinite sum it needs to converge, and preferably be absolutely convergent, implying that we can re-arrange the order of the terms without changing the value. The sum defining $\zeta(s)$ is absolutely convergent when $\mathrm{Re}(s) > 1$.

Applying the *Fundamental Theorem of Arithmetic* to each term in the sum for $\zeta(s)$, we can write

$$\zeta(s) \ = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}. \tag{1}$$

This connection between $\zeta(s)$ and prime numbers was exploited by Riemann: Since the sum defining $\zeta(s)$ is absolutely convergent when $\mathrm{Re}(s) > 1$, it is safe to perform calculus operations on $\zeta(s)$ in this domain. By taking the logarithmic derivative we have

$$-\frac{\zeta'(s)}{\zeta(s)} \ = \ -\frac{d}{ds}\log \zeta(s) \ = \sum_{p \text{ prime}} \frac{d}{ds} \log\left(1 - \frac{1}{p^s}\right) \ = \sum_{\substack{p \text{ prime} \\ m\geq 1}} \frac{\log p}{p^{ms}}.$$

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Notice that the $1/n^s$ term here is non-zero if and only if n is a prime power (that is, $n = p^m$), and so Riemann found a way to identify prime powers (the vast majority of which are primes) via the coefficients of an infinite series.

To count the primes up to x, one can make use of Perron's formula in the form

$$\frac{1}{2i\pi}\int_{\mathrm{Re}(s)=\sigma} \frac{(N/n)^s}{s}\, ds \;=\; \begin{cases} 0 & \text{if } 1 \le n < N; \\ 1/2 & \text{if } n = N; \\ 1 & \text{if } n > N; \end{cases}$$

where $\sigma > 0$, which can be proved via Cauchy's Theorem. Ignoring convergence issues one can then show that for any sequence of complex numbers $(a_n)_{n\ge 1}$ one has

$$\sum_{n<N} a_n + \frac{1}{2}a_N \;=\; \frac{1}{2i\pi}\int_{\mathrm{Re}(s)=\sigma} A(s)\cdot\frac{x^s}{s}\, ds, \text{ where } A(s) := \sum_{n\ge 1}\frac{a_n}{n^s}. \tag{2}$$

In the particular case that $A(s) = -\zeta'(s)/\zeta(s)$, this yields, when x is not itself a prime power,

$$\sum_{\substack{p^m\le x \\ p \text{ prime} \\ m\ge 1}} \log p = \frac{1}{2i\pi}\int_{\mathrm{Re}(s)=\sigma} -\frac{\zeta'(s)}{\zeta(s)}\cdot\frac{x^s}{s}\, ds$$

for $\sigma > 1$ (so as to ensure absolute convergence throughout the argument).

This formula morphs a perfectly understandable question like estimating the number of primes up to x, involving a sum that is easily interpreted, to a rather complicated integral, over an infinitely long line in the complex plane, of a function that is delicate to work with in that it is only well defined when $\mathrm{Re}(s) > 1$. It is by no means obvious how to proceed from here, but Riemann was up to the task.

2. How do we evaluate this integral?

The idea is to pull the integral to the left and to use Cauchy's formula, so that its value is given by a sum of residues. This is not an easy task. First, $\zeta(s)$ is not defined on or to the left of the line $\mathrm{Re}(s) = 1$, so we have to extrapolate $\zeta(s)$ in a meaningful way to the whole complex plane, which is achieved by *analytic continuation*. Then we need to ensure that the integral on the newly chosen contours is actually vanishingly small. And then we need to determine the poles of the integrand, the difficult bit being the function $\zeta'(s)/\zeta(s)$. Indeed its poles are evidently given by the poles and zeros of $\zeta(s)$, and this leads on to the mysterious *Riemann Hypothesis*.

Classical analytic number theory (including important aspects of the theory of elliptic curves and of the Langlands program) takes this type of approach for granted; indeed to proceed one has to assume that the relevant series, $A(s)$, must be meromorphically continuable to the whole complex plane (and one expects that each such $A(s)$ satisfies a whole host of other remarkable properties). Here the coefficients, the a_n, arise naturally in arithmetic problems. However there is a big problem with this – it is not easy to prove in general that $A(s)$ is meromorphically continuable (the great works of Wiles on

Fermat's Last Theorem, and of Taylor on the Sato-Tate conjecture, end up proving, in some of the deepest advances of modern mathematics, just such results) and one can cook up interesting arithmetic examples in which this is not even true. This places significant limitations on which questions one can directly attack with classical analytic techniques.

