

# COMPRESSIVE IMAGING WITH COMPLEX WAVELET TRANSFORM AND TURBO APPROXIMATE MESSAGE PASSING RECONSTRUCTION

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## ABSTRACT

We extend the “turbo” belief propagation framework for compressive imaging to the dual-tree complex wavelet transform (DT-CWT) to exploit both sparsity and dependency across scales. Due to the near shift-invariance property and the improved angular resolution of DT-CWT, better reconstruction can be expected when incorporating with the compressed sensing (CS) algorithms. Two types priors to form the hidden Markov tree structure for the DT-CWT coefficients are considered. One models the real and imaginary components of DT-CWT separately while the other assumes the shared hidden states between the two. Simulation with natural images confirm an improved performance when iterating between the CS reconstruction and the DT-CWT HMT.

**Index Terms**— compressed sensing, dual-tree complex wavelet transform, HMT, turbo decoding, approximate message passing

## 1. INTRODUCTION

The typical compressive imaging problem is to estimate a image  $\mathbf{x} \in \mathbb{R}^n$  or its representation in the transformed domain  $\boldsymbol{\theta} \in \mathbb{R}^n$  by solving an under-determined system

$$\mathbf{y} = \Phi \mathbf{x} = \Phi \Psi \boldsymbol{\theta} \quad (1)$$

where  $\mathbf{y} \in \mathbb{R}^m$  is the compressed sensing (CS) observation,  $\Phi \in \mathbb{R}^{m \times n}$ ,  $m < n$  is known as the measurement matrix and  $\Psi$  is some orthonormal basis (i.e. Fourier, wavelet). The ill-posed problem can be solved by exploiting the sparse property of  $\mathbf{x}$  or  $\boldsymbol{\theta}$  via  $\ell_1$ -minimization [1], greedy methods [2] or approximate message passing (AMP) [3]. Nature images, when expressed in the wavelet domain, have an additional quad-tree structure and some statistical dependency along the branches of the wavelet trees [4]. To better aid the CS reconstruction, several authors have looked beyond the signal sparsity and incorporated wavelet dependencies [5–9], among which [9] has demonstrated the state-of-the-art performance.

The authors in [9] deployed the hidden Markov tree (HMT) [4] to model the interscale dependencies of the discrete wavelet transform (DWT) coefficients. The CS observation system in (1) and the HMT structure can be graphically

represented by the corresponding factor graph [10]. The reconstruction is then based on loopy belief propagation [11] to propagate beliefs for the DWT coefficients on the factor graph. The turbo message passing schedule which alternates between the exploitation of CS observation and the HMT structure is proposed. The introduction of wavelet interscale dependencies has largely reinforced the local beliefs on DWT coefficients thus significantly benefited the reconstruction.

The novelty of [9] is the introduction of the turbo scheme to incorporate additional signal properties into the standard CS reconstruction. Inspired by its success it is natural to consider the utility of richer, overcomplete wavelet representations in this framework. In this paper, we adopt and extend the turbo reconstruction approach to the dual-tree complex wavelet transform (DT-CWT) [12, 13]. The advantages of DT-CWT over the standard DWT have been shown in many image processing applications, such as denoising, detection, segmentation and classification [14, 15]. Compared to the DWT, the DT-CWT is near shift-invariant and more directional sensitive, making it better at characterizing geometric images features like edges. When imposed with the HMT structure, large and small DT-CWT coefficients cascade more consistently along branches of the wavelet trees, which leads to less edge blurring and artifacts.

Two statistical priors to form the HMT have been proposed for the DT-CWT coefficients [14]. The first one simply models the real and imaginary components of the complex coefficients as independent and separate two-state 1-D HMTs. The other is more realistic, which assigns a shared hidden state to the real and imaginary part to form the tree. In this paper, both HMT structures are exploited for the CS imaging. For the former, we directly deploy the factor graph and turbo approach in [9] for reconstruction. For the latter, a new factor graph and turbo scheme are proposed to incorporate the shared states. Numerical simulations demonstrate a significant improvement for the use of DT-CWT, both visually and quantitatively.

