SPECTRAL STABILITY OF METRIC-MEASURE LAPLACIANS

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Abstract. We consider a "convolution mm-Laplacian" operator on metric-measure spaces and study its spectral properties. The definition is based on averaging over small metric balls. For sufficiently nice metric-measure spaces we prove stability of convolution Laplacian's spectrum with respect to metric-measure perturbations and obtain Weyl-type estimates on the number of eigenvalues.

1. Introduction

This paper is motivated by [2] where we approximate a compact Riemannian manifold by a weighted graph and show that the spectra of the Beltrami–Laplace operator on the manifold and the graph Laplace operator are close to each other. The key constructions of [2] can be regarded as a definition of an operator which approximates the Beltrami–Laplace operator. The definition is based on averaging over small balls. The construction makes sense for general metric-measure spaces, which in particular include Riemannian manifolds and weighted graphs.

In this paper we show that an analogue of some results from [2] holds for a large class of metric-measure spaces. Namely we introduce a "convolution Laplacian" operator with a parameter $\rho > 0$ (a radius) and prove that its spectrum enjoys stability under metric-measure approximations.

Recall that a metric-measure space is a triple (X, d, μ) where (X, d) is a metric space and μ is a Borel measure on X. All metric spaces in this paper are compact and all measures are finite. We denote by $B_r(x)$ the metric ball of radius r centered at a point $x \in X$.

Our main object of study is defined as follows.

Definition 1.1. Let $X = (X, d, \mu)$ be a metric-measure space and $\rho > 0$. The *ρ-Laplacian* $\Delta_X^\rho\colon L^2(X) \to L^2(X)$ is defined by

$$\Delta_X^\rho u(x) = \frac{1}{\rho^2 \mu(B_\rho(x))} \int_{B_\rho(x)} \big(u(x) - u(y)\big)\, d\mu(y) \tag{1.1}$$

for $u \in L^2(X)$.

If X is a Riemannian n-manifold, then Δ_X^ρ converges as $\rho \to 0$ (e.g. on smooth functions) to the Beltrami–Laplace operator multiplied by the constant $\frac{-1}{2(n+2)}$. For general metric-measure spaces, it is not clear what should replace the normalizing constant $\frac{1}{2(n+2)}$ so it does not appear in our definition. It is plausible that Δ_X^ρ has

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a meaningful limit as $\rho \to 0$ for a large class of metric-measure spaces X. We hope to address this question elsewhere. In this paper we consider the operator Δ_X^ρ for a fixed "small" value of ρ. Our goal is to study the spectrum of Δ_X^ρ and its stability properties.

Another interesting case is when X is a discrete space. In this case all needed geometric data amounts to weights of points and the information of which pairs of points are within distance ρ. This structure is just a weighted graph (without any lengths assigned to edges) and the ρ-Laplacian defined by (1.1) is just the classic weighted graph Laplacian. Spectral theory of graph Laplacians is a well developed subject, see e.g. [3, 17]. In the case when X is a Riemannian manifold, the spectral properties of ρ-Laplacians are studied in [8] in connection with random walks on the manifold. In this paper we study the topic from a different viewpoint. Namely we are interested in spectral stability under metric-measure perturbations.

As shown in Section 2, Δ_X^ρ is a non-negative self-adjoint operator with respect to a certain scalar product on $L^2(X)$. Hence the spectrum of Δ_X^ρ is a subset of $[0, +\infty)$. Moreover $\operatorname{spec}(\Delta_X^\rho) \subset [0, 2\rho^{-2}]$. The spectrum of a bounded self-adjoint operator divides into the discrete and essential spectrum. The discrete spectrum is the set of isolated eigenvalues of finite multiplicity and the essential spectrum is everything else. It turns out that the essential spectrum of Δ_X^ρ, if nonempty, is the single point $\{\rho^{-2}\}$. In our set-up we are concerned only with parts of the spectrum that are substantially below this value.

The following Theorem 1.2 is a non-technical implications of our main results. It asserts that under suitable conditions lower parts of ρ-Laplacian spectra converge as the metric-measure spaces in question converge. Denote by $\lambda_k(X, \rho)$ the k-th smallest eigenvalue of Δ_X^ρ (counting multiplicities).

Theorem 1.2. *Let a sequence $\{X_n\}$ of metric-measure spaces converge to $X = (X, d, \mu)$ in the sense of Fukaya [6]. Assume that d is a length metric and X satisfies a version of the Bishop–Gromov inequality: there is $\Lambda > 0$ such that*

$$\frac{\mu(B_{r_1}(x))}{\mu(B_{r_2}(x))} \le \left(\frac{r_1}{r_2}\right)^\Lambda \tag{1.2}$$

for all $x \in X$ and $r_1 \ge r_2 > 0$. Then

$$\lambda_k(X, \rho) = \lim_{n\to\infty} \lambda_k(X_n, \rho)$$

for all $\rho > 0$ and all k such that $\lambda_k(X, \rho) < \rho^{-2}$.

Theorem 1.2 follows from more general but more technical Theorem 5.4 which works for larger classes of spaces and provides estimates on the rate of convergence.

Now we discuss hypotheses of Theorem 1.2. We emphasize that "niceness" conditions in Theorem 1.2 are imposed only on the limit space X. The spaces X_n need not satisfy them. In particular, X_n can be discrete approximations of X. Thus, for every "nice" space X, the spectrum of Δ_X^ρ can be approximated by spectra of graph Laplacians.

By definition, a metric is a *length metric* if every pair of points can be connected by a geodesic segment realizing the distance between the points. This condition can be relaxed to an assumption about intersection of balls, see the BIV condition in Definition 5.2.

The classic Bishop–Gromov inequality deals with volumes of balls in Riemannian manifolds with Ricci curvature bounded from below. It implies (1.2) with Λ

depending on the dimension of the manifold, its diameter, and the lower bound for Ricci curvature. (In the case of non-negative Ricci curvature Λ is just equal to the dimension.) The Bishop–Gromov inequality holds for spaces with generalized Ricci curvature bounds in the sense of Lott–Sturm–Villani [14, 16, 9]. Other classes of spaces satisfying (1.2) include Finsler manifolds, dimensionally homogeneous polyhedral spaces, Carnot groups, etc.

The Fukaya convergence combines the Gromov–Hausdorff convergence of metric spaces and weak convergence of measures. See Definition 4.4 for details. Beware of the fact that, unlike most definitions used in this paper, the Fukaya convergence is sensitive to open sets of zero measure. See the example in Section 3.4 for an illustration of this subtle issue.

Actually in this paper we use another notion of metric-measure approximation which is more suitable to the problem. It allows us to obtain nice estimates on the difference of eigenvalues of ρ-Laplacians of close metric-measure spaces.

Structure of the paper. In Section 2 we introduce some notation and collect basic facts about ρ-Laplacians. In Section 3 we discuss some examples.

In Section 4 we introduce a notion of "closeness" of metric-measure spaces, which we call (ε, δ)-closeness. Loosely speaking, metric-measure spaces X and Y are (ε, δ)-close if Y is a result of imprecise measurements in X where distances are measured with a small additive inaccuracy ε and volumes are measured with a small relative inaccuracy δ. The formal definition is a combination of Gromov–Hausdorff distance and a "relative" version of Prokhorov distance between measures. The main results of Section 4 characterize (ε, δ)-closeness in terms of measure transports and Wasserstein distances.

In Section 5 we prove Theorem 5.4 which is a quantitative version of Theorem 1.2. It asserts that, if metric-measure spaces X and Y are (ε, δ)-close and satisfy certain conditions, then the lower parts of the spectra of their ρ-Laplacians are also close. The conditions in Theorem 5.4 can be thought of as "discretized" version of those from Theorem 1.2.

In Section 6 we give a direct construction of a map between $L^2(X)$ and $L^2(Y)$ realizing the spectral closeness in Theorem 5.4. The results of Section 6 complement Theorem 5.4 but they are not used in its proof.

In Section 7 we obtain Weyl-type estimates for the number of eigenvalues in an interval $[0, c\rho^{-2}]$ where $c < 1$ is a suitable constant. See Theorems 7.1 and 7.2. For a Riemannian manifold our estimates are of the same order as those given by Weyl's asymptotic formula for Beltrami–Laplace eigenvalues. However our estimates are formulated in terms of packing numbers rather than the dimension and total volume.

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2. Preliminaries

In the sequel we abbreviate metric-measure spaces as "mm-spaces". We use notation d_X and μ_X for the metric and measure of a mm-space X. In some cases we consider semi-metrics, that is, distances are allowed to be zero. All definitions apply to semi-metrics with no change.

To simplify computations and incorporate constructions from [2] into the present set-up, we introduce weighted ρ-Laplacians. Let $X = (X, d, \mu)$ be a mm-space and $\varphi\colon X \to \mathbb{R}_+$ a positive measurable function bounded away from 0 and ∞ on the support of μ. We call φ the *normalizing function*. We define a *weighted ρ-Laplacian* Δ^ρ_φ by

$$\Delta^\rho_\varphi u(x) = \frac{1}{\varphi(x)} \int_{B_\rho(x)} \big(u(x) - u(y)\big)\, d\mu(y).$$

We regard Δ^ρ_φ as an operator on $L^2(X)$. Note that this operator does not change if one replaces X by the support of its measure.

Definition 1.1 corresponds to the normalizing function $\varphi(x) = \rho^2\mu(B_\rho(x))$. Due to compactness of X, this function is bounded away from 0 and ∞ on the support of μ.

The operator Δ^ρ_φ is self-adjoint on $L^2(X, \varphi\mu)$ where $\varphi\mu$ is the measure with density φ w.r.t. μ. Indeed, for $u, v \in L^2(X)$ we have

$$\begin{aligned}\langle \Delta^\rho_\varphi u, v\rangle_{L^2(X,\varphi\mu)} &= \int_X \varphi(x) v(x) \frac{1}{\varphi(x)} \int_{B_\rho(x)} \big(u(x) - u(y)\big)\, d\mu(y) d\mu(x)\\ &= \iint_{d(x,y)<\rho} v(x)\big(u(x) - u(y)\big)\, d\mu(y) d\mu(x)\end{aligned}$$

and the right-hand side is clearly symmetric in u and v. The corresponding Dirichlet energy form

$$D^\rho_X(u) = \langle \Delta^\rho_\varphi u, u\rangle_{L^2(X,\varphi\mu)}$$

does not depend on φ and is given by

$$D^\rho_X(u) = \frac{1}{2} \iint_{d(x,y)<\rho} \big(u(x) - u(y)\big)^2\, d\mu(x) d\mu(y). \tag{2.1}$$

Note that the Dirichlet form is non-negative.

