An amalgamation of the Banach spaces associated with James and Schreier, Part I: Banach-space structure

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Abstract

We create a new family of Banach spaces, the *James–Schreier spaces,* by amalgamating two important classical Banach spaces: James' quasi-reflexive Banach space on the one hand and Schreier's Banach space giving a counterexample to the Banach–Saks property on the other. We then investigate the properties of these James–Schreier spaces, paying particular attention to how key properties of their 'ancestors' (that is, the James space and the Schreier space) are expressed in them. Our main results include that each James–Schreier space is c_0-saturated and that no James–Schreier space embeds in a Banach space with an unconditional basis.

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1 Introduction

The purpose of this paper is to introduce a new family of Banach spaces which we call the *James–Schreier spaces* because they arise by amalgamating the definitions of the quasi-reflexive James spaces with the Schreier space. The original motivation behind these spaces was to produce a new example of a Banach sequence algebra with a bounded approximate identity, in analogy with Andrew and Green's study of the James space as a Banach algebra [3]. This idea turned out to be successful, as essentially all results about the James space as a Banach algebra carry over to our new spaces; see [9] for details.

Having thus reached our initial goal, we soon realized that a serious problem was lurking in the background, namely: how can we distinguish the James–Schreier spaces from the James spaces? Obviously, if they were isomorphic, our findings would be of no interest. In order to resolve this problem, we turned to Banach-space properties and, as we shall see, at that level differences abound; this is the main theme of the present paper.

We begin with surveys of the James spaces (Section 2) and the Schreier space (Section 3), where we introduce notation and explain ideas and properties which we shall subsequently pursue in the case of the James–Schreier spaces. In fact, as a spin-off of our investigation, we generalize the concept of a Schreier space from just one Schreier space (corresponding to the ℓ_1-norm) to a whole family, one for each $p \in [1, \infty)$ (corresponding to the ℓ_p-norms). The basic theory of these spaces is developed in Section 3; our most important findings are: (i) the standard unit vector basis is a shrinking, 1-unconditional basis for each Schreier space; (ii) those right-shifts which define bounded operators on the Schreier spaces can be characterized (see Corollary 3.17(ii) for details). The former of these results leads to an explicit description of the biduals of the Schreier spaces, while the latter implies that each Schreier space is isomorphic to its Cartesian square.

Section 4 contains the definition of the James–Schreier spaces, one for each $p \in [1, \infty)$, together with an exposition of their basic theory. Results include: (i) the standard basis is a monotone basis, shrinking for $p > 1$, but not unconditional (it will follow from results in Section 6 that *no* basis is); (ii) each James–Schreier space contains a complemented copy of the corresponding Schreier space; (iii) the standard right shift defines a bounded operator on the James–Schreier spaces only in the trivial cases, but a suitably modified version of it turns out to be bounded under conditions similar to those found for the Schreier spaces (see Propositions 4.18(ii) and 4.20).

Sections 5 and 6 contain our two main results: the James–Schreier spaces are c_0-saturated (which means that each of their closed, infinite-dimensional subspaces contains a copy of c_0), and they do not have Pełczyński's property (u), so in particular they do not embed in a Banach space with an unconditional basis.

As a consequence of these results, we can complete the 'comparison theory' of our spaces: firstly, the James spaces are totally incomparable with both the Schreier and the James–Schreier spaces (which means that they have no closed, infinite-dimensional subspaces in common), and secondly, no James–Schreier space embeds in a Schreier space, whereas each Schreier space embeds complementedly in a James–Schreier space, as already mentioned in (ii) above.

We conclude this introduction with some general conventions on notation and terminology. Throughout, $\mathbb{K}$ denotes the scalar field; either $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

By an *operator* we understand a linear mapping between vector spaces. We write I_X for the identity operator on a vector space X. A *functional* is a linear mapping from a vector space to its scalar field. For a functional f on a vector space X and a vector $x \in X$, we usually write $\langle x, f \rangle$ instead of $f(x)$.

Let X be a normed space. For a subset M of X, $\operatorname{span} M$ denotes the linear span of M in X, while $\overline{\operatorname{span}}\, M$ is its closure. We write X' for the (continuous) dual of X, that is, X' consists of the bounded functionals on X. The canonical embedding of X into its bidual X'' is denoted by κ_X, or just κ if reference to X is unnecessary. We say that two normed spaces X and Y are *isomorphic,* written $X \cong Y$, if there is a linear homeomorphism from X onto Y.

By a *basis* for a Banach space X, we shall always understand a Schauder basis, that is, a sequence $(b_n)_{n\in\mathbb{N}}$ in X such that, for each $x \in X$, there is a unique sequence $(\alpha_n)_{n\in\mathbb{N}}$ of scalars such that the series $\sum_{n=1}^{\infty} \alpha_n b_n$ is norm-convergent with sum x. If this series converges unconditionally for each $x \in X$, then the basis $(b_n)_{n\in\mathbb{N}}$ is *unconditional.* A sequence $(b_n)_{n\in\mathbb{N}}$ in a Banach space is a *basic sequence* if $(b_n)_{n\in\mathbb{N}}$ is a basis for its closed linear span, $\overline{\operatorname{span}}\,\{b_n : n \in \mathbb{N}\}$.

For $m \in \mathbb{N}$, the m^{th} *biorthogonal functional* b'_m associated with a basis $(b_n)_{n\in\mathbb{N}}$ for a Banach space X is given by $\langle x, b'_m\rangle := \alpha_m$ for each $x = \sum_{n=1}^{\infty} \alpha_n b_n \in X$. This is a bounded functional on X, and $(b'_m)_{m\in\mathbb{N}}$ is a basic sequence in X'. If $(b'_m)_{m\in\mathbb{N}}$ is a basis for X' (that is, if $\overline{\operatorname{span}}\,\{b'_m : m \in \mathbb{N}\} = X'$), then the basis $(b_n)_{n\in\mathbb{N}}$ is *shrinking.*

We denote by $\mathbb{K}^{\mathbb{N}}$ the vector space of all sequences over $\mathbb{K}$; c_{00} is the subspace of $\mathbb{K}^{\mathbb{N}}$ of finitely supported sequences. The vectors

$$e_n := (0, 0, \ldots, 0, \underset{\text{pos. } n}{1}, 0, 0, \ldots) \qquad (n \in \mathbb{N}) \tag{1.1}$$

form a Hamel basis for c_{00}, called the *standard unit vector basis.* For $m \in \mathbb{N}$, the m^{th} *coordinate functional* is given by

$$f_m\colon\ (\alpha_n)_{n\in\mathbb{N}} \mapsto \alpha_m, \quad \mathbb{K}^{\mathbb{N}} \to \mathbb{K}, \tag{1.2}$$

and for a set A (usually a subset of $\mathbb{N}$), the *natural projection associated with* A is the operator $P_A\colon \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}}$ whose coordinates are given by

$$\langle P_A x, f_m\rangle := \begin{cases} \langle x, f_m\rangle & \text{if } m \in A \\ 0 & \text{otherwise} \end{cases} \qquad (m \in \mathbb{N},\ x \in \mathbb{K}^{\mathbb{N}}). \tag{1.3}$$

This is clearly an idempotent operator. In the special case where $A = \{1, 2, \ldots, m\}$ for some $m \in \mathbb{N}$, we write P_m instead of P_A, that is,

$$P_m\colon\ (\alpha_n)_{n\in\mathbb{N}} \mapsto (\alpha_1, \ldots, \alpha_m, 0, 0, \ldots), \quad \mathbb{K}^{\mathbb{N}} \to c_{00}; \tag{1.4}$$

we call P_m the m^{th} *natural projection.*

For a set A, we define $\chi_A \in \mathbb{K}^{\mathbb{N}}$ by

$$\langle \chi_A, f_m\rangle = \begin{cases} 1 & \text{if } m \in A \\ 0 & \text{otherwise} \end{cases} \qquad (m \in \mathbb{N});$$

the most important cases are

$$\chi_{[n,n+k]} = \sum_{j=n}^{n+k} e_j \in c_{00} \qquad (n, k \in \mathbb{N}) \qquad \text{and} \qquad \chi_{\mathbb{N}} = (1, 1, \ldots, 1, 1, \ldots). \tag{1.5}$$

Given a set B, we write $A \Subset B$ to indicate that A is a finite subset of B. In the case where B is totally ordered (typically $B = \mathbb{N}$), we write $A = \{n_1 < n_2 < \cdots < n_k\}$ to signify that $\{n_1, n_2, \ldots, n_k\}$ is the increasing ordering of A.

We denote by $\operatorname{card} A$ the cardinality of a set A. We shall not be concerned with the subtleties of infinite cardinalities because we shall only consider cardinalities of subsets of $\mathbb{N}$; hence we think of $\operatorname{card} A$ as belonging to $\{0, 1, 2, \ldots, \infty\}$.

2 James' quasi-reflexive Banach spaces

This section contains a brief survey of the James spaces and their most important properties, with special emphasis on the Banach-algebraic aspects; no proofs will be given. However, at the end we shall outline James' original construction because our subsequent work will involve key components of it. An elementary introduction to the James spaces is given in [30], while a much more comprehensive account can be found in [18]; both of these references contain full proofs.

The original *James space* J_2 was introduced by James in [22]. His main result states that this Banach space is isomorphic to its bidual and *quasi-reflexive* (meaning that the canonical image of J_2 in its bidual has codimension 1). This resolved two major open problems at the time:

(i) A Banach space with separable bidual need not be reflexive.

(ii) A separable Banach space which is isomorphic to its bidual need not be reflexive.

In a sequel [23], James defined an equivalent norm on J_2 with respect to which the space even becomes *isometrically* isomorphic to its bidual.

Subsequently, many other interesting properties of the James space have been added to James' list; we mention some of the most important here:

(iii) Bessaga and Pełczyński [8] observed that the quasi-reflexivity of J_2 implies that $J_2 \not\cong J_2 \oplus J_2$, making J_2 the first known example of an infinite-dimensional Banach space which is not isomorphic to its Cartesian square.

(iv) Herman and Whitley [21] proved that J_2 is *ℓ_2-saturated*, that is, every closed, infinite-dimensional subspace of J_2 contains a subspace X which is isomorphic to ℓ_2; Casazza, Lin, and Lohman [14] subsequently showed that it is always possible to choose such an X with the additional property that X is complemented in J_2.

(v) Edelstein and Mityagin [17] calculated the homotopy type of the group of invertible operators on J_2. As part of this calculation, they noted that the quasi-reflexivity of J_2 has the easy, but algebraically very important, consequence that the ideal $\mathscr{W}(J_2)$ of weakly compact operators has codimension 1 in the Banach algebra $\mathscr{B}(J_2)$ of bounded operators on J_2, so that there is a *character* on $\mathscr{B}(J_2)$. No such examples were previously known. Laustsen [25, 26] has subsequently shown that $\mathscr{W}(J_2)$ is the unique maximal ideal in $\mathscr{B}(J_2)$, while the scalar multiples of the character are the only traces on $\mathscr{B}(J_2)$.

(vi) Giesy and James [20] have shown that c_0 is finitely representable in J_2, that is, for each $\varepsilon > 0$ and each $n \in \mathbb{N}$, there is an operator $T\colon \ell_\infty^n \to J_2$ such that

$$(1-\varepsilon)\|x\|_{\ell_\infty^n} \leqslant \|Tx\|_{J_2} \leqslant (1+\varepsilon)\|x\|_{\ell_\infty^n} \qquad (x \in \ell_\infty^n).$$

Consequently, J_2 does not have finite coptype. In contrast, Pisier [33] has proved that the dual J_2' has cotype 2 and, moreover, that J_2 has the Gordon–Lewis property (which means that every 1-summing operator from J_2 to an arbitrary Banach space factors through L_1).

(vii) Casazza [13] has shown that J_2 is *primary*, that is, $X \cong J_2$ or $Y \cong J_2$ whenever X and Y are closed, complementary subspaces of J_2.

(viii) Andrew and Green [3] have shown that J_2 is a Banach algebra with respect to the pointwise product. Further, this Banach algebra is Arens regular, and its multiplier algebra can be identified with J_2'' with the Arens product.

(ix) Loy and Willis [29] have constructed a bounded right approximate identity in the ideal $\mathscr{W}(J_2)$ of weakly compact operators on J_2. This allowed them to deduce that derivations from $\mathscr{B}(J_2)$ are automatically continuous. Willis [35] has subsequently generalized this result by showing that homomorphisms from $\mathscr{B}(J_2)$ are automatically continuous.

(x) Odell and Tylli [31] completed Loy and Willis' work on bounded approximate identities in $\mathscr{W}(J_2)$ by showing that $\mathscr{W}(J_2)$ also has a bounded left approximate identity and hence a bounded two-sided approximate identity.

(xi) Laustsen [24] has calculated the K-groups of the Banach algebra $\mathscr{B}(J_2)$.

(xii) Blanco [11] has proved that the ideal $\mathscr{K}(J_2)$ of compact operators on J_2 is weakly amenable. It is an open problem if $\mathscr{K}(J_2)$ is amenable; Blanco and Grønbæk [12] have recently shown that $\mathscr{K}(J_2 \oplus J_2')$ fails to be amenable.

We shall now take a closer look at the formal definition of the James space and the strategy which James followed when he proved that it is quasi-reflexive.

2.1 The James spaces. Let $1 \leqslant p < \infty$. For $x = (\alpha_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$ and $A \Subset \mathbb{N}$, define

$$\nu_p(x, A) := \begin{cases} 0 & \text{when } \ \operatorname{card} A \leqslant 1; \\ \left(\displaystyle\sum_{j=1}^{k} |\alpha_{n_j} - \alpha_{n_{j+1}}|^p\right)^{\frac{1}{p}} & \text{when } A = \{n_1 < \cdots < n_{k+1}\} \text{ for some } k \in \mathbb{N}. \end{cases}$$

Using Minkowski's inequality, one can easily check that $\nu_p(\,\cdot\,, A)$ is a seminorm on $\mathbb{K}^{\mathbb{N}}$ for each $A \Subset \mathbb{N}$, and

$$\begin{aligned} \|x\|_{J_p} &:= \sup\bigl\{\nu_p(x, A) : A \Subset \mathbb{N}\bigr\} \\ &= \sup\left\{\left(\sum_{j=1}^{k} |\alpha_{n_j} - \alpha_{n_{j+1}}|^p\right)^{\frac{1}{p}} : k, n_1, \ldots, n_{k+1} \in \mathbb{N},\ n_1 < n_2 < \cdots < n_{k+1}\right\} \end{aligned}$$

defines a complete norm on the subspace $J_p := \bigl\{x \in c_0 : \|x\|_{J_p} < \infty\bigr\}$ of $\mathbb{K}^{\mathbb{N}}$. The Banach space $(J_p, \|\cdot\|_{J_p})$ is called the p^{th} *James space.*

2.2 Remark. James' original definition corresponds to the case $p = 2$ above. Edelstein and Mityagin [17] appear to have been the first to observe that it can be generalized to arbitrary $p \geqslant 1$ and, moreover, that James' proof of the quasi-reflexivity of J_2 carries over directly to J_p for each $p > 1$. The proof does not, however, work for $p = 1$ because J_1 is isometrically isomorphic to ℓ_1 via the mapping $(\alpha_n)_{n\in\mathbb{N}} \mapsto (\alpha_n - \alpha_{n+1})_{n\in\mathbb{N}},\ J_1 \to \ell_1$.

Most of the results (iii)–(xii) listed above have generalizations to J_p for $p > 1$. We state the following generalization of (iv) explicitly for later reference.

2.3 Proposition. *Let $p > 1$. Then every closed, infinite-dimensional subspace of J_p contains a subspace which is isomorphic to ℓ_p and complemented in J_p.*

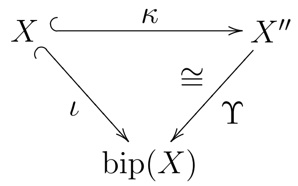
The key element in James' proof [22] that J_2 is quasi-reflexive is the following explicit description of the bidual of a Banach space with a shrinking basis. We follow Megginson's presentation [30, Section 4.5].

