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MINIMAL DECOMPOSITION OF BINARY FORMS WITH RESPECT TO TANGENTIAL PROJECTIONS.

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ABSTRACT. Let $C \subset \mathbb{P}^n$ be a rational normal curve and let $\ell_O : \mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^n$ be any tangential projection form a point $O \in T_A C$ where $A \in C$. In this paper we relate the minimum number r of addenda that are needed to write a binary form p of degree $(n+1)$ and defined over an algebraically closed field of characteristic zero as linear combination of $(n+1)$-th powers of linear binary forms $L_1, \dots, L_r$, with the minimum number of addenda that are required to write $\ell_O(p)$ as linear combination of elements belonging to $\ell_O(C)$.

INTRODUCTION

In many applications, like Biology and Statistics, it turns out to be useful to develop techniques for reducing the dimension of high-dimensional data (like Principal Component Analysis [PCA]) that can be encoded in a tensor. In many cases this tensor turns out to be symmetric and with many entries equal to zero. One of the main problem is to find a minimal decomposition of those tensors in terms of other tensors of the same structure but with the minimal number of entries as possible (in the literature this kind of problems are known either as Structured Tensor Rank Decomposition in the Signal Processing language –see e.g. [6]– or as CANDECOMP/PARAFAC in the Data Analysis context –see e.g. [7]–). We want to address these questions from an Algebraic Geometry point of view (we suggest [9] for a good description about the relation between Biology, Statistics and Algebraic Geometry on these kind of questions).

Let V be a finite dimensional vector space defined over an algebraically closed field K of characteristic zero. A symmetric tensor is an element $T \in S^dV$. Since the space S^dV is isomorphic to the vector space of homogeneous polynomial $K[x_1, \dots, x_{\dim(V)}]_d$ of degree d in $\dim(V)$ variables with the coefficients that take values over K, then one can translate the questions on symmetric tensors into questions on homogeneous polynomials.

In this paper we study the case of homogeneous polynomials of certain fixed degree $n+1$ in 2 variables having one coefficient equal to zero.

Assume for a moment to have fixed an order between the generators of $K[u,t]_{n+1}$ and to have given a corresponding coordinate system, say $\{x_0, \dots, x_{n+1}\}$. A binary form with the coefficient in the i-th position equal to zero can be obtained by projecting a binary form to the hyperplane $H_i \subset K[u,t]_{n+1}$ identified by the equation $x_i = 0$. We will focus on projections ℓ_O from a point $O \in \mathbb{P}(K[u,t]_{n+1}) \simeq \mathbb{P}^{n+1}$ to $\mathbb{P}(H_i) \simeq \mathbb{P}^n$ that corresponds to tangential projections to the rational normal curve that is canonically embedded in $\mathbb{P}^{n+1}$. This will allow to relate the minimal decomposition of a binary form p of degree $n+1$ as sum of $(n+1)$-th powers of linear forms $L_1^{n+1}, \dots, L_r^{n+1} \in K[u,t]_{n+1}$, with the minimal decomposition of the

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projected $\ell_O(p) \in \mathbb{P}(H_i)$ (that is a binary form of the same degree $n+1$ but with the i-th coefficient equal to zero) in terms of $\ell_O(L_1^{n+1}), \ldots, \ell_O(L_r^{n+1})$. Explicitly if r is the minimum number of addenda that are required to write $p \in K[u,t]_{n+1}$ as

$$p = L_1^{n+1} + \cdots + L_r^{n+1}$$

then we will prove in Theorem 1 and in Theorem 2 that there is a dense subset of $\mathbb{P}(H_i) \simeq \mathbb{P}^n$ where r is also the minimum number of addenda that are required to write $\ell_O(p)$ as follows:

$$\ell_O(p) = \ell_O(L_1^{n+1}) + \cdots + \ell_O(L_r^{n+1}).$$

We will also describe which is the relation between the minimal decomposition of p and the minimal decomposition of $\ell_O(p)$ out of this dense subset, and we will prove that the number of addenda required for $\ell_O(p)$ can be only either 1 or 2 less than the number of addenda required in the minimal decomposition of p. The minimal decomposition of a generic binary form of degree $n+1$ in terms of $(n+1)$-th powers of binary linear forms was firstly studied by J. J. Sylvester in [10], then formalized with an algorithm in [5] (see also [3] for a more recent proof).

Actually we will use this language of binary forms only in Section 3 in order to explain what happens if we fix a canonical embedding of a rational normal curve and a particular center of projection. In fact all along the paper we will use a more general setting. Let $C \subset \mathbb{P}^{n+1}$ be a rational normal curve of degree $n+1$ and consider any tangential projection ℓ_O form a point $O \in T_A C \setminus \{A\}$, with $A \in C$, to a $\mathbb{P}^n$. The image of C via ℓ_O is a cuspidal curve $X \subset \mathbb{P}^n$. The elements of C parameterize binary forms that can be written as $(n+1)$-th powers of linear binary forms L^{n+1}'s, and the elements of X correspond to $\ell_O(L^{n+1})$'s. After the preliminary Section 1, that works for any non-degenerate projective variety and not only for rational normal curves, we will give, in Section 2, the already quoted main results of this paper that are Theorem 1 and Theorem 2 that will relate, for any tangential projection ℓ_O, the minimal decomposition of an element $p \in \mathbb{P}^{n+1}$ with respect to C, with the minimal decomposition of its image $\ell_O(p) \in \mathbb{P}^n$ with respect to X.

