ON THE X-RANK WITH RESPECT TO LINEAR PROJECTIONS OF PROJECTIVE VARIETIES

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ABSTRACT. ABSTRACT: In this paper we improve the known bound for the X-rank $R_X(P)$ of an element $P \in \mathbb{P}^N$ in the case in which $X \subset \mathbb{P}^n$ is a projective variety obtained as a linear projection from a general v-dimensional subspace $V \subset \mathbb{P}^{n+v}$. Then, if $X \subset \mathbb{P}^n$ is a curve obtained from a projection of a rational normal curve $C \subset \mathbb{P}^{n+1}$ from a point $O \subset \mathbb{P}^{n+1}$, we are able to describe the precise value of the X-rank for those points $P \in \mathbb{P}^n$ such that $R_X(P) \leq R_C(O) - 1$ and to improve the general result. Moreover we give a stratification, via the X-rank, of the osculating spaces to projective cuspidal projective curves X. Finally we give a description and a new bound of the X-rank of subspaces both in the general case and with respect to integral non-degenerate projective curves.

INTRODUCTION

The subject of this paper is the so called "X-rank" with respect to an integral, projective, non-degenerate variety $X \subset \mathbb{P}^n$, where $\mathbb{P}^n$ is the n-dimensional projective space defined over an algebraically closed field K of characteristic 0.

The notion of X-rank (see Definition 1) arises naturally in applications which want to minimize the number of elements belonging to certain projective variety $X \subset \mathbb{P}^n$ needed to give, with a linear combination of them, a fixed element of $< X >= \mathbb{P}^n$. When the ambient space $\mathbb{P}^n$ is a space of tensors $\mathbb{P}(V_1 \otimes \cdots \otimes V_t)$, with $V_1, \ldots, V_t$ vector spaces, and the variety X parameterizes "completely decomposable tensors", then X is the variety classically known as Segre Variety of t factors; here the notion of X-rank of a tensor coincides with its "rank" which is subject of study of many applications like Computational Biology (eg. [24]), Optimization Problems (eg. [26], [28]), Signal Processing for Telecommunications (eg. [3], [12]), Data Analysis (see [9]).
If $X \subset \mathbb{P}^{\binom{n+d}{d}-1}$ is the n-dimensional d-th Veronese variety that parameterizes symmetric tensors of rank 1, then the X-rank of an element $T \in \mathbb{P}^{\binom{n+d}{d}-1}$ is also called "symmetric rank of T" and it is the minimum number of symmetric completely decomposable tensors of rank one $T_1, \ldots, T_s \in X$ such that $T = T_1 + \cdots + T_s$. For example, Independent Component Analysis (see [16], [11]) was originally introduced for symmetric tensors whose symmetric rank did not exceed dimension (now, it is actually possible to estimate more factors than the dimension, see [20], [21]). More generally, if $\mathbb{P}^n =< X >$ is a space of tensors and the variety X parameterizes tensors of certain structure, then the notion of X-rank has actually a physical

1991 *Mathematics Subject Classification.* Primary: 14N05; Secondary: 14H50.
Key words and phrases. Rank, Secant Varieties, Rational Normal Curves, Projections.
The authors were partially supported by CIRM - FBK (TN - Italy), MIUR and GNSAGA of INdAM (Italy).

meaning and it is also called "structured rank" (this is related to the virtual array concept [2] encountered in sensor array processing). In fact in some applications, tensors may be symmetric only in some modes ([6], [19]), or may not be symmetric nor have equal dimensions ([5], [9]).