We begin with the introduction of the DT-CWT and two HMT models in Section 2. The corresponding factor graphs and turbo reconstruction are explained in Section 3. Image reconstruction examples are shown in Section 4. We finish with conclusion in Section 5.

## 2. COMPLEX WAVELET TRANSFORM AND STATISTICAL HIDDEN MARKOV TREE MODEL

### 2.1. Dual-Tree Complex Wavelet Transform

The DT-CWT is inspired by the shift invariant property of the Fourier transform, namely that the energy in each frequency bin is invariant to any shifts in time or space. As with the Fourier transform, the DT-CWT encodes the signal information in the magnitude and phase of the complex coefficients. For example, a large magnitude indicates the presence of singularity while the phase encodes its position. The DT-CWT decomposition of a 1-D signal is implemented via two parallel DWTs. Each of the DWTs produces a set of real value coefficients  $\{\mu_i\}$  and  $\{v_i\}$ . Together they form the complex wavelet coefficients  $c_i = \mu_i + jv_i, j = \sqrt{-1}$ . With the two DWTs jointly designed, the real and imaginary components are statistically uncorrelated and the DT-CWT is insensitive to small shifts [16].

The extension of DT-CWT to 2D signals is achieved by separately filtering along rows then columns by two parallel DWTs. The resulting coefficients are thus four times redundant. As a bonus, the additional information provides us good directional selectivity with six subbands at each scale, which capture image features strongly oriented at angles of  $\{\pm 15^\circ, \pm 30^\circ, \pm 45^\circ, \pm 75^\circ\}$  [15]. With the more explicit orientation information of singularities, we are able to distinguish signals in a more subtle way.

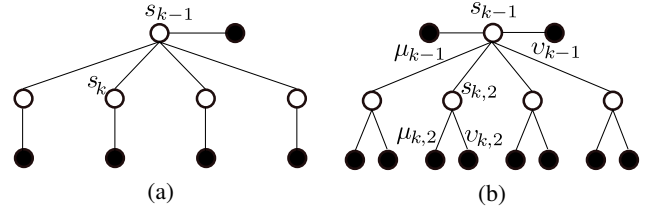
Like the DWT for 2D signals, DT-CWT coefficients for images across scales form a quad-tree structure and exhibit the persistence across scale property [14]. To be specific, if the parent wavelet coefficient is large then the children wavelet coefficients are very likely to be large. Similarly, if the parent coefficient is small, its children are likely to be small. The HMT structure to model the persistence property has also been extended to the DT-CWT. Due to its near shift-invariant, the progressions of the magnitude are better preserved in DT-CWT coefficients.

### 2.2. Prior for independent HMT

The simplest prior is to model the real and imaginary part at each wavelet scale separately with the independent and identically distributed (i.i.d) two-state Gaussian mixture distribution.

$$\begin{aligned} p(\mu_{k,i}) &= p(\mu_{k,i} | s_{\mu,k,i} = 1)p(s_{\mu,k,i} = 1) \\ &\quad + p(\mu_{k,i} | s_{\mu,k,i} = 0)p(s_{\mu,k,i} = 0) \\ &= \lambda_{\mu,k} \mathcal{N}(\mu_{k,i}; 0, \sigma_{k,L}^2) + (1 - \lambda_{\mu,k}) \mathcal{N}(\mu_{k,i}; 0, \sigma_{k,S}^2) \end{aligned} \quad (2)$$

where  $k$  is the wavelet scale index and  $i$  is the coefficient index. In the sequel, we sometimes drop the scale index to imply the generality for all wavelet scales. Each  $\mu_{k,i}$  can be seen as generated from either the large variance Gaussian distribution with variance  $\sigma_{k,L}^2$  or from small variance Gaussian



**Fig. 1.** HMT models for dual-tree complex wavelet coefficients: (a) standard HMT structure for real or imaginary part; (b) assign a shared hidden state for the associated real and imaginary components.

distribution with variance  $\sigma_{k,S}^2$ , depending on the associated hidden state  $s_{\mu,k,i}$  being 1 or 0.

