ON THE VALUE DISTRIBUTION OF TWO DIRICHLET L-FUNCTIONS

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Let ρ denote the non-trivial zeros of the Riemann zeta function. We study the relative value distribution of $L(\rho+\sigma,\chi_1)$ and $L(\rho+\sigma,\chi_2)$, where $\sigma\in[0,1/2)$ is fixed and χ_1, χ_2 are two fixed Dirichlet characters to distinct prime moduli. For $\sigma>0$ we prove that a positive proportion of these pairs of values are linearly independent over $\mathbb{R}$, which implies that the arguments of the values are different. For $\sigma=0$ we show that, up to height T, the values are different for cT of the Riemann zeros for some positive constant c.

1. Introduction

The value distribution of $\zeta(s)$ and $L(s,\chi)$ is a classical problem that has recently attracted attention in for example [6], [10], [22]. In this paper we prove two results relating to the relative distribution of arguments and values of two distinct Dirichlet L-functions. In Theorem 1.1 we compare the arguments of two Dirichlet L-functions at horizontal shifts of the Riemann zeros, $\sigma+i\gamma$, where $\sigma\in(1/2,1)$ is fixed and γ runs over heights of the non-trivial zeros of ζ. This is done by computing discrete averages of the second and fourth moments of Dirichlet L-functions. In Theorem 1.2 we move to the actual zeros of ζ. In this case we are only able to obtain results for the first and second moments. This allows us to compare the *values* of two Dirichlet L-functions.

The most typical focus for investigation for ζ and $L(s,\chi)$ is the distribution of their zeros. Let $N(T)$ denote the number of zeros of $\zeta(s)$ in the region $\operatorname{Re}s\in(0,1)$ and $\operatorname{Im}s\in(0,T)$. The Riemann–von Mangoldt formula states that

$$N(T)=\frac{T}{2\pi}\log T-\frac{T}{2\pi}+O(\log T).$$

We expect that all zeros of Dirichlet L-functions to primitive characters are simple, and that two L-functions with distinct primitive characters do not share any non-trivial zeros at all. This comes from the Grand Simplicity Hypothesis (GSH), which states that the set

$$\{\gamma\geq 0\mid L(\tfrac{1}{2}+i\gamma,\chi)=0\text{ and }\chi\text{ is primitive}\}$$

is linearly independent over $\mathbb{Q}$, see [21]. In 1976 Fujii [4] showed that a positive proportion of zeros of $L(s,\psi)L(s,\chi)$ are distinct, where the characters are primitive and distinct, but not necessarily of distinct moduli. A zero of the product is said to be distinct if it is a zero of only one of the two, or if it is a zero of both then it occurs with different multiplicities for each function. R. Murty and K. Murty [19] proved that two functions of the Selberg class $\mathcal{S}$ cannot share too many zeros (counted with multiplicity). They show that if F, $G\in\mathcal{S}$ then $F=G$ provided that

$$|Z_F(T)\Delta Z_G(T)|=o(T),$$

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where $Z_F(T)$ denotes the set of zeros of $F(s)$ in the region $\operatorname{Re} s \geq 1/2$ and $|\operatorname{Im} s| \leq T$, and Δ is the symmetric difference. In 1986 Conrey et al. [1] proved that the Dedekind zeta function of a quadratic number field has infinitely many simple zeros unconditionally. However, they had to assume the RH in order to extend this to a positive proportion [2].

Apart from looking at the zeros, it is natural to consider other values of ζ and $L(s,\chi)$, that is, the distribution of s such that $\zeta(s) = a$ (or $L(s,\chi) = a$) for some fixed $a \in \mathbb{C}$. In [8] Garunkštis and Steuding proved a discrete average for ζ' over the a-values of ζ, which implies that there are infinitely many simple a-points in the critical strip. On the critical line, however, we do not even know whether there are infinitely many a-points. For further results on the distribution of simple a-points see [11]. On the other hand, we can also look at points where $\zeta(s)$ (or $L(s,\chi)$) has a specific fixed argument. Fix $\varphi \in [0,\pi)$ such that $\varphi \neq \pi/2$. In [14] the authors proved that ζ takes arbitrarily large values with argument φ, that is

$$\max_{\substack{0<t\leq T \\ \operatorname{Arg}(\zeta(1/2+it))=\varphi}} |\zeta(\tfrac{1}{2}+it)| \gg \exp\left(\left(\frac{1}{2}+o(1)\right)\sqrt{\frac{\log T}{\log\log T}}\right).$$

This has recently been improved to angles modulo 2π by Hough [13].

In this paper we will prove the following two theorems.

Theorem 1.1. *Assume the Riemann Hypothesis, i.e. $\beta = \frac{1}{2}$. Let χ_1, χ_2 be two primitive Dirichlet characters modulo distinct primes q and ℓ, respectively. Let $\sigma \in (\frac{1}{2}, 1)$, then, for a positive proportion of the non-trivial zeros of $\zeta(s)$ with $\gamma > 0$, the values of the Dirichlet L-functions $L(\sigma + i\gamma, \chi_1)$ and $L(\sigma + i\gamma, \chi_2)$ are linearly independent over $\mathbb{R}$.*

Remark 1. If the values $L(\sigma + i\gamma, \chi_1)$ and $L(\sigma + i\gamma, \chi_2)$ are linearly independent over $\mathbb{R}$, then in particular their arguments are different.

Theorem 1.2. *Two Dirichlet L-functions with primitive characters modulo distinct primes, attain different values at cT non-trivial zeros of $\zeta(s)$ up to height T, for some positive constant c.*

Remark 2. In Theorem 1.2 we fail to obtain positive proportion and we expect this to be a limitation of the method used. In [5] the authors looked at the mean square of a single Dirichlet L-function at the zeros of another, and showed that it is non-zero for at least cT of the zeros for some explicit $c > 0$. On the other hand, since we are working with two distinct L-functions, and more precisely their *difference*, it seems difficult to introduce a mollifier to improve on the result. Martin and Ng [18] evaluated the mollified first and second moments of $L(s,\chi)$ in arithmetic progressions on the critical line and proved that at least $T(\log T)^{-1}$ of the values are nonzero, which misses the positive proportion (of arithmetic progressions) by a logarithm. This was extended to positive proportion by Li and Radziwiłł [17]. However, their method relies on the strong rigidity of the arithmetic progression and fails when the sequence is slightly perturbed.

Remark 3. We assume that the conductors of χ_1 and χ_2 are primes in order to make the notation simpler. It should be possible to generalise our results to the case when the conductors are coprime or have distinct prime factors.

We will prove Theorem 1.1 in section 2 and Theorem 1.2 in section 3. The main ingredient in the proofs is the Gonek–Landau formula, and results derived from it. In 1912 Landau [15] proved that, for a fixed $x > 1$,

$$\sum_{0<\gamma\leq T} x^{\rho} = -\frac{T}{2\pi}\Lambda(x) + O(\log T), \tag{1.1}$$

as $T \longrightarrow \infty$, where $\Lambda(x)$ is the von Mangoldt function extended to $\mathbb{R}$ by letting $\Lambda(x) = \log p$ if $x = p^k$ for some prime p and integer $k \geq 1$, and $\Lambda(x) = 0$ otherwise. The sum in (1.1) runs over the positive imaginary parts of the Riemann zeros. What is striking in (1.1) is that the right-hand side grows by a factor of T only if x is a prime power. This version of Landau's formula is of limited practical use since the estimate is not uniform in x. Gonek [9] proved a version of Landau's formula which is uniform in both x and T with only small sacrifices to the error term:

Lemma 1.1 (Gonek–Landau Formula). *Let x, $T > 1$. Then*

$$\begin{aligned}\sum_{0<\gamma\leq T} x^{\rho} = -\frac{T}{2\pi}\Lambda(x) &+ O\left(x \log 2xT\, \log\log 3x\right) \\ &+ O\left(\log x\, \min\left(T, \frac{x}{\langle x\rangle}\right)\right) + O\left(\log 2T\, \min\left(T, \frac{1}{\log x}\right)\right),\end{aligned} \tag{1.2}$$

where $\langle x\rangle$ denotes the distance from x to the nearest prime power other than x itself.

If one fixes x then this reduces to the original result of Landau as $T \longrightarrow \infty$. As an application of this result Gonek proves (under the RH) the following mean value for ζ:

$$\sum_{0<\gamma\leq T} |\zeta(\tfrac{1}{2} + i(\gamma + 2\pi\alpha/\log T))|^2 = \left(1 - \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2\right)\frac{T}{2\pi}\log^2 T + O(T\log^{7/4} T),$$

where T is large and α is real with $|\alpha| \leq \frac{1}{2\pi}\log T$.

In order to work on the critical line (or exactly at the Riemann zeros) we need different tools. We replace the classical Gonek–Landau formula with an integrated version, see [6].

Lemma 1.2 (Modified Gonek Lemma). *Suppose that $\sum_{n=1}^{\infty} a(n)n^{-s}$ converges for $\sigma > 1$ and $a(n) = O(n^{\epsilon})$. Let $a = 1 + \log^{-1} T$. Then*

$$\begin{aligned}\frac{1}{2\pi i}\int_{a+i}^{a+iT} \left(\frac{m}{2\pi}\right)^{s} \Gamma(s)\exp\left(\delta\frac{\pi i s}{2}\right)\sum_{n=1}^{\infty}\frac{a(n)}{n^s}\,ds \\ = \begin{cases} \displaystyle\sum_{n\leq \frac{Tm}{2\pi}} a(n)\exp\left(-2\pi i\frac{n}{m}\right) + O(m^a T^{1/2+\epsilon}), & \text{if } \delta = -1, \\ O(m^a), & \text{if } \delta = +1. \end{cases}\end{aligned}$$

We also need the following version of the approximate functional equation for Dirichlet L-functions. First, denote by $G(k,\chi)$ the Gauss sum

$$G(k,\chi) = \sum_{a=1}^{q}\chi(a)e^{2\pi iak/q}.$$

We also write $G(1,\chi) = G(\chi)$.

