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A SOLUTION TO THE 2/3 CONJECTURE*

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Abstract. We prove a vertex domination conjecture of Erdős, Faudree, Gould, Gyárfás, Rousseau, and Schelp that for every n-vertex complete graph with edges colored using three colors there exists a set of at most three vertices which have at least $2n/3$ neighbors in one of the colors. Our proof makes extensive use of the ideas presented in [D. Král' et al., *A new bound for the* 2/3 *conjecture*, Combin. Probab. Comput. 22 (2013), pp. 384–393].

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1. Introduction.

In this paper we prove the 2/3 conjecture of Erdős et al. [6]. Before we discuss this problem we first require some definitions.

A *graph* is a pair of sets $G = (V(G), E(G))$, where $V(G)$ is the set of *vertices* and $E(G)$ is a family of 2-subsets of $V(G)$ called *edges*. A *complete graph* is a graph containing all possible edges.

An r-*coloring* of the edges of a graph G is a map from $E(G)$ to a set of size r. Given an r-coloring of the edges of a complete graph $G = (V(G), E(G))$, a color c, and $A, B \subseteq V(G)$, we say that A c-*dominates* B if for every $b \in B \setminus A$ there exists $a \in A$ such that the edge ab is colored c. We say that A *strongly* c-*dominates* B if for every $b \in B$ there exists an $a \in A$ such that the edge ab is colored c. Note that if A strongly c-dominates B, then it also c-dominates B.

Erdős and Hajnal [8] showed that given a positive integer t, a real value $\epsilon > 0$, and a 2-colored complete graph on $n > n_0$ vertices, there exists a set of t vertices that c-dominate at least $(1-(1+\epsilon)(2/3)^t)n$ vertices for some color c. They asked whether 2/3 could be replaced by 1/2, which was answered by Erdős et al. [7].

THEOREM 1.1 (Erdős et al. [7]). *For any positive integer t and any 2-colored complete graph on n vertices, there exists a color c and a set of at most t vertices that c-dominate at least $(1-1/2^t)n$ of the vertices.*

Erdős, Faudree, Gyárfás, and Schelp went on to ask whether their result could be generalized to say that all r-colored complete graphs contain a set of t vertices that c-dominate at least $(1-(1-1/r)^t)n$ of the vertices. However, in the same paper [7], they presented a construction by Kierstead showing this to be false even for $r = 3$ and $t = 3$. Simply partition the vertices of a complete graph into 3 equal classes V_0, V_1, V_2, and color the edges such that an edge xy with $x \in V_i$ and $y \in V_j$ is colored i if $i = j$ or $i \equiv j + 1 \bmod 3$; see Figure 1. The construction shows that for 3-colorings it is impossible for a small set of vertices to monochromatically dominate significantly more than 2/3 of the vertices. (It also shows that regardless of the size of our dominating set we cannot guarantee that more than $\lceil 2n/3 \rceil$ vertices will be

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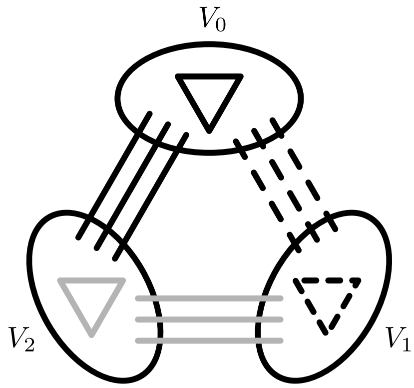

FIG. 1. *Kierstead's construction. The colors* $0, 1, 2$ *are represented by solid black, dashed black, and solid gray lines, respectively.*

strongly c-dominated.) Motivated by this example, Erdős, Faudree, Gould, Gyárfás, Rousseau, and Schelp made the following conjecture [6].

CONJECTURE 1.2 (Erdős et al. [6]). *For any* 3*-colored complete graph, there exists a color* c *and a set of at most* 3 *vertices that* c*-dominates at least* $2/3$ *of the vertices.*

They were able to show that the conjecture holds true when it was relaxed to asking for a dominating set of at most 22 vertices, but they were unable to reduce 22 to 3. We note that 3 is best possible because in a typical random 3-coloring of a complete graph of order n no pair of vertices will monochromatically dominate more than $5n/9 + o(n)$ vertices. (This follows simply from Chernoff's bound.) For completeness we should also mention that in [6] the authors showed there always exist 2 vertices that monochromatically dominate at least $5(n-1)/9$ vertices in a 3-colored complete graph.

Král' et al. [11], made significant progress with Conjecture 1.2 by proving that there exists a color c and set of size at most 4 which not only c-dominates but strongly c-dominates at least $2/3$ of the vertices in a 3-colored complete graph. Their proof makes use of Razborov's semidefinite flag algebra method [12] to show that Kierstead's construction is essentially extremal. We will discuss flag algebras in more detail in section 2.1.