1. Preliminaries

We give here all the definitions and all the notation that we will need in the sequel. We can state them in a general setting even if we will use them in the very particular case of tangential projections of rational normal curves. So for this section we consider $Y \subset \mathbb{P}^N$ to be any non-degenerate projective variety.

Definition 1. The Y-rank $r_Y(P)$ of a point $P \in \langle Y \rangle \simeq \mathbb{P}^N$ with respect to a non degenerate projective variety Y is the minimum integer ρ for which there exists a reduced 0-dimensional sub-scheme $S \subset Y$ of degree ρ whose span contains P.

Definition 2. Let $P \in \langle Y \rangle \simeq \mathbb{P}^N$ be a point of Y-rank equal to ρ. We say that a 0-dimensional sub-scheme $S \subset Y$ computes the Y-rank of P if it is reduced, of degree ρ and such that $P \in \langle S \rangle$.

Notation 1. We indicate with $\sigma_s^0(Y) \subset \mathbb{P}^N$ the set of points $P \in \mathbb{P}^n$ of Y-rank less or equal than s.

Definition 3. The s-th secant variety $\sigma_s(Y) \subset \mathbb{P}^N$ is the Zariski closure of the set $\sigma_s^0(Y)$ of Notation 1.

Remark 1. Observe that if $P \in \sigma_s(Y) \setminus \sigma_s^0(Y)$ then $r_Y(P) > s$.

Definition 4. Let $P \in \langle Y \rangle \subset \mathbb{P}^N$. The Y-border rank $br_Y(P)$ of P is the minimum integer w such that $P \in \sigma_w(Y)$.

Remark 2. Observe that, by Definition 3, the relation between the Y-border rank and the Y-rank of the same point $P \in \mathbb{P}^N$ is the following: $br_Y(P) \leq r_Y(P)$.

We borrow from [4] the following result (we only need the case in which Y is a rational normal curve of $\mathbb{P}^{n+1}$ with $2t \leq n+2$; thus the case we use is a particular case of [4], Lemma 2.1.5).

Lemma 1. *Let $Y \subset \mathbb{P}^N$ be a smooth and non-degenerate subvariety of dimension at most 2. Fix an integer $t \geq 2$ and assume $\dim\langle Z \rangle = \deg(Z) - 1$ for every 0-dimensional sub-scheme $Z \subset Y$ such that $\deg(Z) \leq t$. Fix $P \in \mathbb{P}^N$.*

(i) $P \in \sigma_t(Y)$ if and only if there is a 0-dimensional scheme $Z \subset Y$ such that $\deg(Z) \leq t$ and $P \in \langle Z \rangle$.

(ii) $P \in \sigma_t(Y) \setminus \sigma_{t-1}(Y)$ if and only if t is the first integer such that there is a 0-dimensional sub-scheme $Z \subset Y$ with $\deg(Z) = t$ and $P \in \langle Z \rangle$.

Proof. Since Y is smooth and $\dim(Y) \leq 2$, every 0-dimensional sub-scheme A of Y is smoothable, i.e. it is a flat limit of a family of unions of $\deg(A)$ distinct points ([8]). As remarked in the proof of [4], Lemma 2.1.5, the assumption "$\dim\langle Z \rangle = \deg(Z) - 1$ for every 0-dimensional scheme $Z \subset Y$ such that $\deg(Z) \leq t$" is sufficient to use [3], Proposition 2.8 and get part (i).

Part (ii) follows from part (i) applied to the integers t and $t-1$. □

Remark 3. By Definition 3 of secant varieties of a projective variety $Y \subset \mathbb{P}^N$ we have the following obvious chain of containments:

$$Y = \sigma_1(Y) \subset \sigma_2(Y) \subset \cdots \subset \sigma_{s-1}(Y) \subset \sigma_s(Y) = \mathbb{P}^N$$

for certain integer $s \in \mathbb{N}$.

Definition 5. Let $Y \subset \mathbb{P}^N$ be a smooth and non-degenerate variety of dimension at most 2. Let $P \in \sigma_w(Y) \setminus (\sigma_w^0(Y) \cup \sigma_{w-1}(Y))$, then, by Lemma 1, there exists a non-reduced 0-dimensional sub-scheme $W \subset Y$ such that $P \in \langle W \rangle$. We say that such a W computes the Y-border rank of P.

We can give now a lemma that will be used in the next section in the particular case of tangential projections of rational normal curves, but since it can be stated in a general setting and since it improves Lemma 1.4 in [2] we prefer to state it here in the general setting of a non degenerate projective variety.