From a pure geometrical point of view the notion of the X-rank of an element is not the most natural one to consider since the set $\Sigma_s^0 \subset < X >$ (see Notation 1) parameterizing elements in $< X >$ whose X-rank is precisely s, is not a closed variety. The Zariski closure of it (see also Definition 2) is a projective variety $\sigma_s(X) \subset < X >$ called the s-th secant variety of X. The generic element of $\sigma_s(X)$ has X-rank equal to s and we will say that if $P \in \sigma_s(X)$ then P has "border rank" s (see Definition 5).
A very classical algebraic problem, inspired by a number theory problem posed by Waring in 1770 (see [30]), asks which is the minimum integer s such that a generic element in $K[x_0, \ldots, x_n]_d$ can be written as a sum of s d-th powers of linear forms. This problem is known as the Big Waring Problem and it is equivalent to determining the minimum integer s such that $\sigma_s(X) = \mathbb{P}^{\binom{n+d}{d}-1}$ with X being the d-th Veronese embedding of $\mathbb{P}^n$ (that is also to ask which is the typical rank of X - see Definition 6). We point out this example because the X-typical rank of a projective variety X will play a crucial role in the results of our paper and also because the case of Veronese varieties is the only one case in which the dimensions of their secant varieties are known for all n and d (see [4] for the original proof and [13] and [8] for modern once). Despite the complete knowledge on the dimensions of secant varieties of Veronese varieties, it is not known yet how to determine the border rank of a symmetric tensor: a classification of the equations of the higher secant varieties of Veronese varieties is still an open problem (partially results can be found in [22]). Some algorithms that give the symmetric rank of certain kind of symmetric tensors are actually known (see [27], [15] for the case of X a rational normal curve and [7], [10] and [17] for more general cases). The recent paper [10] analyzes, among the other topics, the perspective of studying the X-ranks of points of a given border rank with respect to a variety X that is the Segre embedding of $\mathbb{P}(A) \times Y$ to $\mathbb{P}(A \otimes W)$, where $Y \subset \mathbb{P}(W)$ and A, W being vector spaces; the new tool introduced in that paper to approach that problem is a generalized notion of rank and border rank to linear subspaces (Sections 3 and 7 of [10]). In our paper (Section 4) we give a contribution to that new tool and in Propositions 3 and 4 we present general results for the X-rank of subspaces with respect to any complex, projective integral and non-degenerate variety $X \subset \mathbb{P}^n$. In the particular case in which X is a curve and the subspace is a line, it is possible to give a more precise result (Theorem 2).

The knowledge of the X-rank with respect to a variety X parameterizing certain kind of tensors is studied also in several recent papers ([15], [10], [23]) for very special varieties X. Among the older papers we point out the examples of smooth space curves X with points of X-rank 3 listed in [25].

On our knowledge the only result that is nowadays known on the X-rank and that holds for any complex, projective, integral and non-degenerate m-dimensional variety $X \subset \mathbb{P}^n$ is due Landsberg and Teitler (see [23], Proposition 5.1): for all $P \in \mathbb{P}^n$:

$$R_X(P) \leq n + 1 - m.$$

In our paper, after the preliminary Section 1, we will refine that bound in the case of $X \subset \mathbb{P}^n$ being a projection of a projective variety $Y \subset \mathbb{P}^{n+v}$ from a linear space of dimension v. The most general result that we present there is the

Theorem 1: Let $Y \subset \mathbb{P}^{n+v}$ be an integral and non-degenerate variety. Let $X_V \subset \mathbb{P}^n$ be the linear projection of Y from a general $(v-1)$-dimensional linear subspace $V \subset \mathbb{P}^{n+v}$. Then for all $P \in \mathbb{P}^n$ we have that

$$R_{X_V}(P) \leq \alpha_Y,$$

where α_Y is the Y-typical rank defined in Definition 6.

Section 2 is entirely devoted to the proof of that theorem, and some example are given.

In Section 3 we can be more precise and realize the bound of Theorem 1 as an equality for the particular case of $X \subset \mathbb{P}^n$ being a projection of a rational normal curve $Y \subset \mathbb{P}^{n+1}$(that can actually be seen as the variety parameterizing either homogeneous polynomials of degree $n+1$ in two variables, or symmetric tensors in $\mathbb{P}(S^nV)$ with the dimension of the vector space V equal to 2) from a point $O \in \mathbb{P}^{n+1}$. We will prove the two following results:

Lemma 2: Let $X \subset \mathbb{P}^n$ be the linear projection of a rational normal curve $C \subset \mathbb{P}^{n+1}$ from a point $O \in \mathbb{P}^{n+1} \setminus C$ (as in (3.2)). If $P \in \mathbb{P}^n$, define $L_P \subset \mathbb{P}^{n+1}$ to be the line $L_P := < O, P >$. Fix $P \in \mathbb{P}^n$ a point such that $R_X(P) \leq R_C(O) - 1$. Then

$$R_X(P) = \min_{A \in L_P \setminus \{O\}} R_C(A).$$

Proposition 1: Fix $P \in \mathbb{P}^n$ and take a zero-dimensional scheme $Z \subset X$ with minimal length such that $P \in \langle Z \rangle$. If X is singular, then assume that Z_{red} does not contain the singular point of X. Set $z := \mathrm{length}(Z)$. If Z is reduced, then $R_X(P) = z$ by definition of X-rank. If Z is not reduced, then $R_X(P) \leq n+3-z$.