2.4 The James representation of the bidual of a Banach space with a shrinking basis. For a Banach space X with a fixed basis $(b_n)_{n\in\mathbb{N}}$, define

$$\|x\|_{\mathrm{bip}(X)} := \sup\left\{\left\|\sum_{n=1}^{m}\alpha_n b_n\right\|_X : m \in \mathbb{N}\right\} \qquad \big(x = (\alpha_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}\big). \tag{2.1}$$

Then $\mathrm{bip}(X) := \{x \in \mathbb{K}^{\mathbb{N}} : \|x\|_{\mathrm{bip}(X)} < \infty\}$ is a subspace of $\mathbb{K}^{\mathbb{N}}$, and $\|\cdot\|_{\mathrm{bip}(X)}$ is a complete norm on it.

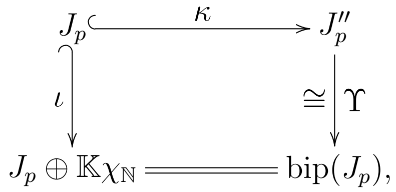
Now suppose that the basis $(b_n)_{n\in\mathbb{N}}$ is shrinking. Then the operator $\Upsilon\colon X'' \to \mathrm{bip}(X)$ given by $\Upsilon(F) := \big(\langle b_n', F\rangle\big)_{n\in\mathbb{N}}$ for each $F \in X''$ is an isomorphism which makes the diagram



commutative, where ι is the operator $x \mapsto \big(\langle x, b_n'\rangle\big)_{n\in\mathbb{N}}$, while κ is the canonical embedding. The isomorphism Υ is isometric if the basis $(b_n)_{n\in\mathbb{N}}$ is monotone.

2.5 Remark. There is no standard notation for the Banach space $\mathrm{bip}(X)$ considered in §2.4; we have chosen the letters bip for 'bounded initial projections'.

2.6 The quasi-reflexivity of the James spaces. The starting point of the proof that J_p is quasi-reflexive for each $p > 1$ is the observation that $(e_n)_{n\in\mathbb{N}}$ is a monotone, shrinking basis for J_p, so that the results of §2.4 apply. One then checks that J_p and $\mathbb{K}\chi_{\mathbb{N}}$ are closed, complementary subspaces of $\mathrm{bip}(J_p)$, and we therefore have a commutative diagram



where ι is the natural inclusion, κ is the canonical embedding, and Υ is the isometric isomorphism from §2.4. This shows immediately that $\kappa(J_p)$ has codimension 1 in J_p'' and, moreover, if we identify J_p'' with $\operatorname{bip}(J_p)$ via Υ, then $\kappa = \iota$, that is, the canonical embedding of J_p into its bidual becomes the natural inclusion of J_p into $J_p \oplus \mathbb{K}\chi_{\mathbb{N}}$.

3 The Schreier spaces

Having proved that, for $1 < p < \infty$, each weakly convergent sequence $(x_n)_{n\in\mathbb{N}}$ in $L_p[0,1]$ has a subsequence $(x_{n_j})_{j\in\mathbb{N}}$ whose associated sequence of arithmetic means, $\big(\frac{1}{m}\sum_{j=1}^m x_{n_j}\big)_{m\in\mathbb{N}}$, is norm-convergent, Banach and Saks [5] went on to ask if a similar result would be true in $C[0,1]$. Schreier [34] answered this question in the negative; his counterexample was based on the following notion.

3.1 Definition. A subset A of $\mathbb{N}$ is *admissible* if $1 \leqslant \operatorname{card} A \leqslant \min A$.

Thus, an admissible set is non-empty and finite, and for $A = \{n_1 < \cdots < n_k\} \Subset \mathbb{N}$, we have

$$A \text{ is admissible} \iff k \leqslant n_1.$$

Each non-empty subset of an admissible set is itself admissible; we shall use this simple observation frequently.

3.2 The unrestricted Schreier spaces. Let $1 \leqslant p < \infty$. For $x = (\alpha_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$ and $A \subseteq \mathbb{N}$, define $\mu_p(x, A) := \big(\sum_{n\in A} |\alpha_n|^p\big)^{\frac{1}{p}}$. (Note: the sum over an empty index set is zero by convention.) Minkowski's inequality implies that $\mu_p(\,\cdot\,, A)$ is a seminorm on $\mathbb{K}^{\mathbb{N}}$ for each $A \Subset \mathbb{N}$, and

$$\begin{aligned} \|x\|_{Z_p} &:= \sup\big\{\mu_p(x, A) : A \subseteq \mathbb{N} \text{ is admissible}\big\} \\ &= \sup\left\{\left(\sum_{j=1}^{k} |\alpha_{n_j}|^p\right)^{\frac{1}{p}} : k, n_1, \ldots, n_k \in \mathbb{N},\ k \leqslant n_1 < n_2 < \cdots < n_k\right\} \end{aligned}$$

then defines a norm on the subspace $Z_p := \big\{x \in \mathbb{K}^{\mathbb{N}} : \|x\|_{Z_p} < \infty\big\}$ of $\mathbb{K}^{\mathbb{N}}$. We call $\|\cdot\|_{Z_p}$ the p^{th} *Schreier norm* and Z_p the p^{th} *unrestricted Schreier space.*

The following lemma records some basic facts about this space; we omit its easy proof.

3.3 Lemma. *Let $p \geqslant 1$. Then:*

(i) *for each $m \in \mathbb{N}$, the restriction to Z_p of the m^{th} coordinate functional f_m given by* (1.2) *is bounded with norm* 1;

(ii) *$\langle x, f_m\rangle \to 0$ as $m \to \infty$ for each $x \in Z_p$;*

(iii) *for each $A \subseteq \mathbb{N}$, Z_p is an invariant subspace for the projection P_A given by* (1.3), *and the restriction to Z_p of P_A is bounded with norm* 1 *(except for $A = \emptyset$, in which case $P_A = 0$);*

(iv) *Z_p is a Banach space with respect to the coordinatewise defined operations inherited from $\mathbb{K}^{\mathbb{N}}$ and the norm $\|\cdot\|_{Z_p}$.*

3.4 The unconditional multiplier constant. Suppose that $(b_n)_{n\in\mathbb{N}}$ is an unconditional basic sequence in a Banach space, and let $(\alpha_n)_{n\in\mathbb{N}}$ be a scalar sequence such that the series $\sum_{n=1}^\infty \alpha_n b_n$ converges. Then, for each scalar sequence $(\beta_n)_{n\in\mathbb{N}}$ with $|\beta_n| \leqslant |\alpha_n|$ for each $n \in \mathbb{N}$, the series $\sum_{n=1}^\infty \beta_n b_n$ converges, and there is a constant $K \geqslant 1$ (independent of $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$) such that $\left\|\sum_{n=1}^\infty \beta_n b_n\right\| \leqslant K \left\|\sum_{n=1}^\infty \alpha_n b_n\right\|$. The smallest such constant K is the *unconditional multiplier constant* of $(b_n)_{n\in\mathbb{N}}$. If $K = 1$ (the smallest possible), then $(b_n)_{n\in\mathbb{N}}$ is 1-*unconditional* (or *monotone unconditional*).

3.5 Proposition. *For each $p \geqslant 1$, the sequence $(e_n)_{n\in\mathbb{N}}$ given by* (1.1) *is a normalized,* 1-*unconditional basic sequence in Z_p.*

Proof. Clearly we have $\|e_n\|_{Z_p} = 1$ for each $n \in \mathbb{N}$. Suppose that $x = \sum_{n=1}^\infty \alpha_n e_n$ and $y = \sum_{n=1}^\infty \beta_n e_n$ are vectors in c_{00} with $|\beta_n| \leqslant |\alpha_n|$ for each $n \in \mathbb{N}$. This implies that $\mu_p(y, A)^p \leqslant \mu_p(x, A)^p$ for each subset A of $\mathbb{N}$, and consequently $\|y\|_{Z_p} \leqslant \|x\|_{Z_p}$. The conclusion now follows from the 1-unconditional counterpart of Banach's fundamental characterization of basic sequences (analogous to [30, Corollary 4.2.33], for instance). □

3.6 The restricted Schreier spaces. Lemma 3.3(iv) and Proposition 3.5 imply that

$$S_p := \overline{\operatorname{span}}\,\{e_n : n \in \mathbb{N}\} \subseteq Z_p$$

is a Banach space with a normalized, 1-unconditional basis $(e_n)_{n\in\mathbb{N}}$ for each $p \geqslant 1$. We call S_p the p^{th} *(restricted) Schreier space* and refer to $(e_n)_{n\in\mathbb{N}}$ as its *standard unit vector basis.* More precisely, we usually call S_p just the p^{th} *Schreier space,* inserting the adjective 'restricted' only if confusion with the unrestricted counterpart Z_p seems likely. We write $\|\cdot\|_{S_p}$ for the restriction to S_p of the norm $\|\cdot\|_{Z_p}$.

Note that the m^{th} biorthogonal functional associated with the standard unit vector basis $(e_n)_{n\in\mathbb{N}}$ for S_p is the restriction to S_p of the m^{th} coordinate functional on $\mathbb{K}^{\mathbb{N}}$, that is, $e'_m = f_m|_{S_p}$; Lemma 3.3(i) implies that this functional has norm 1.

3.7 Remark. The Banach space denoted by S_1 in §3.6 is the one which is usually called *the* Schreier space in the literature. Beauzamy and Lapresté formally introduced it in [6], building on ideas from Baernstein's thesis [4]; an introduction to this work, its context, and many further developments is given in [15, Chapter 0].

We are not aware of any previous studies of the Banach spaces Z_p and S_p for $p > 1$.

3.8 Example. Let $p \geqslant 1$. As we shall see in Corollary 5.6, Z_p contains a subspace isomorphic to ℓ_∞. Hence Z_p is non-separable and cannot, therefore, have a basis, so in particular Z_p is not isomorphic to S_p. The purpose of this example is much simpler, namely to prove that S_p is a *proper* subspace of Z_p. More precisely, we shall show that $z := (n^{-\frac{1}{p}})_{n\in\mathbb{N}} \in Z_p \setminus S_p$.

Suppose that $A = \{n_1 < n_2 < \cdots < n_k\} \subseteq \mathbb{N}$ is admissible. Since z is decreasing and positive, we have

$$\mu_p(z, A)^p = \sum_{j=1}^{k} \frac{1}{n_j} \leqslant \sum_{j=n_1}^{n_1+k-1} \frac{1}{j} \leqslant \sum_{j=n_1}^{2n_1-1} \frac{1}{j} \leqslant n_1 \cdot \frac{1}{n_1} = 1,$$

so $z \in Z_p$ with $\|z\|_{Z_p} \leqslant 1$; in fact, z is a unit vector because $\langle z, f_1\rangle = 1$. On the other hand, $z \notin S_p$ because, for any $y = \sum_{n=1}^{m} \alpha_n e_n \in c_{00}$, the set $B := \mathbb{N} \cap (m, 2m+1]$ is admissible, and therefore

$$\|z-y\|_{Z_p}^p \geqslant \mu_p(z-y, B)^p = \sum_{n=m+1}^{2m+1} \frac{1}{n} \geqslant \int_{m+1}^{2m+2} \frac{\mathrm{d}t}{t} = \log \frac{2m+2}{m+1} = \log 2 > 0.$$

We shall next prove that the standard unit vector basis for S_p is shrinking for each $p \geqslant 1$. As a consequence, we obtain a description of the bidual of S_p in terms of Z_p. Our approach relies on the following well-known characterization of shrinking bases (*e.g.*, see [1, Proposition 3.2.7]).

3.9 Lemma. *A basis $(b_n)_{n\in\mathbb{N}}$ for a Banach space X is shrinking if and only if every normalized block basic sequence of $(b_n)_{n\in\mathbb{N}}$ is a weakly null sequence in X.*

The fact that the standard unit vector basis for S_1 is shrinking is well-known, although we have been unable to trace the original source of this result. We shall outline an elegant proof communicated to us by András Zsák.

3.10 Proposition. *The standard unit vector basis $(e_n)_{n\in\mathbb{N}}$ for S_1 is shrinking.*

Proof. Each subset A of $\mathbb{N}$ induces a functional ω_A on c_{00} via the definition

$$\langle x, \omega_A\rangle := \sum_{n\in A} \langle x, f_n\rangle \qquad (x \in c_{00}),$$

and this functional is contractive with respect to the first Schreier norm if and only if the set A is either empty or admissible. Thus, we may regard

$$\Omega := \{\omega_A : A = \emptyset \text{ or } A \subseteq \mathbb{N} \text{ admissible}\}$$

as a subset of the closed unit ball of S_1', and we have

$$\Omega = \big\{ f \in S_1' : \|f\|_{S_1'} \leqslant 1 \text{ and } \langle e_n, f\rangle \in \{0,1\}\ (n \in \mathbb{N})\big\}$$

which implies that Ω is weak*-closed and hence weak*-compact by the Banach–Alaoglu Theorem.

For each $x \in S_1$, the mapping $Ux\colon f \mapsto \langle x, f\rangle,\ \Omega \to \mathbb{K}$, is continuous with respect to the weak*-topology on Ω, so it induces a mapping $U\colon S_1 \to C(\Omega)$. Straightforward verifications show that U is linear and contractive. Moreover, U is bounded below by

$$\varepsilon := \begin{cases} \frac{1}{2} & \text{for } \mathbb{K} = \mathbb{R} \\ \frac{1}{4} & \text{for } \mathbb{K} = \mathbb{C} \end{cases}$$

as we shall now prove. It suffices to show that, for each $x \in c_{00}$, there is a subset A of $\mathbb{N}$, either admissible or empty, such that $\big|\langle x, \omega_A\rangle\big| \geqslant \varepsilon \|x\|_{S_1}$.

In the case where x has real coordinates, take an admissible subset B of $\mathbb{N}$ such that $\|x\|_{S_1} = \mu_1(x, B)$. The sets $B_+ := \{n \in B : \langle x, f_n\rangle > 0\}$ and $B_- := \{n \in B : \langle x, f_n\rangle < 0\}$ are then admissible or empty, and they satisfy

$$\|x\|_{S_1} = \mu_1(x, B) = \sum_{n\in B_+} \langle x, f_n\rangle - \sum_{n\in B_-} \langle x, f_n\rangle = \big|\langle x, \omega_{B_+}\rangle\big| + \big|\langle x, \omega_{B_-}\rangle\big|,$$

so we conclude that $\big|\langle x, \omega_A\rangle\big| \geqslant \|x\|_{S_1}/2$ holds for either $A := B_+$ or $A := B_-$.

Now suppose that x has complex coordinates and define $y := \sum_{n=1}^{\infty} \operatorname{Re}\langle x, f_n\rangle\, e_n$ and $z := \sum_{n=1}^{\infty} \operatorname{Im}\langle x, f_n\rangle\, e_n$. Then $x = y + \mathrm{i}z$, so that $\|x\|_{S_1} \leqslant \|y\|_{S_1} + \|z\|_{S_1}$ and thus either $\|y\|_{S_1} \geqslant \|x\|_{S_1}/2$ or $\|z\|_{S_1} \geqslant \|x\|_{S_1}/2$. We consider the first case only; the second is similar. As y has real coordinates, the first part of the argument applies, yielding an admissible (or empty) set A such that $\big|\langle y, \omega_A\rangle\big| \geqslant \|y\|_{S_1}/2$, and consequently we have

$$\big|\langle x, \omega_A\rangle\big| = \big|\langle y, \omega_A\rangle + \mathrm{i}\langle z, \omega_A\rangle\big| \geqslant \big|\langle y, \omega_A\rangle\big| \geqslant \frac{\|y\|_{S_1}}{2} \geqslant \frac{\|x\|_{S_1}}{4},$$

as required.