Similarly, the pdf of the imaginary part at scale  $k$  is

$$\begin{aligned} p(v_{k,i}) &= p(v_{k,i} | s_{v,k,i} = 1)p(s_{v,k,i} = 1) \\ &\quad + p(v_{k,i} | s_{v,k,i} = 0)p(s_{v,k,i} = 0) \\ &= \lambda_{v,k} \mathcal{N}(v_{k,i}; 0, \sigma_{k,L}^2) + (1 - \lambda_{v,k}) \mathcal{N}(v_{k,i}; 0, \sigma_{k,S}^2) \end{aligned} \quad (3)$$

Connecting the hidden states across scales for  $\mu$  and  $v$  results in a HMT structure that is the same as the one for the standard DWT, which is illustrated in Fig. 1(a).

### 2.3. Prior for HMT with shared hidden states

A more reasonable model to capture the magnitude persistence would be assigning a two-state hidden variable  $s_{c,i}$  for the complex wavelet coefficient  $c_i$ , taking value 1 when  $|c_i|$  being large and 0 when  $|c_i|$  being small. One statistical model is to consider  $c_i$  as a 2-d random vector  $(\mu_i, v_i)$  and approximate  $p(c_i)$  as a two-state 2-d Gaussian mixture, as proposed in [14].

$$\begin{aligned} p(c_{k,i}) &= p(c_{k,i} | s_{c,k,i} = 1)p(s_{c,k,i} = 1) \\ &\quad + p(c_{k,i} | s_{c,k,i} = 0)p(s_{c,k,i} = 0) \\ &= \frac{\lambda_{c,k}}{\sqrt{2\pi}\sigma_{k,L}} e^{-\frac{\mu_{k,i}^2 + v_{k,i}^2}{2\sigma_{k,L}^2}} + \frac{1 - \lambda_{c,k}}{\sqrt{2\pi}\sigma_{k,S}} e^{-\frac{\mu_{k,i}^2 + v_{k,i}^2}{2\sigma_{k,S}^2}} \end{aligned} \quad (4)$$

Since  $\mu_i$  and  $v_i$  are uncorrelated, this is equivalent to model the real and imaginary parts as the two-state Gaussian mixture with a shared hidden state variable  $s_{c,i}$  [14]. The quad-tree structure with shared hidden states is depicted in Fig. 1(b).

Compared to the independent HMT structure, this model is more realistic: small  $|c_i|$  can only appear when  $|\mu_i|$  and  $|v_i|$  are simultaneously small, which is characterized by  $s_{c,i} = 0$ . While for the independent HMT model, small coefficients generated from a Gaussian mixture with the hidden variable being 1 is also possible.

### 2.4. Complete statistical model for HMT

So far we have presented two priors for DT-CWT coefficients. To complete the statistical model for the HMT structure, we

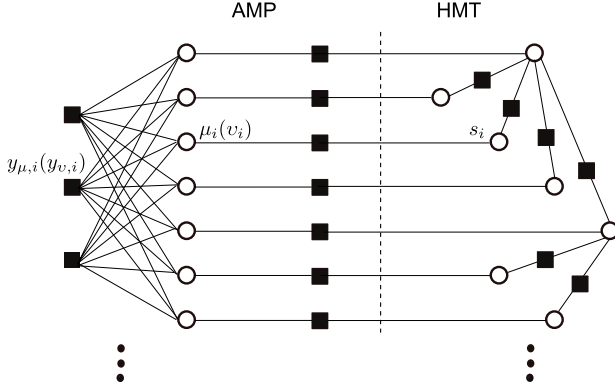


Fig. 2. Factor graph for reconstructing real  $\mu$  and imaginary  $v$  separately.

also need the prior for the hidden states and the transition probabilities  $\xi_k^{00}, \xi_k^{11}$  across scales, where  $\xi_k^{00} = p(s_{k+1} = 0 | s_k = 0)$  and  $\xi_k^{11} = p(s_{k+1} = 1 | s_k = 1)$ . Similar to [9], we take the Bayesian approach and model the remaining statistical parameters as random variables. The appropriate priors are assigned according to the recommendation in [9].