Lemma 2. *Fix an integral and non-degenerate subvariety $Y \subset \mathbb{P}^{n+x}$, $n > 0$, $x > 0$, and a linear $(x-1)$-dimensional subspace $V \subset \mathbb{P}^{n+x}$ such that $V \cap Y = \emptyset$. Set $X := \ell_V(Y)$. Then*

$$r_X(\ell_V(Q)) = \min_{P \in (\langle V \cup \{Q\} \rangle \setminus V)} r_Y(P) \quad \textit{for all } Q \in \mathbb{P}^{n+x} \setminus V. \tag{1}$$

Proof. First of all let us prove that $r_X(\ell_V(Q)) \geq r_Y(P)$ for all $P \in \mathbb{P}^{n+x} \setminus V$. Since $V \cap Y = \emptyset$, then obviously $\ell_V|Y$ is a finite morphism. Now $\ell_V|Y : Y \to X$ is surjective, then for each finite set of points $S \subset X$ we may fix another finite subset $S_V \subset Y$ such that $\ell_V(S_V) = S$ and $\sharp(S_V) = \sharp(S)$. Since $S_V \subseteq Y$, then S_V does not intersect V for all S's, therefore for any such a choice of S_V, the set $S \subset X$ turns out to be linearly independent if and only if S_V is linearly independent and $\langle S_V \rangle \cap V = \emptyset$.
Now fix $Q \in \mathbb{P}^{n+x} \setminus V$ and take $S \subset X$ computing $r_X(\ell_V(Q))$. Thus $\sharp(S) = r_X(\ell_V(Q))$ and S is linearly independent by definition of a set that computes the X-rank of a point (see Definition 2). Since S is linearly independent and since $V \cap \langle S \rangle = \emptyset$, for what proved

above, we also have that S_V is linearly independent and then $\langle S_V\rangle \cap V = \emptyset$. Now $\ell_V(Q)$ is an element of $\langle S\rangle$, then $\langle S_V\rangle \cap \langle V \cup \{Q\}\rangle \neq \emptyset$. Since $\langle S_V\rangle \cap V = \emptyset$, there is a unique $P \in (\langle V \cup \{Q\}\rangle \setminus V)$ such that $\{P\} = \langle S_V\rangle \cap \langle V \cup \{Q\}\rangle$. Therefore since $S_V \subset Y$, we have $r_Y(P) \leq \sharp(S_V) = \sharp(S) = r_X(\ell_V(Q))$.

To get the reverse inequality we may just quote Lemma 14 in [2] but since it is quite easy to be proved, we show here a shorter proof. Fix any $P \in (\langle V \cup \{Q\}\rangle \setminus V)$ and any $A \subset Y$ computing $r_Y(P)$. Since $P \in (\langle V \cup \{Q\}\rangle \setminus V)$ we have $\ell_V(P) = \ell_V(Q)$. Now $\ell_V(P) \in \langle \ell_V(A)\rangle$, then $r_X(\ell_V(Q)) \leq r_Y(P)$. $\square$

2. Theorems

We can now focus on tangential projections $X \subset \mathbb{P}^n$ of rational normal curves $C \subset \mathbb{P}^{n+1}$ for $n \geq 3$. The two theorems that we are going to prove will give a complete description of both the schemes that compute the X-border rank and the schemes that compute the X-rank of a point $P \in \mathbb{P}^n$ with respect to a curve $X \subset \mathbb{P}^n$ obtained as the tangential projection of a rational normal curve $C \subset \mathbb{P}^{n+1}$. Moreover we will explain the relation between the schemes that compute $br_X(P)$ and $r_X(P)$ and the schemes that compute $br_C(B)$ and $r_C(B)$ where $B \in \mathbb{P}^{n+1}$ is a point that is sent into $P \in \mathbb{P}^n$ by the tangential projection.

We fix here the notation that we are going to use. From now on we will always consider the following situation.

Notation 2. Let $C \subset \mathbb{P}^{n+1}$ be a smooth rational normal curve of degree $n+1$. Fix $A \in C$ and consider the line T_AC that sometimes we will also indicate with $\langle 2A\rangle$. Fix also a point $O \in T_AC \setminus \{A\}$ to be the center of the projection $\ell_O : \mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^n$ that sends C into a curve $X := \ell_O(C) \subset \mathbb{P}^n$. The curve X turns out to be a linearly normal curve of $\mathbb{P}^n$ with degree $n+1$, arithmetic genus 1 and the ordinary cusp $\ell_O(A) \in X \subset \mathbb{P}^n$ as its unique singular point.

In the following two theorems we give both a complete description of the schemes that realize the X-border rank (Theorem 1) and the X-rank (Theorem 2) of a point $P \in \mathbb{P}^n$ with respect to a curve X just described, and the precise value of the X-rank of such a point P. In Theorem 1 we give the X-rank of a point $P \in \mathbb{P}^n$ that is the image via ℓ_O of a point $B \in \mathbb{P}^{n+1}$ whose C-border rank is smaller that its C-rank. In Theorem 2 the point $P \in \mathbb{P}^n$ is the image of a point $M\mathbb{P}^{n+1}$ whose C-border rank is equal to its C-rank.