In conclusion, U is an isomorphism of S_1 onto its image inside $C(\Omega)$. Now let $(u_n)_{n\in\mathbb{N}}$ be a normalized block basic sequence of $(e_n)_{n\in\mathbb{N}}$ in S_1. By Lemma 3.9, we must prove that $(u_n)_{n\in\mathbb{N}}$ is weakly null in S_1. This is equivalent to $(Uu_n)_{n\in\mathbb{N}}$ being weakly null in $C(\Omega)$ because the weak topology on the image of U in $C(\Omega)$ is just the restriction of the weak topology on $C(\Omega)$. For each $\omega_A \in \Omega$, we have $(Uu_n)(\omega_A) = \langle u_n, \omega_A\rangle = 0$ eventually because $A \subseteq \mathbb{N}$ is finite, while $(u_n)_{n\in\mathbb{N}}$ is a block basic sequence. Hence $(Uu_n)_{n\in\mathbb{N}}$ converges pointwise to 0 on Ω. As this sequence is also norm-bounded, [16, Corollary IV.6.4] implies that it is weakly null in $C(\Omega)$, as required. □

The proof above does not work for $p > 1$, but fortunately an easier route is available for such p. This relies on the following result.

3.11 Lemma. *Let $p \geqslant 1$, and let $(u_n)_{n\in\mathbb{N}}$ be a normalized block basic sequence of the standard unit vector basis $(e_n)_{n\in\mathbb{N}}$ for S_p. Then the operator*

$$T\colon\ \sum_{n=1}^{\infty} \alpha_n e_n \mapsto \sum_{n=1}^{\infty} \alpha_n u_n, \quad \big(c_{00}, \|\cdot\|_{\ell_p}\big) \to S_p,$$

is contractive, and therefore it extends uniquely to a contractive operator $T\colon \ell_p \to S_p$.

Proof. Choose integers $0 = M_0 < M_1 < \cdots < M_{n-1} < M_n < \cdots$ such that

$$u_n \in \operatorname{span}\{e_j : M_{n-1} < j \leqslant M_n\} \qquad (n \in \mathbb{N}).$$

Given an admissible subset A of $\mathbb{N}$, define $A_n := A \cap (M_{n-1}, M_n]$ for each $n \in \mathbb{N}$. Then A is the disjoint union of $(A_n)_{n\in\mathbb{N}}$, and each of the sets A_n is either empty or admissible.

Hence, for $x = \sum_{n=1}^{\infty} \alpha_n e_n \in c_{00}$, we have

$$\begin{aligned} \|x\|_{\ell_p}^p &= \sum_{n=1}^{\infty} |\alpha_n|^p \geqslant \sum_{n=1}^{\infty} |\alpha_n|^p \mu_p(u_n, A_n)^p = \sum_{n=1}^{\infty} \mu_p(\alpha_n u_n, A_n)^p \\ &= \sum_{n=1}^{\infty} \mu_p\Bigl(\sum_{j=1}^{\infty} \alpha_j u_j, A_n\Bigr)^p = \sum_{n=1}^{\infty} \mu_p(Tx, A_n)^p = \mu_p(Tx, A)^p, \end{aligned}$$

and the result follows by taking the supremum over all admissible sets A. □

3.12 Corollary. *For each $p > 1$, the standard unit vector basis $(e_n)_{n\in\mathbb{N}}$ for S_p is shrinking.*

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a normalized block basic sequence of $(e_n)_{n\in\mathbb{N}}$ in S_p, and consider the contractive operator $T\colon \ell_p \to S_p$ from Lemma 3.11. Since $p > 1$, the standard unit vector basis $(e_n)_{n\in\mathbb{N}}$ is a weakly null sequence in ℓ_p, so the boundedness of T implies that the sequence $(Te_n)_{n\in\mathbb{N}} = (u_n)_{n\in\mathbb{N}}$ is weakly null in S_p. Hence the result follows from Lemma 3.9. □

3.13 Lemma. *Let $p \geqslant 1$. Then $\|x\|_{\mathrm{bip}(S_p)} = \|x\|_{Z_p}$ for each $x \in \mathbb{K}^{\mathbb{N}}$, and so $\mathrm{bip}(S_p) = Z_p$.*

Proof. Comparing the definitions (2.1) and (1.4), we see that

$$\|x\|_{\mathrm{bip}(S_p)} = \sup\bigl\{\|P_m x\|_{S_p} : m \in \mathbb{N}\bigr\} \qquad (x \in \mathbb{K}^{\mathbb{N}}). \tag{3.1}$$

Hence the inequality $\|x\|_{\mathrm{bip}(S_p)} \leqslant \|x\|_{Z_p}$ follows from Lemma 3.3(iii) whenever $x \in Z_p$; otherwise (that is, for $x \in \mathbb{K}^{\mathbb{N}} \setminus Z_p$) it is trivial.

Conversely, let A be an admissible subset of $\mathbb{N}$. Taking $m := \max A \in \mathbb{N}$, we obtain

$$\mu_p(x, A) = \mu_p(P_m x, A) \leqslant \|P_m x\|_{S_p} \leqslant \|x\|_{\mathrm{bip}(S_p)},$$

and therefore $\|x\|_{Z_p} \leqslant \|x\|_{\mathrm{bip}(S_p)}$. □

3.14 The unrestricted Schreier space and the bidual of S_p. For each $p \geqslant 1$, we can combine the results of Proposition 3.10, Corollary 3.12, Lemma 3.13, and §2.4 to obtain a commutative diagram

$$\begin{array}{ccc} S_p & \xrightarrow{\ \kappa\ } & S_p'' \\ {\scriptstyle\iota}\downarrow & & \downarrow{\scriptstyle \Upsilon}\ \cong \\ Z_p & = & \mathrm{bip}(S_p), \end{array}$$

where ι is the natural inclusion, κ is the canonical embedding, and Υ is the isometric isomorphism from §2.4. Thus, if we identify the bidual of S_p with $\mathrm{bip}(S_p)$ via Υ, then $\kappa = \iota$, that is, the canonical embedding of S_p into its bidual becomes the natural inclusion of S_p into Z_p.

3.15 Shift operators. Let $\sigma\colon \mathbb{N} \to \mathbb{N}$ be strictly increasing. We associate with σ two operators on c_{00}, the *left shift* Λ_σ and the *right shift* R_σ, given by

$$\Lambda_\sigma x := \sum_{n=1}^{\infty} \alpha_{\sigma(n)} e_n \qquad \text{and} \qquad R_\sigma x := \sum_{n=1}^{\infty} \alpha_n e_{\sigma(n)} \qquad \Bigl(x = \sum_{n=1}^{\infty} \alpha_n e_n \in c_{00}\Bigr). \tag{3.2}$$

They are clearly linear, and $\Lambda_\sigma R_\sigma = I_{c_{00}}$, while $R_\sigma \Lambda_\sigma$ is the restriction to c_{00} of the projection $P_{\sigma(\mathbb{N})}$ given by (1.3).

Our interest in these operators lies primarily in understanding when they extend to bounded operators on S_p. In order to treat both cases simultaneously (and for later use, see the proof of Proposition 4.20 below), we introduce the *mixed shift operator* $\Xi_{\tau,\omega}$ associated with a pair of injective mappings τ, $\omega\colon \mathbb{N} \to \mathbb{N}$ by

$$\Xi_{\tau,\omega}\colon \ \sum_{n=1}^{\infty} \alpha_n e_n \mapsto \sum_{n=1}^{\infty} \alpha_{\tau(n)} e_{\omega(n)}, \quad c_{00} \to c_{00}.$$

Again, $\Xi_{\tau,\omega}$ is clearly linear, and writing ι for the identity mapping $n \mapsto n$, $\mathbb{N} \to \mathbb{N}$, we see that $\Lambda_\sigma = \Xi_{\sigma,\iota}$ and $R_\sigma = \Xi_{\iota,\sigma}$.

3.16 Lemma. *Let $p \geqslant 1$, let τ, $\omega\colon \mathbb{N} \to \mathbb{N}$ be injective, and suppose that there is a constant $K \in \mathbb{N}$ such that $\omega(n) \leqslant K\tau(n)$ for each $n \in \mathbb{N}$. Then $\|\Xi_{\tau,\omega} x\|_{S_p}^p \leqslant K\|x\|_{S_p}^p$ for each $x \in c_{00}$, and thus $\Xi_{\tau,\omega}$ extends uniquely to a bounded operator on S_p of norm at most $K^{\frac{1}{p}}$.*

Proof. Let $x \in c_{00}$ and an admissible subset A of $\mathbb{N}$ be given. If $A \cap \omega(\mathbb{N}) = \emptyset$, then $\mu_p(\Xi_{\tau,\omega} x, A) = 0$. Otherwise choose $B \subseteq \mathbb{N}$ such that $A \cap \omega(\mathbb{N}) = \omega(B)$, and let $m := \min \tau(B)$. Being a non-empty subset of the admissible set A, the set $\omega(B)$ is itself admissible, and this, together with our assumptions on ω and τ, implies that

$$\operatorname{card} B = \operatorname{card} \omega(B) \leqslant \min \omega(B) \leqslant K \min \tau(B) = Km,$$

so we can express B as the union of K disjoint sets $B_1, \ldots, B_K$, say, each having at most m elements. Thus each of the sets $\tau(B_1), \ldots, \tau(B_K)$ is admissible (or empty), and therefore we have

$$\begin{aligned} K\|x\|_{S_p}^p &\geqslant \sum_{j=1}^{K} \mu_p\bigl(x, \tau(B_j)\bigr)^p = \mu_p\bigl(x, \tau(B)\bigr)^p = \sum_{n \in B} \bigl|\langle x, f_{\tau(n)}\rangle\bigr|^p \\ &= \sum_{n \in B} \bigl|\langle \Xi_{\tau,\omega} x, f_{\omega(n)}\rangle\bigr|^p = \mu_p\bigl(\Xi_{\tau,\omega} x, \omega(B)\bigr)^p = \mu_p(\Xi_{\tau,\omega} x, A)^p. \end{aligned}$$

Taking the supremum over all admissible sets A, we conclude that $K\|x\|_{S_p}^p \geqslant \|\Xi_{\tau,\omega} x\|_{S_p}^p$, and the result follows. ☐

3.17 Corollary. *Let $p \geqslant 1$, and let $\sigma\colon \mathbb{N} \to \mathbb{N}$ be strictly increasing. Then:*

(i) $\|\Lambda_\sigma x\|_{S_p} \leqslant \|x\|_{S_p}$ *for each* $x \in c_{00}$, *so that* Λ_σ *extends uniquely to a contractive operator on* S_p, *no matter what* σ *is;*

(ii) R_σ *is a bounded operator on* $\big(c_{00}, \|\cdot\|_{S_p}\big)$ *if and only if there is a constant* $K \in \mathbb{N}$ *such that* $\sigma(n) \leqslant Kn$ *for each* $n \in \mathbb{N}$. *In the positive case,* R_σ *extends uniquely to a bounded operator of norm at most* $K^{\frac{1}{p}}$ *on* S_p.

Proof. To prove (i), recall from §3.15 that $\Lambda_\sigma = \Xi_{\sigma,\iota}$. Since σ is strictly increasing, we have $\sigma(n) \geqslant n = \iota(n)$ for each $n \in \mathbb{N}$, so that the condition in Lemma 3.16 is satisfied with $K = 1$, and therefore Λ_σ extends to a bounded operator on S_p with norm at most 1.

The implication '$\Leftarrow$' in (ii) and the final clause follow in a similar fashion, so it only remains to prove '$\Rightarrow$' in (ii). Thus, suppose that $R_\sigma \colon \big(c_{00}, \|\cdot\|_{S_p}\big) \to \big(c_{00}, \|\cdot\|_{S_p}\big)$ is bounded, let $n \in \mathbb{N}$ be given, and consider the unit vector $z := (m^{-\frac{1}{p}})_{m\in\mathbb{N}} \in Z_p$ from Example 3.8. If we define $N := n + \sigma(n) - 1$ and $A := \mathbb{N} \cap [n, N]$, then the set $\sigma(A)$ is admissible, and Lemma 3.3(iii) implies that $P_A z \in c_{00}$ with norm at most 1, so that

$$\|R_\sigma\|^p \geqslant \|R_\sigma P_A z\|_{S_p}^p \geqslant \mu_p\big(R_\sigma P_A z, \sigma(A)\big)^p = \sum_{m=n}^{N} \big|\langle R_\sigma P_A z, f_{\sigma(m)}\rangle\big|^p$$
$$= \sum_{m=n}^{N} \big|\langle z, f_m\rangle\big|^p = \sum_{m=n}^{N} \frac{1}{m} \geqslant \int_n^{N+1} \frac{\mathrm{d}t}{t} = \log \frac{N+1}{n} = \log\Big(1 + \frac{\sigma(n)}{n}\Big)$$

from which we deduce that $(e^{\|R_\sigma\|^p} - 1)n \geqslant \sigma(n)$; hence any integer $K \geqslant e^{\|R_\sigma\|^p} - 1$ has the required property. □

The following well-known observation will be called upon repeatedly in the sequel.

3.18 Remark. Let X and Y be normed spaces, and let $\Gamma_1 \colon X \to Y$ and $\Gamma_2 \colon Y \to X$ be bounded operators with $\Gamma_1\Gamma_2 = I_Y$. Then $P := \Gamma_2\Gamma_1$ is a bounded idempotent operator on X whose image is isomorphic to Y. More precisely, we have $\operatorname{im} P = \operatorname{im}\Gamma_2$, and the 'corestriction' of Γ_2 to $\operatorname{im} P$ (that is, the operator $y \mapsto \Gamma_2 y,\ Y \to \operatorname{im} P$) is an isomorphism; its inverse is the restriction of Γ_1 to $\operatorname{im} P$.

3.19 Corollary. *For each* $p \geqslant 1$, *the* p^{th} *Schreier space* S_p *is isomorphic to its Cartesian square* $S_p \oplus S_p$.

Proof. Consider the strictly increasing mappings $\alpha, \beta \colon \mathbb{N} \to \mathbb{N}$ given by

$$\alpha(n) := 2n - 1 \qquad \text{and} \qquad \beta(n) := 2n \qquad (n \in \mathbb{N}). \tag{3.3}$$

Corollary 3.17 shows that the associated shifts Λ_α, R_α, Λ_β, and R_β extend to bounded operators on S_p. Consequently, the identities stated in §3.15 immediately after (3.2) together with Remark 3.18 imply that $Q_\alpha := R_\alpha\Lambda_\alpha$ and $Q_\beta := R_\beta\Lambda_\beta$ are bounded idempotent operators on S_p whose images are isomorphic to S_p, and therefore we have

$$S_p \oplus S_p \cong \operatorname{im} Q_\alpha \oplus \operatorname{im} Q_\beta \cong S_p,$$

where the final isomorphism follows from the fact that $\operatorname{im} Q_\beta = \ker Q_\alpha$, so that $\operatorname{im} Q_\alpha$ and $\operatorname{im} Q_\beta$ are closed, complementary subspaces of S_p. □

3.20 Further results. We conclude this section by listing a few results from the literature on Schreier spaces.

(i) Alspach and Argyros [2] have generalized the notion of admissible subset of $\mathbb{N}$ by introducing the *Schreier families* of finite subsets of $\mathbb{N}$. The Schreier families are indexed by the countable ordinals; the first one is exactly the collection of all admissible subsets of $\mathbb{N}$. In analogy with the definition of the p^{th} Schreier space based on the admissible subsets of $\mathbb{N}$ (as presented in this section), one can associate a p^{th} Schreier space with each Schreier family. There is a substantial body of theory concerning these spaces in the case $p = 1$; the case $p > 1$ appears to be unchartered territory.

(ii) Gasparis and Leung [19] have studied the complemented subspaces of the first Schreier space S_1 (as well as those of the generalized Schreier spaces of Alspach and Argyros described above). Their main result is that there are uncountably many non-isomorphic subspaces among those of the form $\overline{\operatorname{span}}\,\{e_n : n \in A\}$, where A is an infinite subset of $\mathbb{N}$.