Moreover if $X \subset \mathbb{P}^n$ is a cuspidal curve obtained as a projection of a rational normal curve $C \subset \mathbb{P}^{n+1}$, it is also possible to give a stratification of the elements belonging to the osculating spaces to X via the X-rank. The result is the following:

Proposition 2: Let $X \subset \mathbb{P}^n$, $n \geq 3$, be a non-degenerate integral curve such that $\deg(X) = n+1$ and X has a cusp in $Q \in X$. Let $C \subset \mathbb{P}^{n+1}$ be the rational normal curve such that $\ell_O(C) = X$ for $O \in T_{Q'}(C)$ and $Q' \in C$. Moreover let $E_Q(t) \subset \mathbb{P}^n$ be the image by ℓ_O of the t-dimensional osculating space to C in Q' as defined in Definition 4, i.e. $E_Q(t) = \langle (t+1)Q' \rangle_C$. Then

$$R_X(P) = n+2-t$$

for all $P \in E_Q(t) \setminus E_Q(t-1)$ and each point of $E_Q(2) \setminus \{Q\}$ has X-rank n.

1. Preliminaries

Let $X \subset \mathbb{P}^n$ be an integral, projective, non-degenerate variety defined over an algebraically closed field K of characteristic 0. We define the X-rank of $P \in \mathbb{P}^n$ as follows:

Definition 1. Let $P \in \mathbb{P}^n$, the X-rank of P is the minimum integer $R_X(P)$ for which there exist R_X distinct points $P_1, \ldots, P_{R_X} \in X$ such that $P \in < P_1, \ldots, P_{R_X} >$.

Notation 1. We indicate with $\sigma_s^0(X)$ the set of points of $\mathbb{P}^n$ whose X-rank is at most s, and with $\Sigma_s^0(X)$ the set of points of $\mathbb{P}^n$ whose X-rank is actually s, i.e $\Sigma_s^0 := \sigma_s^0(X) \setminus \sigma_{s-1}^0(X)$.

Definition 2. The s-th secant variety $\sigma_s(X) \subset \mathbb{P}^n$ of X is the Zariski closure of $\sigma_s^0(X)$, i.e.

$$\sigma_s(X) = \overline{\bigcup_{P_1,\ldots,P_s \in X} < P_1, \ldots, P_s >}.$$

Remark 1. We observe that $\sigma_1(X) = X$ and also that $\sigma_{s-1}(X) \subset \sigma_s(X)$.

Definition 3. We indicate with $\tau(X)$ the tangential variety of a variety X. Let $T_P^*(X)$ be the Zariski closure of $\cup_{y(t),z(t) \in X y(0)=z(0)=P} \lim_{t \to 0} < y(t), z(t) >$ and then define $\tau(X) = \cup_{P \in X} T_{P(X)}^*$.

Remark 2. Clearly $\tau(X) \subset \sigma_2(X)$.

Definition 4. Let $X \subset \mathbb{P}^n$ be a variety, and let $P \in X$ be a smooth point; we define the k-th osculating space to X at P as the linear space generated by $(k+1)P \cap X$ (i.e. by the k-th infinitesimal neighbourhood of P in X) and we denote it by $< (k+1)P >_X$; hence $< P >_X = P$, and $< 2P >_X = T_P(X)$, the projectivised tangent space to X at P.

Definition 5. If $P \in \sigma_s(X) \setminus \sigma_{s-1}(X)$ we say that P has X-border rank s, and we write $\underline{R_X}(P) = s$.

Remark 3. Observe that $\underline{R_X}(P) \leq R_X(P)$.

Definition 6. The minimum integer α_X such that $\sigma_{\alpha_X}(X) = \mathbb{P}^n$ is called the X-typical rank. The minimum integer α'_X such that $\sigma_{\alpha'_X}(X)$ has non-empty interior is called the X-generic rank.

Remark 4. It is not worthless to point out that those two definitions coincides in the case of $\mathrm{char}(K) = 0$ and $\overline{K} = K$, and moreover that both the X-typical rank and the X-generic rank are uniquely determined (the same is not true if K is not algebraically closed or not of characteristic zero - see e.g. [18]).

We can also define the X-rank of a subspace of $\mathbb{P}^n$.