Theorem 1. *Fix an integer $n \geq 3$. Let $C \subset \mathbb{P}^{n+1}$ be a rational normal curve and let also $X := \ell_O(C) \subset \mathbb{P}^n$ and $O \in T_AC \setminus \{A\}$ for a fixed $A \in C$ be as in Notation 2. Let $w \geq 2$ be the C-border rank of a point $B \in \sigma_w(C) \setminus \sigma_w^0(C) \subset \mathbb{P}^{n+1}$. Then there is a unique 0-dimensional sub-scheme $W \subset C$ that realizes the C-border rank of B.*
Moreover the X-border rank of a point $P := \ell_O(B) \in \mathbb{P}^n$ and the sub-scheme $W \subset X$ are completely classified by the following cases:

1. *If $O \in \langle W\rangle$, then $A \in W_{red}$, A appears with multiplicity 2 in W and $W \setminus 2A$ is reduced. Moreover both the following cases may occur:*
   - *either $r_X(P) = w-1$ and $\ell_O(W_{red})$ computes $r_X(P)$*
   - *or $w \geq 3$, $r_X(P) = w-2$ and $\ell_O(W \setminus 2A)$ computes $r_X(P)$.*
2. *If $O \notin \langle W\rangle$ and $A \notin W_{red}$, then $r_X(P) = n+3-w$.*
3. *If $O \notin \langle W\rangle$ and $A \in W_{red}$, then A appears with multiplicity 1 in W and $r_X(P) = n+2-w$.*

Proof. Since C is a rational normal curve of $\mathbb{P}^{n+1}$, every 0-dimensional sub-scheme $Z \subset C$ such that $\deg(Z) \leq n+2$ is linearly independent, i.e. $\dim\langle Z\rangle = \deg(Z) - 1$, with the usual conventions $\deg(\emptyset) = 0$, $\langle\emptyset\rangle = \emptyset$ and $\dim(\emptyset) = -1$. Thus if Z_1, Z_2 are 0-dimensional sub-schemes of C and $\deg(Z_1) + \deg(Z_2) \leq n+2$, then $\langle Z_1\rangle \cap \langle Z_2\rangle = \langle Z_1 \cap Z_2\rangle$, where $Z_1 \cap Z_2$

denotes the scheme-theoretic intersection. Let $2A$ denotes the degree 2 effective divisor of C with A as its reduction.

First of all observe that since the sub-scheme $W \subset C$ computes the C-border rank of $B \in \mathbb{P}^{n+1}$ then W is not reduced. Let us first prove the uniqueness of such a sub-scheme $W \subset C$. Assume that $W_1 \subset C$ is another such a sub-scheme, then $B \in \langle W \rangle \cap \langle W_1 \rangle$, by Definition 5 the point $B \notin \langle W' \rangle$ for any $W' \subsetneq W$ and $\deg(W) + \deg(W_1) = 2\deg(W) = 2w \leq n+2$. Hence $W_1 \cap W = W$, that means that $W_1 = W$.

Now recall that, since $B \in \sigma_w(C) \setminus \sigma_w^0(C)$ by hypotheis, then $r_C(B) = n+2-w$ (see [3], Theorem 3.8).

(1) Notice that $O \in \langle W \rangle$ if and only if $\langle \{O, B\} \rangle \subseteq \langle W \rangle$. We study now the case of $O \in \langle W \rangle$. Fix any point $Q \in \langle \{O, B\} \rangle \setminus \{O\}$. Since $Q \in \langle W \rangle$, we have $br_C(Q) \leq w$. Thus (by the so called Sylvester algorithm, see e.g. [3]) either $r_C(Q) = br_C(Q)$ or $r_C(Q) = n+3-br_C(Q) \geq n+3-w = r_C(B)$. If the latter case occurs for all $Q \in \langle \{O, B\} \rangle \setminus \{O\}$, then, by Lemma 2, $r_X(\ell_O(B)) = n+3-w$.
Assume the existence of $Q \in \langle \{O, B\} \rangle \setminus \{O\}$ such that $r_C(Q) = br_C(Q)$. Take $S_1 \subset C$ computing $r_C(Q)$. Since $Q \in \langle W \rangle$, the proof of the uniqueness of W gives $S_1 \subseteq W$ but since W is not reduced, then obviously $S_1 \subsetneq W$. Since $Q \neq O$, and $B \notin \langle S_1 \rangle$, Q is the only point of the line $\langle \{O, B\} \rangle$ contained in $\langle S_1 \rangle$. Since $O \in \langle 2A \rangle$, we get $\langle \{O, B\} \rangle \subseteq \langle 2A \cup S_1 \rangle$. Thus the uniqueness of W gives $W \subseteq S_1 \cup 2A$, in which if $A \in S_1$, then $S_1 \cup 2A$ is the only divisor with S_1 as its reduction, reduced outside A and with multiplicity 2 at A. Since $O \in \langle 2A \rangle$, $\deg(2A) + \deg(W) = 2 + w \leq n+2$ and $O \neq A$, we get that A appears with multiplicity at least 2 in W. Thus $W = S_1 \cup 2A$ and either $\sharp(S_1) = w-2$ if $A \notin S_1$, or $\sharp(S_1) = w-1$ if $A \in S_1$. In both cases $2A$ is the only unreduced connected component of the scheme $W \subset C$.
We want to show that both cases occur for certain points B and we also want to describe all points B for which they occur.
If $W = 2A$, then $P = \ell_O(B) \in X$ and hence $r_X(P) = 1$.
Now assume $W \neq 2A$, i.e. $w \geq 3$. Take any $S_2 \subset X$ such that $\sharp(S_2) = w-2$, and $A \notin S_2$. Set $S_1 := S_2 \cup \{A\}$. Set $W := S_2 \cup 2A$. Since $w \leq n+2$, we saw that W is linearly independent. Take as B any point of $\langle W \rangle$ not contained in the linear span of a proper sub-scheme of W . Fix any $S_1 \subset C$ such that $\sharp(S_1) = w-2$. Set $W := S_1 \cup 2A$ and take as B any point of $\langle W \rangle$ not contained in the linear span of a proper sub-scheme of W.