4 Introduction to the James–Schreier spaces

We shall now amalgamate the definitions of the p^{th} James space and the p^{th} Schreier space for each $p \geqslant 1$. Recall from §2.1 that, by definition, the p^{th} James norm of $x \in \mathbb{K}^{\mathbb{N}}$ is the supremum of $\nu_p(x, A)$ over all finite subsets A of $\mathbb{N}$; our idea is to consider only those sets A which satisfy an admissibility-like condition.

4.1 Definition. A subset A of $\mathbb{N}$ is *permissible* if

$$2 \leqslant \operatorname{card} A \leqslant 1 + \min A.$$

Thus a permissible set must contain at least two elements and be finite, and for $A = \{n_1 < n_2 < \cdots < n_{k+1}\} \subseteq \mathbb{N}$, where $k \in \mathbb{N}$, we have

$$A \text{ is permissible} \iff k \leqslant n_1.$$

4.2 Definition. Let $1 \leqslant p < \infty$. For $x = (\alpha_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$, we define

$$\begin{aligned}\|x\|_{W_p} &:= \sup\bigl\{\nu_p(x, A) : A \subseteq \mathbb{N} \text{ is permissible}\bigr\}\\ &= \sup\left\{\left(\sum_{j=1}^{k} |\alpha_{n_j} - \alpha_{n_{j+1}}|^p\right)^{\frac{1}{p}} : k, n_1, \ldots, n_{k+1} \in \mathbb{N},\ k \leqslant n_1 < \cdots < n_{k+1}\right\},\end{aligned}$$

and we then let $W_p := \bigl\{x \in c_0 : \|x\|_{W_p} < \infty\bigr\}$. We call $\|\cdot\|_{W_p}$ the p^{th} *James–Schreier norm* and W_p the p^{th} *unrestricted James–Schreier space.*

The following lemma is the James–Schreier counterpart of Lemma 3.3; we omit its routine proof.

4.3 Lemma. *Let $p \geqslant 1$ and $m \in \mathbb{N}$. Then:*

(i) *the restriction to W_p of the m^{th} coordinate functional f_m given by* (1.2) *is bounded with norm* 1;

(ii) *W_p is an invariant subspace for the m^{th} natural projection P_m given by* (1.4), *and the restriction of P_m to W_p is bounded with norm* 1;

(iii) *W_p is a Banach space with respect to the coordinatewise defined operations inherited from $\mathbb{K}^{\mathbb{N}}$ and the norm $\|\cdot\|_{W_p}$.*

We shall next show that, for each $p \geqslant 1$, W_p contains a complemented subspace which is isomorphic to the unrestricted Schreier space Z_p. In the proof we require the following well-known inequality

$$|\alpha+\beta|^p \leqslant 2^{p-1}\big(|\alpha|^p + |\beta|^p\big) \qquad (p \geqslant 1,\ \alpha, \beta \in \mathbb{K}) \tag{4.1}$$

which is an easy consequence of the convexity of the function $t \mapsto t^p,\ [0,\infty) \to [0,\infty)$.

4.4 Definition. For $x = (\alpha_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$, let

$$\Phi x := (0, \alpha_1, 0, \alpha_2, 0, \alpha_3, \ldots) \in \mathbb{K}^{\mathbb{N}} \qquad \text{and} \qquad \Psi x := (\alpha_{2n} - \alpha_{2n-1})_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}. \tag{4.2}$$

4.5 Proposition. *Let $p \geqslant 1$. The first equation in* (4.2) *defines a bounded operator Φ from Z_p to W_p of norm at most $2^{1+\frac{1}{p}}$, while the second defines a bounded operator Ψ from W_p to Z_p of norm* 1*. These operators satisfy $\Psi\Phi = I_{Z_p}$, and hence $\Phi\Psi$ is idempotent with*

$$\operatorname{im}(\Phi\Psi) = \big\{(\alpha_n)_{n\in\mathbb{N}} \in W_p : \alpha_{2n-1} = 0 \ \ (n \in \mathbb{N})\big\} \cong Z_p.$$

Proof. Considered as mappings on $\mathbb{K}^{\mathbb{N}}$, Φ and Ψ are clearly both linear, so to establish the first part of the proposition, it suffices to show that

(i) $\Phi x \in W_p$ with $\|\Phi x\|_{W_p} \leqslant 2^{1+\frac{1}{p}}\|x\|_{Z_p}$ for each $x \in Z_p$; and

(ii) $\Psi x \in Z_p$ with $\|\Psi x\|_{Z_p} \leqslant \|x\|_{W_p}$ for each $x \in W_p$, with equality for some $x \in W_p$.

To prove (i), let $x \in Z_p$ be given. Lemma 3.3(ii) implies that $x \in c_0$, and therefore $\Phi x \in c_0$ by the definition of Φ. Now suppose that $A = \{n_1 < n_2 < \cdots < n_{k+1}\}$ is a permissible subset of $\mathbb{N}$. If $A \cap 2\mathbb{N} = \emptyset$, then $\nu_p(\Phi x, A) = 0$. Otherwise we choose $\ell, m_1, \ldots, m_\ell \in \mathbb{N}$ such that $A \cap 2\mathbb{N} = \{2m_1 < 2m_2 < \cdots < 2m_\ell\}$. Using the inequality (4.1), we obtain

$$\begin{aligned}\nu_p(\Phi x, A)^p = \sum_{j=1}^{k}\big|\langle \Phi x, f_{n_j} - f_{n_{j+1}}\rangle\big|^p \leqslant 2^{p-1}\sum_{j=1}^{k}\Big(\big|\langle \Phi x, f_{n_j}\rangle\big|^p + \big|\langle \Phi x, f_{n_{j+1}}\rangle\big|^p\Big)\\ \leqslant 2^p\mu_p(\Phi x, A)^p = 2^p\mu_p(\Phi x, A\cap 2\mathbb{N})^p = 2^p\mu_p(x, B)^p,\end{aligned} \tag{4.3}$$

where we have introduced $B := \{m_1 < m_2 < \cdots < m_\ell\} \subseteq \mathbb{N}$. Now let $h := \min\{m_1, \ell\}$, $B_1 := \{m_1 < m_2 < \cdots < m_h\} \subseteq \mathbb{N}$, and $B_2 := B \setminus B_1 \subseteq \mathbb{N}$; we claim that

$$\mu_p(x, B_j) \leqslant \|x\|_{Z_p} \qquad (j = 1, 2). \tag{4.4}$$

This is immediate for $j = 1$ because the definition of h ensures that B_1 is admissible. If $h = \ell$, then $B_2 = \emptyset$, so that (4.4) is satisfied for $j = 2$ by convention. Otherwise $h = m_1 < \ell$ and $B_2 = \{m_{h+1} < m_{h+2} < \cdots < m_\ell\}$, so that B_2 is admissible because

$$\begin{aligned}\min B_2 &= m_{h+1} \geqslant m_h + 1 \geqslant m_{h-1} + 2 \geqslant \cdots \geqslant m_1 + h\\ &= 2m_1 \geqslant \min A \geqslant (\operatorname{card} A) - 1 \geqslant \ell - 1 \geqslant \ell - h = \operatorname{card} B_2,\end{aligned}$$

and (4.4) for $j = 2$ follows. Since B is the disjoint union of B_1 and B_2, we conclude that

$$\mu_p(x, B)^p = \mu_p(x, B_1)^p + \mu_p(x, B_2)^p \leqslant 2\|x\|_{Z_p}^p.$$

Substituting this into (4.3) and then taking the supremum over all permissible sets A, we see that (i) is satisfied.

To prove (ii), let $x \in W_p$, and suppose that $A = \{n_1 < n_2 < \cdots < n_k\} \subseteq \mathbb{N}$ is admissible, so that $k \leqslant n_1$. Then the set $B := \{2n_j - 1, 2n_j : 1 \leqslant j \leqslant k\} \subseteq \mathbb{N}$ is permissible because $\operatorname{card} B = 2k \leqslant 2n_1 = 1 + \min B$, and consequently we have

$$\begin{aligned}\|x\|_{W_p}^p &\geqslant \nu_p(x, B)^p\\ &= \sum_{j=1}^{k-1}\Big(\big|\langle x, f_{2n_j-1} - f_{2n_j}\rangle\big|^p + \big|\langle x, f_{2n_j} - f_{2n_{j+1}-1}\rangle\big|^p\Big) + \big|\langle x, f_{2n_k-1} - f_{2n_k}\rangle\big|^p\\ &\geqslant \sum_{j=1}^{k}\big|\langle x, f_{2n_j-1} - f_{2n_j}\rangle\big|^p = \sum_{j=1}^{k}\big|\langle \Psi x, f_{n_j}\rangle\big|^p = \mu_p(\Psi x, A)^p.\end{aligned}$$

Taking the supremum over all admissible sets A, we conclude that $\|x\|_{W_p} \geqslant \|\Psi x\|_{Z_p}$, as required. Since $\Psi e_1 = -e_1$ and e_1 is a unit vector in both W_p and Z_p, equality does occur. This completes the proof of (ii).

Direct application of the definitions (4.2) shows that $\Psi\Phi = I_{Z_p}$, so Remark 3.18 implies that $P := \Phi\Psi$ is a bounded idempotent operator with image isomorphic to Z_p, and clearly

$$\operatorname{im} P \subseteq \big\{(\alpha_n)_{n\in\mathbb{N}} \in W_p : \alpha_{2n-1} = 0 \ \ (n \in \mathbb{N})\big\}.$$

Conversely, suppose that $x = (\alpha_n)_{n\in\mathbb{N}} \in W_p$ satisfies $\alpha_{2n-1} = 0$ for each $n \in \mathbb{N}$; then we have

$$\operatorname{im} P \ni Px = \Phi\Psi x = \Phi(\alpha_{2n})_{n\in\mathbb{N}} = (0, \alpha_2, 0, \alpha_4, 0, \alpha_6, \ldots) = x,$$

as required. □

4.6 Remark. We can now give an easy proof of Lemma 3.3(iv). Indeed, Proposition 4.5 implies that Z_p is isomorphic to a complemented subspace of W_p, which is a Banach space by Lemma 4.3(iii). Complemented subspaces are automatically closed, hence complete, and therefore Z_p is also complete.

4.7 Proposition. *For each $p \geqslant 1$, the sequence $(e_n)_{n\in\mathbb{N}}$ given by* (1.1) *is a monotone basic sequence in W_p.*

Proof. Lemma 4.3(ii) implies that $\|P_m x\|_{W_p} \leqslant \|x\|_{W_p}$ for each $x \in c_{00}$ and $m \in \mathbb{N}$, and hence the result follows from the monotone version of Banach's fundamental characterization of basic sequences (*e.g.*, see [27, Proposition 1.a.3] or [30, Corollary 4.1.25]). □

4.8 The restricted James–Schreier spaces. Lemma 4.3(iii) and Proposition 4.7 imply that

$$V_p := \overline{\operatorname{span}}\,\{e_n : n \in \mathbb{N}\} \subseteq W_p$$

is a Banach space with a monotone basis $(e_n)_{n\in\mathbb{N}}$ for each $p \geqslant 1$. We call V_p the p^{th} *(restricted) James–Schreier space* and refer to $(e_n)_{n\in\mathbb{N}}$ as its *standard basis.* As in the case of the Schreier spaces, we omit the adjective 'restricted', unless confusion with W_p seems likely. We write $\|\cdot\|_{V_p}$ for the restriction to V_p of the norm $\|\cdot\|_{W_p}$.

Note that the m^{th} biorthogonal functional associated with the standard basis $(e_n)_{n\in\mathbb{N}}$ for V_p is the restriction to V_p of the m^{th} coordinate functional on $\mathbb{K}^{\mathbb{N}}$, that is, $e'_m = f_m|_{V_p}$; this functional has norm 1 by Lemma 4.3(i).

4.9 Remark. (i) Only the first two basis vectors e_1 and e_2 are normalized in the p^{th} James–Schreier norm; for $n \geqslant 3$, we have $\|e_n\|_{V_p} = \nu_p\big(e_n, \{n-1, n, n+1\}\big) = 2^{\frac{1}{p}}$.

(ii) The basis $(e_n)_{n\in\mathbb{N}}$ is not an unconditional basis for V_p because $\chi_{[1,2n]}$ is a unit vector for each $n \in \mathbb{N}$, whereas

$$\Big\|\sum_{j=1}^{2n}(-1)^j e_j\Big\|_{V_p} = \nu_p\Big(\sum_{j=1}^{2n}(-1)^j e_j, \{n, n+1, \ldots, 2n\}\Big) = 2n^{\frac{1}{p}} \to \infty \quad \text{as} \quad n \to \infty.$$

We shall prove a much stronger result in Section 6.

We observe next that Proposition 4.5 carries over to the restricted spaces.

4.10 Proposition. *Let $p \geqslant 1$. The first equation in* (4.2) *defines a bounded operator Φ from S_p to V_p of norm at most $2^{1+\frac{1}{p}}$, while the second defines a bounded operator Ψ from V_p to S_p of norm* 1*. These operators satisfy $\Psi\Phi = I_{S_p}$, and hence $\Phi\Psi$ is idempotent with* $\operatorname{im}(\Phi\Psi) = \overline{\operatorname{span}}\,\{e_{2n} : n \in \mathbb{N}\} \cong S_p$.

Proof. The operators Φ and Ψ given by (4.2) clearly leave c_{00} invariant. Since c_{00} is dense in both S_p and V_p, and Φ and Ψ are bounded on the unrestricted spaces Z_p and W_p, we conclude that $\Phi(S_p) \subseteq V_p$ and $\Psi(V_p) \subseteq S_p$. The result is now easy to deduce from Proposition 4.5. □

The following example is the James–Schreier analogue of Example 3.8.

4.11 Example. Let $p \geqslant 1$. Using the vector $z := (n^{-\frac{1}{p}})_{n\in\mathbb{N}} \in Z_p \setminus S_p$ from Example 3.8, we can easily verify that V_p is a *proper* subspace of W_p. Indeed, Proposition 4.5 implies that

$y := \Phi z \in W_p$, but $y \notin V_p$ because if it were, $\Psi y \in S_p$ by Proposition 4.10, contradicting that $\Psi y = z \notin S_p$.

The much stronger conclusion that $W_p \not\cong V_p$ will follow from Section 5.

We shall next show that the standard basis for V_p is shrinking whenever $p > 1$; this will enable us to describe the bidual of V_p in terms of W_p.

4.12 Proposition. *For each $p > 1$, the standard basis $(e_n)_{n\in\mathbb{N}}$ for V_p is shrinking.*

The *proof* of this result is identical to that of Corollary 3.12 provided that the operator T from Lemma 3.11 is replaced with the operator U from the following lemma throughout.

4.13 Lemma. *Let $p \geqslant 1$, and let $(u_n)_{n\in\mathbb{N}}$ be a normalized block basic sequence of the standard basis $(e_n)_{n\in\mathbb{N}}$ for V_p. Then the operator*

$$U\colon \ \sum_{n=1}^{\infty} \alpha_n e_n \mapsto \sum_{n=1}^{\infty} \alpha_n u_n, \quad \big(c_{00}, \|\cdot\|_{\ell_p}\big) \to V_p,$$

is bounded, and therefore it extends uniquely to a bounded operator $U\colon \ell_p \to V_p$.