We verify Conjecture 1.2 by proving the following theorem.

THEOREM 1.3. *For any* 3*-coloring of the edges of a complete graph on* $n \geq 3$ *vertices, there exists a color* c *and a set of* 3 *vertices that strongly* c*-dominate at least* $2n/3$ *vertices.*

Our proof builds on the work of Král' et al. [11]. The main difference is that by using an idea of Hladky, Král', and Norine [10] we have additional constraints to encode the 2/3 condition when applying the semidefinite flag algebra method (see Lemma 2.4). Another difference is that we conduct our computations on 6 vertex graphs, whereas in [11] they look at only 5 vertex graphs.

2. Proof of Theorem 1.3. For the remainder of this paper we will let $\hat{G}$ be a fixed counterexample to Theorem 1.3 with $|V(\hat{G})| = k$. So $\hat{G}$ is a 3-colored complete graph on $k \geq 3$ vertices such that every set of 3 vertices strongly c-dominates strictly less than $2k/3$ vertices for each color c. We will show that $\hat{G}$ cannot exist by proving that it would have to satisfy two contradicting properties.

Given a 3-colored complete graph and a vertex v, let A_v denote the set of colors of the edges incident to v.

The following lemma is implicitly given in the paper by Král' et al. [11].

LEMMA 2.1 (Král' et al. [11]). *Our counterexample $\hat{G}$ must contain a vertex $v \in V(\hat{G})$ with $|A_v| = 3$.*

Proof. Since $\hat{G}$ is a counterexample it cannot contain a vertex v with $|A_v| = 1$; otherwise any set of 3 vertices containing v will strongly c-dominate all the vertices, for $c \in A_v$. So it is enough to show that if $|A_v| = 2$ for every vertex, then $\hat{G}$ is not a counterexample.

Let the set of colors be $\{1, 2, 3\}$. If every vertex has $|A_v| = 2$ we can partition the vertices into three disjoint classes V_1, V_2, V_3, where $v \in V_i$ if $i \notin A_v$. Without loss of generality we can assume $|V_1| \geq |V_2| \geq |V_3|$. Note that the color of all edges uv with $u \in V_1$ and $v \in V_2$ is 3 because $A_u \cap A_v = \{2, 3\} \cap \{1, 3\} = \{3\}$. Consequently any set of 3 vertices containing a vertex from V_1 and a vertex from V_2 must strongly 3-dominate $V_1 \cup V_2$, which is at least $2/3$ of the vertices.

To complete the proof we need to consider what happens if we cannot choose a vertex from V_1 and V_2. This can occur only if $V_2 = \emptyset$, which implies $V_3 = \emptyset$ and $V_1 = V(\hat{G})$, i.e., $\hat{G}$ is 2-colored. In this case we can apply the result of Erdős, Faudree, Gyárfás, and Schelp [7, Theorem 1.1]. Although technically the theorem is not stated in terms of strongly c-dominating a set, its proof given in [7] is constructive and it can be easily checked that the dominating set it finds is strongly c-dominating (for $t \geq 2$ and $n \geq 2$). ∎

We will show via the semidefinite flag algebra method that $\hat{G}$ cannot contain a vertex v with $|A_v| = 3$ contradicting Lemma 2.1. The flag algebra method is primarily used to study the limit of densities in sequences of graphs. As such we will not apply it directly to $\hat{G}$ but to a sequence of graphs $(G_n)_{n \in \mathbb{N}}$, where G_n is constructed from $\hat{G}$ as follows. G_n is a 3-colored complete graph on nk vertices where each vertex $u \in V(\hat{G})$ has been replaced by a class of n vertices V_u. The edges of G_n are colored as follows: edges between two classes V_u and V_v have the same color as uv in $\hat{G}$, while edges within a class, V_u, say, are colored independently and uniformly at random with the colors from A_u.

We would like to claim that G_n is also a counterexample, but this may not be true. However, there exist particular types of 3-vertex sets which with high probability strongly c-dominate at most $2/3 + o(1)$ of the vertices in G_n for some color c. (Unless otherwise stated $o(1)$ will denote a quantity that tends to zero as $n \to \infty$.)

We note that Chernoff's bound implies that for all $u \in V(\hat{G})$, $c \in A_u$, and $v \in V_u \subset V(G_n)$ we have

$$|\{w \in V_u : vw \text{ is colored } c\}| = \frac{n}{|A_u|} + o(n)$$

with probability $1 - o(1)$.

Given a 3-colored complete graph and a color c we define a *good set for c* to be a set of 3 vertices $\{x, y, z\}$ such that either

(i) at least two of the edges xy, xz, yz are colored c, or
(ii) one of the edges, xy, say, is colored c and the remaining vertex z satisfies $|A_z \cup \{c\}| = 3$.

(Although this definition does not appear particularly natural, it has the advantage of being easily encoded by the semidefinite flag algebra method.)