3. An alternative approach

In typical arithmetic problems the coefficients a_n are not too large, and so we may assume that the series for $A(s)$ is absolutely convergent for s with $\mathrm{Re}(s) > 1$.

The core problem is to develop methods to

Estimate $\sum_{n\leq N} a_n$ via the formula (2),
without assuming that $A(s)$ can be analytically continued.

Indeed, we do not even want to assume that $A(s)$ can be analytically continued on to the line $\mathrm{Re}(s) = 1$. Evidently this must require a careful study of $A(s)$ on the line $\mathrm{Re}(s) = \sigma$ (or a slight variation since one can manipulate contours freely in the domain of absolute convergence). First we need to understand when $|A(s)|$ is large, which can be shown to happen rarely. But if $A(s)$ is large at $s = s_0$ then by continuity it is large nearby, so we need to understand how it varies in a short interval near to s_0. This is the start of the *alternative* (or *pretentious*) approach to analytic number theory. These sort of questions have their roots in classical analytic developments, but have recently been taking on a life of their own.

Let's study this question with an example. Fix $\delta > 0$ very small. When is $|\zeta(1+\delta+it)|$ large? We will examine the value using the Euler product, (1). Since this is convergent the tail should not be too relevant; indeed if $\delta = \frac{1}{\log x}$ then one can truncate the Euler product to the primes $\leq x$, and deduce that

$$\zeta(s) \approx \prod_{\substack{p \text{ prime} \\ p\leq x}} \left(1 - \frac{1}{p^{1+it}}\right)^{-1}.$$

The pth term in this Euler product looks like $(1 - \frac{X(p)}{p})^{-1}$, where $X(p) = p^{-it}$ is some complex number of absolute value 1. This might be, in absolute value, anything between $(1 + \frac{1}{p})^{-1}$ and $(1 - \frac{1}{p})^{-1}$, and so the product is minimized when all of the p^{-it} are close to -1. Therefore $|\zeta(1 + \delta + it)|$ is "large" if and only if most of the p^{it}, with $p \leq x$, are roughly -1. Formulating this usefully takes some work, but allows precise control over the size of $|\zeta(1 + \delta + it)|$, and similar techniques work with $|A(1 + \delta + it)|$ provided the a_n are multiplicative (that is, $a_{mn} = a_m a_n$).

This simple idea suddenly allows us to work with all sorts of L-functions that were not easily accessible to classical techniques and has been a central focus of research for the author and Soundararajan, as well as many collaborators, colleagues, postdocs, and students. We first identified such an approach in improving the long dormant upper bounds for character sums. Recognizing the potential for such methods, our next goal became to show that many of the main results of classical analytic number theory could be reproven without recourse to analytic continuation, a project that is highlighted in recent joint work with Harper and Soundararajan. Moreover estimates of equal strength to the classical results could be proven, most notably in 2013 by Koukoulopoulos, reproving the best bounds known on the error term in the prime number theorem, without recourse to zeta-function zeros. We went on, more recently, to understanding with de la Bréteche precisely when exponential sums can be large, which has relevance to, amongst other things, the circle method. In 2010, Soundararajan showed how such methods could be used to provide "weak subconvexity estimates" for a wide variety of L-functions leading to the completion (with Holowinsky) of the proof of the holomorphic quantum unique ergodicity conjecture (a case in which Lindenstrauss' methods do not work). Much of this will feature in the forthcoming monograph by the speaker and Soundararajan.

These ideas are now beginning to permeate a broad spectrum of questions in analytic number theory. Related recent spectacular advances include Terry Tao's resolution of Erdős's discrepancy conjecture, and Matomäki and Radiziwill's sensational recent work on multiplicative functions in very short intervals.

In this talk we will survey these and other of the more recent developments, including intriguing, developing links with the groundbreaking work of Green, Tao, and Ziegler on prime patterns.