Proof. As in the proof of Lemma 3.11, choose integers $0 = M_0 < M_1 < \cdots < M_n < \cdots$ such that $u_n \in \operatorname{span}\{e_j : M_{n-1} < j \leqslant M_n\}$ for each $n \in \mathbb{N}$. Given a permissible subset A of $\mathbb{N}$, take natural numbers k and $n_1 < n_2 < \cdots < n_k$ such that

$$A \cap (M_{n-1}, M_n] \neq \emptyset \iff n \in \{n_1, n_2, \ldots, n_k\} \qquad (n \in \mathbb{N}),$$

and define $A_j := A \cap (M_{n_j-1}, M_{n_j}]$ for $j = 1, 2, \ldots, k$. Then A is the disjoint union of the sets $A_1, \ldots, A_k$, each of which is either a singleton or permissible, and for each $x = \sum_{n=1}^{\infty} \alpha_n e_n \in c_{00}$, we have

$$\begin{aligned}
\nu_p(Ux, A)^p &= \sum_{j=1}^{k} \nu_p(Ux, A_j)^p + \sum_{j=1}^{k-1} \big|\langle Ux, f_{\max A_j} - f_{\min A_{j+1}}\rangle\big|^p \\
&\underset{\text{by (4.1)}}{\leqslant} \sum_{j=1}^{k} \nu_p(\alpha_{n_j} u_{n_j}, A_j)^p + 2^{p-1} \sum_{j=1}^{k-1} \Big(\big|\langle Ux, f_{\max A_j}\rangle\big|^p + \big|\langle Ux, f_{\min A_{j+1}}\rangle\big|^p\Big) \\
&\leqslant \sum_{j=1}^{k} |\alpha_{n_j}|^p \|u_{n_j}\|_{V_p}^p \\
&\qquad + 2^{p-1} \sum_{j=1}^{k-1} \Big(|\alpha_{n_j}|^p \big|\langle u_{n_j}, f_{\max A_j}\rangle\big|^p + |\alpha_{n_{j+1}}|^p \big|\langle u_{n_{j+1}}, f_{\min A_{j+1}}\rangle\big|^p\Big) \\
&\leqslant (2^p + 1) \sum_{j=1}^{k} |\alpha_{n_j}|^p \leqslant (2^p + 1) \|x\|_{\ell_p}^p.
\end{aligned}$$

The result now follows by taking the supremum over all permissible sets A. $\square$

We have not been able to decide whether or not Proposition 4.12 extends to the case where $p = 1$, that is, the following question is open.

4.14 Question. Is the standard basis $(e_n)_{n\in\mathbb{N}}$ for V_1 shrinking?

Note added in proof. Question 4.14 has been answered in the positive; see [10].

4.15 Lemma. *Let $p \geqslant 1$. Then W_p and $\mathbb{K}\chi_{\mathbb{N}}$ are closed, complementary subspaces of* $\mathrm{bip}(V_p)$, *and $\|w\|_{W_p} = \|w\|_{\mathrm{bip}(V_p)}$ for each $w \in W_p$.*

Proof. We have $W_p \cap \mathbb{K}\chi_{\mathbb{N}} = \{0\}$ because $W_p \subseteq c_0$ by definition.

Lemma 4.3(ii) implies that

$$\|w\|_{\mathrm{bip}(V_p)} \leqslant \|w\|_{W_p} \qquad (w \in W_p), \tag{4.5}$$

so that in particular $W_p \subseteq \mathrm{bip}(V_p)$. Moreover, $\chi_{\mathbb{N}} \in \mathrm{bip}(V_p)$ because $P_m\chi_{\mathbb{N}} = \chi_{[1,m]}$ is a unit vector in V_p for each $m \in \mathbb{N}$, and consequently $W_p + \mathbb{K}\chi_{\mathbb{N}} \subseteq \mathrm{bip}(V_p)$.

Conversely, suppose that $x \in \mathrm{bip}(V_p)$. We claim that the sequence $x = \big(\langle x, f_n\rangle\big)_{n\in\mathbb{N}}$ is convergent in $\mathbb{K}$. If not, x fails to be a Cauchy sequence, so we can find $\varepsilon > 0$ and integers $1 = N_1 < N_2 < \cdots < N_j < N_{j+1} < \cdots$ such that $\big|\langle x, f_{N_j} - f_{N_{j+1}}\rangle\big| \geqslant \varepsilon$ for each $j \in \mathbb{N}$. Take $k \in \mathbb{N}$ such that $k\varepsilon^p > \|x\|^p_{\mathrm{bip}(V_p)}$, and choose $m \in \mathbb{N}$ such that $N_m \geqslant k$. The set $A := \{N_m < N_{m+1} < \cdots < N_{m+k}\}$ is then permissible, and therefore we have

$$\|x\|^p_{\mathrm{bip}(V_p)} \geqslant \nu_p(P_{N_{m+k}}x, A)^p = \sum_{j=m}^{m+k-1} \big|\langle x, f_{N_j} - f_{N_{j+1}}\rangle\big|^p \geqslant k\varepsilon^p > \|x\|^p_{\mathrm{bip}(V_p)}$$

which is clearly absurd. Thus x is convergent with limit $\alpha \in \mathbb{K}$, say.

We now claim that $w := x - \alpha\chi_{\mathbb{N}}$ belongs to W_p; indeed, $w \in c_0$ by the choice of α, and

$$\|w\|_{W_p} \leqslant \|x\|_{\mathrm{bip}(V_p)} \tag{4.6}$$

because, for each permissible subset $A = \{n_1 < n_2 < \cdots < n_{k+1}\}$ of $\mathbb{N}$, we have

$$\nu_p(w, A)^p = \sum_{j=1}^{k} \Big|\big(\langle x, f_{n_j}\rangle - \alpha\big) - \big(\langle x, f_{n_{j+1}}\rangle - \alpha\big)\Big|^p = \nu_p(P_{n_{k+1}}x, A)^p \leqslant \|x\|^p_{\mathrm{bip}(V_p)}.$$

Hence we conclude that $x = w + \alpha\chi_{\mathbb{N}} \in W_p + \mathbb{K}\chi_{\mathbb{N}}$, and therefore $\mathrm{bip}(V_p) = W_p + \mathbb{K}\chi_{\mathbb{N}}$.

To complete the proof, we note that if $x \in W_p$ in the argument given above, then $\alpha = 0$, so that $w = x$, and therefore (4.5)–(4.6) combine to show that $\|w\|_{W_p} = \|w\|_{\mathrm{bip}(V_p)}$ for each $w \in W_p$. In particular, this implies that the subspace W_p is closed in $\mathrm{bip}(V_p)$ because it is complete with respect to the norm $\|\cdot\|_{W_p}$ and thus also with respect to $\|\cdot\|_{\mathrm{bip}(V_p)}$. The subspace $\mathbb{K}\chi_{\mathbb{N}}$ is closed because it is finite-dimensional. □

4.16 Remark. Lemma 4.15 can be rephrased as follows: $\big(\mathrm{bip}(V_p), \|\cdot\|_{\mathrm{bip}(V_p)}\big)$ contains $\big(W_p, \|\cdot\|_{W_p}\big)$ isometrically, and $\lambda\colon (\alpha_n)_{n\in\mathbb{N}} \mapsto \lim_{n\to\infty} \alpha_n$ defines a bounded functional on $\mathrm{bip}(V_p)$ whose kernel is W_p.

4.17 The unrestricted James–Schreier space and the bidual of V_p. For $p > 1$, we can summarize our findings in the commutative diagram

$$\begin{array}{ccccc} V_p & \xhookrightarrow{\quad\quad\kappa\quad\quad} & & & V_p'' \\ \Big\downarrow{\scriptstyle\iota} & & & & {\scriptstyle\cong}\Big\downarrow{\scriptstyle\Upsilon} \\ W_p & \xhookrightarrow{\ \iota\ } & W_p \oplus \mathbb{K}\chi_{\mathbb{N}} & = = = & \mathrm{bip}(V_p), \end{array}$$

where both ι's are the natural inclusions, κ is the canonical embedding, and Υ is the isometric isomorphism from §2.4. Thus, if we identify the bidual of V_p with $\mathrm{bip}(V_p)$ via Υ, then $\kappa = \iota$, that is, the canonical embedding of V_p into its bidual becomes the natural inclusion of V_p into $W_p \oplus \mathbb{K}\chi_{\mathbb{N}}$.

Finally in this section we take a look at shift operators on the James–Schreier spaces. We begin with an easy result that characterizes the increasing mappings $\sigma\colon \mathbb{N} \to \mathbb{N}$ for which the associated left and right shifts Λ_σ and R_σ (as defined in (3.2)) are bounded as operators on $\big(c_{00}, \|\cdot\|_{V_p}\big)$.

4.18 Proposition. *Let $p \geqslant 1$, and let $\sigma\colon \mathbb{N} \to \mathbb{N}$ be strictly increasing. Then:*

(i) $\|\Lambda_\sigma x\|_{V_p} \leqslant \|x\|_{V_p}$ *for each $x \in c_{00}$, so that Λ_σ extends uniquely to a contractive operator on V_p, no matter what σ is;*

(ii) *the operator R_σ is bounded on $\big(c_{00}, \|\cdot\|_{V_p}\big)$ if and only if $\sigma(\mathbb{N})$ is cofinite in $\mathbb{N}$.*

Proof. (i). Given $x \in c_{00}$ and a permissible subset $A = \{n_1 < \cdots < n_{k+1}\}$ of $\mathbb{N}$, we have

$$\begin{aligned}\nu_p(\Lambda_\sigma x, A)^p = \sum_{j=1}^{k}\big|\langle \Lambda_\sigma x, f_{n_j} - f_{n_{j+1}}\rangle\big|^p &= \sum_{j=1}^{k}\big|\langle x, f_{\sigma(n_j)} - f_{\sigma(n_{j+1})}\rangle\big|^p \\ &= \nu_p\big(x, \sigma(A)\big)^p \leqslant \|x\|_{V_p}^p,\end{aligned}$$

where the final inequality follows from the fact that the set $\sigma(A)$ is permissible (because σ is strictly increasing).

(ii). It is easy to see that R_σ is bounded if $\sigma(\mathbb{N})$ is cofinite in $\mathbb{N}$.

Conversely, suppose that the set $\mathbb{N} \setminus \sigma(\mathbb{N})$ is infinite, and take a strictly increasing sequence $(m_j)_{j\in\mathbb{N}}$ of natural numbers such that $\sigma(m_j) + 1 \notin \sigma(\mathbb{N})$ for each $j \in \mathbb{N}$. Then, for given $k \in \mathbb{N}$, we can find $n \in \mathbb{N}$ such that $\sigma(m_n) \geqslant 2k + 1$. This implies that the set $A := \big\{\sigma(m_j), \sigma(m_j) + 1 : n \leqslant j \leqslant n + k\big\}$ is permissible, and therefore we have

$$\|R_\sigma \chi_{[1,m_{n+k}]}\|_{V_p}^p \geqslant \nu_p(R_\sigma \chi_{[1,m_{n+k}]}, A)^p = 2k + 1.$$

Since $\|\chi_{[1,m]}\|_{V_p} = 1$, no matter what $m \in \mathbb{N}$ is, while $(2k+1)^{\frac{1}{p}} \to \infty$ as $k \to \infty$, we conclude that the operator R_σ is unbounded on $\big(c_{00}, \|\cdot\|_{V_p}\big)$. $\square$

Thus, if we want to right-shift elements of V_p in a non-trivial way, a different approach which does not introduce 'gaps' between coordinates is required. This is the motivation behind the following definition.

4.19 Definition. Let $\sigma\colon \mathbb{N}\to\mathbb{N}$ be strictly increasing. We associate with σ the *block right shift*

$$\Theta_\sigma\colon\ \sum_{n=1}^\infty \alpha_n e_n \mapsto \sum_{n=1}^\infty \alpha_n \chi_{(\sigma(n-1),\sigma(n)]},\qquad c_{00}\to c_{00}, \tag{4.7}$$

with the standing convention that $\sigma(0) := 0$.

This block right shift is bounded on $\bigl(c_{00},\|\cdot\|_{V_p}\bigr)$ under exactly the same condition that the usual right shift is bounded on $\bigl(c_{00},\|\cdot\|_{S_p}\bigr)$ (see Corollary 3.17(ii)), as the following proposition shows; this result will be important in the study [9] of V_p as a Banach algebra.

4.20 Proposition. *Let $p\geqslant 1$, and let $\sigma\colon\mathbb{N}\to\mathbb{N}$ be strictly increasing. The block right shift Θ_σ is bounded on $\bigl(c_{00},\|\cdot\|_{V_p}\bigr)$ if and only if there is a constant $K\in\mathbb{N}$ such that $\sigma(n)\leqslant Kn$ for each $n\in\mathbb{N}$. In the positive case, Θ_σ extends uniquely to a bounded operator of norm at most $K^{\frac{1}{p}}$ on V_p.*

Proof. $\Leftarrow$. Suppose that there is a constant $K\in\mathbb{N}$ such that $\sigma(n)\leqslant Kn$ for each $n\in\mathbb{N}$, and let $x\in c_{00}$ and a permissible subset A of $\mathbb{N}$ be given. Take natural numbers k and $m_1<m_2<\cdots<m_k$ and non-empty sets $A_1,\ldots,A_k$ with $A_j\subseteq\bigl(\sigma(m_j-1),\sigma(m_j)\bigr]$ for each $j\in\{1,\ldots,k\}$ (with the convention that $\sigma(0)=0$) such that $A=\bigcup_{j=1}^k A_j$. If $k=1$, then $\nu_p(\Theta_\sigma x, A)=0$, whereas for $k\geqslant 2$, we have

$$\begin{aligned}\nu_p(\Theta_\sigma x,A)^p &= \sum_{j=1}^{k-1}\Bigl(\nu_p(\Theta_\sigma x,A_j)^p + \bigl|\langle\Theta_\sigma x, f_{\max A_j}-f_{\min A_{j+1}}\rangle\bigr|^p\Bigr)+\nu_p(\Theta_\sigma x,A_k)^p\\ &= \sum_{j=1}^{k-1}\bigl|\langle\Theta_\sigma x, f_{\max A_j}-f_{\min A_{j+1}}\rangle\bigr|^p = \sum_{j=1}^{k-1}\bigl|\langle x, f_{m_j}-f_{m_{j+1}}\rangle\bigr|^p.\end{aligned} \tag{4.8}$$

The permissibility of A and our assumption on σ imply that

$$k\leqslant \operatorname{card}A\leqslant 1+\min A\leqslant 1+\sigma(m_1)\leqslant 1+Km_1. \tag{4.9}$$

If this inequality is strict, choose integers $m_{1+Km_1}>m_{Km_1}>\cdots>m_{k+1}$, all greater than m_k. We can then define $B_j:=\{m_h : (j-1)m_1+1\leqslant h\leqslant jm_1+1\}\subseteq\mathbb{N}$ for each $j=\{1,\ldots,K\}$, and this set is permissible because $\operatorname{card}B_j = 1+m_1\leqslant 1+\min B_j$. Hence we conclude that

$$K\|x\|_{V_p}^p\geqslant \sum_{j=1}^K \nu_p\bigl(x,B_j\bigr)^p = \sum_{j=1}^{Km_1}\bigl|\langle x, f_{m_j}-f_{m_{j+1}}\rangle\bigr|^p\geqslant \nu_p(\Theta_\sigma x,A)^p,$$

where the final inequality follows from (4.8)–(4.9). Taking the supremum over all permissible sets A, we obtain $K\|x\|_{V_p}^p \geqslant \|\Theta_\sigma x\|_{V_p}^p$, so that Θ_σ is bounded with norm at most $K^{\frac{1}{p}}$.