(2) Here we assume $O \notin \langle W \rangle$ and that $A \notin W_{red}$. In this case the dimension of $\langle 2A \cup B \rangle$ is $w+1$ because $\deg(2A \cup W) = 2+w$. Hence $\langle 2A \rangle \cap \langle W \rangle = \emptyset$ and $\langle 2A \rangle \cap \langle \{A\} \cup W \rangle = \{A\}$. Since $O \in \langle 2A \rangle$ and $O \neq A$, we get that $O \notin \langle \{A\} \cup W \rangle$. Now fix any point $Q \in \langle \{O, B\} \rangle \setminus \{O\}$. Since $B \in \langle W \rangle$ and $O \in \langle 2A \rangle$, we have $Q \in \langle 2A \cup W \rangle$. Thus, by [3], Proposition 2.8, $br_C(Q) \leq w+2$. Since $B \in \langle W \rangle \subset \langle \{A\} \cup W \rangle$, $O \in \langle 2A \rangle$, $O \notin \langle \{A\} \cup W \rangle$, $Q \neq A$ and $\langle \{A\} \cup W \rangle \cap \langle 2A \rangle = \{A\}$, then $Q \notin \langle \{A\} \cup W \rangle$.
We want to prove now that the C-border rank of Q is actually $w+2$ and that $2A \cup W$ is the scheme that computes it. Assume the existence of a proper sub-scheme $G \subsetneq 2A \cup W$ such that $Q \in \langle G \rangle$. Since $Q \notin \langle \{A\} \cup W \rangle$ there is $G_1 \subsetneq W$ such that $Q \in \langle 2A \cup G_1 \rangle$. Since $O \in \langle 2A \rangle$, we get $B \in \langle 2A \cup G_1 \rangle$. Since $\deg(2A \cup G_1) + \deg(W) \leq 2 + w - 1 + w \leq n+2$, we get $\langle 2A \cup G_1 \rangle \cap \langle W \rangle = \langle G_1 \rangle$, contradicting the assumption $br_C(B) = w$. Thus there is no proper subset G of $2A \cup W$ such that $Q \in \langle G \rangle$. Thus $br_C(Q) = w+2$ ([3], Proposition 2.8). Thus, by [3], Theorem 3.8, $r_C(Q) = n+3-w$. Now our Lemma 2 gives $r_X(P) = n+1-w$.

(3) Assume $A \in W_{red}$ and that A appears with multiplicity 1 in W.
Then $A \in \langle W \rangle$ and $\deg(W \cup 2A) = w+1$. Set $W_1 := W \setminus \{A\}$ and $W_2 := W_1 \cup 2A$. Thus $\deg(W_2) = \deg(W_1) + 2 = w+1$. By step (1) we have $O \notin \langle W \rangle$. Fix any $Q \in \langle \{O, B\} \rangle \setminus \{O\}$. Since $B \in \langle W \rangle$ and $O \notin \langle W \rangle$, then $Q \notin \langle W \rangle$. Since $O \in \langle 2A \rangle$, then $Q \in \langle W_2 \rangle$. Thus $br_C(Q) \le w+1$. Since $2(w+1) \le n+2$, we also know that $br_C(Q)$ is computed by a unique scheme Γ and that $\Gamma \subseteq W_2$. Since $B \in \langle 2A \cup \Gamma \rangle$, we also have $W \subseteq \Gamma \cup 2A$. Hence either $\Gamma = W_2$ or $\Gamma = W$ or $\Gamma = W_1$. Since $Q \notin \langle W \rangle$, we have $\Gamma = W_2$. Thus $br_C(Q) = w+1$. Since $r_C(Q) = n+3-br_C(Q) = n+2-w$ ([3], Theorem 3.8), Lemma 2 gives $r_X(P) = n+1-w$.

Now assume that A appears with multiplicity at least 2 in W. Since $O \in \langle 2A \rangle$, we get $O \in \langle W \rangle$. Hence this case was discussed in step (1). □

The following remark will turn out to be useful in the construction of algorithms for the computation of the X-rank of points $P \in \mathbb{P}^n$ with respect to the arithmetic genus 1 cuspidal linearly normal curve X of degree $n+1$.