Theorem 1.3 (Lavrik [16]). *Let χ be a primitive character mod q. For $s = \sigma + it$ with $0 < \sigma < 1$, $t > 0$, and $x = \Delta\sqrt{\frac{qt}{2\pi}}$, $y = \Delta^{-1}\sqrt{\frac{qt}{2\pi}}$, and $\Delta \geq 1$, $\Delta \in \mathbb{N}$, we have*

$$L(s,\chi) = \sum_{n\leq x}\frac{\chi(n)}{n^s} + \varepsilon(\chi)\left(\frac{q}{\pi}\right)^{\frac{1}{2}-s}\frac{\Gamma\left(\frac{1-s+\mathfrak{a}}{2}\right)}{\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)}\sum_{n\leq y}\frac{\overline{\chi}(n)}{n^{1-s}} + R_{xy}, \tag{1.3}$$

with

$$R_{xy} \ll \sqrt{q}\left(y^{-\sigma} + x^{\sigma-1}(qt)^{1/2-\sigma}\right)\log 2t,$$

and in particular, for $x = y$,

$$R \ll x^{-\sigma}\sqrt{q}\log 2t.$$

Here $\varepsilon(\chi) = q^{-1/2} i^{\mathfrak{a}} G(1,\chi)$, *and*

$$\mathfrak{a} = \frac{1-\chi(-1)}{2}.$$

It is not hard to see that this formula is, in fact, valid for all real $\Delta \geq 0$. Approximate functional equations for imprimitive characters do exist, but they are more complicated. Therefore, we restrict our attention to primitive characters in Theorem 1.1.

2. Proof of Theorem 1.1

The proof will follow the steps of [9, Theorem 2] and [20, Theorem 1.9] for the Riemann zeta function and GL_2 L-functions, respecively.

Two non-zero complex numbers z and w are linearly independent over the reals is equivalent to the quotient z/w being non-real, or that $|z\bar{w} - \bar{z}w| > 0$. For us z and w are values of Dirichlet L-functions. Instead of looking at these functions at a single point, we will average over multiple points with a fixed real part $\sigma \in (\frac{1}{2}, 1)$ and the imaginary part at the height of the Riemann zeros.

We are assuming the RH purely because it makes the proof simpler as expressions of the form x^{ρ} become easier to deal with if we know the real part explicitly. On the other hand, the distribution of these specific points does not seem to have any impact on the rest of the proof. We suspect that the RH is not an essential requirement. In fact, following [10], it might be possible to obtain the result without the RH by integrating

$$\frac{\zeta'}{\zeta}(s-\sigma)B(s,P)L(s,\chi_1)\overline{L(s,\chi_2)}$$

over a suitable contour. This picks the desired points as residues of the integrand yielding the required sum. This idea is also used in the proof of Theorem 1.2.

The proof will be divided into three propositions after which the main result follows easily. In the first proposition we want to calculate discrete mean values of sums of terms of the type $L(\sigma+i\gamma,\chi_1)\overline{L(\sigma+i\gamma,\chi_2)}$ and its complex conjugate. If we subtract one of these mean values from the other then each term is non-zero precisely when the two numbers are linearly independent over the reals. Hence we need to prove that the two mean values are not equal, which is the content of Proposition 2.3. Finally, we get the main result by applying the Cauchy–Schwarz inequality to the difference of the mean values. Because of this we also need to estimate a sum of squares of the absolute values of the above quantities, that is,

$$|L(\sigma+i\gamma,\chi_1)\overline{L(\sigma+i\gamma,\chi_2)} - \overline{L(\sigma+i\gamma,\chi_1)}L(\sigma+i\gamma,\chi_2)|^2.$$

This is done in Proposition 2.2.

The first problem in our proof is that the mean values are complex conjugates. In order to show that the difference is non-zero leads to determining whether $\operatorname{Im} L(2\sigma,\chi_1\overline{\chi}_2) \neq 0$, which does not always hold. Thus we need to introduce some kind of weighting in order to shove these sums off balance. We do this by multiplying by a finite Dirichlet polynomial, $B(s,P)$, which cancels some terms from either of the L-functions, depending on which mean value we are considering. We define

$$B(s,P)=\prod_{p\le P}(1-\chi_1(p)p^{-s})(1-\chi_2(p)p^{-s}) \tag{2.1}$$

for some fixed prime P, depending only on q and ℓ. In Proposition 2.3 we prove that it is sufficient to pick any prime with $P>\max(q,\ell)$. Let us also assume that this Dirichlet polynomial has the expansion

$$B(s,P)=\sum_{n\le R}c_n n^{-s},$$

for some R depending on P. Since $|c_p|\le 2$ for any prime p, we have for all n that

$$|c_n|\le 2^P. \tag{2.2}$$

We prove the following propositions.

Proposition 2.1. *Assume the Riemann Hypothesis. With $s=\sigma+i\gamma$ we have*

$$\sum_{0<\gamma\le T}B(s,P)L(s,\chi_1)\overline{L(s,\chi_2)}\sim N(T)\sum_{n=1}^{\infty}\frac{d_n\overline{\chi}_2(n)}{n^{2\sigma}}, \tag{2.3}$$

and

$$\sum_{0<\gamma\le T}B(s,P)\overline{L(s,\chi_1)}L(s,\chi_2)\sim N(T)\sum_{n=1}^{\infty}\frac{e_n\overline{\chi}_1(n)}{n^{2\sigma}}, \tag{2.4}$$

where

$$B(s,P)L(s,\chi_1)=\sum_{n=1}^{\infty}\frac{d_n}{n^s},\quad B(s,P)L(s,\chi_2)=\sum_{n=1}^{\infty}\frac{e_n}{n^s}. \tag{2.5}$$

Proposition 2.2. *Suppose $s=\sigma+i\gamma$ and let*

$$A(\gamma)=B(s,P)\big(L(s,\chi_1)\overline{L(s,\chi_2)}-\overline{L(s,\chi_1)}L(s,\chi_2)\big).$$

Then, under the Riemann Hypothesis,

$$\sum_{0<\gamma\le T}|A(\gamma)|^2\ll N(T). \tag{2.6}$$

Proposition 2.3. *Under the Riemann Hypothesis we can find a prime P such that*

$$\sum_{0<\gamma\le T}A(\gamma)\sim C\cdot N(T) \tag{2.7}$$

for some non-zero constant C.

Remark 4. Recently, Garunkštis and Laurinčikas [7] considered mean values similar to ours for ζ at horizontal shifts of its zeros. They use the fourth moment estimate to obtain results about the discrete universality of the Riemann zeta function.

Proof of Theorem 1.1. By the Cauchy–Schwarz inequality and Propositions 2.2 and 2.3

$$\sum_{\substack{0<\gamma\le T\\ A(\gamma)\ne 0}}1\ge\frac{\left|\sum_{0<\gamma\le T}A(\gamma)\right|^2}{\sum_{0<\gamma\le T}|A(\gamma)|^2}\gg\frac{|C|^2N(T)^2}{N(T)}=|C|^2N(T). \tag{2.8}$$

This proves that a positive proportion of the $A(\gamma)$'s are non-zero; in particular, for the same γ's, $L(s,\chi_1)$ and $L(s,\chi_2)$ are linearly independent over the reals. $\square$

2.1. **Proof of Proposition 2.1.** It follows directly from the definition (2.5) that the coefficients d_n define a multiplicative arithmetic function and that $d_n = O(1)$ (see proof of Proposition 2.3 for the explicit formula). We define, for a fixed t,

$$B(s,P) \sum_{n \le \sqrt{\frac{q\ell t}{2\pi}}} \chi_1(n) n^{-s} = \sum_{n \le R\sqrt{\frac{q\ell t}{2\pi}}} d'_n n^{-s}.$$

We have

$$d_n = \sum_{n=km} c_k \chi_1(m), \tag{2.9}$$

and hence for $n \le R\sqrt{\frac{q\ell t}{2\pi}}$

$$d'_n = \sum_{\substack{n=km\\ k\le R}} c_k \chi_1(m). \tag{2.10}$$

From this it follows that $d_n = d'_n$ for $n \le \sqrt{\frac{q\ell t}{2\pi}}$. We also need to show that $d'_n \ll 1$. Let $p_1, \dots, p_h$, for some $h > 1$, denote all the primes below P in an increasing order. Define $\widetilde{P} = p_1 \cdots p_h P$. From the product representation of $B(s,P)$, equation (2.1), we see that $c_n = 0$ for $n > 1$, if n contains any prime factors greater than P. Thus, write $n = p_1^{\alpha_1} \cdots p_h^{\alpha_h} P^{\alpha_0} \nu = n'\nu$ for some $\alpha_i \ge 0$. Then

$$d'_n = \sum_{\substack{k\mid n'\\ k<R}} c_k \chi_1\left(\frac{n'}{k}\nu\right) = \chi_1(\nu) d'_{n'}.$$

Thus it suffices to consider n with prime factors only up to P. Since $B(s,P)$ has a finite Euler product of degree two we have $c_{p^j} = 0$ for any prime p and $j \ge 3$. So we can suppose that $n = p_1^{\alpha_1} \cdots p_h^{\alpha_h} P^{\alpha_0}$, where $0 \le \alpha_i \le 2$ for all $i \le h$. The number of summands in (2.10) is then at most 3^{h+1}. By (2.2), we find that $|d'_n| \le 2^P 3^{h+1}$. In particular, $d'_n \ll 1$ as required.