$\Rightarrow$. Conversely, suppose that Θ_σ is bounded on $(c_{00}, \|\cdot\|_{V_p})$, and consider the strictly increasing mappings $\gamma, \tau\colon \mathbb{N} \to \mathbb{N}$ given by $\gamma(n) := 2n+1$ and $\tau(n) := \sigma(2n-1)+1$ for each $n \in \mathbb{N}$. We then see that the diagram

$$\begin{array}{ccc}
(c_{00}, \|\cdot\|_{S_p}) & \xrightarrow{\quad R_\sigma \quad} & (c_{00}, \|\cdot\|_{S_p}) \\
\Phi \downarrow & & \uparrow \Xi_{\tau,\sigma} \\
(c_{00}, \|\cdot\|_{V_p}) & & (c_{00}, \|\cdot\|_{S_p}) \\
\Theta_\sigma \downarrow & & \uparrow \Psi \\
(c_{00}, \|\cdot\|_{V_p}) & \xrightarrow{\quad \Theta_\gamma \quad} & (c_{00}, \|\cdot\|_{V_p})
\end{array}$$

is commutative because

$$\begin{aligned}
\Xi_{\tau,\sigma}\Psi\Theta_\gamma\Theta_\sigma\Phi e_n &= \Xi_{\tau,\sigma}\Psi\Theta_\gamma\Theta_\sigma e_{2n} = \Xi_{\tau,\sigma}\Psi\Theta_\gamma\chi_{(\sigma(2n-1),\sigma(2n)]} \\
&= \Xi_{\tau,\sigma}\Psi\chi_{(2\sigma(2n-1)+1,2\sigma(2n)+1]} = \Xi_{\tau,\sigma}\big(e_{\sigma(2n-1)+1} - e_{\sigma(2n)+1}\big) \\
&= \Xi_{\tau,\sigma}e_{\tau(n)} - \Xi_{\tau,\sigma}e_{\sigma(2n)+1} = e_{\sigma(n)} - 0 = R_\sigma e_n \qquad (n \in \mathbb{N}).
\end{aligned}$$

Moreover, the operators Φ, Θ_σ, Θ_γ, Ψ, and $\Xi_{\tau,\sigma}$ are all bounded with respect to the norms on their domains and codomains specified in the diagram; this follows from Proposition 4.10 for Φ and Ψ, from our assumption for Θ_σ, from the implication '$\Leftarrow$' proved above for Θ_γ (because $\gamma(n) = 2n+1 \leqslant 3n$ for each $n \in \mathbb{N}$), and from Lemma 3.16 for $\Xi_{\tau,\sigma}$ (because $\sigma(n) \leqslant \sigma(2n-1)+1 = \tau(n)$ for each $n \in \mathbb{N}$). Hence the commutativity of the diagram implies that R_σ is bounded, and therefore we have $\sigma(n) \leqslant Kn$ for each $n \in \mathbb{N}$ by Corollary 3.17(ii). □

Recall that the p^{th} James space J_p is not isomorphic to its Cartesian square for any $p > 1$, whereas we saw in Corollary 3.19 that the p^{th} Schreier space S_p is isomorphic to its Cartesian square for each $p \geqslant 1$. We have not been able to answer the corresponding question for the James–Schreier spaces, so we state it formally.

4.21 Question. Let $p \geqslant 1$. Is the p^{th} James–Schreier space V_p isomorphic to its Cartesian square $V_p \oplus V_p$?

5 The James–Schreier spaces are c_0-saturated

5.1 Definition. Let X and Y be infinite-dimensional Banach spaces. We say that X is *Y-saturated* if each closed, infinite-dimensional subspace of X contains a subspace which is isomorphic to Y.

The aim of this section is to prove the following result.

5.2 Theorem. *For each $p \geqslant 1$, the p^{th} James–Schreier space V_p is c_0-saturated.*

We isolate the key step in the proof of Theorem 5.2 in the following lemma.

5.3 Lemma. *Let $p \geqslant 1$. Every block basic sequence of the standard basis for V_p has a block basic sequence which is equivalent to the standard unit vector basis for c_0.*

Proof. Suppose that $(u_n)_{n\in\mathbb{N}}$ is a block basic sequence of $(e_n)_{n\in\mathbb{N}}$, and choose integers $0 = L_0 < L_1 < L_2 < \cdots$ such that $u_n \in \operatorname{span}\{e_j : L_{n-1} < j \leqslant L_n\}$ for each $n \in \mathbb{N}$. By induction, we shall construct sequences $(M_n)_{n\in\mathbb{N}}$ and $(N_n)_{n\in\mathbb{N}}$ of natural numbers with $M_1 \leqslant N_1 < M_2 \leqslant N_2 < \cdots$ and a sequence $(v_n)_{n\in\mathbb{N}}$ of unit vectors in V_p such that

$$v_n \in \operatorname{span}\{u_j : M_n \leqslant j \leqslant N_n\} \qquad \text{and} \qquad \big|\langle v_{n+1}, f_k\rangle\big| \leqslant L_{N_n}^{-\frac{1}{p}} \qquad (n, k \in \mathbb{N}). \tag{5.1}$$

To start the induction, take $M_1 := N_1 := 1$ and $v_1 := u_1/\|u_1\|_{V_p} \in V_p$. Then v_1 is a unit vector which satisfies the first part of (5.1) by definition; the second part is void in this case.

Now let $n \in \mathbb{N}$, and assume that natural numbers $M_1 \leqslant N_1 < \cdots < M_n \leqslant N_n$ and unit vectors $v_1, \ldots, v_n$ in V_p have been chosen in accordance with (5.1).

If there is an integer $m > N_n$ such that $\big|\langle u_m, f_k\rangle\big| \leqslant L_{N_n}^{-\frac{1}{p}} \|u_m\|_{V_p}$ for each $k \in \mathbb{N}$, then we can simply take $M_{n+1} := N_{n+1} := m$ and $v_{n+1} := u_m/\|u_m\|_{V_p} \in V_p$.

Otherwise, for each $m > N_n$, we can find $k_m \in \mathbb{N}$ such that

$$\big|\langle u_m, f_{k_m}\rangle\big| > L_{N_n}^{-\frac{1}{p}} \|u_m\|_{V_p}. \tag{5.2}$$

In particular, $\langle u_m, f_{k_m}\rangle \neq 0$, so that

$$L_{m-1} < k_m \leqslant L_m, \tag{5.3}$$

and therefore the sequence $(k_m)_{m=N_n+1}^{\infty}$ is strictly increasing. We now choose an integer $K \geqslant 2^{-p}(L_{N_n}^2 - 1) + 1$, we then pick an integer $M_{n+1} > N_n$ such that $k_{M_{n+1}} \geqslant K$, and finally we define $N_{n+1} := M_{n+1} + K - 1 \geqslant M_{n+1}$. These choices ensure that the set $A := \{k_m : M_{n+1} \leqslant m \leqslant N_{n+1} + 1\}$ is permissible, and consequently we can estimate the James–Schreier norm of the vector

$$w := \sum_{j=M_{n+1}}^{N_{n+1}} \alpha_j u_j \in V_p, \qquad \text{where} \qquad \alpha_j := \frac{(-1)^j\, \overline{\langle u_j, f_{k_j}\rangle}}{\big|\langle u_j, f_{k_j}\rangle\big| \cdot \|u_j\|_{V_p}} \in \mathbb{K}, \tag{5.4}$$

as follows:

$$\begin{aligned}\|w\|_{V_p}^p \quad &\geqslant \quad \nu_p(w, A)^p = \sum_{m=M_{n+1}}^{N_{n+1}} \left| \sum_{j=M_{n+1}}^{N_{n+1}} \alpha_j \big(\langle u_j, f_{k_m}\rangle - \langle u_j, f_{k_{m+1}}\rangle\big)\right|^p \\ &\underset{\text{by (5.3)}}{=} \sum_{m=M_{n+1}}^{N_{n+1}-1} \big|\alpha_m \langle u_m, f_{k_m}\rangle - \alpha_{m+1}\langle u_{m+1}, f_{k_{m+1}}\rangle\big|^p + \big|\alpha_{N_{n+1}}\langle u_{N_{n+1}}, f_{k_{N_{n+1}}}\rangle\big|^p\end{aligned}$$

$$\underset{\text{by (5.4)}}{=} \sum_{m=M_{n+1}}^{N_{n+1}-1} \left(\frac{\left|\langle u_m, f_{k_m}\rangle\right|}{\|u_m\|_{V_p}} + \frac{\left|\langle u_{m+1}, f_{k_{m+1}}\rangle\right|}{\|u_{m+1}\|_{V_p}} \right)^p + \left(\frac{\left|\langle u_{N_{n+1}}, f_{k_{N_{n+1}}}\rangle\right|}{\|u_{N_{n+1}}\|_{V_p}} \right)^p$$

$$\underset{\text{by (5.2)}}{>} \sum_{m=M_{n+1}}^{N_{n+1}-1} \big(2L_{N_n}^{-\frac{1}{p}}\big)^p + \big(L_{N_n}^{-\frac{1}{p}}\big)^p = \frac{2^p(K-1)+1}{L_{N_n}} \geqslant L_{N_n} \tag{5.5}$$

by the choice of K. In particular we have $w \neq 0$, so $v_{n+1} := w/\|w\|_{V_p}$ defines a unit vector in V_p, and (5.4) implies that this vector satisfies the first part of (5.1). To verify the second part, we use the fact that $(u_j)_{j\in\mathbb{N}}$ is a block basic sequence to obtain

$$\begin{aligned}\left|\langle v_{n+1}, f_k\rangle\right| &= \frac{1}{\|w\|_{V_p}} \max\Big\{\left|\alpha_j \langle u_j, f_k\rangle\right| : M_{n+1} \leqslant j \leqslant N_{n+1}\Big\} \\ &= \frac{1}{\|w\|_{V_p}} \max\left\{\frac{\left|\langle u_j, f_k\rangle\right|}{\|u_j\|_{V_p}} : M_{n+1} \leqslant j \leqslant N_{n+1}\right\} \leqslant \frac{1}{\|w\|_{V_p}} \leqslant L_{N_n}^{-\frac{1}{p}} \qquad (k \in \mathbb{N}),\end{aligned}$$

where the final inequality follows from (5.5). Hence the induction continues.

We shall next show that the block basic sequence $(v_n)_{n\in\mathbb{N}}$ constructed above is equivalent to the standard unit vector basis of c_0; this will clearly complete the proof. To this end, we consider the linear mapping

$$U\colon\ \sum_{n=1}^{\infty} \beta_n e_n \mapsto \sum_{n=1}^{\infty} \beta_n v_n, \quad \big(c_{00}, \|\cdot\|_{c_0}\big) \to V_p.$$

Our aim is to show that U is an isomorphism onto its image, that is, we want to prove that there are constants $C_1, C_2 > 0$ such that

$$C_1\|x\|_{c_0} \leqslant \|Ux\|_{V_p} \leqslant C_2\|x\|_{c_0} \qquad (x \in c_{00}). \tag{5.6}$$

Given $x = \sum_{n=1}^{\infty} \beta_n e_n \in c_{00}$, take $\ell \in \mathbb{N}$ such that $\|x\|_{c_0} = |\beta_\ell|$. Then we have

$$\begin{aligned}\|x\|_{c_0} &= \|\beta_\ell v_\ell\|_{V_p} \\ &= \begin{Bmatrix} \|P_{L_{N_1}}(Ux)\|_{V_p} \leqslant \|P_{L_{N_1}}\|\,\|Ux\|_{V_p} & \text{for } \ell = 1 \\ \big\|(P_{L_{N_\ell}} - P_{L_{N_{\ell-1}}})Ux\big\|_{V_p} \leqslant \|P_{L_{N_\ell}} - P_{L_{N_{\ell-1}}}\|\,\|Ux\|_{V_p} & \text{for } \ell \geqslant 2 \end{Bmatrix} \leqslant 2\|Ux\|_{V_p}\end{aligned}$$

by Lemma 4.3(ii). Hence the first inequality in (5.6) holds with $C_1 := 1/2$.

To prove the second, let a permissible subset $A = \{n_1 < n_2 < \cdots < n_{k+1}\}$ of $\mathbb{N}$ be given. Choose $h \in \mathbb{N}$ minimal such that $n_1 \leqslant L_{N_h}$, and then take $g \in \{1,\ldots,k+1\}$ maximal such that $n_g \leqslant L_{N_h}$. If $g = k+1$, then $A \subseteq (L_{N_{h-1}}, L_{N_h}]$ (with $N_0 := 0$ if $h = 1$), and we have

$$\nu_p(Ux, A)^p = \nu_p(\beta_h v_h, A)^p \leqslant \|\beta_h v_h\|_{V_p}^p = |\beta_h|^p \leqslant \|x\|_{c_0}^p. \tag{5.7}$$

Otherwise (that is, when $g \leqslant k$) we write

$$\nu_p(Ux, A)^p = \sum_{j=1}^{g-1} \left|\langle Ux, f_{n_j} - f_{n_{j+1}}\rangle\right|^p + \left|\langle Ux, f_{n_g} - f_{n_{g+1}}\rangle\right|^p + \sum_{j=g+1}^{k} \left|\langle Ux, f_{n_j} - f_{n_{j+1}}\rangle\right|^p,$$

where we ignore the left-hand sum if $g = 1$ and the right-hand sum if $g = k$. Application of the inequality (4.1) to the middle term and the right-hand sum (but not the left-hand sum) yields

$$\begin{aligned}\nu_p(Ux, A)^p &\leqslant \sum_{j=1}^{g-1} \big|\langle Ux, f_{n_j} - f_{n_{j+1}}\rangle\big|^p + 2^{p-1}\Big(\big|\langle Ux, f_{n_g}\rangle\big|^p + \big|\langle Ux, f_{n_{g+1}}\rangle\big|^p\Big) \\ &\qquad + 2^{p-1}\Big(\sum_{j=g+1}^{k} \big|\langle Ux, f_{n_j}\rangle\big|^p + \big|\langle Ux, f_{n_{j+1}}\rangle\big|^p\Big) \\ &\leqslant 2^{p-1}\Big(\sum_{j=1}^{g-1} \big|\langle Ux, f_{n_j} - f_{n_{j+1}}\rangle\big|^p + \big|\langle Ux, f_{n_g}\rangle\big|^p\Big) + 2^p \sum_{j=g+1}^{k+1} \big|\langle Ux, f_{n_j}\rangle\big|^p.\end{aligned}$$

We estimate the two terms on the right-hand side of this inequality separately. Firstly, the set $B := \{n_1 < n_2 < \cdots < n_g < L_{N_h} + 1\}$ is permissible because $g \leqslant k \leqslant n_1$, and therefore we have

$$\sum_{j=1}^{g-1} \big|\langle Ux, f_{n_j} - f_{n_{j+1}}\rangle\big|^p + \big|\langle Ux, f_{n_g}\rangle\big|^p = \nu_p(\beta_h v_h, B)^p \leqslant \|\beta_h v_h\|_{V_p}^p = |\beta_h|^p \leqslant \|x\|_{c_0}^p.$$

Secondly, we observe that

$$\begin{aligned}\sum_{j=g+1}^{k+1} \big|\langle Ux, f_{n_j}\rangle\big|^p &\leqslant (k - g + 1) \sup\Big\{\big|\langle Ux, f_n\rangle\big|^p : n \geqslant n_{g+1}\Big\} \\ &\leqslant k \sup\Big\{\big|\langle Ux, f_n\rangle\big|^p : n > L_{N_h}\Big\} \\ &= k \sup\Big\{|\beta_j|^p \big|\langle v_j, f_n\rangle\big|^p : n > L_{N_h},\ j > h\Big\} \leqslant k\, \|x\|_{c_0}^p \big(L_{N_h}^{-\frac{1}{p}}\big)^p \leqslant \|x\|_{c_0}^p\end{aligned}$$

because $k \leqslant n_1 \leqslant L_{N_h}$. Hence we conclude that

$$\nu_p(Ux, A)^p \leqslant 2^{p-1}\|x\|_{c_0}^p + 2^p\, \|x\|_{c_0}^p = 3 \cdot 2^{p-1}\|x\|_{c_0}^p,$$

and this, together with (5.7), shows that the second inequality in (5.6) is satisfied with $C_2 := (3 \cdot 2^{p-1})^{\frac{1}{p}} = 3^{\frac{1}{p}} \cdot 2^{1-\frac{1}{p}}$. □

Proof of Theorem 5.2. Let X be a closed, infinite-dimensional subspace of V_p. By a theorem of Bessaga and Pełczyński (see [7], or [27, Proposition 1.a.11] for an exposition), X contains a sequence $(x_n)_{n\in\mathbb{N}}$ which is equivalent to a block basic sequence $(u_n)_{n\in\mathbb{N}}$ of the standard basis for V_p. Lemma 5.3 implies that $(u_n)_{n\in\mathbb{N}}$ has a block basic sequence $(v_n)_{n\in\mathbb{N}}$ which is equivalent to the standard unit vector basis for c_0. Since $(u_n)_{n\in\mathbb{N}}$ is equivalent to $(x_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ is equivalent to a block basic sequence of $(x_n)_{n\in\mathbb{N}}$, say $(y_n)_{n\in\mathbb{N}}$, and consequently we have

$$X \supseteq \overline{\operatorname{span}}\,\{x_n : n \in \mathbb{N}\} \supseteq \overline{\operatorname{span}}\,\{y_n : n \in \mathbb{N}\} \cong \overline{\operatorname{span}}\,\{v_n : n \in \mathbb{N}\} \cong c_0,$$

as desired. □

5.4 Corollary. *For each $p \geqslant 1$, the p^{th} Schreier space S_p is c_0-saturated.*

Proof. This is immediate from the fact that V_p contains a subspace isomorphic to S_p (see Proposition 4.10). □

5.5 Remark. (i) Corollary 5.4 is known in the case $p = 1$ (*e.g.*, see [15]).

(ii) Proposition 2.3, which is the counterpart of Theorem 5.2 and Corollary 5.4 for James spaces, appears to be formally stronger as it states that every closed, infinite-dimensional subspace of J_p contains a *complemented* subspace which is isomorphic to ℓ_p; however, Sobczyk's Theorem (*e.g.,* see [27, Theorem 2.f.5]) implies that *every* subspace of V_p or S_p (or any other separable Banach space) which is isomorphic to c_0 is complemented.