Notation 2. Let $G(v-1, n+v)$ denote the Grassmannian of all $(v-1)$-dimensional projective linear subspaces of $\mathbb{P}^{n+v}$.

Definition 7. Let $V \subseteq \mathbb{P}^n$ be a non-empty linear subspace. The X-rank $R_X(V)$ of V is the minimal cardinality of a finite set $S \subset X$ such that $V \subseteq < S >$.

Remark 5. Obviously $\dim(V)+1 \leq R_X(V) \leq n+1$ for any $V \subset \mathbb{P}^n$ and $R_X(\mathbb{P}^n) = n+1$.

2. X-rank with respect to linear projections of projective varieties

Let $X \subset \mathbb{P}^n$ be an irreducible variety of dimension m not contained in a hyperplane. On our knowledge, the only known general result on a bound for the X-rank in the general case is due to Landsberg and Teitler (see [23], Proposition 5.1) who proved that

$$R_X(P) \le n + 1 - m \tag{2.1}$$

for all $P \in \mathbb{P}^n$. In this section we restrict to the case of the a variety $X_V \subset \mathbb{P}^n$ obtained from the following construction.

Let $Y \subset \mathbb{P}^{n+v}$ be an m-dimensional projective variety and let $V \subset \mathbb{P}^{n+v}$ be a general $(v-1)$-dimensional projective linear subspace of $\mathbb{P}^{n+v}$. Let $V \subset \mathbb{P}^{n+v}$ be a v-dimensional projective linear subspace of $\mathbb{P}^{n+v}$. Consider the linear projection ℓ_V of $\mathbb{P}^{n+v}$ onto $\mathbb{P}^n$ from V, i.e.

$$\ell_V : \mathbb{P}^{n+v} \setminus V \to \mathbb{P}^n, \tag{2.2}$$

and define the variety $X_V \subset \mathbb{P}^n$ to be the projective variety obtained as $\ell_V(Y)$:

$$X_V := \ell_V(Y) \subset \mathbb{P}^n. \tag{2.3}$$

We prove that, for a general $V \subset \mathbb{P}^{n+v}$, another stronger upper bound for $R_{X_V}(P)$, with $P \in \mathbb{P}^n$ can be given.

Theorem 1. *Let $Y \subset \mathbb{P}^{n+v}$ be an integral and non-degenerate variety. Let $X_V \subset \mathbb{P}^n$ be the linear projection of Y from a general $(v-1)$-dimensional linear subspace $V \subset \mathbb{P}^{n+v}$. Then for all $P \in \mathbb{P}^n$*

$$R_{X_V}(P) \le \alpha_Y$$

where α_Y is the Y-typical rank defined in Definition 6.

We point out the following examples in which the integer α_Y is known. Hence each of these examples gives a corollary of Theorem 1 which the interested reader may quote.

Example 1. If $m = 1$, i.e. if $Y \subset \mathbb{P}^{n+v}$ is an integral and non-degenerate curve, we have that

$$\alpha_Y := \lfloor (n + v + 2)/2 \rfloor$$

(see [1], Remark 1.6). Here the general linear projection induces an isomomorphism $\alpha : Y \to X$ such that $\alpha^*(\mathcal{O}_X(1)) \cong \mathcal{O}_Y(1)$.

Example 2. Fix integers $m > 0$ and $d > 0$. Let $Y = Y_{m,d} \subset \mathbb{P}^{\binom{m+d}{m}-1}$ be the d-th Veronese embedding of $\mathbb{P}^m$. We have

$$\alpha_{Y_{m,d}} \le \left\lceil \binom{m+d}{m} / (m+1) \right\rceil$$

and equality holds, except in few exceptional cases listed in [4], [13], [8]. This is a deep theorem by J. Alexander and A. Hirschowitz (they proved it in [4]; another proof can be found in [13]; see [8] for a recent reformulation of it). Here the general linear projection $\ell_V(Y) = X_V$ is isomorphic to Y if $\dim(\sigma_2(Y)) \le n$, i.e. if $2m + 1 \le n$.

Example 3. Let $Y \subset \mathbb{P}^{n+v}$ be an integral and non-degenerate subvariety of dimension m. Let $b := \dim(\mathrm{Sing}(Y))$ with the convention $b = -1$ if Y is smooth. Assume $m > (2n + 2v + b)/3 - 1$. Then $\alpha_Y = 2$ (see [29], Theorem II.2.8). the general linear projection $\ell_V(Y) = X_V$ of Y is birational to Y, but not isomorphic to it.