The approximate functional equation (1.3) for χ_1 with $\Delta = \sqrt{\ell}$ gives

$$L(s,\chi_1) = \sum_{n \le \sqrt{\frac{q\ell t}{2\pi}}} \chi_1(n) n^{-s} + X(s,\chi_1) \sum_{n \le \sqrt{\frac{qt}{2\pi\ell}}} \overline{\chi}_1(n) n^{s-1} + O(t^{-\sigma/2}\log t + t^{-1/4}),$$

where

$$X(s,\chi) = \varepsilon(\chi)\left(\frac{q}{\pi}\right)^{1/2-s} \frac{\Gamma\left(\frac{1-s+\mathfrak{a}}{2}\right)}{\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)}.$$

Similarly for χ_2 with $\Delta = \sqrt{q}R$ we get

$$L(s,\chi_2) = \sum_{n \le R\sqrt{\frac{q\ell t}{2\pi}}} \chi_2(n) n^{-s} + X(s,\chi_2) \sum_{n \le \frac{1}{R}\sqrt{\frac{\ell t}{2\pi q}}} \overline{\chi}_2(n) n^{s-1} + O(t^{-\sigma/2}\log t + t^{-1/4}).$$

We can now expand the left-hand side in (2.3) to

$$\begin{aligned} &\sum_{0<\gamma\le T} B(s,P) \\ &\quad\times \left(\sum_{n \le \sqrt{\frac{q\ell\gamma}{2\pi}}} \chi_1(n) n^{-s} + X(s,\chi_1) \sum_{n \le \sqrt{\frac{q\gamma}{2\pi\ell}}} \overline{\chi}_1(n) n^{s-1} + O(\gamma^{-\sigma/2}\log\gamma + \gamma^{-1/4}) \right) \\ &\quad\times \left(\sum_{n \le R\sqrt{\frac{q\ell\gamma}{2\pi}}} \overline{\chi}_2(n) n^{-\overline{s}} + \overline{X(s,\chi_2)} \sum_{n \le \frac{1}{R}\sqrt{\frac{\ell\gamma}{2\pi q}}} \chi_2(n) n^{\overline{s}-1} + O(\gamma^{-\sigma/2}\log\gamma + \gamma^{-1/4}) \right). \end{aligned} \tag{2.11}$$

Denote the sum with χ_1 and $\overline{\chi}_2$ by $M(T)$. We will take care of the other sums at the end of the proof. The main term comes from the diagonal entries of $M(T)$. First, write

$$\begin{aligned} M(T) &= \sum_{0<\gamma\le T} B(s,P) \sum_{n\le\sqrt{\frac{q\ell\gamma}{2\pi}}} \chi_1(n)n^{-s} \sum_{m\le R\sqrt{\frac{q\ell\gamma}{2\pi}}} \overline{\chi}_2(m)m^{-\overline{s}} \\ &= \sum_{0<\gamma\le T} \sum_{n\le R\sqrt{\frac{q\ell\gamma}{2\pi}}} d'_n n^{-\sigma-i\gamma} \sum_{m\le R\sqrt{\frac{q\ell\gamma}{2\pi}}} \overline{\chi}_2(m)m^{-\sigma+i\gamma}. \end{aligned}$$

Then, we separate the diagonal terms

$$\begin{aligned} M(T) &= \sum_{0<\gamma\le T}\Bigg(\sum_{n\le R\sqrt{\frac{q\ell\gamma}{2\pi}}} \frac{d'_n\overline{\chi}_2(n)}{n^{2\sigma}} + \sum_{n\ne m}^{R\sqrt{\frac{q\ell\gamma}{2\pi}}} \frac{d'_m\overline{\chi}_2(n)}{(nm)^{\sigma}} \left(\frac{n}{m}\right)^{i\gamma}\Bigg) \\ &= Z_1 + Z_2. \end{aligned} \tag{2.12}$$

The asymptotics in (2.3) come from Z_1. We have

$$\begin{aligned} Z_1 &= \sum_{0<\gamma\le T}\Bigg(\sum_{n=1}^{\infty} \frac{d_n\overline{\chi}_2(n)}{n^{2\sigma}} - \sum_{n> R\sqrt{\frac{q\ell\gamma}{2\pi}}} \frac{d_n\overline{\chi}_2(n)}{n^{2\sigma}} + \sum_{n\le R\sqrt{\frac{q\ell\gamma}{2\pi}}} \frac{(d'_n-d_n)\overline{\chi}_2(n)}{n^{2\sigma}}\Bigg) \\ &= N(T)\sum_{n=1}^{\infty} \frac{d_n\overline{\chi}_2(n)}{n^{2\sigma}} + C_1 + C_2. \end{aligned}$$

We need to estimate C_1 and C_2. For C_1 we have

$$C_1 \ll \sum_{0<\gamma\le T}\sum_{n>\sqrt{\gamma}} n^{-2\sigma} \ll \sum_{0<\gamma\le T} \gamma^{1/2-\sigma} = o(N(T)).$$

Similarly,

$$C_2 \ll \sum_{0<\gamma\le T}\sum_{n>\sqrt{\frac{q\ell\gamma}{2\pi}}} n^{-2\sigma} = o(N(T)).$$

To estimate Z_2 we wish to exchange the order of summation and apply the Gonek–Landau formula (1.2). Splitting and rewriting Z_2 in terms of the zeros of ζ we get

$$\begin{aligned} Z_2 &= \sum_{0<\gamma\le T}\sum_{n\le R\sqrt{\frac{q\ell\gamma}{2\pi}}}\sum_{m<n}\left(\frac{d'_m\overline{\chi}_2(n)}{n^{\sigma+1/2}m^{\sigma-1/2}}\left(\frac{n}{m}\right)^{1/2+i\gamma} + \frac{d'_n\overline{\chi}_2(m)}{n^{\sigma+1/2}m^{\sigma-1/2}}\overline{\left(\frac{n}{m}\right)^{1/2+i\gamma}}\right) \\ &= \sum_{n\le R\sqrt{\frac{q\ell T}{2\pi}}}\sum_{m<n}\sum_{\frac{2\pi n^2}{q\ell R^2}\le\gamma\le T}\left(\frac{d'_m\overline{\chi_2}(n)}{n^{\sigma+1/2}m^{\sigma-1/2}}\left(\frac{n}{m}\right)^{\rho} + \frac{d'_n\overline{\chi}_2(m)}{n^{\sigma+1/2}m^{\sigma-1/2}}\overline{\left(\frac{n}{m}\right)^{\rho}}\right). \end{aligned}$$

To apply the Gonek–Landau formula we split the innermost sum to $0<\gamma\le T$ and $0<\gamma\le 2\pi n^2/q\ell R^2$. Hence, we can write

$$Z_2 = Z_{21} + Z_{22} + Z_{23} + Z_{24} + Z_{25},$$

with

$$\begin{aligned} Z_{21} &= -\frac{T}{2\pi}\sum_{n\le R\sqrt{\frac{q\ell T}{2\pi}}}\sum_{m<n} \frac{d'_m\overline{\chi}_2(n)+d'_n\overline{\chi}_2(m)}{n^{\sigma+1/2}m^{\sigma-1/2}}\Lambda\left(\frac{n}{m}\right), \\ Z_{22} &\ll \sum_{n\le R\sqrt{\frac{q\ell T}{2\pi}}}\sum_{m<n} \frac{n^2\Lambda(n/m)}{n^{\sigma+1/2}m^{\sigma-1/2}}, \end{aligned}$$

$$Z_{23} \ll \sum_{n \le R\sqrt{\frac{q\ell T}{2\pi}}} \sum_{m<n} \frac{1}{n^{\sigma+1/2}m^{\sigma-1/2}} \frac{n}{m} \log \frac{2nT}{m} \ \log\log \frac{3n}{m},$$

$$Z_{24} \ll \sum_{n \le R\sqrt{\frac{q\ell T}{2\pi}}} \sum_{m<n} \frac{1}{n^{\sigma+1/2}m^{\sigma-1/2}} \log \frac{n}{m} \ \min\left(T, \frac{n/m}{\langle n/m \rangle}\right),$$

and

$$Z_{25} \ll \log T \sum_{n \le R\sqrt{\frac{q\ell T}{2\pi}}} \sum_{m<n} \frac{1}{n^{\sigma+1/2}m^{\sigma-1/2}} \min\left(T, \frac{1}{\log(n/m)}\right).$$

We begin by estimating Z_{21}. The only non-vanishing terms are with $m|n$. Thus we write $n = km$ and obtain

$$Z_{21} \ll \frac{T}{2\pi} \sum_{k \le R\sqrt{\frac{q\ell T}{2\pi}}} \sum_{m < \frac{R}{k}\sqrt{\frac{q\ell T}{2\pi}}} \frac{\Lambda(k)}{k^{\sigma+1/2}m^{2\sigma}} \ll \frac{T}{2\pi} \sum_{k \le R\sqrt{\frac{q\ell T}{2\pi}}} k^{\epsilon-\sigma-1/2} \sum_{m < \frac{R}{k}\sqrt{\frac{q\ell T}{2\pi}}} m^{-2\sigma},$$

since $\Lambda(k) \ll k^{\epsilon}$ for any $\epsilon > 0$. Since both sums are partial sums of convergent series we get $Z_{21} = O(T)$. Working similarly with Z_{22} gives

$$\begin{aligned} Z_{22} &\ll \sum_{k \le R\sqrt{\frac{q\ell T}{2\pi}}} \sum_{m < \frac{R}{k}\sqrt{\frac{q\ell T}{2\pi}}} \frac{\Lambda(k)}{k^{\sigma-3/2}m^{2\sigma-2}} \ll \sum_{k \le R\sqrt{\frac{q\ell T}{2\pi}}} k^{3/2-\sigma+\epsilon} \sum_{m < \frac{R}{k}\sqrt{\frac{q\ell T}{2\pi}}} m^{2-2\sigma} \\ &\ll \sum_{k \le R\sqrt{\frac{q\ell T}{2\pi}}} k^{3/2-\sigma+\epsilon} \left(\left(\frac{T^{1/2}}{k} \right)^{3-2\sigma} + 1 \right) \ll T^{\frac{3-2\sigma}{2}} \sum_{k \ll T^{1/2}} k^{\sigma-3/2+\epsilon} = O(T). \end{aligned}$$