Remark 4. Let $S \subset C$ be a reduced 0-dimensional sub-scheme that computes the C-rank of a point $B \in \mathbb{P}^{n+1}$ with respect to a rational normal curve $C \subset \mathbb{P}^{n+1}$. Let also $\ell_O : \mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^n$ be the tangential projection from $O \in T_AC$ for a fixed $A \in C$, $A \neq B$, that sends C into an arithmetic genus 1 cuspidal linearly normal curve $X \subset \mathbb{P}^n$ of degree $n+1$. Then the sub-scheme $S' := \ell_O(S) \subset X$ computes the X-rank of the point $P = \ell_O(B)$. Moreover all the sub-schemes $S' \subset X$ that compute the X-rank of a point $P \in \mathbb{P}^n$ are the projections via ℓ_O of schemes $S \subset C$ that compute the C-rank of a point $B \in \mathbb{P}^{n+1}$ such that $\ell_O(B) = P$.

Theorem 2. *Fix an integer $n \ge 3$. Let $C \subset \mathbb{P}^{n+1}$ be a rational normal curve and let also $X := \ell_O(C) \subset \mathbb{P}^n$ and $O \in T_AC \setminus \{A\}$ for a fixed $A \in C$ be as in Notation 2. Let $M \in \mathbb{P}^{n+1} \setminus \{O\}$ be such that $r_C(M) = \rho$ with ρ such that $2 \le \rho \le \lfloor (n+2)/2 \rfloor$. Let $E \subset C$ be a finite set that computes the C-rank of M. Then the scheme E is unique if $2\rho \le n+1$. Call $P \in \mathbb{P}^n$ the point obtained as $\ell_O(M) =: P$. Therefore the following hold:*

- (i) *If $2\rho \le n$, then $r_X(P) = \rho$ and $\ell_O(E)$ is the unique subset of X computing $r_X(P)$.*
- (ii) *If $n+1 \le 2\rho \le n+2$, then $\rho - 1 \le r_X(P) \le \rho$.*
- (iii) *If n is even and $2\rho = n+2$, then there is a non-empty open subset $\mathcal{U}$ of $\mathbb{P}^{n+1}$ such that $r_C(M) = \rho$ and $r_X(P) = \rho - 1$ for all $M \in \mathcal{U}$.*

Proof. Let us first check that $O \notin \langle E \rangle$.
Assume $O \in \langle E \rangle$. Thus $O \in \langle 2A \rangle \cap \langle E \rangle$. Since $\deg(2A) + \deg(E) = 2 + \rho \le n+2$, we get $O \in \langle 2A \rangle \cap \langle E \rangle = \{A\}$. Since E is reduced and $O \neq A$, we got a contradiction. Therefore $O \notin \langle E \rangle$.

Assume just for now that $2\rho \le n+1$.
By [3], Theorem 3.8, we have $br_C(M) = \rho$. The proof of the uniqueness of a non-reduced 0-dimensional scheme $W \subset C$ that computes the C-border rank of a point $M \in \mathbb{P}^{n+1}$ in Theorem 1, gives that $E \subset C$ is the unique 0-dimensional sub-scheme T of C such that $\deg(T) \le \rho$ and $M \in \langle T \rangle$. Since $\ell_O|C$ is injective, $\sharp(\ell_O(E)) = \rho$. Obviously, $\ell_O(B) = P \in \langle \ell_O(E) \rangle$. Thus $r_X(P) \le \rho$. But we have just proved that $O \notin \langle E \rangle$, i.e. $\dim(\langle \ell_O(E) \rangle) = \rho$.

(i) Here we assume $2\rho \le n$. Take $S \subset X$ computing $r_X(\ell_O(M))$. In this case it is sufficient to prove that $S = \ell_O(E)$. Since $\ell_O|C$ is injective, there is a unique $S' \subset C$ such that $\ell_O(S') = S$. Since $P = \ell_O(M) \in \langle S \rangle$, we have $M \in \langle \{O\} \cup S' \rangle \subset \langle 2A \cup S' \rangle$. Thus $M \in \langle 2A \cup S' \rangle \cap \langle E \rangle$. Since $\deg(2A \cup S') + \deg(E) \le 2 + 2\rho \le n+2$, the scheme $2A \cup S' \cup E$ is linearly independent. Thus $\langle 2A \cup S' \rangle \cap \langle E \rangle$ is the linear span of the scheme-theoretic intersection $(2A \cup S') \cap E$. Since E is reduced and $M \notin \langle E' \rangle$ for any $E' \subsetneqq E$, we get that either $S' = E$ or $S' \cup \{A\} = E$. If $A \notin E$, then we get $S' = E$, as wanted.

Assume $A \in E$, if $S' = E$ we are done, so assume that $S' \neq E$, i.e. $S' = E \setminus \{A\}$. In the ρ-dimensional linear space $\langle 2A \cup S' \rangle$, the linear subspace $\langle E \rangle$ and $\langle \{O\} \cup S' \rangle$ are different hyperplanes, because $O \notin \langle E \rangle$. Hence the line $\langle \{O, M\} \rangle \subset \langle 2A \cup S' \rangle$ intersects E in a unique point. Since $S' \subsetneqq E$, we have $M \notin \langle S' \rangle$. Thus $P \neq M$. Since $P \in \langle S' \rangle$ and $M \in \rangle E \rangle$, we get $\langle \{O, M\} \rangle \subseteq \langle E \rangle$, that is a contradiction.