2.1. **The proof.** Lel $G(v-1, n+v)$ denote the Grassmannian of all $(v-1)$-dimensional linear subspaces of $\mathbb{P}^{n+v}$ as in Notation 2. For any $V \in G(v-1, n+v)$ let $\ell_V : \mathbb{P}^{n+v} \backslash V \to \mathbb{P}^n = M$ denote the linear projection from V as in (2.2). For any $P \in M$, set $V_P := \langle \{P\} \cup V \rangle \subset \mathbb{P}^{n+v}$. If $V \cap M = \emptyset$ and $P \in M$, then $\dim(V_P) = v$ and we identify the fiber $\ell_V^{-1}(P) \subset \mathbb{P}^{n+v} \backslash V$ with $V_P \backslash V$. Now we fix an integral and non-degenerate variety $Y \subset \mathbb{P}^{n+v}$ and let $X_V := \ell_V(Y)$ be as in (2.3). In the statement of Theorem 1 we only need the linear projection from a general $V \in G(v-1, n+v)$. We consider only linear projections ℓ_V from $V \in G(v-1, n+v)$ such that $V \cap Y = \emptyset$. For these subspaces $\ell_V|Y$ is a finite morphism.

Lemma 1. *Fix $P \in \mathbb{P}^n$ and let $X_V = \ell_V(Y) \subset \mathbb{P}^n$ as above. Then*

$$R_X(P) \le \min_{Q \in V_P \backslash V} R_Y(Q). \tag{2.4}$$

where V and $V_P \subset \mathbb{P}^{n+v}$ are as above.

Proof. Fix $Q \in V_P \backslash V$ and $S \subset Y$ computing $R_Y(Q)$. Thus $Q \in \langle S \rangle$ and $\sharp(S) = R_Y(Q)$. Since $S \subset Y$ and $Y \cap V = \emptyset$, the linear projection ℓ_V is defined at each point of S. Thus $\ell_V(S)$ is a finite subset of $\ell_V(Y) = X$ and $\sharp(\ell_V(S)) \le \sharp(S)$. Since $Q \notin V$, $Q \in \langle S \rangle$ and $\ell_V(Q) = P$, we have $P \in \langle \ell_V(S) \rangle$. Thus $R_X(P) \le \sharp(\ell_V(S)) \le R_Y(Q)$ for all $Q \in V_P \backslash V$. □

Proof of Theorem 1. By definition $\sigma_{\alpha_Y}(Y) = \mathbb{P}^{n+v}$, and there exists a non-empty open subset $U \subset \mathbb{P}^{n+v} \backslash \sigma_{\alpha_Y - 1}(Y)$ such that and $R_Y(Q) = \alpha_Y$ for all $Q \in U$. Take as V any element of $G(v-1, n+v)$ such that $V \cap Y = \emptyset$, $V \cap M = \emptyset$ and $V \cap U \neq \emptyset$ (the space $M = \mathbb{P}^n$ is as above the image of $\mathbb{P}^{n+v}$ via ℓ_V). Thus $V \cap U$ is a non-empty open subset of V. Thus $V_P \cap U$ is a non-empty open subset of V_P for all $P \in M$. Thus $\min_{Q \in V_P \backslash V} R_Y(Q) \le \alpha_Y$ for all $P \in M$ by applying Lemma 1. □

Remark 6. We stress that in Theorem 1 we may make less vague the words "general $V \in G(v-1, n+v)$" and say "take any $V \in G(v-1, n+v)$ such that $V \cap Y = \emptyset$ and $V \cap U \neq \emptyset$ ", where U is the non-empty open subset of $\mathbb{P}^{n+v}$ such that $U \subset \mathbb{P}^{n+v} \setminus \sigma_{\alpha_Y - 1}(Y)$ and $R_Y(Q) = \alpha_Y$ for all $Q \in U$. Moreover, it is sufficient first to know the integer α_Y and then find a point $O \in \mathbb{P}^{n+v}$ such that $R_Y(O) = \alpha_Y$ (just one point with this "general" Y-rank !); then we may take any $V \in G(v-1, n+v)$ such that $V \cap Y = \emptyset$ and $O \in V$.