For Z_{23} we get

$$\begin{aligned} Z_{23} &\ll \log T \ \log\log T \sum_{n \le R\sqrt{\frac{q\ell T}{2\pi}}} \frac{1}{n^{\sigma-1/2}} \sum_{m<n} \frac{1}{m^{\sigma+1/2}} \\ &\ll \log T \ \log\log T \sum_{n \le R\sqrt{\frac{q\ell T}{2\pi}}} \frac{1}{n^{\sigma-1/2}} = o(N(T)). \end{aligned}$$

In order to estimate Z_{24} we write $n = um + r$, where $-m/2 < r \le m/2$. Hence

$$\left\langle u + \frac{r}{m} \right\rangle = \begin{cases} \frac{|r|}{m}, & \text{if } u \text{ is a prime power and } r \ne 0, \\ \ge \frac{1}{2}, & \text{otherwise.} \end{cases} \tag{2.13}$$

Let $c = R\sqrt{q\ell/2\pi}$ then $n/m \le c\sqrt{T}$, and so

$$\begin{aligned} Z_{24} &\ll \log T \sum_{n \le cT^{1/2}} \sum_{m<n} \frac{1}{n^{\sigma+1/2}m^{\sigma-1/2}} \frac{n}{m} \frac{1}{\langle n/m \rangle} \\ &\ll \log T \sum_{m \le cT^{1/2}} \sum_{u \le \lfloor cT^{1/2}/m \rfloor + 1} \sum_{-\frac{m}{2} < r \le \frac{m}{2}} \frac{1}{m^{\sigma+1/2}(um+r)^{\sigma-1/2}} \frac{1}{\langle u + \frac{r}{m} \rangle}, \end{aligned}$$

and then evaluate the sum over r depending on whether u is a prime power or not to get

$$\begin{aligned} &\ll \log T \sum_{m \le cT^{1/2}} \sum_{u \le \lfloor cT^{1/2}/m \rfloor + 1} \left(\Lambda(u) \frac{m \log m}{m^{\sigma+1/2}(um)^{\sigma-1/2}} + \frac{m}{m^{\sigma+1/2}(um)^{\sigma-1/2}} \right) \\ &\ll \log T \sum_{m \le cT^{1/2}} \frac{\log m}{m^{2\sigma-1}} \sum_{u \ll cT^{1/2}/m} \frac{u^{\epsilon}}{u^{\sigma-1/2}} = O(T). \end{aligned}$$

Finally, for Z_{25} set $m = n - r$, $1 \le r \le n-1$. So in particular $\log(n/m) = -\log(1-r/n) > r/n$. Hence,

$$\begin{aligned} Z_{25} &\ll \log T \sum_{n \le cT^{1/2}} \sum_{1 \le r < n} \frac{1}{n^{\sigma+1/2}(n-r)^{\sigma-1/2}} \frac{n}{r} \\ &\ll \log T \sum_{n \le cT^{1/2}} \frac{1}{n^{\sigma-1/2}} \sum_{r \le n-1} \frac{1}{r} = O(T). \end{aligned}$$

It remains to estimate all the other terms in (2.11). By repeating the analysis done for Z_1 and Z_2 we obtain the following estimates

$$\sum_{0<\gamma\le T} \left| \sum_{n \le R\sqrt{\frac{q\ell\gamma}{2\pi}}} d'_n n^{-\sigma-i\gamma} \right|^2 \ll N(T), \tag{2.14}$$

and

$$\sum_{0<\gamma\le T} \left| \sum_{n \le R\sqrt{\frac{q\ell\gamma}{2\pi}}} \overline{\chi}_2(n) n^{-\sigma+i\gamma} \right|^2 \ll N(T). \tag{2.15}$$

With trivial changes to the above argument we get,

$$\sum_{0<\gamma\le T} \left| \sum_{n \le \sqrt{\frac{q\gamma}{2\pi\ell}}} \chi_1(n) n^{\sigma-1+i\gamma} \right|^2 \ll T^{\sigma-1/2} N(T), \tag{2.16}$$

and

$$\sum_{0<\gamma\le T} \left| \sum_{n \le \frac{1}{R}\sqrt{\frac{\ell\gamma}{2\pi q}}} \overline{\chi}_2(n) n^{\sigma-1-i\gamma} \right|^2 \ll T^{\sigma-1/2} N(T). \tag{2.17}$$

We also need to estimate the order of growth of the derivative in t of $|X(s,\chi)|^2$. First, notice that $|\varepsilon(\chi)| = 1$, so

$$|X(s,\chi)| = \left(\frac{q}{\pi}\right)^{1/2-\sigma} \left|\Gamma\left(\frac{1-s+\mathfrak{a}}{2}\right)\right| \left|\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)\right|^{-1}.$$

By Stirling asymptotics

$$|X(s,\chi)|^2 \sim A \left(\frac{q}{\pi}\right)^{1-2\sigma} \gamma^{1-2\sigma}, \tag{2.18}$$

as $\overline{\Gamma(z)} = \Gamma(\overline{z})$, where A is some non-zero constant. Thus, with $\psi = \Gamma'/\Gamma$,

$$\begin{aligned} \frac{d}{d\gamma}|X(s,\chi)|^2 &= |X(s,\chi)|^2 \frac{i}{2}\left(\psi\left(\overline{\frac{1-s+\mathfrak{a}}{2}}\right) - \psi\left(\frac{1-s+\mathfrak{a}}{2}\right) + \psi\left(\overline{\frac{s+\mathfrak{a}}{2}}\right) - \psi\left(\frac{s+\mathfrak{a}}{2}\right)\right) \\ &= |X(s,\chi)|^2 \frac{i}{2}\left(2i\left(\arg\left(\overline{\frac{1-s+\mathfrak{a}}{2}}\right) - \arg\left(\frac{s+\mathfrak{a}}{2}\right)\right) + O(\gamma^{-2})\right) \\ &\ll \gamma^{1-2\sigma}\Big(O(\gamma^{-1}) + O(\gamma^{-2})\Big) \ll \gamma^{-2\sigma}, \end{aligned} \tag{2.19}$$

by a standard estimate on ψ [12, p. 902, 8.361(3)] and the Taylor expansion of arccot. Let

$$S(T) = \sum_{0<\gamma\le T} |X(\sigma+i\gamma,\chi_1)|^2 \left|\sum_{n\le R} c_n n^{-\sigma-i\gamma}\right|^2 \left| \sum_{n \le \sqrt{\frac{q\gamma}{2\pi\ell}}} \chi_1(n) n^{\sigma-1+i\gamma} \right|^2.$$

We use summation by parts, (2.19), (2.16), and (2.17) to see that

$$S(T) = |X(\sigma+iT,\chi_1)|^2 \sum_{0<\gamma\le T} \Bigg| \sum_{n\le\sqrt{\frac{q\gamma}{2\pi\ell}}} \chi_1(n) n^{\sigma-1+i\gamma}\Bigg|^2 \\ - \int_1^T \sum_{0<\gamma\le t} \Bigg| \sum_{n\le\sqrt{\frac{q\gamma}{2\pi\ell}}} \chi_1(n) n^{\sigma-1+i\gamma}\Bigg|^2 \frac{d}{dt}|X(\sigma+it,\chi_1)|^2\, dt,$$

which simplifies to

$$S(T) \ll T^{1-2\sigma} T^{\sigma-1/2} N(T) + \int_1^T t^{\sigma-1/2} N(t) t^{-2\sigma}\, dt.$$

The first term is clearly $o(N(T))$. For the integral we use the fact that $N(t) = O(t\log t)$ to estimate it as

$$\int_1^T t^{1/2-\sigma}\log t\, dt \ll T^{3/2-\sigma+\epsilon}.$$

Hence we have that $S(T) = o(N(T))$, and similarly

$$\sum_{0<\gamma\le T} |X(\sigma+i\gamma,\overline{\chi}_2)|^2 \Bigg| \sum_{n\le\frac{1}{R}\sqrt{\frac{\ell\gamma}{2\pi q}}} \overline{\chi}_2(n) n^{\sigma-1-i\gamma}\Bigg|^2 \ll T^{3/2-\sigma+\epsilon} = o(N(T)).$$

Finally we use the Cauchy–Schwarz inequality, (2.14), (2.15), and the above two equations to estimate all other terms in (2.11) as $o(N(T))$. □

2.2. **Proof of Proposition 2.2.** Since $B(s,P)$ is a finite Dirichlet polynomial it is bounded independently of T. Thus, to estimate $\sum_{\gamma\le T}|A(\gamma)|^2$, it suffices to estimate

$$\sum_{0<\gamma\le T} |L(s,\chi_1)|^2 |L(s,\chi_2)|^2 = O(N(T)). \tag{2.20}$$

The approximate functional equation for $L(s,\chi_1)$, as in the proof of Proposition 2.1, gives

$$\begin{aligned} L(s,\chi_1) &= \sum_{n\le\sqrt{\frac{q\ell t}{2\pi}}} \chi_1(n) n^{-s} + X(s,\chi_1) \sum_{n\le\sqrt{\frac{qt}{2\pi\ell}}} \overline{\chi}_1(n) n^{s-1} + O(t^{-\sigma/2}\log t + t^{-1/4}) \\ &= W_1 + X(s,\chi_1) W_2 + O(t^{-\sigma/2}\log t) + O(t^{-1/4}). \end{aligned}$$