### 3. TURBO RECONSTRUCTION

With the statistical model chosen, we now discuss the implementation of the turbo reconstruction method. Throughout, we assume the CS measurements are taken in the DT-CWT domain, which is

$$\mathbf{y} = \Phi \mathbf{c} = \Phi \boldsymbol{\mu} + j \Phi \mathbf{v} \quad (5)$$

where  $\mathbf{c} = \{c_{k,i}\}$ ,  $\boldsymbol{\mu} = \{\mu_{k,i}\}$ , and  $\mathbf{v} = \{v_{k,i}\}$  for all  $k, i$ .

Separating the real and imaginary parts of the DT-CWT coefficients, we have

$$\mathbf{y}_\mu = \Phi \boldsymbol{\mu}, \quad \mathbf{y}_v = \Phi \mathbf{v} \quad (6)$$

Ideally, we would like to calculate the posterior pdf  $p(\mathbf{c}|\mathbf{y})$  to infer  $\mathbf{c}$ . However, exact computation of  $p(\mathbf{c}|\mathbf{y})$  is intractable due to the dense measurement matrix. Alternatively, we can resort to the marginal posterior  $p(c_i|\mathbf{y})$  which can be effectively approximated using loopy belief propagation on the factor graph.

#### 3.1. Reconstruction for Independent HMT

When  $\boldsymbol{\mu}$  and  $\mathbf{v}$  are modeled separately with two-state 1-D HMT structure, the reconstruction can be performed independently in parallel by calculating  $p(\mu_i|\mathbf{y}_\mu)$  and  $p(v_i|\mathbf{y}_v)$ . When dealing with complicated functions involving many variables, efficient algorithms may be derived by exploiting the factorization of the global function: expressing it as a product of several simpler “local” functions, each of which depends on only a subset of the variables. A factor graph is

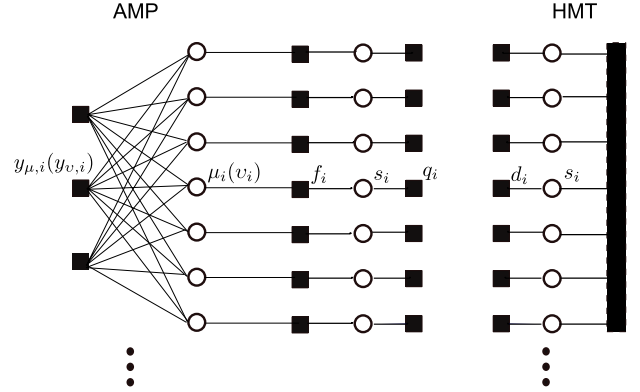


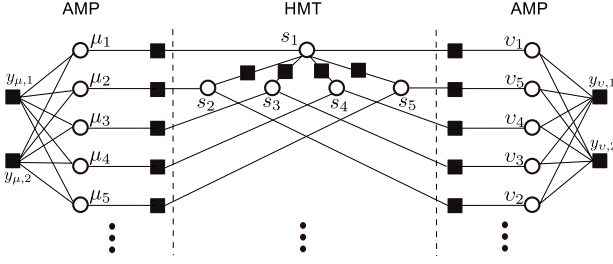
Fig. 3. The decoupled two subgraphs for the original factor graph in Fig. 2.

graphical representation of such factorization, which connects local functions with their related argument variables [10]. The factor graph for  $p(\boldsymbol{\mu}|\mathbf{y}_\mu)$  or  $p(\mathbf{v}|\mathbf{y}_v)$  is illustrated in Fig. 2.

We use round nodes to denote variables (i.e.  $\mu_i, v_i, s_i$ ), square nodes for operations (i.e. transition function among hidden states, assigning priors) and lines to connect operations with the associated augments in the factor graph. “Messages” flowing to/from a variable node along edges (as illustrated by arrows in Fig. 5) can be calculated according to the sum-product algorithm [10]. The loopy belief propagation is conducted by exchanging messages passing along nodes and edges until they converge. In our context, messages take the form of pdfs and represent local beliefs for variables.