(iii) Corollary 5.4 implies that ℓ_1 does not embed in S_p for each $p \geqslant 1$. Since $(e_n)_{n\in\mathbb{N}}$ is an unconditional basis for S_p, a well-known theorem of James [22] shows that $(e_n)_{n\in\mathbb{N}}$ is shrinking, thus giving an alternative proof of Proposition 3.10 and Corollary 3.12.

5.6 Corollary. *For each $p \geqslant 1$, the p^{th} unrestricted Schreier space Z_p contains a subspace which is isomorphic to ℓ_∞; the same is true for the p^{th} unrestricted James–Schreier space W_p.*

Proof. Since S_p contains a subspace which is isomorphic to c_0, its bidual S_p'' contains a subspace which is isomorphic to $c_0'' \cong \ell_\infty$; this proves the first clause because $Z_p \cong S_p''$ by §3.14. The second follows immediately from this because Proposition 4.5 implies that W_p contains a subspace which is isomorphic to Z_p. □

5.7 Definition. Two infinite-dimensional Banach spaces X and Y are *totally incomparable* if no closed, infinite-dimensional subspace of X is isomorphic to a subspace of Y.

Any two distinct spaces from the family $\{\ell_p : 1 \leqslant p < \infty\} \cup \{c_0\}$ are totally incomparable, as is well-known (*e.g.,* see [27, p. 54]).

5.8 Lemma. *Let X_1, X_2, Y_1, and Y_2 be infinite-dimensional Banach spaces, and suppose that X_j is Y_j-saturated for $j = 1, 2$ and that Y_1 and Y_2 are totally incomparable. Then X_1 and X_2 are totally incomparable.*

Proof. Assume towards a contradiction that E is a closed, infinite-dimensional subspace of X_1 which is isomorphic to a subspace of X_2. Then E contains a subspace F which is isomorphic to Y_1. Since F is isomorphic to a subspace of X_2, it contains a subspace which is isomorphic to Y_2, contradicting that Y_1 and Y_2 are totally incomparable. □

5.9 Corollary. *Let $p, q \geqslant 1$. Then:*

(i) *ℓ_p and S_q are totally incomparable;*

(ii) *ℓ_p and V_q are totally incomparable;*

(iii) J_p *and* S_q *are totally incomparable;*

(iv) J_p *and* V_q *are totally incomparable.*

Proof. We apply Lemma 5.8 with $X_1 := \ell_p$ or $X_1 := J_p$, $X_2 := S_q$ or $X_2 := V_q$, and $Y_1 := \ell_p$ and $Y_2 := c_0$. As mentioned above, Y_1 and Y_2 are totally incomparable, while [27, Proposition 2.a.2], Proposition 2.3, Corollary 5.4, and Theorem 5.2 imply that X_j is Y_j-saturated for $j = 1, 2$. □

Evidence from other sequence spaces suggests that no two Schreier spaces with distinct indices should be isomorphic and, likewise, no two James–Schreier spaces with distinct indices should be isomorphic. We have, unfortunately, been unable to verify any of these conjectures, so the following question is open.

5.10 Question. Is it true that $S_p \not\cong S_q$ and $V_p \not\cong V_q$ whenever $1 \leqslant p < q < \infty$?

Note added in proof. Question 5.10 has been answered in the positive; see [10].

6 The p^{th} James–Schreier space does not embed in a Banach space with an unconditional basis for $p > 1$

The aim of this section is to prove the result stated in the title; in particular, this will imply that, for $p > 1$, the p^{th} James–Schreier space does not embed in any of the Schreier spaces.

Our proof relies on the following technical notion which was originally introduced by Pełczyński [32] who, motivated by work of Orlicz, was studying the relationship between weak completeness and weak unconditional convergence of series in Banach spaces.

6.1 Definition. A Banach space X has *Pełczyński's property* (u) if, for every weak Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in X, there is a sequence $(y_n)_{n\in\mathbb{N}}$ in X such that

$$\sum_{n=1}^{\infty}\bigl|\langle y_n, f\rangle\bigr| < \infty \quad\text{and}\quad \Bigl\langle x_n - \sum_{j=1}^{n} y_j, f\Bigr\rangle \to 0 \quad\text{as}\quad n\to\infty \qquad (f \in X'). \tag{6.1}$$

The reason that this notion is relevant for our purposes is immediately explained by parts (i) and (ii) of the following result, which was stated in [32].

6.2 Theorem. (Pełczyński.)

(i) *Every Banach space with an unconditional basis has Pełczyński's property* (u)*.*

(ii) *Every closed subspace of a Banach space with Pełczyński's property* (u) *has Pełczyński's property* (u)*.*

(iii) *The James space* J_2 *does not have Pełczyński's property* (u)*.*

Detailed proofs of all three parts of Theorem 6.2 are given in [1, Section 3.5]. A more general statement than Theorem 6.2(i) can be found in [28, Proposition 1.c.2], namely: every order continuous Banach lattice has Pełczyński's property (u).

We can now state the main result of this section, which is the counterpart of Theorem 6.2(iii) for James–Schreier spaces.

6.3 Theorem. *For each $p > 1$, the p^{th} James–Schreier space V_p does not have Pełczyński's property* (u).

The proof of Theorem 6.3 goes via two lemmas. The first is easy, so we omit its proof.

6.4 Lemma. *Let X be a Banach space with a shrinking basis $(b_n)_{n\in\mathbb{N}}$, and let $(b'_m)_{m\in\mathbb{N}}$ be the associated biorthogonal functionals. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is a weak Cauchy sequence if and only if $(x_n)_{n\in\mathbb{N}}$ is bounded and the sequence $\bigl(\langle x_n, b'_m\rangle\bigr)_{n\in\mathbb{N}}$ converges for each $m \in \mathbb{N}$.*

6.5 Lemma. *Let $p \geqslant 1$, and suppose that $(y_n)_{n\in\mathbb{N}}$ is a sequence in V_p such that*

$$\Bigl\langle \chi_{[1,n]} - \sum_{j=1}^{n} y_j, f_m \Bigr\rangle \to 0 \quad \text{as} \quad n \to \infty \qquad (m \in \mathbb{N}). \tag{6.2}$$

Then the series $\sum_{n=1}^{\infty} \bigl|\langle y_n, f\rangle\bigr|$ diverges for some $f \in V'_p$.

Before we engage with the proof of Lemma 6.5, let us see how these two lemmas combine to establish our main result.

Proof of Theorem 6.3. Assume towards a contradiction that V_p has Pełczyński's property (u), and let $x_n := \chi_{[1,n]}$ for each $n \in \mathbb{N}$. Since $p > 1$, the standard basis for V_p is shrinking, so Lemma 6.4 implies that $(x_n)_{n\in\mathbb{N}}$ is a weak Cauchy sequence in V_p. Then, by assumption, we can choose a sequence $(y_n)_{n\in\mathbb{N}}$ in V_p which satisfies (6.1). This, however, contradicts Lemma 6.5. ☐

6.6 Remark. Our proof of Theorem 6.3, like that of Theorem 6.2(iii) given in [1], starts by taking $x_n := \chi_{[1,n]}$ and then proceeds by arguing that there is no sequence $(y_n)_{n\in\mathbb{N}}$ which satisfies (6.1). This follows quite easily for J_2 from general structure theorems because no subspace of J_2 is isomorphic to c_0. As we saw in Section 5, this is no longer true for V_p, and so a different and, as it turns out, more delicate, argument is required.

The following family of functionals will play a key role in the proof of Lemma 6.5.

6.7 Definition. Given a strictly increasing mapping $\sigma\colon \mathbb{N} \to \mathbb{N}$, we associate with it the *alternating harmonic functional*

$$\xi_\sigma\colon\ x \mapsto \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \langle x, f_{\sigma(n)}\rangle, \quad c_{00} \to \mathbb{K}. \tag{6.3}$$

As one might guess, our interest in ξ_σ stems from the fact that it extends to a bounded functional on V_p. This can be conveniently proved using the following lemma which, roughly speaking, states that in order to establish the boundedness of a functional on V_p, it suffices to consider its action on finite, positive vectors. To make this statement precise, we introduce the notation c_{00}^+ for the positive cone of c_{00}, that is,

$$c_{00}^+ := \Big\{ \sum_{j=1}^n \alpha_j e_j : n \in \mathbb{N},\ \alpha_1, \ldots, \alpha_n \in [0, \infty) \Big\}.$$

6.8 Lemma. *Let $p \geqslant 1$, let f be a functional on c_{00}, and suppose that there is a constant $C \geqslant 0$ such that $\big|\langle x, f\rangle\big| \leqslant C\|x\|_{V_p}$ for each $x \in c_{00}^+$. Then f extends uniquely to a bounded functional on V_p, and the extension has norm at most $4C$.*

Proof. Given $x \in c_{00}$, let $x^+ := \big(\max\{\mathrm{Re}\langle x, f_n\rangle, 0\}\big)_{n\in\mathbb{N}} \in c_{00}^+$. Then, for each permissible subset $A = \{n_1 < n_2 < \cdots < n_{k+1}\}$ of $\mathbb{N}$, we have

$$\begin{aligned} \|x\|_{V_p}^p &\geqslant \sum_{j=1}^k \big|\langle x, f_{n_j} - f_{n_{j+1}}\rangle\big|^p \geqslant \sum_{j=1}^k \big|\mathrm{Re}\langle x, f_{n_j}\rangle - \mathrm{Re}\langle x, f_{n_{j+1}}\rangle\big|^p \\ &\geqslant \sum_{j=1}^k \big|\max\{\mathrm{Re}\langle x, f_{n_j}\rangle, 0\} - \max\{\mathrm{Re}\langle x, f_{n_{j+1}}\rangle, 0\}\big|^p = \nu_p(x^+, A)^p, \end{aligned} \tag{6.4}$$

where the final inequality is an immediate consequence of the elementary estimate

$$|\alpha - \beta| \geqslant \big|\max\{\alpha, 0\} - \max\{\beta, 0\}\big| \qquad (\alpha, \beta \in \mathbb{R}).$$

Taking the supremum over all permissible sets A in (6.4), we obtain $\|x\|_{V_p} \geqslant \|x^+\|_{V_p}$.

Now define $x^- := (-x)^+$, $x^{\mathrm{i}+} := (-\mathrm{i}x)^+$, and $x^{\mathrm{i}-} := (\mathrm{i}x)^+$. (Strictly speaking, the latter two definitions make sense only if the scalar field is $\mathbb{C}$; if $\mathbb{K} = \mathbb{R}$, let $x^{\mathrm{i}\pm} := 0$, no matter what x is.) Then we have $x = x^+ - x^- + \mathrm{i}(x^{\mathrm{i}+} - x^{\mathrm{i}-})$, where $x^\pm, x^{\mathrm{i}\pm} \in c_{00}^+$, and each of them has James–Schreier norm at most $\|x\|_{V_p}$ by the first part of the proof. Thus the assumption on f implies that

$$\big|\langle x, f\rangle\big| \leqslant \big|\langle x^+, f\rangle\big| + \big|\langle x^-, f\rangle\big| + \big|\langle x^{\mathrm{i}+}, f\rangle\big| + \big|\langle x^{\mathrm{i}-}, f\rangle\big| \leqslant 4C\|x\|_{V_p},$$

and the result follows from the density of c_{00} in V_p. □

6.9 Lemma. *For each $p \geqslant 1$ and each strictly increasing mapping $\sigma\colon \mathbb{N} \to \mathbb{N}$, the functional ξ_σ given by* (6.3) *extends uniquely to a bounded functional on V_p of norm at most $4 \cdot 2^{\frac{1}{p}}(2^{\frac{1}{p}} - 1)^{-1}$.*

Proof. We begin with an elementary observation for later reference:

$$\sum_{j=1}^n |\alpha_j| \leqslant n^{1-\frac{1}{p}} \Big(\sum_{j=1}^n |\alpha_j|^p \Big)^{\frac{1}{p}} \qquad (n \in \mathbb{N},\ \alpha_1, \ldots, \alpha_n \in \mathbb{K}); \tag{6.5}$$

indeed, this inequality is trivial for $p = 1$, whereas for $p > 1$, application of Hölder's inequality to the n-tuples $(1, 1, \ldots, 1)$ and $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ gives the required estimate. We note in passing that (4.1) is a special case of (6.5) corresponding to $n = 2$.

For each $m \in \mathbb{N}$, let $A_m := \mathbb{N} \cap [2^m - 1, 2^{m+1} - 2]$; this is a permissible set because $\operatorname{card} A_m = 2^m = \min A_m + 1$, and therefore, as σ is increasing, $\sigma(A_m)$ is also permissible. Now, for given $x \in c_{00}^+$, choose $N \in \mathbb{N}$ such that $\langle x, f_{\sigma(n)} \rangle = 0$ whenever $n > 2(2^N - 1)$. Then we have

$$\langle x, \xi_\sigma \rangle = \sum_{n=1}^{2(2^N-1)} \frac{(-1)^n}{n} \langle x, f_{\sigma(n)} \rangle = \sum_{n=1}^{2^N-1} \left(-\frac{\langle x, f_{\sigma(2n-1)} \rangle}{2n-1} + \frac{\langle x, f_{\sigma(2n)} \rangle}{2n} \right), \tag{6.6}$$

which is real because $x \in c_{00}^+$; the following estimates yield an upper bound:

$$\begin{aligned} \langle x, \xi_\sigma \rangle &\leqslant \sum_{n=1}^{2^N-1} \frac{\langle x, f_{\sigma(2n)} - f_{\sigma(2n-1)} \rangle}{2n} \leqslant \sum_{m=1}^{N} \left(\sum_{n=2^{m-1}}^{2^m-1} \frac{\left| \langle x, f_{\sigma(2n-1)} - f_{\sigma(2n)} \rangle \right|}{2^m} \right) \\ &\leqslant \sum_{m=1}^{N} \frac{1}{2^m} \sum_{j=2^m-1}^{2^{m+1}-3} \left| \langle x, f_{\sigma(j)} - f_{\sigma(j+1)} \rangle \right| \\ &\underset{\text{by (6.5)}}{\leqslant} \sum_{m=1}^{N} \frac{(2^m-1)^{1-\frac{1}{p}}}{2^m} \left(\sum_{j=2^m-1}^{2^{m+1}-3} \left| \langle x, f_{\sigma(j)} - f_{\sigma(j+1)} \rangle \right|^p \right)^{\frac{1}{p}} \\ &= \sum_{m=1}^{N} \frac{(2^m-1)^{1-\frac{1}{p}}}{2^m} \nu_p\big(x, \sigma(A_m)\big) \leqslant \sum_{m=1}^{N} (2^m)^{-\frac{1}{p}} \|x\|_{V_p} \leqslant \frac{\|x\|_{V_p}}{2^{\frac{1}{p}} - 1} \end{aligned}$$

by summation of the geometric progression.