Similarly,

$$L(s,\chi_2) = Y_1 + X(s,\chi_2) Y_2 + O(t^{-\sigma/2}\log t) + O(t^{-1/4}),$$

where

$$Y_1 = \sum_{n\le\sqrt{\frac{q\ell t}{2\pi}}} \chi_2(n) n^{-s}, \quad Y_2 = \sum_{n\le\sqrt{\frac{\ell t}{2\pi q}}} \overline{\chi}_2(n) n^{s-1}.$$

We have

$$\sum_{0<\gamma\le T} Y_1 \overline{Y}_1 W_1 \overline{W}_1 = \sum_{0<\gamma\le T} \sum_{m,n,\mu,\nu\le\sqrt{\frac{q\ell\gamma}{2\pi}}} \frac{\chi_1(m)\chi_2(n)\overline{\chi}_1(\mu)\overline{\chi}_2(\nu)}{(mn\mu\nu)^\sigma} \left(\frac{\mu\nu}{mn}\right)^{i\gamma}. \tag{2.21}$$

Again, we consider the diagonal terms separately from the rest of the sum. The number of solutions to $mn = \mu\nu = r$ is at most the square of the number of divisors of r, $d(r)^2$. Thus

$$\sum_{0<\gamma\le T} \sum_{mn=\mu\nu}^{(q\ell\gamma/2\pi)^{1/2}} \frac{\chi_1(m)\chi_2(n)\overline{\chi}_1(\mu)\overline{\chi}_2(\nu)}{(mn)^{2\sigma}} \ll \sum_{0<\gamma\le T} \sum_{r=1}^{\infty} \frac{d(r)^2}{r^{2\sigma}} \ll N(T), \tag{2.22}$$

since the inner series converges. For the off-diagonal terms set $\mu\nu = r$ and $mn = s$. We can treat the cases $s < r$ and $s > r$ separately. In the following analysis we assume m, n, μ, $\nu \le (q\ell T/2\pi)^{1/2}$. Consider first the terms with $s < r$ in (2.21). We have that

$$Z_2 = \sum_{r\le q\ell T/2\pi} \sum_{s<r} \sum_{m|s,\,\mu|r} \frac{\chi_1(m)\chi_2(s/m)\overline{\chi}_1(\mu)\overline{\chi}_2(r/\mu)}{r^{\sigma}s^{\sigma}} \sum_{K\le\gamma\le T} \left(\frac{r}{s}\right)^{i\gamma}, \tag{2.23}$$

where $K = \min(T, (2\pi/q\ell)\max(m^2, s^2/m^2, \mu^2, r^2/\mu^2))$. Applying Gonek-Landau Formula (1.1) to Z_2 gives

$$\begin{aligned} Z_2 &= \sum_{r\ll T} \sum_{s<r} \sum_{m|s,\,\mu|r} \frac{\chi_1(m)\chi_2(s/m)\overline{\chi}_1(\mu)\overline{\chi}_2(r/\mu)}{r^{\sigma+1/2}s^{\sigma-1/2}} \left(\sum_{0<\gamma\le T} \left(\frac{r}{s}\right)^{\rho} - \sum_{0<\gamma<K} \left(\frac{r}{s}\right)^{\rho} \right) \\ &= Z_{21,2} + Z_{23} + Z_{24} + Z_{25}, \end{aligned}$$

with

$$\begin{aligned} Z_{21,2} &= \sum_{r\ll T} \sum_{s<r} \sum_{m|s,\,\mu|r} \frac{\chi_1(m)\chi_2(s/m)\overline{\chi}_1(\mu)\overline{\chi}_2(r/\mu)}{r^{\sigma+1/2}s^{\sigma-1/2}} \frac{K-T}{2\pi} \Lambda\left(\frac{r}{s}\right), \\ Z_{23} &\ll \sum_{r\ll T} \sum_{s<r} \sum_{m|s,\,\mu|r} \frac{1}{r^{\sigma+1/2}s^{\sigma-1/2}} \frac{r}{s} \log\frac{2Tr}{s} \log\log\frac{3r}{s}, \\ Z_{24} &\ll \sum_{r\le cT} \sum_{s<r} \sum_{m|s,\mu|r} \frac{\chi_1(m)\chi_2(s/m)\overline{\chi}_1(\mu)\overline{\chi}_2(r/\mu)}{r^{\sigma+1/2}s^{\sigma-1/2}} \log\frac{r}{s} \min\left(T, \frac{r/s}{\langle r/s\rangle}\right), \end{aligned}$$

and

$$Z_{25} \ll \sum_{r\ll T} \sum_{s<r} \sum_{m|s,\,\mu|r} \frac{1}{r^{\sigma+1/2}s^{\sigma-1/2}} \log 2T \min\left(T, \frac{1}{\log(r/s)}\right).$$

For $Z_{21,2}$ we set $r = sk$. Since $d(x) \ll x^{\epsilon}$ and $K \le T$, we get

$$Z_{21,2} \ll T \sum_{k\ll T} \sum_{s\ll T/k} \frac{\Lambda(k)k^{\epsilon}s^{\epsilon}}{k^{\sigma+1/2}s^{2\sigma}} = O(T).$$

We also have

$$\begin{aligned} Z_{23} &\ll \log T \log\log T \sum_{r\ll T} \frac{r^{\epsilon}}{r^{\sigma-1/2}} \sum_{s<r} \frac{s^{\epsilon}}{s^{\sigma+1/2}} \\ &\ll \log T \log\log T \sum_{r\ll T} \frac{r^{\epsilon}}{r^{\sigma-1/2}} = o(N(T)). \end{aligned}$$

We can rewrite Z_{24} as

$$\sum_{r\le cT} \frac{\overline{(\chi_1 * \chi_2)}(r)}{r^{\sigma+1/2}} \sum_{s<r} \frac{(\chi_1 * \chi_2)(s)}{s^{\sigma-1/2}} \log\frac{r}{s} \min\left(T, \frac{r/s}{\langle r/s\rangle}\right), \tag{2.24}$$

where $*$ denotes the Dirichlet convolution. Let $r = us + t$, where $-s/2 < t \le s/2$, and separate the terms where u is not a prime power to $Z_{24,1}$, and denote the remaining terms by $Z_{24,2}$. We use (2.13) to see that

$$Z_{24,1} \ll \sum_{s \le cT} \sum_{u \ll cT/s+1} \sum_{|t|<s/2} \frac{s^{\epsilon}(us+t)^{\epsilon}}{s^{\sigma+1/2}(us+t)^{\sigma-1/2}} \log\left(u+\frac{t}{s}\right). \tag{2.25}$$

Rewriting yields

$$Z_{24,1} \ll \log T \sum_{s \le cT} \frac{s^{2\epsilon}}{s^{2\sigma}} \sum_{u \ll cT/s+1} \sum_{|t|<s/2} \left(u+\frac{t}{s}\right)^{1/2-\sigma+\epsilon}.$$

The terms in u can be bound from above by $(u-1)^{1/2-\sigma+\epsilon}$. Thus

$$\begin{aligned} Z_{24,1} &\ll \log T \sum_{s \le cT} s^{1+2\epsilon-2\sigma} \left(\frac{cT}{s}\right)^{3/2-\sigma+\epsilon} \\ &\ll T^{3/2-\sigma+\epsilon} T^{1/2-\sigma+\epsilon} \log T = O(N(T)). \end{aligned}$$

For $Z_{24,2}$ let $'$ in summation denote that the sum extends only over prime powers. We need to estimate

$$\sum_{s \le cT} \sideset{}{'}\sum_{u \le \lfloor \frac{cT}{s} \rfloor + 1} \sum_{0 \ne |t| < s/2} \frac{\overline{(\chi_1 * \chi_2)}(us+t)(\chi_1 * \chi_2)(s)}{(us+t)^{\sigma+1/2} s^{\sigma-1/2}} \log\left(u+\frac{t}{s}\right) \min\left(T, \frac{us+t}{|t|}\right)$$

as $O(N(T))$. This can be rewritten as

$$\sum_{s \le cT} \frac{(\chi_1 * \chi_2)(s)}{s^{2\sigma}} \sideset{}{'}\sum_{u \le \lfloor \frac{cT}{s} \rfloor + 1} \sum_{0 \ne |t| < s/2} \frac{\overline{(\chi_1 * \chi_2)}(us+t)}{(u+\frac{t}{s})^{\sigma+1/2}} \log\left(u+\frac{t}{s}\right) \min\left(T, \frac{us+t}{|t|}\right).$$

By taking absolute values and using the triangle inequality we find that

$$\begin{aligned} Z_{24,2} &\ll \log^2 T \sum_{s \ll T} s^{2\epsilon-2\sigma+1} \sum_{u \ll T/s} u^{1/2-\sigma+\epsilon} \\ &\ll T^{3/2-\sigma+\epsilon} \log^2 T \sum_{s \ll T} s^{\epsilon-\sigma-1/2} = O(N(T)), \end{aligned}$$

as required. It remains to estimate Z_{25}. We use the same method as in Proposition 2.1. Let $s = r-k$, and $1 \le k < r$ to get

$$\begin{aligned} Z_{25} &\ll \log T \sum_{r \ll T} \sum_{k<r} \frac{1}{r^{\sigma+1/2-\epsilon}(r-k)^{\sigma-1/2-\epsilon}} \frac{r}{k} \\ &\ll \log T \sum_{r \ll T} \frac{1}{r^{\sigma-1/2-\epsilon}} \sum_{k<r} \frac{1}{k} = o(N(T)). \end{aligned}$$