Unsurprisingly, the factor graph in Fig. 2 has the same structure as the one in [9] since we essentially treat the reconstruction of  $\mathbf{c}$  as two DWT CS problems. Thus we can directly deploy the same turbo approach in [9] to obtain  $p(\mu_i|\mathbf{y}_\mu)$  and  $p(v_i|\mathbf{y}_v)$ . The essence is to decouple the factor graph into two subgraphs (reconstruction parts) along the dash line in Fig. 2 and exchange the local belief on the hidden states  $\{s_i\}$  between two subgraphs alternately, by treating the likelihood from one part as the prior for the other. More specifically, at the first turbo iteration, by assuming some initial prior  $p(s_i = 1)$  for the hidden states, we start with the reconstruction on the left subgraph. Performing the loopy belief propagation on the left subgraph of Fig. 2 is equivalent to the AMP algorithm [3] and produces the “local likelihood” on hidden states. These local beliefs are then treated as priors for  $\{s_i\}$  in the right subgraph for the HMT reconstruction. The belief propagation for factor graphs without loops has exact solution and yields the commonly known forward-backward algorithm [10]. The local likelihood for  $\{s_i\}$  from the right subgraph are then fed back to the left subgraph as an updated prior for the AMP algorithm and trigger the new turbo iteration.

The decoupled factor graph is shown in Fig. 3, where the quad-tree structure are abstracted as the supernode H in the



**Fig. 4.** Factor graph for modeling  $\mu$  and  $v$  with shared hidden states.

right subgraph. Let  $t$  denote the turbo iteration index,  $\{q_i(\cdot)\}$  are hidden states priors acting on the left subgraph,  $\{d_i(\cdot)\}$  are hidden states priors acting on the right subgraph. Let  $m_{A \rightarrow B}^{(t)}$  represents the message sent from node A to node B in the  $t$ th turbo iteration. At  $t = 1$ , initialize  $q_i(s_i) = p(s_i = 1)$ . Then the turbo message passing can be summarized as

$$d_i^{(t)}(s_i) = m_{f_i \rightarrow s_i}^{(t)}(s_i) \quad (7)$$

$$q_i^{(t+1)}(s_i) = m_{H \rightarrow s_i}^{(t)}(s_i) \quad (8)$$

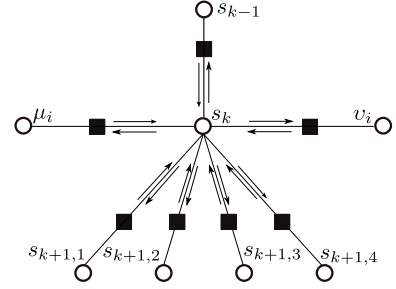
for  $t = 2, 3, \dots$ . We refer to this reconstruction as the Turbo-CHMT1 algorithm.

### 3.2. Reconstruction for HMT with shared hidden states

When connecting  $\mu$  and  $v$  with the shared hidden states, the reconstruction is no longer separable. The new factor graph for the system is illustrated in Fig. 4.

To perform the belief propagation, we also deploy the turbo method and split the loopy factor graph along the two dash lines into two AMP reconstruction parts and one HMT reconstruction part. The turbo iteration starts with the AMP reconstruction on the left and right subgraph simultaneously and produces two partial information for the hidden state  $m_{\mu_i \rightarrow s_i}(s_i)$  and  $m_{v_i \rightarrow s_i}(s_i)$ , respectively. The forward-backward algorithm is then performed for the middle subgraph on the quad-tree structure, which yields two output messages  $m_{s_i \rightarrow \mu_i}(s_i)$  and  $m_{s_i \rightarrow v_i}(s_i)$ . In the next turbo iteration these are used as the updated prior for  $s_i$  for the separate AMP reconstructions.