3. The X-rank with respect to projections of rational normal curves

In this section we want to apply the results of the previous one to the particular case in which $Y = C \subset \mathbb{P}^{n+1}$ is a rational normal curve and the subspace $V \subset \mathbb{P}^{n+1}$ is a point $O \in \mathbb{P}^{n+1} \setminus Y$. The linear projection (2.2) becomes:

$$\ell_O : \mathbb{P}^{n+1} \to \mathbb{P}^n. \tag{3.1}$$

Each point $P \in \mathbb{P}^n$ corresponds to a line $L_P := \{O\} \cup \ell_O^{-1}(P)$, and each line L through O intersects $\mathbb{P}^n$ in a unique point. Now

$$X := \ell_O(C) \subset \mathbb{P}^n. \tag{3.2}$$

Remark 7. Since the center of the projection $O \notin C$, the curve $X \subset \mathbb{P}^n$ turns out to be an integral and non-degenerate subcurve of $\mathbb{P}^n$ of degree $\deg(X) = n+1$ and $\ell_O|C \to X$ is the normalization map.

Remark 8. The curve X is smooth if and only if $O \notin \sigma_2(C)$. If $O \in \sigma_2(C)$, the arithmetic genus $p_a(X)$ of X is $p_a(X) = 1$. Moreover if $O \in \sigma_2(C)\backslash\tau(C)$ (where $\tau(C)$ is the tangential variety of C defined in Definition 3) than X has an ordinary node (and in this case $R_C(O) = 2$). If $O \in \tau(C)$ then X has and an ordinary cusp (in this case if $R_C(O) = n+1$, see [27], [15], [17] and [7]).

Remark 9. Fix $P \in \mathbb{P}^n$ and let $S \subset X \subset \mathbb{P}^n$ be a finite subset computing $R_X(P)$, i.e. $\sharp(S) = R_X(P)$ and $P \in \langle S \rangle$. The set $S' \subset C \subset \mathbb{P}^{n+1}$ such that $\sharp(S') = \sharp(S)$ and $\ell_O(S') = S$ is uniquely determined by S, unless $X \subset \mathbb{P}^n$ is nodal and the node belongs to S. If the node belongs to S, then S' is uniquely determined if the preimages of the node are prescribed.

If $O \in \sigma_2(C)\backslash\tau(C)$, call $Q \in X$ the singular point of X and call $Q', Q'' \in C$ the points of C mapped onto Q by ℓ_O. Since $P \in \langle S \rangle$ then $L_P \cap \langle S' \rangle \neq \emptyset$. Since ℓ_O is a linear projection, $\ell_O|S'$ is injective and $\ell_O(S')$ is linearly independent, S' is linearly independent and $O \notin L_P \cap \langle S' \rangle \neq \emptyset$. Thus $L_P \cap \langle S' \rangle$ is a unique point $P_S \in \mathbb{P}^{n+1}$ and $P_S \neq O$. Conversely, if we take any linearly independent $S_1 \subset C$ (with the restriction that if X is nodal, then $Q'' \notin S_1$) and $O \notin \langle S_1 \rangle$, then $\ell_O|S_1$ is injective and $\ell_O(S_1)$ is linearly independent. Hence S' computes $R_C(P_S)$ unless $R_C(P_S)$ is computed only by subsets whose linear span contains O or, in the nodal case, by subsets containing Q''. Thus, except in these cases, $R_X(P) = R_C(P_S)$. In the latter case for a fixed P and S we could exchange the role of Q' and Q'', but still we do not obtain in this way the rank.

Lemma 2. *Let $X \subset \mathbb{P}^n$ be the linear projection of a rational normal curve $C \subset \mathbb{P}^{n+1}$ from a point $O \in \mathbb{P}^{n+1} \setminus C$ (as in (3.2)). If $P \in \mathbb{P}^n$, define $L_P \subset \mathbb{P}^{n+1}$ to be the line $L_P := < O, P >$. Fix $P \in \mathbb{P}^n$ a point such that $R_X(P) \leq R_C(O) - 1$. Then*

$$R_X(P) = \min_{A \in L_P \backslash \{O\}} R_C(A). \tag{3.3}$$

Proof. Let $A \in L_P\backslash\{O\}$ and take $S_A \subset C$ computing $R_C(A)$. Now $P \in < \ell_O(S_A) >$ for all $A \in L_P \setminus \{O\}$, then $R_X(P) \leq \min_{A \in L_P \backslash \{O\}} R_C(A)$. The other inequality is done above in Remark 9. □

Remark 9, together with the knowledge of the C-ranks of a rational normal curve (see [7] Theorem 3.8) immediately give the following result.