LEMMA 2.2. *Any good set for c in G_n strongly c-dominates at most $2/3 + o(1)$ of the vertices with probability $1 - o(1)$.*

Proof. For $u \in V(\hat{G})$ recall that V_u is the corresponding class of n vertices in G_n. Given $S = \{x, y, z\}$ a good set for c in G_n, we will consider its "preimage" in $\hat{G}$ which we denote S', i.e., $S' \subseteq V(\hat{G})$ is minimal such that $S \subseteq \bigcup_{u \in S'} V_u$. Let the strongly c-dominated sets be D_S and $D_{S'}$ for S in G_n and S' in $\hat{G}$, respectively. For $v \in S$ there exists some $u \in V(\hat{G})$ such that $v \in V_u$; let us define W_v to be the vertices that lie within V_u that are strongly c-dominated by v.

We first consider the case where $|S'| = 3$ (which occurs when no two members of S lie in the same vertex class). Since $\hat{G}$ is a counterexample we have $|D_{S'}| < 2k/3$ or equivalently $|D_{S'}| \leq 2k/3 - 1/3$. It is easy to check that

$$D_S = \left(\bigcup_{d \in D_{S'}} V_d \right) \cup \left(\bigcup_{v \in S} W_v \right).$$

We can split the problem into two cases depending on which type of good set S is. If S is of type (i), then it is easy to see that $W_v \subseteq \bigcup_{d \in D_{S'}} V_d$ for every $v \in S$. Consequently $D_S = \bigcup_{d \in D_{S'}}, V_d$ implying $|D_S| = n|D_{S'}| \leq 2nk/3 - n/3$; hence D_S contains at most $2/3$ of the vertices in G_n as required.

If S is of type (ii), with xy colored c, then $W_x, W_y \subseteq \bigcup_{d \in D_{S'}} V_d$. If $c \in A_z$, then $|A_z| = 3$ and Chernoff's bound implies that $|W_z| \leq n/3 + o(n)$ holds with probability $1 - o(1)$. Note that this also holds when $c \notin A_z$ as $W_z = \emptyset$. So $D_S = W_z \cup \bigcup_{d \in D_{S'}} V_d$ and

$$|D_S| \leq |W_z| + n|D_{S'}| \leq n/3 + o(n) + n(2k/3 - 1/3) = (2/3 + o(1))nk$$

with probability $1 - o(1)$ as claimed.

To complete the proof we need to consider what happens when $|S'| < 3$. This can occur only if in G_n there exists a vertex class, V_m, say (with $m \in V(\hat{G})$), that contains two or three members of S. By the definition of a good set at least two of the vertices in S are incident with an edge of color c, so at least one such vertex is present in V_m, implying $c \in A_m$. This shows that we can always choose a set T of three vertices in $\hat{G}$ that contains both S' and a vertex u (possibly contained in S') with the property that um is colored c. The fact that S' is a subset of T gives us

$$D_S \subseteq \left(\bigcup_{d \in D_T} V_d \right) \cup \left(\bigcup_{v \in S} W_v \right),$$

where D_T is the set strongly c-dominated by T in $\hat{G}$, and having $u \in T$ ensures $W_v \subseteq \bigcup_{d \in D_T} V_d$ for every $v \in S \cap V_m$. Consequently we can apply the same argument we used for $|S'| = 3$. We note that (since $\hat{G}$ is a counterexample) $|D_T| \leq 2k/3 - 1/3$ and that if S is of type (i) we have $D_S \subseteq \bigcup_{d \in D_T} V_d$; otherwise S is of type (ii) and $D_S \subseteq W_z \cup \bigcup_{d \in D_T} V_d$. In either case we get the desired result that D_S contains at most $2/3 + o(1)$ of the vertices with high probability. ∎

We note that there are other types of 3-vertex sets that we could potentially utilize other than the "good sets"; however, our proof does not require them and so we will not discuss them here. Also, similar results can be proved for sets larger than 3, which may be of use in other problems. For a more general treatment we refer the reader to Král' et al. [11].

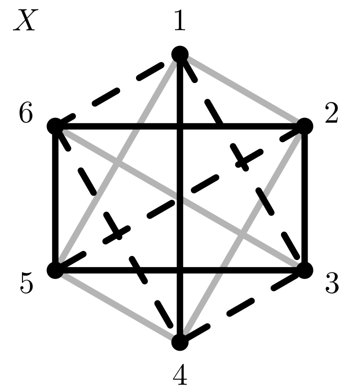

FIG. 2. *The 3-colored graph X.*

Given two 3-colored complete graphs F, G with $|V(F)| \leq |V(G)|$, we define $d_F(G)$, the *density* of F in G, to be the proportion of sets of size $|V(F)|$ in G that induce a 3-colored complete graph that is identical to F up to a reordering of vertices.