Fig. 5 features a hidden state node  $s_k$  with one parent node  $s_{k-1}$ , four children nodes  $\{s_{k+1,i}\}_{i=1}^4$  and the associated real and imaginary coefficients  $\mu_i, v_i$ . Compared to the HMT structure in Fig. 3, the key modification is the extra input message for the hidden states, which needs to be taken care of when performing the forward-backward to calculate the messages exchanging between  $s_k$  and its parent and children. According to the sum-product algorithm, the output



**Fig. 5.** One typical hidden variable node for the factor graph in Fig. 4.

message from  $s_k$  to  $\mu_i$  and  $v_i$  are

$$m_{s_k \rightarrow \mu_i}(s_k) = m_{v_i \rightarrow s_k}(s_k) m_{s_{k-1} \rightarrow s_k}(s_k) \quad (9)$$

$$\times \prod_{i=1}^4 m_{s_{k+1,i} \rightarrow s_k}(s_k)$$

$$m_{s_k \rightarrow v_i}(s_k) = m_{s_k \rightarrow \mu_i}(s_k) m_{s_{k-1} \rightarrow s_k}(s_k) \quad (10)$$

$$\times \prod_{i=1}^4 m_{s_{k+1,i} \rightarrow s_k}(s_k)$$

The benefit of sending messages from both real and imaginary parts to the hidden states is twofold. For one thing, it reinforces the confidence of estimation for  $s_i$  when the messages agree with each other (both imply  $s_i$  is more likely to be 1 or 0). For another, it softens the decision for  $s_i$  when they indicate different state. We denote this turbo approach as the Turbo-CHMT2 algorithm. Numerical simulation with natural images indicates an accelerated convergence for the reconstruction with Turbo-CHMT2.

## 4. NATURAL IMAGE EXAMPLE

In this section, the proposed Turbo-CHMT1 and Turbo-CHMT2 algorithm are tested on three  $128 \times 128$  grayscale images using 4 level Q-shift DT-CWT<sup>1</sup>, and compared with the original Turbo algorithm with db2 wavelet decomposition<sup>2</sup> and the EM-GM-GAMP algorithm<sup>3</sup>, which does not exploit the wavelet dependency. In all cases, the entries of the measurement matrix are i.i.d Gaussian elements. The CS measurements are taken in the wavelet domain. We used the normalized mean squared error (NMSE)  $\|\mathbf{x} - \hat{\mathbf{x}}\|_2^2 / \|\mathbf{x}\|_2^2$  for performance comparison.

Table 1 reports the quantitative comparison of the reconstruction algorithms under different sampling ratios for three standard images. The reconstruction of the lena images in Fig. 6 gives us a visual comparison. We can see that both Turbo-CHMT1 and Turbo-CHMT2 provide us a significant

<sup>1</sup><http://www-sigproc.eng.cam.ac.uk/Main/NGK>

<sup>2</sup><http://www2.ece.ohio-state.edu/~schniter/turboAMPimaging/>

<sup>3</sup><http://www2.ece.ohio-state.edu/~vilaj/EMGMAMP/EMGMAMP.html>



Fig. 6. Reconstruction with 30% CS observations of the  $128 \times 128$  lena image using various algorithms.

reconstruction improvement (about 1.5 dB) over the original Turbo algorithm. We attribute the improvement to the near shift-invariant property and good directional resolution of the DT-CWT. The best performance belongs to the Turbo-CHMT2, which validates the HMT model with shared hidden states and the proposed turbo reconstruction approach.

Table 1. NMSE (dB) for image reconstruction

Image $m/n$	Cameraman		Lena		Boat	
	0.2	0.3	0.2	0.3	0.2	0.3
EM-GAMP	-15.10	-19.03	-14.11	-17.43	-16.74	-19.29
Turbo	-17.96	-20.85	-15.60	-18.47	-18.36	-21.08
Turbo-CHMT1	-18.63	-23.32	-17.64	-20.36	-19.38	-21.61
Turbo-CHMT2	-19.12	-23.98	-18.06	-20.88	-19.76	-22.09

## 5. CONCLUSION

Given the advantageous near shift-invariant property and directional selectivity of the DT-CWT, we are motivated to exploit wavelet dependencies for the DT-CWT coefficients to aid the CS imaging. Two types of Gaussian mixture priors are considered for the DT-CWT coefficients with two different graphical representations for reconstruction. Inspired by the turbo decoding mechanism for the DWT coefficients, we propose two turbo algorithms for the loopy factor graphs. As the reconstruction examples have demonstrated, our algorithms have successfully leveraged the interscale dependencies among DT-CWT coefficients for CS reconstruction. Further work might involve the exploitation of the phase information encoded in the DT-CWT coefficients.

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