(ii) Here we assume $n + 1 \leq 2\rho \leq n + 2$. We can almost reproduce the same proof of the step (i) above: the only part that failed in step (i) is that one in which we use the inequality $\deg(2A \cup S') + \deg(E) \leq 2 + 2\rho \leq n + 2$. However, to apply the first part of the proof of the theorem it is sufficient to have $\deg(2A \cup S' \cup E) \leq n + 2$. That is true because $\deg(2A) = 2$, $\deg(E) = \rho$ and $\deg(S') \leq \rho - 2$, and $2A$ is independent both with E and with S'.

(iii) Assume n even and $2\rho = n + 2$. A general $P \in \mathbb{P}^{n+1}$ satisfies $r_C(P) = br_C(P) = (n + 2)/2$. A general $P' \in \mathbb{P}^n$ satisfies $r_X(P') = br_X(P') = n/2$. A general $P' \in \mathbb{P}^n$ is of the form $\ell_O(P)$ with P general in $\mathbb{P}^{n+1}$. □

Remark 5. In the same setting of the two theorems above with the respective hypothesis, we can observe that, since all the points $Q \neq O$ contained in the line $\langle \{O, B\} \rangle \subset \mathbb{P}^{n+1}$ are sent by ℓ_O into the same point $P = \ell_O(B) \subset \mathbb{P}^n$, then all the points $Q \in \langle \{O, B\} \rangle \setminus \{O\}$ have the same C-rank.

Remark 6. Take the set-up of part (1) of Theorem 1. Notice that the closure C_A of $\ell_A(C \setminus \{A\})$ in $\mathbb{P}^n$ is a rational normal curve of $\mathbb{P}^n$. Hence both the C_A-border rank and the C_A-rank of the point $\ell_A(B) \in \mathbb{P}^n$ are algorithmically computable via the so called Sylvester algorithm (see e.g. [3], §3).

Notice that the closure C_{2A} of $\ell_{\langle 2A \rangle}(C \setminus \{A\})$ in $\mathbb{P}^{n-1}$ is a rational normal curve of $\mathbb{P}^{n-1}$. Hence both the border C_{2A}-rank and the C_{2A}-rank of the point $\ell_{\langle 2A \rangle}(B) \in \mathbb{P}^{n-1}$ are algorithmically computable (again via the Sylvester algorithm, [3], §3) and to know their value it is not necessary to know W.

Thus to check these ranks and these border ranks it is not necessary to know W.

If $B \in \langle 2A \rangle$, then $r_X(\ell_O(B)) = 1$. Thus from now on we assume $B \notin \langle 2A \rangle$. Now we will check that $br_{C_A}(\ell_A(B)) = w$ if and only if $A \notin W_{red}$ (and if and only if $br_{C_A}(\ell_A(B)) \geq w$) and that $br_{C_{2A}}(\ell_A(B)) = w - 1$ if and only if A appears with multiplicity 1 in W.

Since $\deg(A) + \deg(W) = 1 + w \leq n + 2$, we saw that $A \notin W_{red}$ if and only if $\dim(\langle \{A\} \cup W \rangle) = w$. Thus if $A \notin W_{red}$, then $\ell_A(W)$ is a degree w zero-dimensional sub-scheme of the rational normal curve C_A such that $\ell_A(B) \in \langle \ell_A(W) \rangle$ and $\ell_A(B) \notin \langle Z \rangle$ for every $Z \subsetneqq \ell_A(B)$. Since $w = \deg(\ell_A(W)) \leq n + 1 = \deg(C_A) + 1$ and C_A is a rational normal curve, we have $br_{C_A}(\ell_A(B)) = w$. Thus if $A \notin W_{red}$, then $\ell_A(B)$ has C_A-border rank w.

Conversely, if $A \in W_{red}$, then $\ell_A(B)$ is contained in the linear span of the degree $w - 1$ scheme $\ell_A(W - A)$. Since $w - 1 \leq \deg(C_A) + 1$, we get $br_{C_A}(\ell_A(B)) \leq w - 1$.

Similarly, A appears with multiplicity 1 in W if and only if $\ell_{2A}(B)$ is contained in the linear span of a degree $w-1$ sub-scheme of C_{2A} (i.e. $\ell_{2A}(B) \in \langle \ell_{2A}(W - A) \rangle$), but $\ell_{2A}(B) \notin \langle Z \rangle$ for any $Z \subsetneqq \ell_{2A}(W - A)$. Thus A appears with multiplicity 1 in W if and only if $br_{C_{2A}}(\ell_A(B)) = w - 1$.