In [11] the authors bound the density in G_n of a family of graphs in order to contradict Lemma 2.1. In particular they chose their family to consist of all 3-colored complete graphs on 5 vertices that contain a vertex v with $|A_v| = 3$. We will instead bound the density of a single 6 vertex graph X, whose colored edge sets are given by

$$\{14, 23, 35, 56, 62\}, \{25, 34, 46, 61, 13\}, \{36, 45, 51, 12, 24\};$$

see Figure 2.

Observe that for our counterexample $\hat{G}$, Lemma 2.1 implies that there exists a vertex u with $|A_u| = 3$, which in turn implies there exists a class of n vertices V_u in G_n with the edges colored uniformly at random. By considering the probability of finding X in V_u we have the following simple bound for $d_X(G_n)$.

COROLLARY 2.3. *With probability* $1 - o(1)$,

$$d_X(G_n) \geq k^{-|V(X)|} 3^{-|E(X)|} + o(1).$$

By encoding Lemma 2.2 using flag algebras we will show that with high probability, $d_X(G_n) = o(1)$, a contradiction proving that no counterexample exists.

2.1. Flag algebras. Razborov's semidefinite flag algebra method introduced in [12] and [13] has proved to be an invaluable tool in extremal graph theory. Many results have been found through its application; see, for example, [1], [3], [4], [9], [10], [11]. We also refer interested readers to [2] for a minor improvement to the general method. Our notation and description of the method for 3-colored graphs are largely adapted from the explanation given by Baber and Talbot in [3].

We will say that two 3-colored complete graphs are *isomorphic* if they can be made identical by permutating their vertices. Let $\mathcal{H}$ be the family of all 3-colored complete graphs on l vertices, up to isomorphism. If l is sufficiently small we can explicitly determine $\mathcal{H}$ (by computer search if necessary). For $H \in \mathcal{H}$ and a large 3-colored complete graph K, we define $p(H; K)$ to be the probability that a random set of l vertices from K induces a 3-colored complete graph isomorphic to H.

Using this notation and averaging over l vertex sets in G_n (with $l \geq |V(X)|$), we can show

$$d_X(G_n) = \sum_{H\in\mathcal{H}} d_X(H)p(H;G_n), \tag{2.1}$$

and hence $d_X(G_n) \leq \max_{H\in\mathcal{H}} d_X(H)$. This bound is unsurprisingly extremely poor. We will rectify this by creating a series of inequalities from Lemma 2.2 that we can use to improve (2.1). To do this we first need to consider how small pairs of 3-colored complete graphs can intersect. We will use Razborov's method and his notion of flags and types to formally do this.

A *flag*, $F = (K', \theta)$, is a 3-colored complete graph K' together with an injective map $\theta : \{1, \dots, s\} \to V(K')$. If θ is bijective (and so $|V(K')| = s$) we call the flag a *type*. For ease of notation given a flag $F = (K', \theta)$ we define its order $|F|$ to be $|V(K')|$. Given a type σ we call a flag $F = (K', \theta)$ a *σ-flag* if the induced labeled 3-colored subgraph of K' given by θ is σ.

For a type σ and an integer $m \geq |\sigma|$, let $\mathcal{F}^\sigma_m$ be the set of all σ-flags of order m, up to isomorphism. For a nonnegative integer s and 3-colored complete graph K, let $\Theta(s, K)$ be the set of all injective functions from $\{1, \dots, s\}$ to $V(K)$. Given $F \in \mathcal{F}^\sigma_m$ and $\theta \in \Theta(|\sigma|, K)$ we define $p(F, \theta; K)$ to be the probability that an m-set V' chosen uniformly at random from $V(K)$ subject to $\mathrm{im}(\theta) \subseteq V'$ induces a σ-flag $(K[V'], \theta)$ that is isomorphic to F.

If $F_1 \in \mathcal{F}^\sigma_{m_1}$, $F_2 \in \mathcal{F}^\sigma_{m_2}$, and $\theta \in \Theta(|\sigma|, K)$, then $p(F_1, \theta; K)p(F_2, \theta; K)$ is the probability that two sets $V_1, V_2 \subseteq V(K)$ with $|V_1| = m_1$, $|V_2| = m_2$, chosen independently at random subject to $\mathrm{im}(\theta) \subseteq V_1 \cap V_2$, induce σ-flags $(K[V_1], \theta)$, $(K[V_2], \theta)$ that are isomorphic to F_1, F_2, respectively. We define the related probability, $p(F_1, F_2, \theta; K)$, to be the probability that two sets $V_1, V_2 \subseteq V(K)$ with $|V_1| = m_1$, $|V_2| = m_2$, chosen independently at random subject to $\mathrm{im}(\theta) = V_1 \cap V_2$, induce σ-flags $(K[V_1], \theta)$, $(K[V_2], \theta)$ that are isomorphic to F_1, F_2, respectively. Note that the difference in definitions between $p(F_1, \theta; K)p(F_2, \theta; K)$ and $p(F_1, F_2, \theta; K)$ is that of choosing the two sets with or without replacement. It is easy to show that $p(F_1, \theta; K)p(F_2, \theta; K) = p(F_1, F_2, \theta; K) + o(1)$, where the $o(1)$ term vanishes as $|V(K)|$ tends to infinity.