When dealing with ρ-Laplacians from Definition 1.1, that is when $\varphi(x) = \rho^2\mu(B_\rho(x))$, we denote the measure $\varphi\mu$ by μ^ρ. We denote the scalar product and norm in $L^2(X, \mu^\rho)$ by $\langle\cdot,\cdot\rangle_{X^\rho}$ and $\|\cdot\|_{X^\rho}$, resp. That is,

$$d\mu^\rho(x)/d\mu(x) = \rho^2\mu(B_\rho(x)), \tag{2.2}$$

$$\langle u, v\rangle_{X^\rho} = \rho^2 \int_X \mu(B_\rho(x)) u(x) v(x)\, d\mu(x), \tag{2.3}$$

$$\|u\|^2_{X^\rho} = \rho^2 \int_X \mu(B_\rho(x)) u(x)^2\, d\mu(x). \tag{2.4}$$

The norm of Δ^ρ_X in $L^2(X, \mu^\rho)$ is bounded by $2\rho^{-2}$. Indeed,

$$\begin{aligned}D^\rho_X(u) &= \frac{1}{2} \iint_{d(x,y)<\rho} (u(x) - u(y))^2\, d\mu(x) d\mu(y)\\ &\le \iint_{d(x,y)<\rho} (u(x)^2 + u(y)^2)\, d\mu(x) d\mu(y)\\ &= 2\int_X \mu(B_\rho(x)) u(x)^2\, d\mu(x) = 2\rho^{-2}\|u\|^2_{X^\rho}.\end{aligned}$$

Thus the spectrum of Δ^ρ is contained in $[0, 2\rho^{-2}]$.

The ρ-Laplacian Δ_X^ρ can be rewritten in the form $\Delta_X^\rho u = \rho^{-2}u - Au$ where

$$Au(x) = \frac{1}{\rho^2\mu(B_\rho(x))}\int_{B_\rho(x)} u(y)\,d\mu(y).$$

Observe that A is an integral operator with a bounded kernel. Hence it is a compact operator on $L^2(X)$. It follows that the essential spectrum of Δ_X^ρ is the same as that of the operator $u \mapsto \rho^{-2}u$. Namely it is empty if $L^2(X)$ is finite-dimensional and the single point $\{\rho^{-2}\}$ otherwise.

A similar argument shows that the essential spectrum of Δ_φ^ρ is located between the infimum and supremum of the function $x \mapsto \mu(B_\rho(x))/\varphi(x)$.

Notation 2.1. Let $\lambda_\infty = \lambda_\infty(X, \rho, \varphi)$ be the infimum of the essential spectrum of Δ_φ^ρ. If there is no essential spectrum (that is, if $L^2(X)$ is finite-dimensional), we set $\lambda_\infty = \infty$. For every $k \in \mathbb{N}$ we define $\lambda_k = \lambda_k(X, \rho, \varphi) \in [0, +\infty]$ as follows. First let $0 = \lambda_1 \le \lambda_2 \le \dots$ be the eigenvalues of Δ_φ^ρ (with multiplicities) which are smaller than λ_∞. If there are only finitely many of such eigenvalues, we set $\lambda_k = \lambda_\infty$ for all larger values of k.

We abuse the language and refer to $\lambda_k(X, \rho, \varphi)$ as the *k-th eigenvalue* of Δ_φ^ρ even though it may be equal to λ_∞.

For the ρ-Laplacian Δ_X^ρ we drop φ from the notation and denote the k-th eigenvalue by $\lambda_k(X, \rho)$.

By the standard Min-Max Theorem, for every $k \in \mathbb{N}$ we have

$$\lambda_k(X, \rho, \varphi) = \inf_{H^k} \sup_{u \in H^k \setminus \{0\}} \left(\frac{D_X^\rho(u)}{\|u\|^2_{L^2(X, \varphi\mu)}} \right) \tag{2.5}$$

and in particular

$$\lambda_k(X, \rho) = \inf_{H^k} \sup_{u \in H^k \setminus \{0\}} \left(\frac{D_X^\rho(u)}{\|u\|^2_{X^\rho}} \right) \tag{2.6}$$

where the infima are taken over all k-dimensional subspaces H^k of $L^2(X)$. This formula is our main tool for eigenvalue estimates. We emphasize that it holds in both cases $\lambda_k < \lambda_\infty$ and $\lambda_k = \lambda_\infty$.

As an immediate application, we observe that the eigenvalues are stable with respect to small relative changes of the normalizing function and measure. If μ_1 and μ_2 are measures on X satisfying $a\mu_1 \le \mu_2 \le b\mu_1$ where a and b are positive constants, then for the corresponding mm-spaces $X_1 = (X, d, \mu_1)$ and $X_2 = (X, d, \mu_2)$ we have

$$\frac{a^2}{b^2} \le \frac{\lambda_k(X_2, \rho)}{\lambda_k(X_1, \rho)} \le \frac{b^2}{a^2} \tag{2.7}$$

for every $k \in \mathbb{N}$. This follows from (2.6) and the inequalities

$$a^2 \le D^\rho_{X_2}(u)/D^\rho_{X_1}(u) \le b^2,$$
$$a^2 \le \|u\|^2_{X_2^\rho}/\|u\|^2_{X_1^\rho} \le b^2,$$

which hold for all $u \in L^2(X)$. Note that multiplying the measure by a constant does not change the ρ-Laplacian.

For any two normalizing functions φ_1 and φ_2 (2.5) implies that

$$\inf_{x\in X} \frac{\varphi_1(x)}{\varphi_2(x)} \le \frac{\lambda_k(X, \rho, \varphi_2)}{\lambda_k(X, \rho, \varphi_1)} \le \sup_{x \in X} \frac{\varphi_1(x)}{\varphi_2(x)}. \tag{2.8}$$

For nice spaces such as Riemannian manifolds, the volume of small ρ-balls is almost constant as a function of the center of the ball. In such cases one can consider a weighted ρ-Laplacian with a constant normalizing function and conclude that its spectrum is close to that of Δ^ρ_X (cf. Section 3.1).

3. Examples

3.1. **Riemannian manifolds.** The paper [2] deals with the case of X being a closed Riemannian n-manifold M or a discrete approximation of M. In the terminology of Section 2, the object studied in [2] is a weighted ρ-Laplacian with constant normalization function $\varphi(x) = \varphi_\rho := \frac{\nu_n \rho^{n+2}}{2n+4}$. Here ν_n is the volume of the unit ball in $\mathbb{R}^n$. As $\rho \to 0$, we have $\mu(B_\rho(x)) \sim \nu_n \rho^n$ uniformly in $x \in M$. Hence $\varphi_\rho/\rho^2\mu(B_\rho(x)) \to \frac{1}{2n+4}$. Thus, by (2.8), the spectrum of Δ^ρ_X is close to that of Δ^ρ_φ multiplied by $\frac{1}{2n+4}$.

The results of [8] imply that the spectrum of Δ^ρ_X, where X is a Riemannian manifold, converges as $\rho \to 0$ to the Beltrami–Laplace spectrum multiplied by $\frac{1}{2n+4}$. In [2] similar convergence is shown for graph Laplacians arising from discrete approximations of a Riemannian manifold. Theorem 1.2 generalizes this result.

Note that the scalar product $\langle \cdot, \cdot \rangle_{X^\rho}$ and Dirichlet form D^ρ_X tend to 0 as $\rho \to 0$. To make them comparable with the Riemannian counterparts one multiplies them by ρ^{-n-2}.

3.2. **Finsler manifolds.** Let X be a closed Finsler manifold M with smooth and quadratically convex Finsler structure. First recall that there are many reasonable notions of volume for Finsler manifolds, see e.g. [18]. Different volume definitions obviously lead to different ρ-Laplacians. Still the issues we study in this paper are not sensitive to the choice of volume.

Consider a tangent space $V = T_xM$ at a point $x \in M$. It is equipped with a norm $\|\cdot\| = \|\cdot\|_x$ which is the restriction of the Finsler structure. Let B be the unit ball of $\|\cdot\|$. There is a unique ellipsoid $E \subset V$ such that for every quadratic form the integrals of it over B and E coincide. Rescaling E by a suitable factor (depending on the chosen Finsler volume definition) and regarding the resulting ellipsoid as the unit ball of a Euclidean metric, one obtains a Euclidean metric $|\cdot|$ on V whose ρ-Laplacian coincides with that of $\|\cdot\|$ on the set of quadratic forms on V.

Applying this construction to every $x \in M$ one obtains a family of quadratic forms on the tangent spaces thus defining a Riemannian metric on M. It is very likely that the spectra of ρ-Laplacians of the Finsler metric converge as $\rho \to 0$ to the Beltrami–Laplace spectrum of this Riemannian metric.

3.3. **Piecewise Riemannian polyhedra.** Let X be a finite simplicial complex whose faces are equipped with Riemannian metrics which agree on the intersections of faces. First assume that X is dimensionally homogeneous of dimension n. In this case one can mostly follow the analysis of the Riemannian case. The difference is that, due to boundary terms, the Riemannian Dirichlet energy $\int_X \|du\|^2$ is not always equal to $\langle \Delta u, u\rangle$ where Δ is the Beltrami–Laplace operator. They are however equal on the subspace of functions satisfying Kirchhoff's condition. This condition says that, at every point in an $(n-1)$-dimensional face, the sum of normal derivatives in the adjacent n-dimensional faces equals 0. For instance, if

X is a manifold with boundary, this boils down to the Neumann boundary condition. It is plausible that the spectra of Δ_X^ρ converge as $\rho \to 0$ to the spectrum of Beltrami–Laplace operator with Kirchhoff's condition.

The problem can also be studied for polyhedral spaces with varying local dimension. For instance, consider a two-dimensional membrane with a one-dimensional string attached. One can equip this space with a measure which is one-dimensional on the string and two-dimensional on the membrane. Unlike the previous examples, we cannot apply our results to this example because it does not satisfy the doubling condition. It is violated near the point where the string is attached to the membrane. It is rather intriguing if Theorem 1.2 still holds in this situation.

3.4. **Disappearing measure support.** The following example shows that one has to be careful with limits of mm-spaces if the limit measure does not have full support.

Let X be a disjoint union of two compact Riemannian manifolds M_1 and M_2. Define a distance d on X as follows: in each component it is the standard Riemannian distance, and the distance between the components is a large constant. For each $t \geq 0$ define a measure μ_t on X by $\mu_t = \mathrm{vol}_{M_1} + t\,\mathrm{vol}_{M_2}$ where vol_{M_i}, $i = 1, 2$, are Riemannian volumes on the components. Then μ_t weakly converges to $\mu_0 = \mathrm{vol}_{M_1}$ as $t \to 0$.

For every $t > 0$, locally constant functions form a two-dimensional subspace in $L^2(X)$. Hence the zero eigenvalue of $\Delta^\rho_{X_t}$ has multiplicity 2. Thus $\lambda_2(X_t, \rho) = 0$ for all $t > 0$. On the other hand, $\lambda_2(X_0, \rho) > 0$ since the ρ-Laplacian of X_0 is the same as that of the component M_1. Thus $\lambda_1(M_0) \neq \lim_{t\to 0} \lambda_1(M_t)$.

A formal reason for the failure of Theorem 1.2 in this example is that the Bishop–Gromov condition (1.2) is not satisfied. Another issue is that d is not a length metric. The latter can be fixed by connecting M_1 and M_2 by a long segment and taking the induced intrinsic metric.