Remark 7. Take the set-up of part (2) of Theorem 1. The closure C_A of $\ell_A(C \setminus \{A\})$ in $\mathbb{P}^{n-1}$ is a rational normal curve of $\mathbb{P}^n$. Hence both the C_A-border rank and the C_A-rank of the point $\ell_A(P) \in \mathbb{P}^n$ are algorithmically computable and to compute it, it is not necessary to know E ([3], §3).

Notice that $A \in E$ if and only if $r_{C_A}(\ell_A(B)) < \sharp(E)$ and that in this case $r_{C_A}(\ell_A(B)) = \sharp(E) - 1$.

3. Polynomial description

We give here a description of what we have proved in the previous section in terms of binary forms of degree $n+1$.

Let $C \subset \mathbb{P}^{n+1}$ be the rational normal curve parameterized by the map

$$\begin{array}{ccc} \mathbb{P}^1 & \to & \mathbb{P}^{n+1} \\ [u,t] & \mapsto & [u^{n+1}, u^n t, u^{n-1}t^2, \ldots, ut^n, t^{n+1}]. \end{array}$$

Let $O = [0, 1, 0, \ldots, 0] \in \mathbb{P}^{n+1}$. Then the curve $X \subset \mathbb{P}^n$ obtained as a tangential projection $\ell_O : \mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^n$ of C form O is parameterized by

$$\begin{array}{ccc} \mathbb{P}^1 & \to & \mathbb{P}^{n} \\ [u,t] & \mapsto & [u^{n+1}, u^{n-1}t^2, u^{n-2}t^3 \ldots, ut^n, t^{n+1}]. \end{array}$$

Now the $\mathbb{P}^{n+1} = \langle C \rangle$ can be interpreted as the projectivization of the vector space of binary forms in degree $n+1$ defined over an algebraically closed field K of characteristic 0, i.e. $\mathbb{P}^{n+1} = \langle C \rangle \simeq \mathbb{P}(K[u,t]_{n+1})$. If we think the elements of $\mathbb{P}^{n+1}$ as projectivization of binary forms of degree $n+1$, i.e.

$$p = \sum_{i=0}^{n+1} a_i u^i t^{n+1-i} \in \mathbb{P}^{n+1}, \tag{2}$$

then the elements of the $\mathbb{P}^n$ obtained with the projection ℓ_O can be described as projectivization of binary forms of degree $n+1$ over K without the terms in $u^n t$, i.e. if $\{x_0, \ldots, x_{n+1}\}$ is a system of coordinates of $\mathbb{P}^{n+1}$, then $\mathbb{P}^n = \langle X \rangle$ is the following:

$$\langle X \rangle = \{[x_0, \ldots, x_n] \in \mathbb{P}^{n+1} \mid x_1 = 0\} = \mathbb{P}(\langle u^{n+1}, u^{n-1}t^2, u^{n-2}t^3 \ldots, ut^n, t^{n+1} \rangle). \tag{3}$$

Hence if p is as in (2), then

$$\tilde{p} := \ell_O(P) = \sum_{i=0,2,3,\ldots n+1} a_i u^i t^{n+1-i} \in \mathbb{P}^n \tag{4}$$

Moreover, with this language, the elements of $C \subset \mathbb{P}^{n+1}$ are all of the type $[L^{n+1}]$ where L is a binary linear form, and the elements of $X \subset \mathbb{P}^n$ are all projectivization of binary forms of degree $n+1$ obtained from $(n+1)$-th powers of binary linear forms by dropping the term in $u^n t$; i.e. if

$$L^{n+1} = (au + bt)^{n+1} = \sum_{i=0}^{n+1} \binom{n+1}{i} (au)^i (bt)^{n+1-i} \in C,$$

then

$$\widetilde{L^{n+1}} = \ell_O(L^{n+1}) = \sum_{i=0,2,3,\ldots n+1} \binom{n+1}{i} (au)^i (bt)^{n+1-i} \in X \tag{5}$$

and all the elements of X are of this type.

Saying that a binary form $p \in K[u,t]_{n+1}$ has C-rank equal to r means that r is the minimum number of binary linear forms such that a linear combination of their $(n+1)$-th powers gives p:

$$p = L_1^{n+1} + \cdots + L_r^{n+1} \tag{6}$$

and such an r is minimal.

Saying that a binary form $\tilde{p} \in \langle X \rangle$ (as in (3)) has X-rank equal to s means that s is the minimum number of binary linear forms such that a linear combination of the projection via ℓ_O of their $(n+1)$-th powers gives $\tilde{p}$:

$$\tilde{p} = \widetilde{L_1^{n+1}} + \cdots + \widetilde{L_s^{n+1}} \tag{7}$$

and such an s is minimal.

What is proved in Theorem 1 and in Theorem 2 is that if the minimal decomposition of $p \in K[u,t]_{n+1}$ via $(n+1)$-th powers of linear forms is made by r addenda as in (6), then there is an open subset of $\langle X \rangle = \mathbb{P}^n$ (as in (3)) where $\tilde{p}$ can be minimally decomposed as in (7) with precisely r addenda. Moreover we precisely describe which is the X-rank of $\tilde{p}$ out of that open subset: what it turns out in Theorem 1 and in Theorem 2 is that it can drop only by 1 or 2 with respect to the the C-rank of p.

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