Taking the expectation over a uniformly random choice of $\theta \in \Theta(|\sigma|, K)$ gives

$$\mathbf{E}_{\theta\in\Theta(|\sigma|,K)}\left[p(F_1, \theta; K)p(F_2, \theta; K)\right] = \mathbf{E}_{\theta\in\Theta(|\sigma|,K)}\left[p(F_1, F_2, \theta; K)\right] + o(1).$$

Furthermore the expectation on the right-hand side can be rewritten in terms of $p(H; K)$ by averaging over l-vertex subgraphs of K, provided $m_1 + m_2 - |\sigma| \leq l$ (i.e., F_1 and F_2 intersecting on σ fits inside an l vertex graph). Hence

$$\begin{aligned}\mathbf{E}_{\theta\in\Theta(|\sigma|,K)}&\left[p(F_1, \theta; K)p(F_2, \theta; K)\right]\\ &= \sum_{H\in\mathcal{H}} \mathbf{E}_{\theta\in\Theta(|\sigma|,H)}\left[p(F_1, F_2, \theta; H)\right]p(H; K) + o(1).\end{aligned} \tag{2.2}$$

Observe that the right-hand side of (2.2) is a linear combination of $p(H; K)$ terms whose coefficients can be explicitly calculated using just $\mathcal{H}$; this will prove useful as (2.1) is of a similar form.

Given $\mathcal{F}^\sigma_m$ with $2m - |\sigma| \leq l$ and a positive semidefinite matrix $Q = (q_{ab})$ of dimension $|\mathcal{F}^\sigma_m|$, let $\mathbf{p}_\theta = (p(F, \theta; K) : F \in \mathcal{F}^\sigma_m)$ for $\theta \in \Theta(|\sigma|, K)$. Using (2.2) and the linearity of expectation we have

$$0 \leq \mathbf{E}_{\theta\in\Theta(|\sigma|,K)}[\mathbf{p}_\theta^T Q \mathbf{p}_\theta] = \sum_{H\in\mathcal{H}} a_H(\sigma, m, Q) p(H; K) + o(1), \tag{2.3}$$

where

$$a_H(\sigma, m, Q) = \sum_{F_a, F_b \in \mathcal{F}_m^\sigma} q_{ab} \mathbf{E}_{\theta\in\Theta(|\sigma|,H)}[p(F_a, F_b, \theta; H)].$$

Note that $a_H(\sigma, m, Q)$ is independent of K and can be explicitly calculated. Combining (2.3) when $K = G_n$ with (2.1) gives

$$\begin{aligned} d_X(G_n) &\leq \sum_{H\in\mathcal{H}} (d_X(H) + a_H(\sigma, m, Q)) p(H; G_n) + o(1) \\ &\leq \max_{H\in\mathcal{H}} (d_X(H) + a_H(\sigma, m, Q)) + o(1). \end{aligned}$$

Since some of the $a_H(\sigma, m, Q)$ values may be negative (for a careful choice of Q) this may be a better bound (asymptotically) for $d_X(G_n)$. To help us further reduce the bound we can of course create multiple inequalities of the form given by (2.3) by choosing different types σ_i, orders of flags m_i, and positive semidefinite matrices Q_i. Let $\alpha_H = \sum_i a_H(\sigma_i, m_i, Q_i)$ and hence we can say $d_X(G_n) \leq \max_{H\in\mathcal{H}}(d_X(H)+\alpha_H)+o(1)$. Finding the optimal choice of matrices Q_i which lowers the bound as much as possible is a convex optimization problem, in particular a semidefinite programming problem. As such we can use freely available software such as CSDP [5] to find the Q_i.

So far the bound on $d_X(G_n)$ is valid for any 3-colored complete graph; we have not yet made any use of the fact that G_n comes from our counterexample $\hat{G}$. Král' et al. remedy this (see Lemma 3.3 in [11]) by constructing a small set of constraints that G_n must satisfy, but a general 3-colored complete graph may not. By using an idea of Hladky, Král', and Norine [10] we can significantly increase the number of such constraints.

We say that a σ-flag F is *c-good* if the coloring of F and the size of σ imply that σ is a good set for c in F.

LEMMA 2.4. *Given a color c and a c-good σ-flag F, the following holds with probability $1 - o(1)$:*

$$\mathbf{E}_{\theta\in\Theta(|\sigma|,G_n)}\left[p(F,\theta;G_n)\left(\frac{2}{3}p(\sigma,\theta;G_n) - \sum_{F'\in\mathcal{D}} p(F',\theta;G_n)\right)\right] + o(1) \geq 0,$$

where $\mathcal{D} \subseteq \mathcal{F}_{|\sigma|+1}^\sigma$ is the set of all σ-flags on $|\sigma|+1$ vertices where the vertex not in σ is c-dominated by the type.