4. Relative Prokhorov and Wasserstein closeness

This section is devoted to the notion of (ε, δ)-closeness that we use in our spectrum stability results. This notion is introduced in Definition 4.2. The main results of this section are Proposition 4.6 and Corollary 4.7 which characterize (ε, δ)-closeness in terms of measure transport.

We use the following notation. For a metric space (X, d) and a set $A \subset X$ and $r \geq 0$, we denote by A^r the closed r-neighborhood of A. That is, $A^r = \{x \in X : d(x, A) \leq r\}$.

Definition 4.1 (relative Prokhorov closeness)**.** Let Z be a metric space, μ_1, μ_2 finite Borel measures on Z, and $\varepsilon, \delta \geq 0$. We say that μ_1 and μ_2 are *relative* (ε, δ)*-close* if for every Borel set $A \subset Z$,

$$e^\delta \mu_1(A^\varepsilon) \geq \mu_2(A) \quad \text{and} \quad e^\delta \mu_2(A^\varepsilon) \geq \mu_1(A).$$

This definition is similar to that of Prokhorov's distance on the space of measures [13]. The crucial difference is that we use multiplicative corrections rather than additive ones.

The topology arising from Definition 4.1 is stronger than the standard weak topology on the space of measures on X. If however we restrict ourselves to the subspace of measures with full support, then the topologies are the same.

We combine Definition 4.1 with the notion of Gromov–Hausdorff (GH) distance analogously to the definition of Gromov–Wasserstein distances as in e.g. [15]. Recall that metric spaces (X, d_X) and (Y, d_Y) are ε-close in the GH distance iff the disjoint union $X \sqcup Y$ can be equipped with a (semi-)metric d extending d_X and d_Y and such that X and Y are contained in the ε-neighborhoods of each other with respect to d. For discussion of GH distance see e.g. [1].

Definition 4.2. Let $\varepsilon, \delta \geq 0$. We say that mm-spaces $X = (X, d_X, \mu_X)$ and $Y = (Y, d_Y, \mu_Y)$ are *mm-relative* (ε, δ)*-close* if there exists a semi-metric d on $X \sqcup Y$ extending d_X and d_Y and such that μ_X and μ_Y are relative (ε, δ)-close in $(X \sqcup Y, d)$ in the sense of Definition 4.1.

In the sequel we abbreviate "mm-relative (ε, δ)-close" to just (ε, δ)-close.

Observe that, if the measures have full support, then (ε, δ)-closeness of mm-spaces (X, d_X, μ_X) and (Y, d_Y, μ_Y) implies that the metric spaces (X, d_X) and (Y, d_Y) are ε-close in the sense of Gromov–Hausdorff distance.

The following example motivated Definition 4.2 as well as a number of other definitions and assumptions in this paper.

Example 4.3 (discretization, cf. [2])**.** Let X be a mm-space and Y a finite ε-net in X. We can associate a small basin in X to every point of Y and move all measure from each basin to its point. More precisely, there is a partition of X into measurable sets V_y, $y \in Y$, such that each V_y is contained in the ball $B_\varepsilon(y)$. We assign the weight equal to $\mu_X(V_y)$ to each y thus defining a measure μ_Y on Y. If we regard μ_Y as a measure on X, then it is relative $(\varepsilon, 0)$-close to μ_X in the sense of Definition 4.1. We can also regard Y equipped with μ_Y as a separate mm-space. Then it is $(\varepsilon, 0)$-close to X in the sense of Definition 4.2.

Now consider a result of some "measurement errors" in Y. Namely, let $Y' = (Y, d'_Y, \mu'_Y)$ be a mm-space with the same point set Y and such that $|d'_Y - d_Y| < \varepsilon$ and $e^{-\delta} \leq \mu'_Y/\mu_Y \leq e^{\delta}$. Then Y' is $(2\varepsilon, \delta)$-close to X.

Now we show that Fukaya convergence (used in Theorem 1.2) implies convergence with respect to (ε, δ)-closeness, provided that the limit measure has full support. Recall that the Fukaya convergence is defined as follows.

Definition 4.4 (cf. [6, (0.2)])**.** A sequence $X_n = (X_n, d_n, \mu_n)$ of mm-spaces converges to a mm-space $X = (X, d, \mu)$ in the sense of Fukaya if the following holds. There exist a sequence $\sigma_n \to 0$ of positive numbers and a sequence $f_n \colon X_n \to X$ of measurable maps such that

(1) $f_n(X_n)$ is an σ_n-net in X;
(2) $|d(f_n(x), f_n(y)) - d_n(x, y)| < \sigma_n$ for all $x, y \in X_n$;
(3) the push-forward measures $(f_n)_*\mu_n$ weakly converge to μ.

Proposition 4.5. *Let X_n converge to X in the sense Fukaya and assume that μ_X has full support. Then there exist sequences $\varepsilon_n, \delta_n \to 0$ such that X_n is $(\varepsilon_n, \delta_n)$-close to X for all n.*

Proof. Let X, X_n, σ_n, f_n be as above. The existence of f_n implies that X_n is $2\sigma_n$-close to X in the GH distance, see e.g. [1, Cor. 7.3.28]. Moreover there is a metric d'_n on the disjoint union $X \sqcup X_n$ such that d'_n extends $d \cup d_n$ and

$$d'_n(x, f_n(x)) \leq \sigma_n \tag{4.1}$$

for all $x \in X_n$.

It suffices to prove that for every $\varepsilon, \delta > 0$ the spaces X_n eventually get (ε, δ)-close to X. Fix ε and δ. Let $\nu = \nu(\varepsilon, \delta) > 0$ be so small that

$$(e^{\delta} - 1)(\mu(B_{\varepsilon/3}(x)) - \nu) \geq \nu$$

for all $x \in X$. Such ν exists since the measures of $(\varepsilon/3)$-balls in X are bounded away from 0. This is where we use the assumption that μ has full support.

Since $(f_n)_*\mu_n$ weakly converges to μ, the Prokhorov distance between $(f_n)_*\mu_n$ and μ tends to 0, see [13]. This implies that for all sufficiently large n we have

$$\mu(A^{\varepsilon/3}) + \nu > \mu_n(f_n^{-1}(A)), \tag{4.2}$$

$$\mu_n(f_n^{-1}(A^{\varepsilon/3})) + \nu > \mu(A) \tag{4.3}$$

for every Borel set $A \subset X$.

Now consider the disjoint union $Z_n = X \sqcup X_n$ equipped with the metric d'_n. The measures μ and μ_n can be regarded as measures on Z_n. If $\sigma_n < \varepsilon/3$ then by (4.1) we have $f_n^{-1}(A^{\varepsilon/3}) \subset A^{\varepsilon}$ for all $A \subset X$ and $(f_n(B))^{\varepsilon/3} \subset B^{\varepsilon}$ for all $B \subset X_n$. Here the neighborhoods are taken in (Z_n, d'_n). These inclusions along with (4.2) and (4.3) imply that

$$\mu(A^{\varepsilon}) + \nu > \mu_n(A), \tag{4.4}$$

$$\mu_n(A^{\varepsilon}) + \nu > \mu(A) \tag{4.5}$$

for every Borel set $A \subset Z_n$ provided that n is large enough.

Let $A \subset Z_n$ be a nonempty set and $\sigma_n < \varepsilon/6$. Then there exists $x \in X$ such that A^{ε} contains the ball $B_{5\varepsilon/6}(x)$. This fact is trivial if $A \cap X \neq \emptyset$, otherwise it follows from (4.1). Let $D = B_{\varepsilon/3}(x) \cap X$. By (4.1) we have

$$f_n^{-1}(D^{\varepsilon/3}) \subset f_n^{-1}(B_{2\varepsilon/3}(x)) \subset B_{5\varepsilon/6}(x) \subset A^{\varepsilon}.$$

Therefore

$$\mu_n(A^{\varepsilon}) \geq \mu_n(f_n^{-1}(D^{\varepsilon/3})) > \mu(D) - \nu \tag{4.6}$$

by (4.3). Since D is an $(\varepsilon/3)$-ball in X, by the definition of ν we have

$$(e^{\delta} - 1)(\mu(D) - \nu) \geq \nu.$$

This and (4.6) imply that $(e^{\delta} - 1)\mu_n(A^{\varepsilon}) \geq \nu$ and therefore

$$e^{\delta}\mu_n(A^{\varepsilon}) \geq \mu_n(A^{\varepsilon}) + \nu > \mu(A) \tag{4.7}$$

by (4.5). Similarly, since $D \subset A^{\varepsilon}$, we have $\mu(A^{\varepsilon}) \geq \mu(D)$. This inequality and (4.4) imply that

$$e^{\delta}\mu(A^{\varepsilon}) \geq \mu_n(A) \tag{4.8}$$

in the same way as (4.6) and (4.5) imply (4.7). Now (4.7) and (4.8) imply that X_n and X are (ε, δ)-close. The proposition follows. □

Now we reformulate (ε, δ)-closeness in terms of measure transport. Recall that a *measure coupling* (or a *measure transportation plan*) between measure spaces (X, μ_X) and (Y, μ_Y) is a measure γ on $X \times Y$ whose marginals on X and Y coincide with μ_X and μ_Y, resp. The *marginals* are push-forwards of γ by the coordinate projections from $X \times Y$ to the factors. Obviously a measure coupling exists if and only if $\mu_X(X) = \mu_Y(Y)$.

In our set-up X and Y are compact subsets of a metric space (Z, d) and all measures are finite Borel. In this case μ_X and μ_Y can be regarded as measures on

Z and, assuming that $\mu_X(X) = \mu_Y(Y)$, one defines the L^∞-Wasserstein distance $W_\infty(\mu_X, \mu_Y)$ as the minimum of all $\varepsilon \geq 0$ such that there exists a coupling γ between μ_X and μ_Y such that $d(x, y) \leq \varepsilon$ for γ-almost all pairs $(x, y) \in X \times Y$. (The minimum exists due to the weak compactness of the space of measures.) For discussion of Wasserstein distances, see e.g. [19].

Proposition 4.6 (approximate coupling)**.** *Let Z be a compact metric space and μ_X, μ_Y finite Borel measures on Z. Then the following two conditions are equivalent:*

(i) *μ_X and μ_Y are relative (ε, δ)-close (see Definition 4.1);*
(ii) *There exist measures $\widetilde{\mu}_X$ and $\widetilde{\mu}_Y$ on Z such that*

$$e^{-\delta}\mu_X \leq \widetilde{\mu}_X \leq \mu_X, \qquad e^{-\delta}\mu_Y \leq \widetilde{\mu}_Y \leq \mu_Y$$

and $W_\infty(\widetilde{\mu}_X, \widetilde{\mu}_Y) \leq \varepsilon$.

In particular, μ_X and μ_Y are relative $(\varepsilon, 0)$-close iff $W_\infty(\mu_X, \mu_Y) \leq \varepsilon$

For comparison of mm-spaces we have the following corollary, which avoids explicit mentioning of metrics on disjoint unions.