Finally, if $s > r$ we can consider the complex conjugate of (2.21) to obtain the same estimate. The rest of the proof proceeds in the same way as in Proposition 2.1. We obtain trivially the estimates

$$\sum_{0<\gamma\le T} |W_1|^4 \ll N(T), \tag{2.26}$$

$$\sum_{0<\gamma\le T} |Y_1|^4 \ll N(T). \tag{2.27}$$

Also, by modifying the argument slightly we find that

$$\sum_{0<\gamma\le T} |W_2|^4 \ll T^{2\sigma-1+\epsilon} N(T), \tag{2.28}$$

$$\sum_{0<\gamma\le T} |Y_2|^4 \ll T^{2\sigma-1+\epsilon} N(T). \tag{2.29}$$

We also need to estimate the derivative of $|X(s,\chi)|^4$. By estimate (2.19) from Proposition 2.1 we get

$$\frac{d}{d\gamma}|X(s,\chi)|^4 \ll \gamma^{1-2\sigma}\gamma^{-2\sigma} \ll \gamma^{1-4\sigma}.$$

The rest of the proof now follows from estimating

$$\sum_{0<\gamma\le T} |X(s,\chi_1)|^4 |W_2|^4 = O(T^{1-2\sigma+\epsilon}N(T)),$$

and similarly for Y_2, and applying the Cauchy–Schwarz to the remaining terms in the expansion of the product in (2.20). □

2.3. **Proof of Proposition 2.3.** Let

$$D = \sum_{n=1}^{\infty} \frac{d_n \overline{\chi}_2(n)}{n^{2\sigma}}, \quad E = \sum_{n=1}^{\infty} \frac{e_n \overline{\chi}_1(n)}{n^{2\sigma}}.$$

By Proposition 2.2 it is sufficient to show that $D - E \neq 0$. First, we need to compute the d_n and e_n's explicitly. Let us denote the set of primes smaller than P by $\mathscr{P} = \{p_1, p_2, \ldots, p_h, P\}$. Suppose P is large enough so that q, $\ell \in \mathscr{P}$. The coefficients d_n are defined by the Euler product

$$\prod_{p\le P}(1-\chi_2(p)p^{-s}) \times \prod_{p>P}\sum_{n=0}^{\infty}\frac{\chi_1(p^n)}{p^{ns}}.$$

If $p^2 \mid n$, $p \in \mathscr{P}$, then n disappears from the expansion, i.e. $d_n = 0$. If n has no prime factors from the set $\mathscr{P}$, then we just get the usual coefficient from $L(s,\chi_1)$. On the other hand, if some prime $p \in \mathscr{P}$ divides n exactly once then it contributes $-\chi_2(p)$. Hence

$$d_n = \begin{cases} \chi_1(n), & \text{if } p \nmid n \text{ for all } p \in \mathscr{P}, \\ (-1)^k \chi_1\big(\frac{n}{p_{i_1}\cdots p_{i_k}}\big)\chi_2(p_{i_1}\cdots p_{i_k}), & \text{if } p_{i_j} \parallel n \text{ for } p_{i_j} \in \mathscr{P} \text{ for all } j, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly for e_n. Hence for $p > P$ the Euler factors of D are of the form

$$(1 - \chi_1(p)\overline{\chi}_2(p)p^{-2\sigma})^{-1},$$

while for E one obtains the complex conjugate. On the other hand, for $p \le P$ we have

$$1 + d_p\overline{\chi}_2(p)p^{-2\sigma} + d_{p^2}\overline{\chi}_2(p^2)p^{-4\sigma} + \cdots = 1 - \chi_2\overline{\chi}_2(p)p^{-2\sigma} = 1 - p^{-2\sigma},$$

unless $p = \ell$, and similarly for the second series. Now, suppose that $D = E$, then

$$\prod_{\substack{p\le P\\ p\ne \ell}}(1-p^{-2\sigma})\prod_{p>P}(1-(\chi_1\overline{\chi}_2)(p)p^{-2\sigma})^{-1} = \prod_{\substack{p\le P\\ p\ne q}}(1-p^{-2\sigma})\prod_{p>P}(1-(\overline{\chi}_1\chi_2)(p)p^{-2\sigma})^{-1}.$$

We cancel out the common terms in the product over $p < P$, which yields

$$(1-q^{-2\sigma})z = (1-\ell^{-2\sigma})\bar{z},$$

where
$$z = \prod_{p>P} (1 - (\chi_1\overline{\chi}_2)(p)p^{-2\sigma})^{-1}.$$

Hence,
$$\frac{1-q^{-2\sigma}}{1-\ell^{-2\sigma}} = \frac{\bar{z}}{z}.$$

Taking absolute values yields
$$\frac{1-q^{-2\sigma}}{1-\ell^{-2\sigma}} = 1,$$

which is a contradiction. □

3. Proof of Theorem 1.2

We now sample the values of $L(s,\chi)$ at precisely the non-trivial zeros of ζ. In this case we do not assume RH. Off the critical line we used the method of Gonek–Landau to prove linear independence. On the critical line, however, this becomes very difficult. This is mainly because of the corresponding Z_{24} term in the first proposition. We would need to control sums of the form
$$\sum_{n\le X}\sum_{m<n} \frac{\chi_1(m)\overline{\chi}_2(n)}{n} \log\left(\frac{n}{m}\right) \min\left(T, \frac{n/m}{\langle n/m\rangle}\right),$$
where $X = qT/2\pi\sqrt{\log T}$. This should be $o(N(T)\log T)$, which we cannot prove. In the proof of Conrey et al. [1] they make a reduction to the discrete mean values of one L-function at a time. We have been unable to find such a reduction in our case. Garunkštis et al. [6] presented a more suitable method through contour integration and a modified Gonek Lemma (see Lemma 1.2).

Denote the characters in Theorem 1.2 by χ_1 and χ_2 with distinct prime moduli q and ℓ. For any Dirichlet character χ modulo n, we denote the principal character modulo n by χ_0. Moreover, put $B(s,p) = p^s$ for some prime p to be determined later. Then,
$$A(\gamma) := B(\rho,p)\,(L(\rho,\chi_1) - L(\rho,\chi_2))$$
is non-zero precisely when the two L-functions assume distinct values.

Proposition 3.1. *Let $\mathfrak{C}$ be the rectangular contour with vertices at $a+i$, $a+iT$, $1-a+iT$, and $1-a+i$ with positive orientation, where $a = 1 + (\log T)^{-1}$. Then we have*
$$\frac{1}{2\pi i}\int_{\mathfrak{C}} \frac{\zeta'}{\zeta}(s)B(s,p)L(s,\chi_1)\,ds \sim \frac{\overline{C}_{\chi_1}T}{2\pi}\log\frac{T}{2\pi}, \tag{3.1}$$
where
$$C_{\chi_1} = \frac{G(1,\overline{\chi}_1)G(-p,\chi_1)}{q},$$
and similarly for χ_2.

Then, by the residue theorem, we get
$$\sum_{0<\gamma\le T} A(\gamma) = \frac{1}{2\pi i}\int_{\mathfrak{C}} \frac{\zeta'}{\zeta}(s)B(s,p)\,(L(s,\chi_1) - L(s,\chi_2))\,ds.$$

Proposition 3.2. *With the same contour as in Proposition 3.1 we have for $j, j' \in \{1,2\}$ that*
$$\frac{1}{2\pi i}\int_{\mathfrak{C}} \frac{\zeta'}{\zeta}(s)L(s,\chi_j)L(1-s,\overline{\chi}_{j'})\,ds \ll T\log^2 T.$$

This proposition gives us estimates for all the terms in $\sum|A(\gamma)|^2$, since $B(s,p)$ can be bounded independently of T. Finally, we have to prove that the difference coming from Proposition 3.1 is non-zero.

Proposition 3.3. *There is a prime p, different from q and ℓ, such that $C_{\chi_1} - C_{\chi_2} \neq 0$.*

With these propositions we can prove Theorem 1.2 in the same way as in (2.8). In the proofs below we make extensive use of the following facts about Gauss sums, see [3, pg. 65 (2)].

$$G(n,\chi_1) = \overline{\chi_1}(n)G(1,\chi_1), \tag{3.2}$$

since q is a prime, and (see [3, pg. 66 (5)])

$$G(1,\overline{\chi_1})G(-1,\chi_1) = q, \tag{3.3}$$

and similarly for χ_2.

3.1. **Proof of Proposition 3.1.** We prove the proposition for χ_1 as the case of χ_2 is identical. Denote the integral in (3.1) by $\mathcal{I}$. Then

$$\begin{aligned}\mathcal{I} &= \left(\int_{a+i}^{a+iT} + \int_{a+iT}^{1-a+iT} + \int_{1-a+iT}^{1-a+i} + \int_{1-a+i}^{a+i}\right)\frac{\zeta'}{\zeta}(s)B(s,p)L(s,\chi_1)\,ds\\ &= \mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4.\end{aligned}$$

We can evaluate $\mathcal{I}_1$ explicitly to get

$$\begin{aligned}\mathcal{I}_1 &= \int_{a+i}^{a+iT}\frac{\zeta'}{\zeta}(s)L(s,\chi_1)p^s\,ds\\ &= -i\sum_{n,\,m}\frac{\Lambda(n)\chi_1(m)}{(mnp^{-1})^a}\int_1^T\left(\frac{p}{mn}\right)^{it}dt \ll \frac{\zeta'}{\zeta}(a)\zeta(a) + T = O(T),\end{aligned}$$

where the second term comes from the case $mn = p$. For $\mathcal{I}_2$ we use the following bounds (see [3, pg. 108]):

$$\frac{\zeta'}{\zeta}(\sigma+iT) \ll \log^2 T, \quad \text{if } -1\le\sigma\le 2, \quad |T|\ge 1, \tag{3.4}$$

and

$$L(\sigma+iT,\chi_1) \ll |T|^{1/2}\log|T+2|, \quad \text{if } 1-a\le\sigma\le a, \quad |T|\ge 1. \tag{3.5}$$