We note that when $F = \sigma$, Lemma 2.4 is equivalent to Lemma 3.3 in [11].

Proof. For a fixed $\theta \in \Theta(|\sigma|, G_n)$, if $p(F, \theta; G_n) = 0$, then trivially we get

$$p(F,\theta;G_n)\left(\frac{2}{3}p(\sigma,\theta;G_n) - \sum_{F'\in\mathcal{D}} p(F',\theta;G_n)\right) \geq 0.$$

If $p(F, \theta; G_n) > 0$, then there exists a copy of F in G_n and so the image of θ is σ (or equivalently $p(\sigma, \theta; G_n) = 1$) and σ must be a good set for c. By Lemma 2.2 we know that with probability $1 - o(1)$,

$$\frac{2}{3} + o(1) \geq \sum_{F'\in\mathcal{D}} p(F',\theta;G_n),$$

which implies

$$p(F,\theta;G_n)\left(\frac{2}{3}p(\sigma,\theta;G_n)-\sum_{F'\in\mathcal{D}}p(F',\theta;G_n)\right)+o(1)\geq 0.$$

Taking the expectation completes the proof. ∎

Given a c-good flag F, (2.2) tells us that provided $|F|+1\leq l$ we can express the inequality given in Lemma 2.4 as

$$\sum_{H\in\mathcal{H}} b_H(c,F)p(H;G_n)+o(1)\geq 0, \tag{2.4}$$

where $b_H(c,F)$ can be explicitly calculated from c, F, and H. Equation (2.4) is of the same form as (2.3) and as such we can use it in a similar way to improve the bound on $d_X(G_n)$. Moreover, observe that we can multiply (2.4) by any nonnegative real value without changing its form.

Let $\mathcal{C}$ be a set of pairs of colors c and c-good flags F satisfying $|F|+1\leq l$. For $(c,F)\in\mathcal{C}$ let $\mu(c,F)\geq 0$ be a real number (whose value we will choose later to help us improve the bound on $d_X(G_n)$). To ease notation we define $\beta_H=\sum_{(c,F)\in\mathcal{C}}\mu(c,F)b_H(c,F)$. It is easy to check $\sum_{H\in\mathcal{H}}\beta_H p(H;G_n)+o(1)\geq 0$; thus combining it with (2.1) and terms such as (2.3) gives

$$\begin{aligned} d_X(G_n) &\leq \sum_{H\in\mathcal{H}}(d_X(H)+\alpha_H+\beta_H)p(H;G_n)+o(1)\\ &\leq \max_{H\in\mathcal{H}}(d_X(H)+\alpha_H+\beta_H)+o(1). \end{aligned}$$

Finding an optimal set of nonnegative coefficients $\mu(c,F)$ and semidefinite matrices Q_i can still be posed as a semidefinite programming problem.

We complete the proof of Theorem 1.3 with the following lemma, which contradicts Corollary 2.3.

LEMMA 2.5. *With probability $1-o(1)$ we have $d_X(G_n)=o(1)$.*

Proof. By setting $l=6$ (the order of the graphs H) and solving a semidefinite program we can find coefficients $\mu(c,F)$ and semidefinite matrices Q_i such that $\max_{H\in\mathcal{H}}(d_X(H)+\alpha_H+\beta_H)=0$. The relevant data needed to check this claim can be found in the data file `2-3.txt`. There is too much data to check by hand so we also provide the C++ program DominatingDensityChecker to check the data file. Both data file and proof checker may be downloaded from http://arxiv.org/e-print/1306.6202v1.

It is worth noting that in order to get a tight bound we used the methods described in section 2.4.2 of [1] to remove the rounding errors from the output of the semidefinite program solvers.

We end by mentioning that when $l=6$, the computation has to consider 25,506 nonisomorphic graphs which form $\mathcal{H}$, and as a result solving the semidefinite program is very time-consuming. However, our method for proving the result makes no preferences between the colors. Consequently it is quite easy to see that if there exists a solution, then there must also exist a solution which is invariant under the permutations of the colors. So if $H_1,H_2\in\mathcal{H}$ are isomorphic after a permutation of their colors, then in an "invariant solution" $d_X(H_1)+\alpha_{H_1}+\beta_{H_1}=d_X(H_2)+\alpha_{H_2}+\beta_{H_2}$ must necessarily hold. Therefore by restricting our search to invariant solutions we need only worry about those H in $\mathcal{H}'$ the set of 3-colored complete graphs on l vertices that are nonisomorphic even after a permutation of colors. For $l=6$, $|\mathcal{H}'|=4300$, which results in a significantly easier computation. ∎