Proposition 1. *Fix $P \in \mathbb{P}^n$ and take a zero-dimensional scheme $Z \subset X$ with minimal length such that $P \in \langle Z \rangle$. If X is singular, then assume that Z_{red} does not contain the singular point of X. Set $z :=$ length(Z). If Z is reduced, then $R_X(P) = z$ by definition of X-rank. If Z is not reduced, then $R_X(P) \leq n+3-z$.*

Proposition 2. *Let $X \subset \mathbb{P}^n$, $n \geq 3$, be a non-degenerate integral curve such that $\deg(X) = n+1$ and X has a cusp in $Q \in X$. Let $C \subset \mathbb{P}^{n+1}$ be the rational normal curve such that $\ell_O(C) = X$ for $O \in T_{Q'}(C)$ and $Q' \in C$. Moreover let $E_Q(t) \subset \mathbb{P}^n$ be the image by ℓ_O of the t-dimensional osculating space to C in Q' as defined in Definition 4, i.e. $E_Q(t) = \langle (t+1)Q' \rangle_C$. Then*

$$R_X(P) = n + 2 - t$$

for all $P \in E_Q(t)\backslash E_Q(t-1)$ and each point of $E_Q(2) \setminus \{Q\}$ has X-rank n.

Proof. First of all observe that the definition of $X \subset \mathbb{P}^n$ implies that $p_a(X) = 1$, the point Q is an ordinary cusp and there is a rational normal curve $C \subset \mathbb{P}^{n+1}$ such that for $Q' \in C$ and $O \in T_{Q'}C \backslash \{Q'\}$ the curve X turns out to be $X = \ell_O(C)$. Remark that $\dim(E_Q(t)) = t-1$ and $E_Q(1) = \{Q\}$. The line $E_Q(2)$ is the reduction of the tangent cone of X at Q.
Now fix an integer $t \geq 2$. Since $R_C(O) = n+1$ (see e.g. Thoerem 3.8 in [7]) and $R_C(P') = n+2-t$ for all $P' \in \langle (t+1)Q' \rangle \setminus \langle tQ' \rangle$, the Teorem 3.13 in [7] gives that $R_X(P) = n+2-t$ for all $P \in E_Q(t) \backslash E_Q(t-1)$. In particular each point of $E_Q(2) \setminus \{Q\}$ has X-rank n. □

4. X-rank of subspaces

In this section we study the X-rank of subspaces as we defined it in Definition 7 with respect to any integral, non-degenerate projective variety $X \subset \mathbb{P}^n$ and we will get the bound (4.4). Then we will discuss the case of the X-rank of lines with respect to a curve $X \subset \mathbb{P}^n$ for $n \geq 4$ in which we can give a precise statement.

Proposition 3. *Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate m-dimensional subvariety. Let $V \subset \mathbb{P}^n$ be a projective linear subspace such that $V \cap X = \emptyset$. Then*

$$R_X(V) \leq n+1-m$$

where $R_X(V)$ is defined as in Definition 7.

Proof. Since $V \cap X = \emptyset$, the linear system Γ cut out on X by the set of all hyperplanes containing V has no base points. Hence, by Bertini's theorem, if $H \in \Gamma$ is general, the scheme $X \cap H$ is reduced and of pure dimension $m-1$; moreover if $m \geq 2$ then $X \cap H$ is also an integral scheme. Since X is connected, the exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{I}_X(1) \to \mathcal{I}_{X \cap H}(1) \to 0 \tag{4.1}$$

shows that $X \cap H$ spans H.
To get the case $m = 1$ it is sufficient to take any $S \subset X \cap H$ with $\sharp(S) = n$ and spanning H.
Now we can proceed by induction on m and n. Assume first that $m' := 1$ and $n' := n-m+1$ and get the statement when X is a curve. Now assume for the induction procedures for $m \geq 2$ that the statement is true for $(m-1)$-dimensional varieties in $\mathbb{P}^{n-1}$, and use (4.1) to show that the proposition is true also for $\dim(X) = m$ and $X \subset \mathbb{P}^n$. Hece we can take $n+1-m$ of the $\deg(X)$ points of $X \cap H$ spanning $H = \mathbb{P}^{n'-1}$ and conclude. □

Now we want to study the X-rank of a line $L \subset \mathbb{P}^n$ with respect to an integral and non-degenerate curve $X \subset \mathbb{P}^n$.