These yield $\mathcal{I}_2 = O(T^{1/2}\log^3 T)$. Next we consider $\mathcal{I}_3$. Changing variables $s\mapsto 1-\bar{s}$ gives

$$\mathcal{I}_3 = \frac{-1}{2\pi i}\int_{a+i}^{a+iT}\frac{\zeta'}{\zeta}(1-\bar{s})L(1-\bar{s},\chi_1)p^{1-\bar{s}}\,ds.$$

Conjugating and applying the functional equation of ζ and $L(s,\chi_1)$ yields

$$\overline{\mathcal{I}}_3 = \frac{p}{2\pi i}\int_{a+i}^{a+iT}\left(\frac{\zeta'}{\zeta}(s)+\frac{\gamma'}{\gamma}(s)\right)L(s,\chi_1)\Delta(s,\chi_1)p^{-s}\,ds,$$

where

$$\gamma(s) = \pi^{1/2-s}\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)},$$

and

$$\Delta(s,\chi_1) = \left(\frac{q}{2\pi}\right)^s\frac{1}{q}G(1,\overline{\chi}_1)\Gamma(s)\left(e^{-\pi is/2}+\overline{\chi}_1(-1)e^{\pi is/2}\right).$$

Using the definition of Δ to expand the above we find that

$$\overline{\mathcal{I}}_3 = p\left(\mathcal{F}_1+\cdots+\mathcal{F}_4\right),$$

where

$$\mathcal{F}_1 = \frac{G(1,\overline{\chi}_1)}{q}\frac{1}{2\pi i}\int_{a+i}^{a+iT}\frac{\gamma'}{\gamma}(s)\left(\frac{q}{2\pi p}\right)^s\Gamma(s)\exp\left(-\frac{\pi i s}{2}\right)L(s,\chi_1)\,ds,$$
$$\mathcal{F}_2 = \frac{\overline{\chi}_1(-1)G(1,\overline{\chi}_1)}{q}\frac{1}{2\pi i}\int_{a+i}^{a+iT}\frac{\gamma'}{\gamma}(s)\left(\frac{q}{2\pi p}\right)^s\Gamma(s)\exp\left(+\frac{\pi i s}{2}\right)L(s,\chi_1)\,ds,$$
$$\mathcal{F}_3 = \frac{G(1,\overline{\chi}_1)}{q}\frac{1}{2\pi i}\int_{a+i}^{a+iT}\frac{\zeta'}{\zeta}(s)\left(\frac{q}{2\pi p}\right)^s\Gamma(s)\exp\left(-\frac{\pi i s}{2}\right)L(s,\chi_1)\,ds,$$
$$\mathcal{F}_4 = \frac{\overline{\chi}_1(-1)G(1,\overline{\chi}_1)}{q}\frac{1}{2\pi i}\int_{a+i}^{a+iT}\frac{\zeta'}{\zeta}(s)\left(\frac{q}{2\pi p}\right)^s\Gamma(s)\exp\left(+\frac{\pi i s}{2}\right)L(s,\chi_1)\,ds.$$

We rewrite $\mathcal{F}_1$ in the following way

$$\mathcal{F}_1 = \frac{G(1,\overline{\chi}_1)}{q}\int_1^T\frac{\gamma'}{\gamma}(a+i\tau)\,d\left(\frac{1}{2\pi i}\int_{a+i}^{a+i\tau}\left(\frac{q}{2\pi p}\right)^s\Gamma(s)\exp\left(-\frac{\pi i s}{2}\right)L(s,\chi_1)\,ds\right). \tag{3.6}$$

By Lemma 1.2,

$$\frac{1}{2\pi i}\int_{a+i}^{a+i\tau}\left(\frac{q}{2\pi p}\right)^s\Gamma(s)\exp\left(-\frac{\pi i s}{2}\right)L(s,\chi_1)\,ds = \sum_{n\le\frac{\tau q}{2\pi p}}\chi_1(n)\exp\left(-2\pi i\frac{np}{q}\right)+O(\tau^{1/2+\epsilon}).$$

We can separate the periods to write the sum as

$$\sum_{a=1}^{q}\chi_1(a)\exp\left(-2\pi i\frac{ap}{q}\right)\sum_{\substack{n\le\frac{\tau q}{2\pi p}\\ n\equiv a \bmod q}}1 = \frac{\tau}{2\pi p}G(-p,\chi_1)+O(1).$$

We integrate by parts in (3.6) and use the standard estimate

$$\frac{\gamma'}{\gamma}(s) = \log\frac{|s|}{2\pi}+O(|t|^{-1}), \quad |t|\ge 1,$$

to see that

$$\begin{aligned}\mathcal{F}_1 &= \frac{C_{\chi_1}}{2\pi p}\int_1^T\left(\log\frac{\tau}{2\pi}+O(\tau^{-1})\right)\,d\left(\tau+O(\tau^{1/2+\epsilon})\right)\\ &= \frac{C_{\chi_1}T}{2\pi p}\log\frac{T}{2\pi}+O(T^{1/2+\epsilon}).\end{aligned}$$

Similarly by Lemma 1.2, $\mathcal{F}_2$ is $O(\log T)$, while $\mathcal{F}_4 = O(1)$. For $\mathcal{F}_3$ we have

$$\begin{aligned}\mathcal{F}_3 &= \frac{G(1,\overline{\chi}_1)}{q}\frac{1}{2\pi i}\int_{a+i}^{a+iT}\left(\frac{q}{2\pi p}\right)^s\Gamma(s)\exp\left(-\frac{\pi i s}{2}\right)\frac{\zeta'}{\zeta}(s)L(s,\chi_1)\,ds\\ &= \frac{-G(1,\overline{\chi}_1)}{q}\sum_{mn\le\frac{Tq}{2\pi p}}\Lambda(m)\chi_1(n)\exp\left(-2\pi i\frac{mnp}{q}\right)+O(T^{1/2+\epsilon}).\end{aligned}$$

Looking at the summation we decompose it as

$$\sum_{m\le\frac{Tq}{2\pi p}}\Lambda(m)\sum_{n\le\frac{Tq}{2\pi pm}}\chi_1(n)\exp\left(-2\pi i\frac{mnp}{q}\right).$$

We separate the periods in the same way as for $\mathcal{F}_1$ and write the above sum as

$$\sum_{m\le\frac{Tq}{2\pi p}}\Lambda(m)G(-mp,\chi_1)\frac{T}{2\pi pm}+O(T).$$

We will show that the summation over m in fact converges. This means that we have $\mathcal{F}_3 = O(T)$. To do this it suffices to consider

$$\sum_{m\leq X} \frac{\Lambda(m)\overline{\chi}_1(m)}{m}.$$

Let

$$\psi(X,\overline{\chi}_1) = \sum_{m\leq X} \Lambda(m)\overline{\chi}_1(m).$$

Then, by [3, pg. 123 (8)],

$$\psi(X,\overline{\chi}_1) = -\frac{X^\beta}{\beta} + O(X\exp(-c(\log X)^{1/2})),$$

where the term with β comes from the Siegel zero of χ_1 and c is some positive absolute constant. However, since our q is fixed, we know that β is bounded away from 1. Hence, with summation by parts we obtain

$$\sum_{m\leq X} \frac{\Lambda(m)\overline{\chi}_1(m)}{m} = \frac{\psi(X,\overline{\chi}_1)}{X} + \int_1^X \frac{\psi(t,\overline{\chi}_1)}{t^2}\,dt = O(1)$$

as required. Finally $\mathcal{I}_4 = O(1)$ as the integrand is analytic in a neighbourhood of the line of integration. □

3.2. **Proof of Proposition 3.2.** We prove the case $j = j' = 1$ as the other cases are either similar or easier. Now, denote the integral by $\mathcal{I}$, i.e.

$$\mathcal{I} = \frac{1}{2\pi i}\int_{\mathfrak{C}} \frac{\zeta'}{\zeta}(s)L(s,\chi_1)L(1-s,\overline{\chi}_1)\,ds,$$

and split it in the same way as in the proof of Proposition 3.1, so that

$$\mathcal{I} = \mathcal{I}_1 + \cdots + \mathcal{I}_4.$$

We can write $\mathcal{I}_1$ as

$$\begin{aligned}
\mathcal{I}_1 &= \frac{1}{2\pi i}\int_{a+i}^{a+iT} \frac{\zeta'}{\zeta}(s)L(s,\chi_1)^2\Delta(s,\chi_1)\,ds \\
&= \frac{G(\overline{\chi}_1)}{q}\frac{1}{2\pi i}\int_{a+i}^{a+iT} \left(\frac{q}{2\pi}\right)^s\Gamma(s)\left(\exp\left(\frac{-\pi i s}{2}\right) + \overline{\chi}_1(-1)\exp\left(\frac{\pi i s}{2}\right)\right)\frac{\zeta'}{\zeta}(s)L(s,\chi_1)^2\,ds \\
&= \mathcal{E}_1 + \mathcal{E}_2.
\end{aligned}$$

By Lemma 1.2, $\mathcal{E}_2 = O(1)$. Let us now estimate $\mathcal{E}_1$. We have

$$\mathcal{E}_1 = -\frac{G(\overline{\chi}_1)}{q}\sum_{mn\leq \frac{Tq}{2\pi}} d(n)\chi_1(n)\Lambda(m)\exp\left(-2\pi i\frac{nm}{q}\right) + O(T^{1/2+\epsilon}).$$

Denote the sum over m and n by S. As before, we first separate the periods

$$S = \sum_{a,\,b=1}^{q} \chi_1(a)\exp\left(-2\pi i\frac{ab}{q}\right)\sum_{\substack{mn\leq\frac{Tq}{2\pi}\\ n\equiv a \bmod q\\ m\equiv b \bmod q}} d(n)\Lambda(m).$$