On the other hand, pairing up neighbouring terms differently in (6.6), we obtain a lower bound:

$$\begin{aligned} \langle x, \xi_\sigma \rangle &= -\langle x, f_{\sigma(1)} \rangle + \sum_{n=1}^{2^N-1} \left(\frac{\langle x, f_{\sigma(2n)} \rangle}{2n} - \frac{\langle x, f_{\sigma(2n+1)} \rangle}{2n+1} \right) \\ &\geqslant -\|x\|_{V_p} + \sum_{n=1}^{2^N-1} \frac{\langle x, f_{\sigma(2n)} - f_{\sigma(2n+1)} \rangle}{2n} \geqslant -\frac{2^{\frac{1}{p}} \|x\|_{V_p}}{2^{\frac{1}{p}} - 1}, \end{aligned}$$

where the final inequality follows from estimates very similar to those used in the calculation above.

Combining the upper and lower bound, we obtain $\left| \langle x, \xi_\sigma \rangle \right| \leqslant 2^{\frac{1}{p}} (2^{\frac{1}{p}} - 1)^{-1} \|x\|_{V_p}$ for each $x \in c_{00}^+$, and hence Lemma 6.8 gives the result. □

6.10 Remark. It would suffice to prove Lemma 6.9 for $\sigma = \iota$ (the identity mapping on $\mathbb{N}$) because $\xi_\sigma = \Lambda'_\sigma \xi_\iota$, where Λ_σ is the left shift defined in (3.2). The proof of the boundedness of ξ_ι is not, however, significantly simpler than the general proof given above.

We are now ready to prove Lemma 6.5.

Proof of Lemma 6.5. We split in two cases. Suppose first that, for some $k \in \mathbb{N}$ and $\varepsilon > 0$, we have $\sum_{j=n}^{\infty} \bigl|\langle y_j, f_k\rangle\bigr| > \varepsilon$ for each $n \in \mathbb{N}$. Then the series $\sum_{j=1}^{\infty} \bigl|\langle y_j, f_k\rangle\bigr|$ diverges, so in this case we can simply take $f := f_k \in V_p'$.

Otherwise, for each $k \in \mathbb{N}$ and $\varepsilon > 0$, we can find $n \in \mathbb{N}$ such that

$$\sum_{j=n}^{\infty} \bigl|\langle y_j, f_k\rangle\bigr| \leqslant \varepsilon. \tag{6.7}$$

In this case our strategy is to construct a strictly increasing mapping $\sigma\colon \mathbb{N} \to \mathbb{N}$ such that the series $\sum_{n=1}^{\infty} \bigl|\langle y_n, \xi_\sigma\rangle\bigr|$ diverges. Fix a summable, decreasing sequence $(\varepsilon_m)_{m\in\mathbb{N}}$ in $(0,\infty)$. By induction, we choose integers $0 \leqslant N_0 < \sigma(0) \leqslant N_1 < \sigma(1) \leqslant \cdots$ such that the following three conditions are satisfied:

$$\Bigl|\sum_{j=1}^{n} \langle y_j, f_{\sigma(m-1)}\rangle\Bigr| \geqslant 1 - \varepsilon_m \qquad (m \in \mathbb{N},\ n \geqslant N_m); \tag{6.8}$$

$$\Bigl\|(I_{V_p} - P_k) \sum_{j\in A} y_j\Bigr\|_{V_p} \leqslant \varepsilon_m \qquad \bigl(m \in \mathbb{N},\ k \geqslant \sigma(m) - 1,\ A \subseteq [1, N_m] \cap \mathbb{N}\bigr); \tag{6.9}$$

$$\sum_{j=N_m+1}^{\infty} \bigl|\langle y_j, f_{\sigma(k)}\rangle\bigr| \leqslant \frac{k\varepsilon_m}{m-1} \qquad (m \geqslant 2,\ 1 \leqslant k < m). \tag{6.10}$$

To start the induction, we observe that (6.8)–(6.10) are vacuous for $m = 0$, so we can simply take $N_0 := 0$ and $\sigma(0) := 1$. Now let $\ell \in \mathbb{N}$, and assume inductively that integers $0 \leqslant N_0 < \sigma(0) \leqslant N_1 < \sigma(1) \leqslant \cdots \leqslant N_{\ell-1} < \sigma(\ell-1)$ have been chosen such that (6.8)–(6.10) are satisfied for each $m \leqslant \ell - 1$. For $n \geqslant \sigma(\ell-1)$, our assumption (6.2) implies that

$$1 - \sum_{j=1}^{n} \langle y_j, f_{\sigma(\ell-1)}\rangle = \Bigl\langle \chi_{[1,n]} - \sum_{j=1}^{n} y_j, f_{\sigma(\ell-1)} \Bigr\rangle \to 0 \quad \text{as} \quad n \to \infty,$$

so we can find $N_\ell' \geqslant \sigma(\ell-1)$ such that (6.8) is satisfied for $m = \ell$ and each $n \geqslant N_\ell'$. In the case where $\ell \geqslant 2$, $\ell - 1$ applications of (6.7) yield a number $N_\ell'' \in \mathbb{N}$ such that

$$\sum_{j=N_\ell''+1}^{\infty} \bigl|\langle y_j, f_{\sigma(k)}\rangle\bigr| \leqslant \frac{k\varepsilon_\ell}{\ell-1} \qquad \bigl(k \in \{1, \ldots, \ell-1\}\bigr).$$

Thus (6.8) and (6.10) will both be satisfied for each $m \leqslant \ell$ provided that we define

$$N_\ell := \begin{cases} N_\ell' & \text{if } \ell = 1,\\ \max\{N_\ell', N_\ell''\} & \text{otherwise.} \end{cases}$$

Since there are only finitely many subsets of $[1, N_\ell] \cap \mathbb{N}$, and $\bigl\|(I_{V_p} - P_k)x\bigr\|_{V_p} \to 0$ as $k \to \infty$ for each $x \in V_p$ because $(e_n)_{n\in\mathbb{N}}$ is a basis for V_p, we can find $\sigma(\ell) > N_\ell$ such that (6.9) is satisfied for $m = \ell$, and hence the induction continues.

Now let $z_m := \sum_{j=N_m+1}^{N_{m+1}} y_j \in V_p$ for each $m \in \mathbb{N}$; we then have

$$\sum_{m=1}^{\infty}\big|\langle y_m, \xi_\sigma\rangle\big| \geqslant \sum_{m=1}^{\infty}\left|\sum_{j=N_m+1}^{N_{m+1}}\langle y_j, \xi_\sigma\rangle\right| = \sum_{m=1}^{\infty}\big|\langle z_m, \xi_\sigma\rangle\big|. \tag{6.11}$$

Seeking a lower bound for the right-hand side of this inequality, we observe that

$$\begin{aligned}
\big|\langle z_m, \xi_\sigma\rangle\big| &\geqslant \big|\langle P_{\sigma(m+1)-1} z_m, \xi_\sigma\rangle\big| - \big|\langle (I_{V_p} - P_{\sigma(m+1)-1}) z_m, \xi_\sigma\rangle\big| \\
&\geqslant \left|\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\langle P_{\sigma(m+1)-1} z_m, f_{\sigma(k)}\rangle\right| - \|\xi_\sigma\|_{V_p'}\big\|(I_{V_p} - P_{\sigma(m+1)-1}) z_m\big\|_{V_p} \\
&\geqslant \left|\sum_{k=1}^{m}\frac{(-1)^k}{k}\langle z_m, f_{\sigma(k)}\rangle\right| - \|\xi_\sigma\|_{V_p'}\,\varepsilon_{m+1} \\
&\geqslant \frac{\big|\langle z_m, f_{\sigma(m)}\rangle\big|}{m} - \sum_{k=1}^{m-1}\frac{\big|\langle z_m, f_{\sigma(k)}\rangle\big|}{k} - \|\xi_\sigma\|_{V_p'}\,\varepsilon_{m+1}
\end{aligned} \tag{6.12}$$

for each $m \in \mathbb{N}$, where the penultimate inequality follows from (6.9) and the fact that

$$P'_{\sigma(m+1)-1} f_{\sigma(k)} = \begin{cases} f_{\sigma(k)} & \text{if } k \leqslant m, \\ 0 & \text{otherwise.} \end{cases}$$

If $m = 1$, we can ignore the second term on the right-hand side of (6.12). Otherwise it has the following upper bound

$$\sum_{k=1}^{m-1}\frac{\big|\langle z_m, f_{\sigma(k)}\rangle\big|}{k} \leqslant \sum_{k=1}^{m-1}\sum_{j=N_m+1}^{N_{m+1}}\frac{\big|\langle y_j, f_{\sigma(k)}\rangle\big|}{k} \leqslant \sum_{k=1}^{m-1}\frac{k\varepsilon_m}{k(m-1)} = \varepsilon_m \tag{6.13}$$

by (6.10). We find a lower bound for the first term on the right-hand side of (6.12) by writing $z_m = \sum_{j=1}^{N_{m+1}} y_j - \sum_{j=1}^{N_m} y_j$ and using the fact that $P'_{\sigma(m)-1} f_{\sigma(m)} = 0$:

$$\begin{aligned}
\big|\langle z_m, f_{\sigma(m)}\rangle\big| &= \left|\sum_{j=1}^{N_{m+1}}\langle y_j, f_{\sigma(m)}\rangle - \left\langle \sum_{j=1}^{N_m} y_j, (I_{V_p} - P_{\sigma(m)-1})' f_{\sigma(m)}\right\rangle\right| \\
&\geqslant \left|\sum_{j=1}^{N_{m+1}}\langle y_j, f_{\sigma(m)}\rangle\right| - \|f_{\sigma(m)}\|_{V_p'}\left\|(I_{V_p} - P_{\sigma(m)-1})\sum_{j=1}^{N_m} y_j\right\|_{V_p} \\
&\geqslant 1 - \varepsilon_{m+1} - \varepsilon_m
\end{aligned} \tag{6.14}$$

by (6.8), (6.9), and Lemma 4.3(i). Substituting (6.13) and (6.14) into (6.12), we obtain

$$\big|\langle z_m, \xi_\sigma\rangle\big| \geqslant \frac{1 - \varepsilon_{m+1} - \varepsilon_m}{m} - \varepsilon_m - \|\xi_\sigma\|_{V_p'}\,\varepsilon_{m+1} \geqslant \frac{1}{m} - \big(3 + \|\xi_\sigma\|_{V_p'}\big)\varepsilon_m.$$

Since the harmonic series diverges, whereas the series $\sum_{m=1}^{\infty}\varepsilon_m$ converges, we conclude that the series $\sum_{m=1}^{\infty}\big|\langle z_m, \xi_\sigma\rangle\big|$ diverges, and therefore, by (6.11), the same is true for the series $\sum_{m=1}^{\infty}\big|\langle y_m, \xi_\sigma\rangle\big|$, as desired. □

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References

[1] F. Albiac and N. J. Kalton, *Topics in Banach space theory,* Grad. Texts Math. 233, Springer-Verlag, 2006.

[2] D. E. Alspach and S. A. Argyros, *Complexity of weakly null sequences*, Dissertationes Math. 321 (1992).

[3] A. D. Andrew and W. L. Green, *On James' quasi-reflexive space as a Banach algebra,* Can. J. Math. 32 (1980), 1080–1101.

[4] A. Baernstein II, *On reflexivity and summability,* Studia Math. 42 (1972), 91–94.

[5] S. Banach and S. Saks, *Sur la convergence forte dans les champs L^p,* Studia Math. 2 (1930), 51–57.

[6] B. Beauzamy and J.-T. Lapresté, *Modèles étalés des espaces de Banach,* Publ. Dép. Math. (Lyon) 1983.

[7] C. Bessaga and A. Pełczyński, *On bases and unconditional convergence of series in Banach spaces,* Studia Math. 17 (1958), 151–164.

[8] C. Bessaga and A. Pełczyński, *Banach spaces non-isomorphic to their Cartesian squares. I,* Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 8 (1960), 77–80.

[9] A. Bird, *An amalgamation of the Banach spaces associated with James and Schreier, Part II: Banach-algebra structure,* this volume.

[10] A. Bird, N. J. Laustsen, and A. Zsák, *Some remarks on James–Schreier spaces,* J. Math. Anal. Appl. 371 (2010), 609–613.

[11] A. Blanco, *Weak amenability of $\mathscr{A}(E)$ and the geometry of E,* J. London Math. Soc. 66 (2002), 721–740.

[12] A. Blanco and N. Grønbæk, *Amenability of algebras of approximable operators,* Israel J. Math. 171 (2009), 127–156.

[13] P. G. Casazza, *James' quasi-reflexive space is primary,* Israel J. Math. 26 (1977), 294–305.

[14] P. G. Casazza, Bor-Luh Lin, and R. H. Lohman, *On James' quasi-reflexive Banach space,* Proc. Amer. Math. Soc. 67 (1977), 265–271.

[15] P. G. Casazza and T. J. Shura, *Tsirelson's space,* Springer Lecture Notes in Mathematics 1363, Springer-Verlag, 1989.

[16] N. Dunford and J. T. Schwartz, *Linear Operators,* Vol. I, Pure and Applied Mathematics 7, Interscience, 1958.

[17] I. S. Edelstein and B. S. Mityagin, *Homotopy type of linear groups of two classes of Banach spaces,* Functional Anal. Appl. 4 (1970), 221–231.

[18] H. Fetter and B. Gamboa de Buen, *The James forest,* LMS Lecture Notes 236, Cambridge University Press, 1997.

[19] I. Gasparis and D. H. Leung, *On the complemented subspaces of the Schreier spaces,* Studia Math. 141 (2000), 273–300.

[20] D. P. Giesy and R. C. James, *Uniformly non-ℓ_1 and B-convex Banach spaces,* Studia Math. 48 (1973), 61–69.

[21] R. Herman and R. Whitley, *An example concerning reflexivity,* Studia Math. 28 (1967), 289–294.

[22] R. C. James, *Bases and reflexivity of Banach spaces,* Ann. of Math. 52 (1950), 518–527.

[23] R. C. James, *A non-reflexive Banach space isometric with its second conjugate space,* Proc. Nat. Acad. Sci. 37 (1951), 174–177.

[24] N. J. Laustsen, *K-theory for the Banach algebra of operators on James's quasi-reflexive Banach spaces,* K-Theory 23 (2001), 115–127.

[25] N. J. Laustsen, *Maximal ideals in the algebra of operators on certain Banach spaces,* Proc. Edin. Math. Soc. 45 (2002), 523–546.

[26] N. J. Laustsen, *Commutators of operators on Banach spaces,* J. Operator Theory 48 (2002), 503–514.

[27] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces,* Vol. I, Ergeb. Math. Grenzgeb. 92, Springer-Verlag, 1977.

[28] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces,* Vol. II, Ergeb. Math. Grenzgeb. 97, Springer-Verlag, 1979.

[29] R. J. Loy and G. A. Willis, *Continuity of derivations on $\mathscr{B}(E)$ for certain Banach spaces E,* J. London Math. Soc. 40 (1989), 327–346.

[30] R. E. Megginson, *An introduction to Banach space theory*, Grad. Texts Math. 183, Springer-Verlag, New York, 1998.

[31] E. W. Odell and H.-O. Tylli, *Weakly compact approximation in Banach spaces,* Trans. Amer. Math. Soc. 357 (2005), 1125–1159.

[32] A. Pełczyński, *A connection between weakly unconditional convergence and weakly completeness of Banach spaces,* Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 6 (1958), 251–253.

[33] G. Pisier, *The dual J^* of the James space has cotype* 2 *and the Gordon–Lewis property,* Math. Proc. Camb. Phil. Soc. 103 (1988), 323–331.

[34] J. Schreier, *Ein Gegenbeispiel zur Theorie der schwachen Konvergentz,* Studia Math. 2 (1930), 58–62.

[35] G. A. Willis, *Compressible operators and the continuity of homomorphisms from algebras of operators,* Studia Math. 115 (1995), 251–259.

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