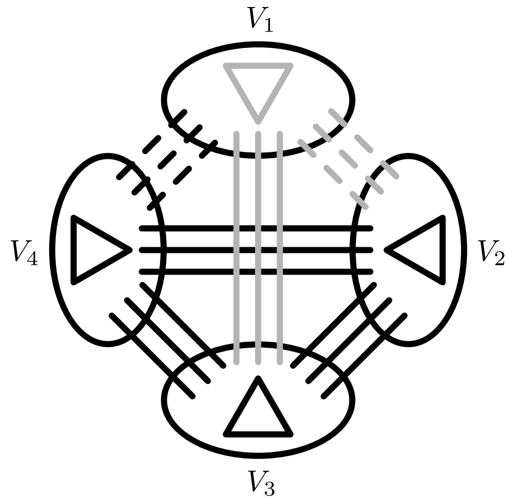

FIG. 3. *A generalization of Kierstead's construction to* 4 *colors. The* 4 *colors are represented by solid black, dashed black, solid gray, and dashed gray lines. The classes* V_1, V_2, V_3, *and* V_4 *contain* $2/5, 1/5, 1/5$, *and* $1/5$ *of the vertices, respectively.*

3. Open problems. Erdős, Faudree, Gould, Gyárfás, Rousseau, and Schelp ask in [6] whether every 4-colored complete graph always contains a small set of vertices that monochromatically dominate at least $3/5$ of the vertices. The value of $3/5$ comes from considering the 4-color equivalent of Kierstead's construction given in Figure 3. It shows that we cannot hope to find a small set of vertices that monochromatically dominate significantly more than $3/5$ of the vertices. Also, regardless of the size of the dominating set we can at most guarantee that $\lceil 3n/5 \rceil$ vertices will be strongly monochromatically dominated in an n vertex graph.

By applying Chernoff's bound it is easy to see that a typical random 4-coloring on an n vertex graph contains no 3-sets that monochromatically dominate more than $(1-(3/4)^3)n + o(n)$ vertices, which is less than $3n/5$ when n is large. So the minimal possible dominating set size is 4.

We could not prove that there always exists a 4-set that strongly monochromatically dominates $3/5$ of the vertices in a complete 4-colored graph. However, by generalizing the method given in section 2, and replacing X with a specific family of graphs, we were able to show that there exist 4-sets that strongly monochromatically dominate 0.5711 of the vertices. The family of graphs we chose to bound instead of X are the 48 graphs on 5 vertices that are, up to a permutation of colors, isomorphic to one of those given in Figure 4. It is not immediately obvious why such a family should have a positive density in a counterexample, so we will outline why this is the case.

LEMMA 3.1. *Any* 4*-colored complete graph* $\hat{G}$ *on* k *vertices that has the property that every set of* 3 *vertices strongly* c*-dominates strictly less than* $3/5$ *of the vertices for every color* c *must contain either*

(i) *a vertex* u *with* $|A_u| = 4$ *or*

(ii) *two vertices* v, w *with* $|A_v| = |A_w| = 3$ *and* $A_v \neq A_w$.

Proof. Let the set of colors be $\{1, 2, 3, 4\}$. Trivially $\hat{G}$ cannot contain a vertex u with $|A_u| = 1$; otherwise any set containing u will strongly dominate all the vertices. It is therefore sufficient to show that no $\hat{G}$ can exist in which every vertex u satisfies $|A_u| = 2$, or $A_u = \{1, 2, 3\}$.

We can take our vertices and partition them into disjoint classes based on their value of A_v. For ease of notation we will refer to these classes by V_S, where S is a subset of the colors; for example, V_{13} contains all the vertices v which have $A_v = \{1, 3\}$. We will split our argument into multiple cases depending on whether a class is empty. Throughout we will make use of the fact that $\hat{G}$ cannot be 3-colored; otherwise, by Theorem 1.3 we can find a 3-set which dominates over 3/5 of the vertices. Also note that if $S, T \subseteq \{1, 2, 3, 4\}$ and $S \cap T = \emptyset$, then either V_S or V_T must be empty as any edge that goes between the two classes must have a color in $S \cap T$.

Suppose $V_{123} = \emptyset$. Without loss of generality we can assume $V_{12} \neq \emptyset$ (implying $V_{34} = \emptyset$). There must be another nonempty class; otherwise, $\hat{G}$ is 3-colored. Without loss of generality we may assume $V_{13} \neq \emptyset$ (implying $V_{24} = \emptyset$). To avoid being 3-colored we must have $V_{14} \neq \emptyset$ (implying $V_{23} = \emptyset$). There are no more classes we could add and $\hat{G}$ has all its vertices strongly 1-dominated by a 2-set containing a vertex from V_{12} and a vertex from V_{13}, which is a contradiction.