Corollary 4.7. *Let X, Y be compact mm-spaces and $\varepsilon, \delta \geq 0$. Then the following two conditions are equivalent:*

(i) *X and Y are mm-relative (ε, δ)-close (see Definition 4.2).*
(ii) *There exist measures $\widetilde{\mu}_X$ on X and $\widetilde{\mu}_Y$ on Y such that*

$$e^{-\delta}\mu_X \leq \widetilde{\mu}_X \leq \mu_X, \qquad e^{-\delta}\mu_Y \leq \widetilde{\mu}_Y \leq \mu_Y \tag{4.9}$$

and a measure coupling γ between $(X, \widetilde{\mu}_X)$ and $(Y, \widetilde{\mu}_Y)$ such that

$$|d_X(x_1, x_2) - d_Y(y_1, y_2)| \leq 2\varepsilon \tag{4.10}$$

for all pairs $(x_1, y_1), (x_2, y_2) \in \operatorname{supp}(\gamma)$.

In particular, $(\varepsilon, 0)$-closeness of mm-spaces is equivalent to ε-closeness with respect to the L_∞ Gromov–Wasserstein distance.

The proof of Proposition 4.6 and Corollary 4.7 occupies the rest of this section. We prove the proposition by means of discrete approximations. We begin with a version of it for bipartite graphs.

Let $G = (V, E)$ be a bipartite graph with partite sets M and W. That is, the set V of vertices is the union of disjoint sets M and W and each edge connects a vertex from M to a vertex from W. (Exercise: guess where the notations M and W came from.) For a set $A \subset V$ we denote by $N_G(A)$ its graph neighborhood, i.e., the set of vertices adjacent to at least one vertex from A. A *matching* in G is a set of pairwise disjoint edges.

The classic Hall's Marriage Theorem [7] states the following. If for every set $A \subset M$ one has $|N_G(A)| \geq |A|$, then there exists a matching that covers M (that is, the set of endpoints of the matching contains M). For discussion of Hall's Theorem and related topics see e.g. [11, Ch. 7]. We need the following generalization of Hall's Theorem.

Lemma 4.8 (Dulmage-Mendelsohn [4])**.** *Let $G = (V, E)$ be a bipartite graph with partite sets M and W. Let $M_0 \subset M$ and $W_0 \subset W$ be sets such that, for every subset A of either M_0 or W_0 one has $|N_G(A)| \geq |A|$. Then G contains a matching that covers $M_0 \cup W_0$.*

This lemma is proven as Theorem 1 in [4]. It can also be seen as a combination of Hall's Theorem and Ore's Mapping Theorem, see [11, Theorem 7.4.1] or [10, Theorem 2.3.1].

The next lemma is a "continuous" generalization of Lemma 4.8 where finite sets M and W are replaced by metric spaces X and Y, and a closed set $E \subset X \times Y$ plays the role of the set of edges of the graph.

Lemma 4.9. *Let X and Y be compact metric spaces. Let μ_X, μ'_X be finite Borel measures on X and μ_Y, μ'_Y finite Borel measures on Y such that $\mu_X \ge \mu'_X$ and $\mu_Y \ge \mu'_Y$.*

Let $E \subset X \times Y$ be a closed set. Suppose that, for any Borel sets $A \subset X$ and $B \subset Y$ one has

$$\mu_Y(A^E) \ge \mu'_X(A), \quad \mu_X(B^E) \ge \mu'_Y(B), \tag{4.11}$$

where

$$A^E = \{y \in Y : \text{there is } x \in A \text{ such that } (x,y) \in E\},$$
$$B^E = \{x \in X : \text{there is } y \in B \text{ such that } (x,y) \in E\}.$$

Then there exist measures $\widetilde{\mu}_X$, $\widetilde{\mu}_Y$ such that

$$\mu'_X \le \widetilde{\mu}_X \le \mu_X, \quad \mu'_Y \le \widetilde{\mu}_Y \le \mu_Y, \tag{4.12}$$

and a measure coupling γ between $\widetilde{\mu}_X$ and $\widetilde{\mu}_Y$ such that $\operatorname{supp}(\gamma) \subset E$.

Proof. First we prove the lemma in the special case when X and Y are finite sets. By means of approximation we may assume that all values of the measures $\mu_X, \mu'_X, \mu_Y, \mu'_Y$ are rational numbers. Multiplying by a common denominator we make them integers. Then we derive the statement from Lemma 4.8 as follows.

Split each point $x \in X$ into $\mu_X(x)$ points of unit weight (do not forget that $\mu_X(x) \in \mathbb{Z}$). Paint $\mu'_X(x)$ of these points in red and the remaining $\mu_X(x) - \mu'_X(x)$ points in green. Similarly, split each point $y \in Y$ into $\mu_Y(y)$ points of which $\mu'_Y(y)$ are red and the rest are green. Let M and W be the sets of points descending from points of X and Y, resp. Let M_0 and W_0 be the sets of red points from M and W, resp.

Now construct a bipartite graph G with partite sets M and W as follows. For $x \in X$ and $y \in Y$ such that $(x,y) \in E$, connect every descendant of x to every descendant of y by an edge in G. If $(x,y) \notin E$ then there are no edges between descendants of x and y.

The relation (4.11) implies that the graph G satisfies the assumptions of Lemma 4.8. Therefore G contains a matching E_0 covering $M_0 \cup W_0$. For each pair $(x,y) \in X \times Y$ define a point measure $\gamma(x,y)$ equal to the number of edges from E_0 connecting descendants of x and y. Then γ is a desired coupling between some measures $\widetilde{\mu}_X$ and $\widetilde{\mu}_Y$ satisfying (4.12). Thus we are done with the discrete case.

Passing to the general case, fix a sequence $\sigma_n \to 0$ of positive numbers. For each n, divide X and Y into a finite number of Borel subsets Ω^i_X, Ω^j_Y with $\operatorname{diam}(\Omega^i_X) < \sigma_n$ and $\operatorname{diam}(\Omega^j_Y) < \sigma_n$. Choose points $x_i \in \Omega^i_X$, $y_j \in \Omega^j_Y$ and associate to them point measures $\mu_{i,n} = \mu_X(\Omega^i_X)$, $\mu'_{i,n} = \mu'_X(\Omega^i_X)$ and $\mu_{j,n} = \mu_Y(\Omega^j_Y)$, $\mu'_{j,n} = \mu'_Y(\Omega^j_Y)$. This defines atomic measures $\mu_{X,n}, \mu'_{X,n}$ on X and $\mu_{Y,n}, \mu'_{Y,n}$ on Y and the relation (4.11) holds for these discrete measures with E_n in place of E, where E_n is the $2\sigma_n$-neighborhood of E with respect to the product distance on $X \times Y$.

By the discrete case proven above, there is a measure γ_n on $X \times Y$ whose marginals $\widetilde{\mu}_{X,n}$ and $\widetilde{\mu}_{Y,n}$ satisfy

$$\mu'_{X,n} \le \widetilde{\mu}_{X,n} \le \mu_{X,n}, \quad \mu'_{Y,n} \le \widetilde{\mu}_{Y,n} \le \mu_{Y,n} \tag{4.13}$$

and such that $\operatorname{supp}(\gamma_n) \subset E_n$. By the weak compactness of the space of measures we may assume that the sequences $\widetilde{\mu}_{X,n}$, $\widetilde{\mu}_{Y,n}$ and γ_n weakly converge to some measures $\widetilde{\mu}_X$, $\widetilde{\mu}_Y$ and γ, resp. Then $\operatorname{supp}(\gamma) \subset E$ and γ is a measure coupling between $\widetilde{\mu}_X$ and $\widetilde{\mu}_Y$. Also observe that the measures $\mu_{X,n}$, $\mu'_{X,n}$, $\mu_{Y,n}$, $\mu'_{Y,n}$ weakly converge to μ_X, μ'_X, μ_Y, μ'_Y, resp. This and (4.13) imply that $\widetilde{\mu}_X$ and $\widetilde{\mu}_Y$ satisfy (4.12). □

Proof of Proposition 4.6. Let $X = \operatorname{supp}(\mu_X)$ and $Y = \operatorname{supp}(\mu_Y)$. The implication (i)⇒(ii) follows from Lemma 4.9 by substituting $\mu'_X = e^{-\delta}\mu_X$, $\mu'_Y = e^{-\delta}\mu_Y$, and

$$E = \{(x,y) \in X \times Y : d(x,y) \le \varepsilon\}.$$

To prove the implication (ii)⇒(i), let $\widetilde{\mu}_X$ and $\widetilde{\mu}_Y$ be as in Proposition 4.6(ii), and let γ be a measure coupling between $\widetilde{\mu}_X$ and $\widetilde{\mu}_Y$ realizing the L^∞-Wasserstein distance. Then $\operatorname{supp}(\gamma) \subset E$. This implies that, for every Borel set $A \subset X$,

$$\widetilde{\mu}_X(A) = \gamma(A \times Y) \le \gamma(X \times (A^\varepsilon \cap Y)) = \widetilde{\mu}_Y(A^\varepsilon),$$

where the inequality follows from the inclusion

$$(A \times Y) \cap E \subset X \times (A^\varepsilon \cap Y).$$

Therefore

$$\mu_X(A) \le e^\delta\, \widetilde{\mu}_X(A) \le e^\delta\, \widetilde{\mu}_Y(A^\varepsilon) \le e^\delta \mu_Y(A^\varepsilon).$$

Similarly $\mu_Y(B) \le e^\delta \mu_X(B^\varepsilon)$ for every Borel set $B \subset Y$. Thus μ_X and μ_Y are relative (ε,δ)-close. □

Proof of Corollary 4.7. (i)⇒(ii): By definition, there exists a semi-metric d on the disjoint union $Z = X \sqcup Y$ such that μ_X and μ_Y, regarded as measures on Z, are relative (ε,δ)-close. Proposition 4.6 implies that there exist measures $\widetilde{\mu}_X$ and $\widetilde{\mu}_Y$ satisfying (4.9) and a measure coupling γ between them such that $d(x,y) \le \varepsilon$ for all $(x,y) \subset \operatorname{supp}\gamma$. This property and the triangle inequality implies (4.10).

(ii)⇒(i): The proof is similar to that of [1, Theorem 7.3.25]. Let γ be a measure coupling between $\widetilde{\mu}_X$ and $\widetilde{\mu}_Y$ such that (4.9) and (4.10) are satisfied. Define a semi-metric d on $X \sqcup Y$ by setting $d|_{X\times X} = d_X$, $d|_{Y\times Y} = d_Y$, and

$$d(x,y) = \inf_{(x',y')\in\operatorname{supp}(\gamma)} \left\{ d_X(x,x') + d_Y(y,y') + \varepsilon \right\}.$$

The triangle inequality for d easily follows from (4.10), thus d is indeed a semi-metric. The definition of d implies that $d(x,y) = \varepsilon$ if $(x,y) \in \operatorname{supp}(\gamma)$. Therefore $W_\infty(\widetilde{\mu}_X, \widetilde{\mu}_Y) \le \varepsilon$ where $\widetilde{\mu}_X$ and $\widetilde{\mu}_Y$ are regarded as measures on Z. By Proposition 4.6 this implies that μ_X and μ_Y are relative (ε,δ)-close and hence the mm-spaces X and Y are (ε,δ)-close. □

5. Stability of eigenvalues

In this section we formulate and prove Theorem 5.4 which is one of the main results of this paper. Informally it says that if two mm-spaces are close then the lower parts of spectra of their ρ-Laplacians are close. First we introduce conditions on mm-spaces needed in the theorem.

Definition 5.1 (SLV condition)**.** Let X be a mm-space and $\Lambda, \rho, \varepsilon > 0$. We say that X satisfies the *spherical layer volume condition* with parameters Λ, ρ, ε, if for every $x \in \operatorname{supp}(\mu_X)$,

$$\frac{\mu(B_{\rho+\varepsilon}(x) \setminus B_\rho(x))}{\mu(B_\rho(x))} \le \Lambda \frac{\varepsilon}{\rho}. \tag{5.1}$$

We abbreviate this condition as $SLV(\Lambda, \rho, \varepsilon)$.

Definition 5.2 (BIV condition)**.** Let X be a mm-space, $0 < \varepsilon \le \rho/2$ and $\Lambda > 0$. We say that X satisfies the *ball intersection volume condition* with parameters Λ, ρ, and ε, if for all $x, y \in \operatorname{supp}(\mu_X)$ such that $d_X(x, y) \le \rho + \varepsilon$,

$$\mu(B_\rho(x) \cap B_\rho(y)) \ge \Lambda^{-1} \mu(B_{\rho+\varepsilon}(x)).$$

We abbreviate this condition as $BIV(\Lambda, \rho, \varepsilon)$.

Note that the Bishop–Gromov inequality (1.2) implies $SLV(\Lambda', \rho, \varepsilon)$ for all $\rho \ge \varepsilon > 0$ with Λ' depending on the parameter Λ of (1.2). For $\varepsilon = \rho$, the SLV condition (5.1) turns into a doubling condition:

$$\mu(B_{2\rho}(x)) \le 2\Lambda\mu(B_\rho(x)).$$

If d is a length metric and this doubling condition holds for all $\rho > 0$, then X satisfies $BIV(\Lambda', \rho, \varepsilon)$ for all $\rho > 0$ and $\varepsilon \le \rho/2$, where Λ' depends only on Λ. This follows from the fact that the intersection $B_\rho(x) \cap B_\rho(y)$ contains a ball of radius $\frac{\rho-\varepsilon}{2}$.

The next lemma shows that the conditions SLV and BIV are in a sense stable with respect to (ε, δ)-closeness introduced in Section 4.

Lemma 5.3. *Let X and Y be (ε, δ)-close mm-spaces (see Definition 4.2) where $0 < \varepsilon \le \rho/12$. Then:*

1. *If X satisfies $SLV(\Lambda, \rho - 2\varepsilon, 5\varepsilon)$, then Y satisfies $SLV(6e^{2\delta}\Lambda, \rho, \varepsilon)$.*
2. *If X satisfies $BIV(\Lambda, \rho - 2\varepsilon, 5\varepsilon)$, then Y satisfies $BIV(e^{2\delta}\Lambda, \rho, \varepsilon)$.*

Proof. We may assume that μ_X and μ_Y have full support. By definition, there is a metric d on the disjoint union $Z = X \sqcup Y$ such that μ_X and μ_Y are relative (ε, δ)-close in (Z, d). Throughout this proof all balls, neighborhoods, etc, are considered in the space (Z, d). Since the measures have full support, the Hausdorff distance between X and Y is no greater than ε. That is, for every $y \in Y$ there exists $x \in X$ such that $d(x, y) \le \varepsilon$, and vice versa.

Let $y \in Y$. Take $x \in X$ such that $d(x, y) \le \varepsilon$. Recall that A^ε denotes the closed ε-neighborhood of a set A. The triangle inequality implies that

$$(B_{\rho-2\varepsilon}(x))^\varepsilon \subset B_\rho(y)$$

and

$$(B_{\rho+\varepsilon}(y) \setminus B_\rho(y))^\varepsilon \subset B_{\rho+3\varepsilon}(x) \setminus B_{\rho-2\varepsilon}(x).$$

These inclusions and the relative (ε, δ)-closeness of μ_X and μ_Y imply that

$$\mu_Y(B_\rho(y)) \ge e^{-\delta}\mu_X(B_{\rho-2\varepsilon}(x))$$

and

$$\mu_Y(B_{\rho+\varepsilon}(y) \setminus B_\rho(y)) \le e^\delta \mu_X(B_{\rho+3\varepsilon}(x) \setminus B_{\rho-2\varepsilon}(x)).$$

Therefore

$$\frac{\mu(B_{\rho+\varepsilon}(y) \setminus B_\rho(y))}{\mu(B_\rho(y))} \le \frac{\mu(B_{\rho+3\varepsilon}(x) \setminus B_{\rho-2\varepsilon}(x))}{\mu(B_{\rho-2\varepsilon}(x))} \le e^{2\delta}\Lambda \frac{5\varepsilon}{\rho - 2\varepsilon} \le 6\Lambda \frac{\varepsilon}{\rho}$$

and the first claim of the proposition follows.

To prove the second claim, consider points $y_1, y_2 \in Y$ such that $d(y_1, y_2) \le \rho + \varepsilon$. We have to prove that

$$Q := \frac{\mu_Y(B_\rho(y_1) \cap B_\rho(y_2))}{\mu_Y(B_{\rho+\varepsilon}(y_1))} \ge (e^{2\delta}\Lambda)^{-1}.$$

Choose $x_1, x_2 \in X$ such that $d(x_1, y_1) \le \varepsilon$ and $d(x_2, y_2) \le \varepsilon$. The triangle inequality implies that $d(x_1, x_2) \le \rho + 3\varepsilon$,

$$(B_{\rho+\varepsilon}(y_1))^\varepsilon \subset B_{\rho+3\varepsilon}(x_1)$$

and

$$(B_{\rho-2\varepsilon}(x_1) \cap B_{\rho-2\varepsilon}(x_2))^\varepsilon \subset B_\rho(y_1) \cap B_\rho(y_2).$$

Therefore, by relative (ε, δ)-closeness of μ_X and μ_Y,

$$\mu_Y(B_{\rho+\varepsilon}(y_1)) \le e^\delta \mu_X(B_{\rho+3\varepsilon}(x_1))$$

and

$$\mu_Y(B_\rho(y_1) \cap B_\rho(y_2)) \ge \varepsilon^{-\delta} \mu_X(B_{\rho-2\varepsilon}(x_1) \cap B_{\rho-2\varepsilon}(x_2)).$$

Hence

$$Q \ge e^{-2\delta} \frac{\mu_X(B_{\rho-2\varepsilon}(x_1) \cap B_{\rho-2\varepsilon}(x_2))}{\mu_X(B_{\rho+3\varepsilon}(x_1))} \ge e^{-2\delta}\Lambda^{-1}$$

where the last inequality follows from the BIV condition for X. This finishes the proof of Lemma 5.3. $\square$

Now we are in a position to state our main theorem.

Theorem 5.4. *For every $\Lambda > 0$ there exists $C = C(\Lambda) > 0$ such that the following holds. If X and Y are mm-spaces which are (ε, δ)-close and satisfy the conditions $SLV(\Lambda, \rho, 2\varepsilon)$ and $BIV(\Lambda, \rho, 2\varepsilon)$, $0 \le \varepsilon \le \rho/4$, $\delta \ge 0$, then*

$$e^{-4\delta}(1 + C\varepsilon/\rho)^{-1} \le \frac{\lambda_k(X, \rho)}{\lambda_k(Y, \rho)} \le e^{4\delta}(1 + C\varepsilon/\rho) \tag{5.2}$$

for all k such that $\lambda_k(X, \rho) < e^{-4\delta}(1 + C\varepsilon/\rho)^{-1}\rho^{-2}$.

The proof of Theorem 5.4 occupies the rest of this section. First we prove the theorem for $\delta = 0$ (see Proposition 5.7). In this case Corollary 4.7 implies that the mm-spaces X and Y in question admit a measure coupling γ satisfying (4.10).

To estimate the difference between eigenvalues of Δ_X^ρ and Δ_Y^ρ, we transform X to Y in three steps. In the case when X and Y are discrete spaces these steps can be described as follows. First, we split each atom of X into several points and distribute the measure between them. The distances between the descendants of each atom is set to be zero, so we obtain a semi-metric-measure space. Second, we "transport" the points to their destinations in Y. The formal meaning of this is that we keep the point set and the measure but change distances between points. Finally, we glue together some points to obtain Y. The last step is inverse to the first one with Y in place X.

After we provided this intuition in the discrete case, let us proceed with a formal construction of "splitting". It is slightly more cumbersome.

Let γ a measure coupling between mm-spaces X and Y. Recall that γ is a measure on $X \times Y$ and for every Borel set $A \subset X$,

$$\mu_X(A) = \gamma(A \times Y). \tag{5.3}$$

Define a semi-metric $d_{X|X\times Y}$ on $X \times Y$ by

$$d_{X|X\times Y}((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2).$$

The desired splitting of X is the mm-space $X_\gamma = (X \times Y, d_{X|X\times Y}, \gamma)$.

We do not use the (non-Hausdorff) topology arising from the semi-metric $d_{X|X\times Y}$. We equip $X \times Y$ with the standard product Borel σ-algebra.

An interested reader may check that the arguments below also apply if one replaces the semi-metric $d_{X|X\times Y}$ by a genuine metric d defined by

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), c\rho^{-2} d_Y(y_1, y_2)\}$$

where c is a sufficiently small constant, $0 < c < 1/\operatorname{diam}(Y)$.

Applying Definition 1.1 to X_γ we define the associated ρ-Laplacian $\Delta^\rho_{X_\gamma}$. Even though X_γ is almost the same space as X, the spectrum of $\Delta^\rho_{X_\gamma}$ may slightly differ from that of Δ^ρ_X. We compare the two spectra in the following lemma:

Lemma 5.5. *Let X and X_γ be as above. Then*

$$\operatorname{spec}(\Delta^\rho_{X_\gamma}) \subset \operatorname{spec}(\Delta^\rho_X) \cup \{\rho^{-2}\}.$$

Furthermore, every eigenvalue smaller than ρ^{-2} has the same multiplicity in the two spectra.

Proof. Consider a subspace $L \subset L^2(X \times Y, \gamma)$ given by $L = \pi_X^*(L^2(X))$ where $\pi_X : X \times Y \to X$ is the coordinate projection. In other words, L consists of functions which are constant on every fiber $\{x\} \times Y$, $x \in X$. Due to (5.3), π_X^* is a Hilbert space isomorphism between $L^2(X)$ and L. We decompose $L^2(X \times Y, \gamma)$ into a direct sum $L \oplus L^\perp$. Loosely speaking, $L^\perp$ consists of functions which are orthogonal to constants in every fiber. More precisely, if $u \in L^\perp$ then

$$\int_{A\times Y} u \, d\gamma = 0 \tag{5.4}$$

for every Borel set $A \subset X$.

The statement of the lemma is a consequence of the following three facts:

(1) L and $L^\perp$ are invariant under $\Delta^\rho_{X_\gamma}$;
(2) π_X^* provides an equivalence between Δ^ρ_X and $\Delta^\rho_{X_\gamma}|_L$;
(3) for every $u \in L^\perp$ we have $\Delta^\rho_{X_\gamma} u = \rho^{-2} u$.

To prove these facts, observe that a ρ-ball $B^{X_\gamma}_\rho(x, y)$ of the semi-metric $d_{X|X\times Y}$ is of the form

$$B^{X_\gamma}_\rho(x, y) = B^X_\rho(x) \times Y.$$

Hence for a function $u = \pi_X^*(v) \in L$, where $v \in L^2(X)$, we have

$$\begin{aligned}
\Delta^\rho_{X_\gamma} u(x, y) &= \frac{\rho^{-2}}{\gamma(B^X_\rho(x) \times Y)} \int_{B^X_\rho(x)\times Y} [u(x, y) - u(x_1, y_1)] \, d\gamma(x_1, y_1) \\
&= \frac{\rho^{-2}}{\mu_X(B^X_\rho(x))} \int_{B^X_\rho(x)} [v(x) - v(x_1)] \, d\mu_X(x_1) = \Delta^\rho_X v(x)
\end{aligned}$$

where the second identity follows from (5.3). Thus $\Delta^\rho_{X_\gamma}(\pi_X^*(v)) = \pi_X^*(\Delta^\rho_X v)$, proving (2) and the first part of (1).

For every $u \in L^\perp$ we have

$$\begin{aligned}\Delta^\rho_{X_\gamma} u(x,y) &= \frac{\rho^{-2}}{\gamma(B^X_\rho(x)\times Y)} \int_{B^X_\rho(x)\times Y} [u(x,y) - u(x_1,y_1)]\, d\gamma(x_1,y_1) \\ &= \rho^{-2}u(x,y) - \frac{\rho^{-2}}{\gamma(B^X_\rho(x)\times Y)} \int_{B^X_\rho(x)\times Y} u(x_1,y_1)\, d\gamma(x_1,y_1) \\ &= \rho^{-2}u(x,y)\end{aligned}$$

where the last identity follows from (5.4). This proves (3) and the second part of (1). ☐

The next lemma serves the step where we handle the difference between distances.

Lemma 5.6. *Let $X_1 = (X, d_1, \mu)$, $X_2 = (X, d_2, \mu)$ be two mm-spaces with the same point set X and measure μ. Let $\Lambda \ge 1$, $0 < \varepsilon \le \rho/2$, and assume that X_1, X_2 satisfy the conditions $SLV(\Lambda, \rho, \varepsilon)$ and $BIV(\Lambda, \rho, \varepsilon)$. Also assume that*

$$|d_1(x,y) - d_2(x,y)| \le \varepsilon \tag{5.5}$$

for all $x, y \in \mathrm{supp}(\mu)$. Then, for every $k \in \mathbb{N}$,

$$(1 + C\varepsilon/\rho)^{-1} \le \frac{\lambda_k(X_2,\rho)}{\lambda_k(X_1,\rho)} \le 1 + C\varepsilon/\rho \tag{5.6}$$

where C is a constant depending only on Λ.

Proof. We may assume that (5.5) holds for all $x, y \in X$, otherwise just replace X by $\mathrm{supp}(\mu)$. We estimate the eigenvalues by means of the min-max formula (2.6). For $i = 1, 2$, let $B^i_\rho(x)$ denote the ρ-ball of d_i centered at $x \in X$, $\|\cdot\|_i = \|\cdot\|_{X^\rho_i}$ (see (2.4)), and $D_i = D^\rho_{X_i}$ (see (2.1)). The only difference as we pass from X_1 to X_2 is that the balls $B^i_\rho(x)$ are different.

The assumption (5.5) implies that $B^1_\rho(x) \subset B^2_{\rho+\varepsilon}(x)$ for every $x \in X$. This and the condition $SLV(\Lambda, \rho, \varepsilon)$ for X_2 imply that

$$\frac{\mu(B^1_\rho(x))}{\mu(B^2_\rho(x))} \le 1 + \frac{\mu(B^2_{\rho+\varepsilon}(x) \setminus B^2_\rho(x))}{\mu(B^2_\rho(x))} \le 1 + \Lambda\varepsilon/\rho.$$

This and (2.4) imply that

$$\|u\|_1^2 \le (1 + \Lambda\varepsilon/\rho)\|u\|_2^2 \tag{5.7}$$

for every $u \in L^2(X)$.

For the Dirichlet forms we have

$$\begin{aligned} D_2(u) &= \iint_{d_2(x,y)<\rho} |u(x) - u(y)|^2\, d\mu(x)d\mu(y) \\ &\le \iint_{d_1(x,y)<\rho+\varepsilon} |u(x) - u(y)|^2\, d\mu(x)d\mu(y) \\ &= D_1(u) + \iint_L |u(x) - u(y)|^2\, d\mu(x)d\mu(y), \end{aligned}$$

where

$$L = \{(x,y) \in X \times X : \rho \le d_1(x,y) < \rho + \varepsilon\}.$$

Hence

$$D_2(u) - D_1(u) \le \iint_L |u(x) - u(y)|^2 \, d\mu(x)d\mu(y). \tag{5.8}$$

Let us estimate the right-hand side of (5.8). For every $(x, y) \in L$ consider the set $U(x, y) = B^1_\rho(x) \cap B^1_\rho(y)$. Recall that

$$\mu(U(x, y)) \ge \Lambda^{-1} \max\{\mu(B^1_\rho(x)), \mu(B^1_\rho(y))\} \tag{5.9}$$

by the condition $BIV(\Lambda, \rho, \varepsilon)$ for X_1. For every $z \in U(x, y)$ we have

$$|u(x) - u(y)|^2 \le 2(|u(x) - u(z)|^2 + |u(z) - u(y)|^2).$$

Integrating this inequality and taking into account (5.9) yields that

$$\begin{aligned} |u(x) - u(y)|^2 &\le \frac{2}{\mu_1(U(x, y))} \int_{U(x,y)} (|u(x) - u(z)|^2 + |u(z) - u(y)|^2) \, d\mu_1(z) \\ &\le 2\Lambda\big(Q(x) + Q(y)\big) \end{aligned}$$

where

$$Q(x) = \frac{1}{\mu(B^1_\rho(x))} \int_{B^1_\rho(x)} |u(x) - u(z)|^2 \, d\mu_1(z).$$

This and (5.8) imply that

$$\begin{aligned} D_2(u) - D_1(u) &\le 2\Lambda \iint_L (Q(x) + Q(y)) \, d\mu(x)d\mu(y) \\ &= 4\Lambda \iint_L Q(x) \, d\mu(x)d\mu(y) \\ &= 4\Lambda \int_X \mu(B^1_{\rho+\varepsilon}(x) \setminus B^1_\rho(x)) \, Q(x) \, d\mu(x) \\ &\le \frac{4\Lambda^2\varepsilon}{\rho} \int_X \mu(B^1_\rho(x)) \, Q(x) \, d\mu(x) \\ &= \frac{4\Lambda^2\varepsilon}{\rho} D_1(u) \end{aligned}$$

where the second inequality follows from the condition $SLV(\Lambda, \rho, \varepsilon)$ for X_1. Thus

$$D_2(u) \le (1 + 4\Lambda^2\varepsilon/\rho) D_1(u). \tag{5.10}$$

This and (5.7) imply that

$$\frac{D_2(u)}{\|u\|_2^2} \le (1 + \Lambda\varepsilon/\rho)(1 + 4\Lambda^2\varepsilon/\rho) \frac{D_1(u)}{\|u\|_1^2}$$

for every $u \in L^2(X) \setminus \{0\}$. By the min-max formula (2.6) this implies the second inequality in (5.6) with $C = \Lambda + 4\Lambda^2 + 4\Lambda^3$. Then the first inequality in (5.6) follows by swapping X_1 and X_2. □

The following proposition deals with the case of $\delta = 0$ of Theorem 5.4.

Proposition 5.7. *For every $\Lambda > 0$ there exists $C = C(\Lambda) > 0$ such that the following holds. Let $0 < \varepsilon \le \rho/4$ and let X, Y be mm-spaces that are $(\varepsilon, 0)$-close and satisfy the conditions $SLV(\Lambda, \rho, 2\varepsilon)$ and $BIV(\Lambda, \rho, 2\varepsilon)$. Then*

$$(1 + C\varepsilon/\rho)^{-1} \le \frac{\lambda_k(X, \rho)}{\lambda_k(Y, \rho)} \le 1 + C\varepsilon/\rho$$

for every $k \in \mathbb{N}$ such that $\lambda_k(X, \rho) < (1 + C\varepsilon/\rho)^{-1}\rho^{-2}$.

Proof. By Corollary 4.7, there exists a measure coupling γ between X and Y satisfying (4.10). With this coupling, we construct mm-spaces

$$X_\gamma = (X \times Y, d_{X|X\times Y}, \gamma) \quad \text{and} \quad Y_\gamma = (X \times Y, d_{Y|X\times Y}, \gamma)$$

as explained in the text before Lemma 5.5. Then Lemma 5.5 implies that $\lambda_k(X_\gamma, \rho) = \lambda_k(X, \rho)$ provided that $\lambda_k(X, \rho) < \rho^{-2}$.

The spaces X_γ and Y_γ inherit the conditions $SLV(\Lambda, \rho, 2\varepsilon)$ and $BIV(\Lambda, \rho, 2\varepsilon)$ from X and Y. Due to (4.10), X_γ and Y_γ satisfy the assumptions of Lemma 5.6 with 2ε in place of ε. Hence

$$(1 + C\varepsilon/\rho)^{-1} \le \frac{\lambda_k(X_\gamma, \rho)}{\lambda_k(Y_\gamma, \rho)} \le 1 + C\varepsilon/\rho$$

where C is a constant depending only on Λ. If $\lambda_k(X_\gamma, \rho) < (1 + C\varepsilon/\rho)^{-1}\rho^{-2}$, this implies that $\lambda_k(Y_\gamma, \rho) < \rho^{-2}$ and therefore $\lambda_k(Y_\gamma, \rho) = \lambda_k(Y, \rho)$ by Lemma 5.5. The proposition follows. □

Proof of Theorem 5.4. Let X, Y be as in Theorem 5.4. By Corollary 4.7, there exist measures $\widetilde{\mu}_X$ and $\widetilde{\mu}_Y$ satisfying (4.9) and such that the mm-spaces $\widetilde{X} = (X, d_X, \widetilde{\mu}_X)$ and $\widetilde{Y} = (Y, d_Y, \widetilde{\mu}_Y)$ are $(\varepsilon, 0)$-close. By (2.7) we have

$$e^{-2\delta} \le \frac{\lambda_k(\widetilde{X}, \rho)}{\lambda_k(X, \rho)} \le e^{2\delta} \tag{5.11}$$

and

$$e^{-2\delta} \le \frac{\lambda_k(\widetilde{Y}, \rho)}{\lambda_k(Y, \rho)} \le e^{2\delta}. \tag{5.12}$$

Now we estimate the ratio $\lambda_k(\widetilde{X}, \rho)/\lambda_k(\widetilde{Y}, \rho)$. Due to (4.9), $\widetilde{X}$ and $\widetilde{Y}$ satisfy the conditions $SLV(\Lambda', \rho, 2\varepsilon)$ and $BIV(\Lambda', \rho, 2\varepsilon)$ with $\Lambda' = e^\delta \Lambda$. By Proposition 5.7 applied to $\widetilde{X}$ and $\widetilde{Y}$ we have

$$(1 + C\varepsilon/\rho)^{-1} \le \frac{\lambda_k(\widetilde{X}, \rho)}{\lambda_k(\widetilde{Y}, \rho)} \le 1 + C\varepsilon/\rho \tag{5.13}$$

provided that $\lambda_k(\widetilde{X}, \rho) < (1 + C\varepsilon/\rho)^{-1}\rho^{-2}$. Here C is a constant depending only on Λ. The desired estimate (5.2) follows from (5.13), (5.11) and (5.12). □

Proof of Theorem 1.2. Let $X = (X, d, \mu)$ and $X_n = (X_n, d_n, \mu_n)$ be as in Theorem 1.2. The Bishop–Gromov condition (1.2) implies that μ has full support. By Proposition 4.5 it follows that X_n is $(\varepsilon_n, \delta_n)$-close to X where $\varepsilon_n, \delta_n \to 0$.

Fix $\rho > 0$ and assume that $\varepsilon_n < \rho/24$. As explained after Definition 5.2, the assumption that d is a length metric and (1.2) imply that X satisfies $SLV(\Lambda', r, \varepsilon)$ and $BIV(\Lambda', r, \varepsilon)$ for all $r > 0$ and $\varepsilon \le r/2$, where Λ' depends only on Λ. By Lemma 5.3 it follows that X_n satisfies $SLV(\Lambda'', \rho, 2\varepsilon_n)$ and $BIV(\Lambda'', \rho, 2\varepsilon_n)$ for some Λ'' depending only on Λ. Now Theorem 5.4 implies that, for some $C = C(\Lambda)$,

$$e^{-4\delta_n}(1 + C\varepsilon_n/\rho)^{-1} \le \frac{\lambda_k(X_n, \rho)}{\lambda_k(X, \rho)} \le e^{4\delta_n}(1 + C\varepsilon_n/\rho)$$

for all n, k such that $\lambda_k(X, \rho) < e^{-4\delta_n}(1 + C\varepsilon_n/\rho)^{-1}\rho^{-2}$. Thus $\lambda_k(X_n, \rho) \to \lambda_k(X, \rho)$ as $n \to \infty$. □

6. Transport of ρ-Laplacians

In this section we further analyze the structures appeared in the proof of Theorem 5.4. Our goal is to construct a map $T_{XY}\colon L^2(X) \to L^2(Y)$ which shows "almost equivalence" of ρ-Laplacians Δ_X^ρ and Δ_Y^ρ. See Proposition 6.1 for a precise formulation.

Let X, Y be as in Theorem 5.4. As in the proof of Theorem 5.4, let γ be a measure coupling provided by Corollary 4.7 and $\widetilde{\mu}_X$, $\widetilde{\mu}_Y$ marginals of γ. The coordinate projection $\pi_X\colon X \times Y \to X$ determines two maps

$$I_X\colon L^2(X, \widetilde{\mu}_X) \to L^2(X \times Y, \gamma)$$

and

$$P_X\colon L^2(X \times Y, \gamma) \to L^2(X, \widetilde{\mu}_X)$$

which are dual to each other. Namely $I_X = \pi_X^*$ is a map given by

$$(I_X u)(x, y) = u(x), \qquad u \in L^2(X),\ x \in X,\ y \in Y.$$

Note that I_X is an isometric embedding of $L^2(X, \widetilde{\mu}_X)$ to $L^2(X \times Y, \gamma)$. Let $L_X \subset L^2(X \times Y, \gamma)$ be the image of I_X. Then P_X is the composition of the orthogonal projection onto L_X and the map $I_X^{-1}\colon L_X \to L^2(X)$.

Loosely speaking, P_X sends each function on $X \times Y$ to the family of its average values over the fibers $\{x\} \times Y$, $x \in X$. More precisely, by disintegration theorem (see [12] or [5, Theorem 452I]), for a.e. $x \in X$, there is a measure ν_x on Y such that

$$\gamma(A \times B) = \int_A \nu_x(B) d\widetilde{\mu}_X, \quad A \subset X, B \subset Y.$$

Then, for a.e. $x \in X$,

$$(P_X \varphi)(x) = \int_Y \varphi(x, y)\, d\nu_x, \qquad x \in X. \tag{6.1}$$

Then, since $\widetilde{\mu}_X$ is a marginal measure of γ, (6.1) implies that $P_X \circ I_X = id_X$.

Similarly one defines maps I_Y, P_Y and a subspace L_Y. We introduce a map $T_{XY}\colon L^2(X) \to L^2(Y)$ by $T_{XY} = P_Y \circ I_X$. By (6.1), for $u \in L^2(X)$,

$$(T_{XY} u)(y) = \int_X u(x)\, d\nu_y,$$

where ν_y is defined similarly to ν_x and the integral in the right-hand side exists for a.e. $y \in Y$. The main result of this section is the following proposition.

Proposition 6.1. *For every $\Lambda > 0$ there exists $C = C(\Lambda) > 0$ such that the following holds. Let X, Y, ρ, ε, Λ be as in Theorem 5.4. Then for every $u \in L^2(X)$ the map T_{XY} defined above satisfies*

$$A^{-1}\|u\|_{X^\rho}^2 - A\rho^2 D_X^\rho(u) \le \|T_{XY} u\|_{Y^\rho}^2 \le A\|u\|_{X^\rho}^2, \tag{6.2}$$

$$D_Y^\rho(T_{XY} u) \le A D_X^\rho(u), \tag{6.3}$$

and

$$\|T_{YX}(T_{XY} u) - u\|_{X^\rho}^2 \le A\rho^2 D_X^\rho(u), \tag{6.4}$$

where

$$A = e^\delta(1 + C\varepsilon/\rho).$$

We are interested in the situation when δ and ε/ρ are small. Then A is close to 1 and the cumbersome formulas (6.2) and (6.4) can be informally interpreted in the following way. At not too high energy levels (that is, if the Dirichlet energy of a unit vector u is substantially smaller than ρ^{-2}), the operator T_{XY} almost preserves the norm and the inner product by (6.2) and $T_{YX} \circ T_{XY}$ is close to identity by (6.4).

Proof of Proposition 6.1. Let γ, $\widetilde{\mu}_X$, $\widetilde{\mu}_Y$ be as above. Consider mm-spaces $\widetilde{X} = (X, d_X, \widetilde{\mu}_X)$ and $\widetilde{Y} = (Y, d_Y, \widetilde{\mu}_Y)$, the corresponding ρ-Laplacians, norms $\|\cdot\|_{\widetilde{X}^\rho}$ and $\|\cdot\|_{\widetilde{Y}^\rho}$, and Dirichlet forms $D^\rho_{\widetilde{X}}$ and $D^\rho_{\widetilde{Y}}$ (see (2.4) and (2.1)).

As in Section 5 we equip $X \times Y$ with two semi-distances $d_{X|X\times Y}$ and $d_{Y|X\times Y}$ and denote by $\widetilde{X}_\gamma$ and $\widetilde{Y}_\gamma$ the corresponding mm-spaces (see the proof of Proposition 5.7). These mm-spaces determine ρ-Laplacians $\Delta^\rho_{\widetilde{X}_\gamma}$ and $\Delta^\rho_{\widetilde{Y}_\gamma}$, scalar products $\langle,\rangle_{\widetilde{X}^\rho_\gamma}$ and $\langle,\rangle_{\widetilde{Y}^\rho_\gamma}$, and Dirichlet forms $D^\rho_{\widetilde{X}_\gamma}$ and $D^\rho_{\widetilde{Y}_\gamma}$ (see (2.3) and (2.1)). The structures introduced above satisfy the following properties (see the proof of Lemma 5.5):

- L_X and $L_X^\perp$ are orthogonal with respect to $\langle\cdot,\cdot\rangle_{\widetilde{X}^\rho_\gamma}$;
- I_X is an isometric embedding with respect to norms $\|\cdot\|_{\widetilde{X}^\rho}$ and $\|\cdot\|_{\widetilde{X}^\rho_\gamma}$;
- I_X preserves the Dirichlet form $D^\rho_{\widetilde{X}}$, that is $D^\rho_{\widetilde{X}_\gamma}(I_X u) = D^\rho_{\widetilde{X}}(u)$ for all $u \in L^2(X)$;
- L_X and $L_X^\perp$ are invariant under $\Delta^\rho_{\widetilde{X}_\gamma}$ and hence they are orthogonal with respect to $D^\rho_{\widetilde{X}_\gamma}$.

Recall that P_X is the composition of the orthogonal projection to L_X and the map I_X^{-1}. Hence P_X does not increase the norms and Dirichlet forms. Similar properties hold for Y in place of X.

As in the proof of Lemma 5.6, for every $v \in L^2(X\times Y, \gamma)$ we have (see (5.7) and (5.10))

$$A_1^{-1} \le \|v\|^2_{\widetilde{X}^\rho} \big/ \|v\|^2_{\widetilde{Y}^\rho} \le A_1 \tag{6.5}$$

and

$$A_2^{-1} \le D^\rho_{\widetilde{X}}(v) \big/ D^\rho_{\widetilde{Y}}(v) \le A_2 \tag{6.6}$$

where $A_1 = 1 + \Lambda\varepsilon/\rho$ and $A_2 = 1 + 4\Lambda^2\varepsilon/\rho$.

Let $u \in L^2(X)$ and $v = I_X(u)$. Then

$$\|T_{XY}u\|^2_{\widetilde{Y}^\rho} = \|P_Y v\|^2_{\widetilde{Y}^\rho} \le \|v\|^2_{\widetilde{Y}^\rho_\gamma} \le A_1 \|v\|^2_{\widetilde{X}^\rho_\gamma} = A_1 \|u\|^2_{\widetilde{X}^\rho}. \tag{6.7}$$

Now we estimate $\|T_{XY}u\|^2_{\widetilde{Y}^\rho}$ from below. Decompose v as $v = v_1 + v_2$ where $v_1 \in L_Y$ and $v_2 \in L_Y^\perp$. As shown in the proof of Lemma 5.5, the ρ-Laplacian $\Delta^\rho_{\widetilde{Y}_\gamma}$ acts on $L_Y^\perp$ by multiplication by ρ^{-2}. Hence

$$D^\rho_{\widetilde{Y}_\gamma}(v) = D^\rho_{\widetilde{Y}_\gamma}(v_1) + D^\rho_{\widetilde{Y}_\gamma}(v_2) \ge D^\rho_{\widetilde{Y}_\gamma}(v_2) = \rho^{-2}\|v_2\|^2_{\widetilde{Y}^\rho_\gamma} \tag{6.8}$$

and therefore

$$\|v_1\|^2_{\widetilde{Y}^\rho_\gamma} = \|v\|^2_{\widetilde{Y}^\rho_\gamma} - \|v_2\|^2_{\widetilde{Y}^\rho_\gamma} \ge \|v\|^2_{\widetilde{Y}^\rho_\gamma} - \rho^2 D^\rho_{\widetilde{Y}_\gamma}(v).$$

Thus

$$\|T_{XY}u\|^2_{\widetilde{Y}^\rho} = \|v_1\|^2_{\widetilde{Y}^\rho_\gamma} \geq \|v\|^2_{\widetilde{Y}^\rho_\gamma} - \rho^2 D^\rho_{\widetilde{Y}_\gamma}(v) \geq A_1^{-1}\|v\|^2_{\widetilde{X}^\rho_\gamma} - A_2\rho^2 D^\rho_{\widetilde{X}_\gamma}(v) = A_1^{-1}\|u\|^2_{\widetilde{X}^\rho} - A_2\rho^2 D^\rho_{\widetilde{X}}(u) \tag{6.9}$$

by (6.5) and (6.6). Now (6.2) follows from (6.7), (6.9) and the bounds (4.9) for $\widetilde{\mu}_X$ and $\widetilde{\mu}_Y$.

To estimate the Dirichlet form of $T_{XY}u$, observe that

$$D^\rho_{\widetilde{Y}}(T_{XY}u) = D^\rho_{\widetilde{Y}_\gamma}(v_1) \leq D^\rho_{\widetilde{Y}_\gamma}(v)$$

and

$$D^\rho_{\widetilde{Y}_\gamma}(v) \leq A_2 D^\rho_{\widetilde{X}_\gamma}(v) = A_2 D^\rho_{\widetilde{X}}(u) \tag{6.10}$$

by (6.6). These estimates and (4.9) imply (6.3).

To prove (6.4), observe that $I_Y(T_{XY}u) = v_1$ and therefore

$$T_{YX}(T_{XY}u) - u = P_X(v_1) - u = P_X(v_1 - v) = P_X(v_2).$$

Further,

$$\|P_X(v_2)\|^2_{\widetilde{X}^\rho} \leq \|v_2\|^2_{\widetilde{X}^\rho_\gamma} \leq A_1\|v_2\|^2_{\widetilde{Y}^\rho_\gamma} \leq A_1\rho^2 D^\rho_{\widetilde{Y}_\gamma}(v) \leq A_1A_2\rho^2 D^\rho_{\widetilde{X}}(u)$$

by (6.5), (6.8), and (6.10). Thus

$$\|T_{YX}(T_{XY}u) - u\|^2_{\widetilde{X}^\rho} \leq A_1A_2\rho^2 D^\rho_{\widetilde{X}}(u).$$

This and (4.9) imply (6.4). □

After we obtained estimates on closeness of eigenvalues in Theorem 5.4 we certainly would like to show that corresponding eigenspaces are also close.

The most naive formulation definitely fails. If we have an eigenvalue of multiplicity 2 then there is a two-dimensional eigenspace. Then a small perturbation would generically result in splitting the eigenspace into two orthogonal one-dimensional eigenspaces. An original eigenvector may fail to be close to either of the new eigenspaces. It is still close to a linear combination of new eigenvectors.

In our case we have a similar situation. Let u be an eigenvector of Δ^ρ_X with eigenvalue λ which is substantially smaller that ρ^{-2}. Then $T_{XY}(u)$ is close to a linear combination of eigenvectors of Δ^ρ_Y with eigenvalues close to λ. We don't give a precise formulation of the statement. It is a direct reformulation of Theorem 3 in [2]. The proof is an application of Proposition 6.1 and straightforward linear algebra.

7. Weyl-type estimates

In this section we prove Theorems 7.1 and 7.2. Theorem 7.1 gives us a Weyl-type upper bound on the number of eigenvalues in a lower part of the spectrum of Δ^ρ_X. Theorem 7.2 provides a similar lower bound.

To formulate the theorems we need notation for packing numbers. For a compact metric space X and $r > 0$ we denote by $N_X(r)$ the maximum number of points in an r-separated set in X. Recall that a set $Y \subset X$ is *r-separated* if $d_X(y_1, y_2) \geq r$ for all $y_1, y_2 \in Y$.

For $R > 0$, we denote by $\#^\rho_X(R)$ the number of eigenvalues of Δ^ρ_X in the interval $[0, R]$, counted with multiplicities. Equivalently,

$$\#^\rho_X(R) = \sup\{k \in \mathbb{N} : \lambda_k(X, \rho) \le R\}.$$

Note that $\#^\rho_X(R) = \infty$ if $R \ge \lambda_\infty(X, \rho)$.

Theorem 7.1. *For every $\Lambda \ge 1$ there exists $c = c(\Lambda) > 0$ such that the following holds. Let X be a mm-space satisfying the condition $BIV(\Lambda, \frac{5}{6}\rho, \frac{5}{12}\rho)$ and the following restricted doubling condition:*

$$\mu_X(B_{5\rho/3}) \le \Lambda \mu_X(B_{5\rho/6})$$

for all $x \in \operatorname{supp}(\mu_X)$. Then

$$\#^\rho_X(c\rho^{-2}) \le N_X(\rho/24).$$

If X is a Riemannian manifold then $N_X(r) \sim C_n \mu(X) r^{-n}$ as $r \to 0$, where n is the dimension of X. In this case the conclusion of Theorem 7.1 can be restated as follows: for $R = c\rho^{-2}$, we have

$$\#^\rho_X(R) \le C(n, \Lambda)\mu(X)R^{n/2}.$$

The reader is invited to compare the right-hand side of this formula with the classic Weyl's asymptotics for the Beltrami–Laplace spectrum.

Proof of Theorem 7.1. Let X be a mm-space satisfying the assumptions of the theorem. Fix $\varepsilon = \rho/24$. Let Y be a maximal ε-separated set in X and $N = N_X(\varepsilon)$ the cardinality of Y. Then Y is an ε-net in X. Equip Y with a measure μ_Y as in Example 4.3 so that the resulting mm-space is $(\varepsilon, 0)$-close to X.

By the assumptions of the theorem, X satisfies the conditions $SLV(\Lambda, \rho-4\varepsilon, 10\varepsilon)$ and $BIV(\Lambda, \rho - 4\varepsilon, 10\varepsilon)$. By Lemma 5.3 it follows that Y satisfies $SLV(6\Lambda, \rho, 2\varepsilon)$ and $BIV(\Lambda, \rho, 2\varepsilon)$. Therefore Theorem 5.4 applies to X and Y with 6Λ in place of Λ. By Theorem 5.4, for every $k \ge 1$ at least one of the following holds: either

$$\lambda_k(X, \rho) > C^{-1}\rho^{-2}$$

or

$$C^{-1} \le \frac{\lambda_k(X, \rho)}{\lambda_k(Y, \rho)} \le C$$

where C is a constant depending only on Λ. Since $\dim L^2(Y) = N$, we have $\lambda_k(Y, \rho) = \infty$ for all $k > N$. Hence for $k = N + 1$ the second alternative above cannot occur unless $\lambda_k(X, \rho) = \infty$. We conclude that $\lambda_{N+1}(X, \rho) > C^{-1}\rho^{-2}$. Therefore for $c = C^{-1}$ we have $\#^\rho_X(c\rho^{-2}) \le N$. □

Theorem 7.2. *Let $X = (X, d, \mu)$ be a mm-space whose measure has full support. Let μ^ρ be the measure defined by (2.2). Let $r \ge \rho$ and $N = N_X(3r)$. Then*

$$\lambda_N(x, \rho) \le 4Q(r)r^{-2}$$

where

$$Q(r) = \sup_{x \in X} \frac{\mu^\rho(B_{2r}(x))}{\mu^\rho(B_{r/2}(x))}.$$

For spaces satisfying reasonable assumptions, Theorem 7.2 complements Theorem 7.1 by giving a lower bound on $\#^\rho_X(c\rho^{-2})$ of the same order of magnitude as in Theorem 7.1. Indeed, let c be the constant from Theorem 7.1 and assume that $Q(r) \le Q_{\max}$ for all $r \ge \rho$. Then, applying Theorem 7.2 to $r = 2\sqrt{c^{-1}Q_{\max}}\,\rho$ we get

$$\#^\rho_X(c\rho^{-2}) = \#^\rho_X(4Q_{\max}r^{-2}) \ge N_X(3r) = N_X(C_1\rho)$$

where $C_1 = 6\sqrt{c^{-1}Q_{\max}}$.

Proof of Theorem 7.2. By the min-max formula (2.6), it suffices to construct a linear subspace $H \subset L^2(X)$ such that $\dim H = N$ and

$$D^\rho_X(u) \le 4Q(r)r^{-2}\|u\|^2_{L^2(X,\mu^\rho)} \tag{7.1}$$

for every $u \in H$. Here D^ρ_X is the Dirichlet form given by (2.1).

Let $\{x_1, \dots, x_N\}$ be a $3r$-separated set in X. For each i, define a function $u_i\colon X \to \mathbb{R}$ by

$$u_i(x) = \max\left\{1 - \frac{d(x,x_i)}{r}, 0\right\}.$$

Let H be the linear span of $u_1, \dots, u_N$. We are going to show that (7.1) is satisfied for all $u \in H$. The supports of u_i's are separated by distance at least ρ. Hence $u_i \perp u_j$ and $\Delta^\rho_X(u_i) \perp u_j$ in $L^2(X,\mu^\rho)$ for all $i \ne j$. Therefore it suffices to verify (7.1) for $u = u_i$ only.

Since $u_i(x) \ge \frac{1}{2}$ for all $x \in B_{r/2}(x_i)$, we have

$$\begin{aligned}\|u_i\|^2_{L^2(X,\mu^\rho)} &= \rho^2 \int_X \mu(B_\rho(x))u_i^2(x)d\mu(x) \\ &\ge \frac{\rho^2}{4}\int_{B_{r/2}(x_i)} \mu(B_\rho(x))d\mu(x) = \frac{\rho^2}{4}\mu^\rho(B_{r/2}(x)).\end{aligned}$$

Since $u_i(x) = 0$ if $x \notin B_r(x_i)$ and u_i is $(1/r)$-Lipschitz, we have

$$\begin{aligned}D^\rho_X(u_i) &= \int_{B_{r+\rho}(x_i)}\int_{B_\rho(x)} |u_i(x) - u_i(y)|^2\, d\mu(y)d\mu(x) \\ &\le \frac{\rho^2}{r^2}\int_{B_{r+\rho}(x_i)} \mu(B_\rho(x))\, d\mu(x) = \frac{\rho^2}{r^2}\mu^\rho(B_{r+\rho}(x_i)).\end{aligned}$$

Thus

$$\frac{D^\rho_X(u_i)}{\|u_i\|^2_{L^2(X,\mu^\rho)}} \le \frac{4}{r^2}\frac{\mu^\rho(B_{r+\rho}(x_i))}{\mu^\rho(B_{r/2}(x))} \le 4r^{-2}Q(r).$$

The theorem follows. ☐

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