Now sum over characters η of modulus q to get

$$= \frac{1}{\varphi(q)} \sum_{\eta \bmod q} \sum_{a,\, b=1}^{q} \chi_1(a)\overline{\eta}(a) \exp\left(-2\pi i \frac{ab}{q}\right) \sum_{\substack{mn \le \frac{Tq}{2\pi} \\ m \equiv b \bmod q}} d(n)\eta(n)\Lambda(m)$$

$$= \frac{1}{\varphi(q)} \sum_{\eta \bmod q} \sum_{b=1}^{q} G(-b, \chi_1\overline{\eta}) \sum_{\substack{mn \le \frac{Tq}{2\pi} \\ m \equiv b \bmod q}} d(n)\eta(n)\Lambda(m),$$

and as before

$$= \frac{1}{\varphi(q)^2} \sum_{\eta,\, \omega \bmod q} G(-1, \chi_1\overline{\eta}) \left(\sum_{mn \le \frac{Tq}{2\pi}} d(n)\eta(n)\Lambda(m)\omega(m) \right) \sum_{b=1}^{q} \overline{\chi}_1(b)\eta(b)\overline{\omega}(b).$$

The sum over b is non-zero if and only if $\omega = \omega_0$ and $\eta = \chi_1$; or $\omega = \overline{\chi_1}$ and $\eta = \eta_0$; or $\omega \neq \omega_0$ and $\eta = \chi_1\omega$. By Perron's formula

$$-\sum_{mn \le \frac{Tq}{2\pi}} d(n)\eta(n)\Lambda(m)\omega(m) = \frac{1}{2\pi i} \int_{a-iU}^{a+iU} \frac{L'}{L}(s,\omega)L(s,\eta)^2 \left(\frac{Tq}{2\pi}\right)^s \frac{ds}{s} + O\left(\frac{T\log^3 T}{U}\right)$$

for some U with $|U| \le T$. Since our characters are fixed, we can use Vinogradov-type zero-free region [6, pg. 296]. That is, let $b_1 = 1 - c_1/(\log t)^{3/4+\epsilon}$ (in fact, any power smaller than 1 would do), then $L(\sigma + it, \chi_1)$ has no zeros in the region $\sigma \ge b_1$. Here c_1 is some positive absolute constant. By the approximate functional equation and Stirling asymptotics we have uniformly for $0 < \sigma < 1$ and $|t| > 1$ that

$$L(s,\chi_1) \ll |t|^{\frac{1-\sigma}{2}} \log(|t|+1). \tag{3.7}$$

Then, by shifting the contour we get

$$\begin{aligned} -\sum_{mn \le \frac{Tq}{2\pi}} d(n)\eta(n)\Lambda(m)\omega(m) &= \operatorname*{Res}_{s=1} \frac{L'}{L}(s,\omega)L(s,\eta)^2 \left(\frac{Tq}{2\pi}\right)^s \frac{1}{s} \\ &- \frac{1}{2\pi i}\left(\int_{a+iU}^{b_1+iU} + \int_{b_1+iU}^{b_1-iU} + \int_{b_1-iU}^{a-iU} \right) \frac{L'}{L}(s,\omega)L(s,\eta)^2 \left(\frac{Tq}{2\pi}\right)^s \frac{ds}{s} + O\left(\frac{T\log^3 T}{U}\right). \end{aligned} \tag{3.8}$$

We need to find the residues in each of the three cases.

$$\begin{aligned} \operatorname*{Res}_{s=1} \frac{L'}{L}(s,\omega_0)L(s,\chi_1)^2 \left(\frac{Tq}{2\pi}\right)^s \frac{1}{s} &= -L(1,\chi_1)^2 \frac{Tq}{2\pi}, \\ \operatorname*{Res}_{s=1} \frac{L'}{L}(s,\overline{\chi}_1)L(s,\eta_0)^2 \left(\frac{Tq}{2\pi}\right)^s \frac{1}{s} &= \frac{Tq}{2\pi}\left(\frac{\varphi(q)}{q}\right)^2 \frac{L'}{L}(1,\overline{\chi}_1) \log\frac{Tq}{2\pi} + O(T), \\ \operatorname*{Res}_{s=1} \frac{L'}{L}(s,\omega)L(s,\chi_1\omega)^2 \left(\frac{Tq}{2\pi}\right)^s \frac{1}{s} &= 0. \end{aligned}$$

It remains to estimate the integrals on the right-hand side of (3.8). By (3.4) and (3.7) we see that the first and third integrals yield $O(T^a U^{-b_1} \log^4 U)$. We split the second integral and estimate it as

$$\begin{aligned} &\left(\int_{b_1+iU}^{b_1+i} + \int_{b_1+i}^{b_1-i} + \int_{b_1-i}^{b_1-iU} \right) \frac{L'}{L}(s,\overline{\chi}_1)L(s,\eta_0)^2 \left(\frac{Tq}{2\pi}\right)^s \frac{ds}{s} \\ &\qquad = O(T^{b_1}U^{1-b_1}\log^4 U + T^{b_1}|b_1 - 1|^{-2}), \end{aligned}$$

where the second error term comes from the integral over the constant segment. It suffices to choose $U = T^{1/2}$ as then

$$T^a U^{-b_1} \log^4 U \ll T e^{-\frac{1}{2}\log T + \frac{c_1}{2}(\log T)^{1/4-\epsilon} + 4\log\log T},$$
$$T^{b_1} U^{1-b_1} \log^4 U \ll T e^{-\frac{c_1}{2}(\log T)^{1/4-\epsilon} + 4\log\log T},$$

and

$$T^{b_1}|b_1 - 1|^{-2} \ll T e^{-c_1(\log T)^{1/4-\epsilon} + (3/2+2\epsilon)\log\log T},$$

which are all $O(T)$. Therefore we conclude that $\mathcal{I}_1 = O(T\log T)$.

Next up is $\mathcal{I}_2$. We use again the convexity bound writing it as

$$L(\sigma + it, \chi_1) \ll_\epsilon |t|^{\mu_0(\sigma)+\epsilon},$$

where $\epsilon > 0$, $-1 < \sigma < 2$ (say), $|t| > 1$, and

$$\mu_0(\sigma) = \begin{cases} 0, & \text{if } \sigma > 1, \\ \frac{1-\sigma}{2}, & \text{if } 0 < \sigma < 1, \\ \frac{1}{2} - \sigma, & \text{if } \sigma < 0. \end{cases}$$

With this we can write

$$\begin{aligned} \mathcal{I}_2 &= \frac{1}{2\pi i}\int_{a+iT}^{1-a+iT} \frac{\zeta'}{\zeta}(s) L(s,\chi_1) L(1-s,\overline{\chi}_1)\, ds \\ &\ll \left(\int_{1-a}^{0} + \int_0^1 + \int_1^a\right) \log^2 T\, T^{\mu_0(\sigma)+\epsilon} T^{\mu_0(1-\sigma)+\epsilon}\, d\sigma. \end{aligned}$$

Keeping in mind that $\sigma \le a$ we get

$$\mathcal{I}_2 \ll T^{a-1/2+\epsilon}\log^2 T + T^{1/2+\epsilon}\log^2 T = O(T).$$

For $\mathcal{I}_3$ we do the usual trick of mapping $s \mapsto 1 - \bar{s}$. Taking complex conjugates leads to

$$\overline{\mathcal{I}}_3 = \frac{1}{2\pi i}\int_{a+i}^{a+iT} \left(\frac{\zeta'}{\zeta}(s) + \frac{\gamma'}{\gamma}(s)\right) L(s,\chi_1)^2 \Delta(s,\chi_1)\, ds.$$

As in Proposition 3.1 we split this up into $\mathcal{F}_1, \ldots, \mathcal{F}_4$. Adding up $\mathcal{F}_3$ and $\mathcal{F}_4$ gives $\mathcal{I}_1$, which is $O(T\log T)$. As before, $\mathcal{F}_2$ does not contribute. So we have to estimate $\mathcal{F}_1$, that is

$$\mathcal{F}_1 = \frac{G(1,\overline{\chi}_1)}{q}\int_1^T \left(\log\frac{\tau}{2\pi} + O(\tau^{-1})\right) d\left(\frac{1}{2\pi i}\int_{a+i}^{a+i\tau} L(s,\chi_1)^2 \Gamma(s)\exp\left(-\frac{\pi i s}{2}\right) ds\right).$$

Working as in Proposition 3.1 we can write the inner integral (plus an error term) as

$$\sum_{n \le \frac{\tau q}{2\pi}} \chi_1(n) d(n) \exp\left(-2\pi i \frac{n}{q}\right).$$

This is $O(\tau\log\tau)$, which gives $\mathcal{I}_3 = O(T\log^2 T)$. It is not difficult to extend this to an asymptotic estimate, but for our purposes the upper bound is sufficient. Trivially we also have that $\mathcal{I}_4 = O(1)$. Hence $\mathcal{I} = O(T\log^2 T)$. □

3.3. **Proof of Proposition 3.3.** By (3.2) and (3.3) we see that

$$C_{\chi_1} = C_{\chi_2}$$

if and only if $\chi_1(p) = \chi_2(p)$. By Chinese Remainder Theorem and Dirichlet's Theorem we can find a prime p different from q and ℓ that satisfies

$$\begin{cases} p \equiv 1 \bmod q, \\ p \equiv a \bmod \ell, \end{cases}$$

such that $\chi_2(p) = \chi_2(a) \neq 1$, since χ_2 is non-principal. This gives $1 = \chi_1(p) = \chi_2(p) \neq 1$, which is a contradiction. $\square$

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