Suppose $V_{123} \neq \emptyset$. To avoid being 3-colored at least one of V_{14}, V_{24}, V_{34} must be nonempty. Without loss of generality assume $V_{14} \neq \emptyset$ (implying $V_{23} = \emptyset$). To avoid having every vertex strongly 1-dominated by a 2-set containing a vertex from V_{123} and a vertex from V_{14}, either V_{24} or V_{34} must be nonempty. Without loss of generality assume $V_{24} \neq \emptyset$ (implying $V_{13} = \emptyset$). If $V_{34} = \emptyset$, then the vertices in $\hat{G}$ are partitioned into three disjoint classes $V_{123} \cup V_{12}$, V_{14}, and V_{24}. We can strongly c-dominate at least 2/3 of the vertices by choosing c to be the color of the edges that go between the largest two of the classes and by choosing our dominating set to contain a vertex from each of the largest two classes.

The only case left to consider is when $V_{123} \neq \emptyset$, $V_{14} \neq \emptyset$, $V_{24} \neq \emptyset$, and $V_{34} \neq \emptyset$. (Note that $\hat{G}$ resembles Figure 3.) We may suppose that $|V_{14}| \geq |V_{24}| \geq |V_{34}|$ and so either $|V_{14} \cup V_{24} \cup V_{34}| \geq 3k/5$ (where k is the order of $\hat{G}$) and a 2-set containing a vertex from each of V_{14} and V_{24} will strongly 4-dominate at least 3/5 of the vertices or $|V_{123} \cup V_{14}| \geq 3k/5$ and a 2-set containing a vertex from each of V_{123} and V_{14} will strongly 1-dominate at least 3/5 of the vertices. ∎

COROLLARY 3.2. *Let $\hat{G}$ be a 4-colored complete graph on k vertices that has the property that every set of 3 vertices strongly c-dominates strictly less than 3/5 of the vertices for every color c. If G_n is constructed from $\hat{G}$ as before then, with probability $1 - o(1)$,*

$$d_F(G_n) \geq k^{-|V(F)|}4^{-|E(F)|} + o(1)$$

holds for some graph F that is, up to a permutation of colors, isomorphic to one of the graphs given in Figure 4.

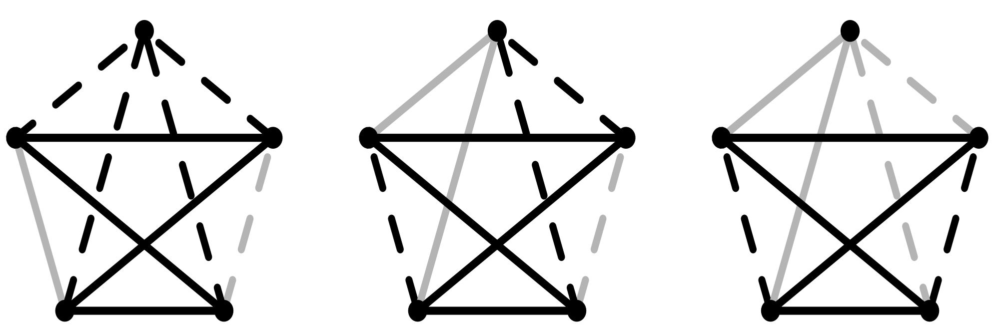

FIG. 4. *Canonical members of a family of 4-colored complete graphs that replace X. The 4 colors are represented by solid black, dashed black, solid gray, and dashed gray lines.*

Proof. By Lemma 3.1 $\hat{G}$ must have a vertex u with $|A_u| = 4$ or two vertices v, w with $|A_v| = |A_w| = 3$ and $A_v \neq A_w$. If we have a vertex u with $|A_u| = 4$ there will be a vertex class of size n in G_n with all its edges colored uniformly at random. The result trivially holds by considering the density of any 5-vertex graph inside this vertex class.

Suppose instead there exist two vertices v, w in $\hat{G}$ with $|A_v| = |A_w| = 3$ and $A_v \neq A_w$. To ease notation let c_{xy} be the color of the edge xy. Note that by the definition of $\hat{G}$ we know that every vertex is not strongly c_{vw}-dominated by the set $\{v, w\}$. Consequently there must exist a vertex z such that $c_{vz} \neq c_{vw}$ and $c_{wz} \neq c_{vw}$. Now consider the vertex classes V_v, V_w, and V_z in G_n. There are 9 possible 5-vertex graphs that could be formed from taking one vertex in V_z and two vertices in V_v and V_w. Only one of these 9 graphs has the property that the two vertices v_1, v_2 chosen from V_v satisfy $A_{v_1} = A_{v_2} = A_v$ and the two vertices w_1, w_2 chosen from V_w satisfy $A_{w_1} = A_{w_2} = A_w$. This graph is, up to a permutation of colors, isomorphic to one of those given in Figure 4. The result trivially follows by considering its density in $V_v \cup V_w \cup V_z$. ∎

Although our discussion has centred on 4-colorings, it would also be interesting to know what happens for complete graphs which are r-colored for $r \geq 5$.

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