Consider the following constructions. Let

$$\ell_Q : \mathbb{P}^n \setminus Q \to \mathbb{P}^{n-1} \tag{4.2}$$

be the linear projection of $\mathbb{P}^n$ onto $\mathbb{P}^{n-1}$ from a point $Q \in X$ and call $C_Q \subset \mathbb{P}^{n-1}$ the closure in $\mathbb{P}^{n-1}$ of the integral curve $\ell_Q(X \setminus \{Q\})$.

Analogously let

$$\ell_L : \mathbb{P}^n \setminus L \to \mathbb{P}^{n-2} \tag{4.3}$$

be the linear projection of $\mathbb{P}^n$ onto $\mathbb{P}^{n-2}$ from a line $L \subset \mathbb{P}^n$ and call $C_L \subset \mathbb{P}^{n-2}$ the closure in $\mathbb{P}^{n-2}$ of the integral curve $\ell_L(X \setminus (X \cap L))$.

Theorem 2. *Let $X \subset \mathbb{P}^n$, for $n \geq 4$, be an integral non-degenerate curve of degree d and $L \subset \mathbb{P}^n$ be a line. Let ℓ_Q and ℓ_L be the linear projections defined in (4.2) and (4.3) respectively such that $b_L := \deg(\ell_L|_{X\backslash\{L\}})$and $b_Q := \deg(\ell_Q|_{X\backslash\{Q\}})$, and let $C_Q := \overline{\ell_Q(X)} \subset \mathbb{P}^{n-1}$ and $C_L := \overline{\ell_L(X)} \subset \mathbb{P}^{n-2}$ as above. Then*

(1) *If $L \cap X = \emptyset$, then $R_X(L) \leq n$.*
(2) *If $L \cap X \neq \emptyset$, then $\sharp((X \cap L)_{red}) \geq 2$ if and only if $R_X(L) = 2$. If $\sharp((X \cap L)_{red}) = \{Q\}$ then*

$$lenght(L \cap X) + b_L \cdot \deg(C_L) = d = m_X(Q) + b_Q \cdot \deg(C_Q)$$

where with $m_X(Q)$ we indicate the multiplicity of X at Q. Moreover
(a) *if $R_X(L) = n$, then C_L is a rational normal curve and $b_Q = b_L$.*
(b) *if C_L is a rational normal curve and $b_L = 1$, then $R_X(L) = n$.*

Proof. Part (1) is a consequence of Proposition 3 applied for $m = \dim(V) = 1$.

Part (2) for the case $\sharp((X\cap L)_{red}) \geq 2$ is obvious. Assume therefore $(X\cap L)_{red} = \{Q\}$. Since we are in characteristic zero, the X-rank of a point $O \in \mathbb{P}^n$ is $R_X(O) \leq n$ for all $O \in \mathbb{P}^n$ (see [23], Proposition 4.1). Hence if $P \in L\backslash\{Q\}$, we get that $L =< P, Q >$ and then clearly $R_X(L) \leq n + 1$.

Here we prove part (2a). First assume that C_L is not a rational normal curve. Since $n \geq 4$, there is a finite set of points $A \subset C_L$ such that $a := \sharp(A) \leq n$ and $\dim(< A >) = a - 2$. Let $A' \subset X\backslash\{Q\}$ such that $\sharp(A') = a$ and $\ell_L(A') = A$. Since the points of A are linearly dependent, the definition of ℓ_L implies $L \subseteq< \{Q\}\cup A' >$. Hence $R_X(L) \leq a$.
Now assume that C_L is a rational normal curve and that $b_Q < b_L$. Hence there are $A_1, A_2 \in X\backslash\{Q\}$ such that $\ell_L(A_1) = \ell_L(A_2)$ and $\ell_Q(A_1) \neq \ell_Q(A_2)$. Since $Q \notin \{A_1, A_2\}$ and $\ell_Q(A_1) \neq \ell_Q(A_2)$, the subspace $< \{Q, A_1, A_2\} >$ is actually a plane. But $\ell_L(A_1) = \ell_L(A_2)$, hence $P \in< \{Q, A_1, A_2\} >$ for any $P \in L$. Hence $R_X(L) \leq 3 < n$.

Here we prove part (2b). Since $R_X(L) \leq n+1$, it is sufficient to prove $R_X(L) \geq n + 1$. