Non-Reflecting Boundary Conditions for Traffic Flow Schrödinger Equation

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Abstract

Urban traffic flow can be roughly approximated by a Schrödinger equation. For a simulation of the Schrödinger equation as well as for analytical computations it is useful that waves of traffic which travel along a road are not reflected at the boundaries of the simulated region. Here, we present the non-reflecting boundary condition for the Schrödinger equation and prove it via numerical computations.

1 Theory

One way to describe urban traffic is to use the following macroscopic equations:

$$\dot{\rho} + \nabla q = 0 \ , \qquad (1)$$

$$\dot{v} = -v\nabla v + \mu\nabla^2 v + \sum_j F_j(t) \ . \qquad (2)$$

The first equation is the equation of continuity, the second one includes external forces $\sum_j F_j(t)$ representing the effects of traffic light signals at intersections. A second order viscosity like term like $\mu\nabla^2 v$ often is introduced in order to smooth shock waves [1,2]. A relaxation term for a velocity adaptation as well as an anticipation term are neglected which is possible as a rough approximation for urban traffic in contrast to, e.g., highway traffic.

The equations can be written in the following way

$$i\eta\ \dot{\Psi} = \left(-\frac{\eta^2}{2}\nabla^2 + U\right)\Psi \qquad (3)$$

with $\nabla U = -\sum_j F_j(t)$, $\Psi = \sqrt{\rho} \cdot e^{\frac{i}{\eta}\Phi}$, and $\nabla\Phi = v$ when generalizing the viscosity term as

$$\frac{\eta^2}{2}\nabla\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} \ .$$

The interpretation of a fluid dynamical equation as a Schrödinger equation first was pointed out by Madelung [3].

To derive the non-reflecting boundary conditions consider a road for x from $-\infty$ to $+\infty$. A right boundary of the system shall be established at the point $x = 0$. For $x < -b$, $b > 0$, there are assumed to be one or some positive potential barriers, and zero potential for $x \geq -b$. There exist eigenfunctions with positive energy eigenvalues E_m. Waves travelling from inside the system from $x < 0$ to the boundary $x = 0$ are assumed, and no waves in the other direction. No reflections of waves means for values x between $-b$ and 0 for zero mean waves:

$$\Psi(x,t) = \sum_{k_m>0} a_m e^{ik_m x - i\frac{E_m}{\eta}t} \ . \qquad (4)$$

The positive values k_m are given by

$$E_m = \eta^2 k_m^2/2 \ . \qquad (5)$$

In order to simplify computations we restrict m to fulfil $m \in \{1,2,3,...\}$ and we define $k_m < k_{m+1}$ for all m. In order to obtain the corresponding boundary conditions, the function

$$J(t) = \int_{-\infty}^{t} \Psi(0,t') \cdot (t-t')^{-1/2} dt' \qquad (6)$$

is considered. Replacing $t - t'$ by T , and replacing the integral over T from 0 to ∞ by a sum of three integrals in the complex plane $T = T' + iT''$ starting from close to zero with $T'' = 0$ along a circle to $T' = 0$ into the part of the plane with positive T'', then from this point to close to $T = +i\infty$, and from there along a circle back to $T'' = 0$ the integral finally arrives close to $T = T' = \infty$, as desired. Both integrals along the parts of the circles converge to zero when taking the limit of the radius of the former circle to zero, and that of the latter to infinity. Within the relevant region, i.e. for positive T' and positive T'' , there are no singularities. Therefore, using the Gamma function for the remaining integral, $J(t)$ can be computed directly which yields

$$J(t) = e^{i\pi/4}\sqrt{\pi} \cdot \sum_{k_m>0} \frac{a_m}{\sqrt{E_m/\eta}} \cdot e^{-i\frac{E_m}{\eta}t} \ . \qquad (7)$$

From here it follows that the function $\frac{d}{dt}J(t)$ is proportional to $\nabla_x \Psi(x,t) \mid_{x=0}$, i.e.

$$\frac{d}{dt}\int_{-\infty}^{t} \Psi(0,t') \cdot (t-t')^{-1/2} dt' = -\sqrt{\eta\pi/2} \cdot e^{i\pi/4} \cdot \nabla_x \Psi(x,t) \mid_{x=0} \qquad (8)$$

which are the desired non-reflecting boundary conditions.

2 Simulation results

In order to check the validity of the above boundary conditions we compared the temporal evolution of a system with and without non-reflecting boundaries. The value $\Psi(0,t)$ influences $\nabla\Psi(x,t) \mid_{x=0}$ and, therefore, also $\dot{\Psi}(x,t)$ (Eq. 3). So, the boundary conditions can be proven in simulations by comparing the temporal evolution of $\Psi(x<0,t)$ with and without boundary.

Eq. (8) gives a relation between $dJ(t)/dt$ and $d\Psi(x,t)/dx \mid_{x=0}$. From this, it is possible to calculate the wave function values $\Psi(0,t)$ at the boundary from the previous values of Ψ at the boundary and the values of the wave function $\Psi(x<0,t)$ which are defined by Eq. 3.

Numerical problems might result from that $\Psi(0,t)$ and, therefore, $J(t)$ have to be calculated from previous values of $\Psi(0,t^*)$ at $t^* < t$ which themselves are obtained from the boundary condition equation. Therefore, we compute $J(t)$ not from the values of $\Psi(t)$ at the boundary $x=0$, but as close as possible before the boundary at $x = 0 - dx$.

Fig. 1 shows the simulation results. As example, we define an initial wave function in the interval $[-5.2, -2.8]$ (blue line in Fig. 1) and set all values outside this interval to zero initially. Within the interval the function is approximately $e^{-ik_m x}$ with $k_m = 4\pi$ smoothed around -5.2 and -2.8. The parameter η is set to 0.1. The discretisation steps have been chosen as follows: $dx = 0.1$, $dt = 0.001$. The temporal evolution of this function has been studied with and without the non-reflecting boundary described above. The result in Fig. 1 demonstrates a nice correspondence between the non-reflecting boundary conditions at $x = 0$ (green line) and the situation without boundary (red line).

3 Conclusions

Generally, the solution of the Schrödinger equation (Eq. 3) leads for initially localized functions to a wave function that is extended over any given interval

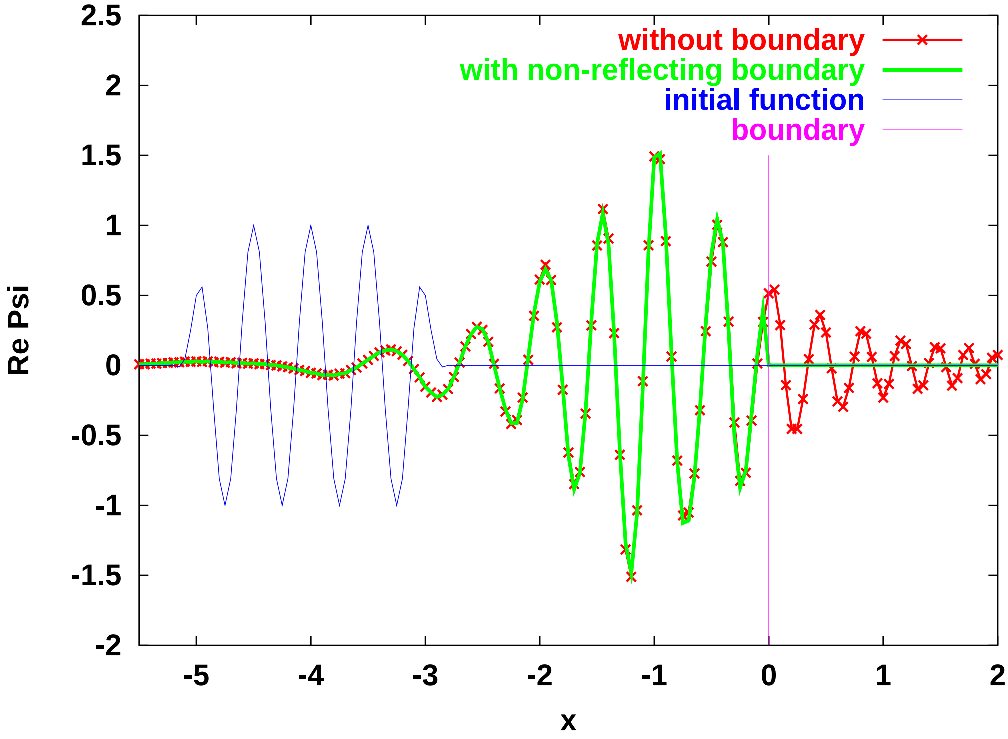

Figure 1: Real part of $\Psi(x,t)$ for $t = 0$ (blue), $t = 5$ without boundary (red), t=5 with non-reflecting boundary (green) at $x = 0$. The parameters for the Schrödinger equation have been chosen as $k_m = 4\pi$, $\eta = 0.1$.

after a certain time. In order to allow long–time simulations, it is necessary to limit the definition area. The problem, however, is that the introduction of a boundary, especially a reflecting one, falsifies the shape of the function. We introduced a non-reflecting boundary condition where the characteristics of the function are maintained. This has been proven in simulations.

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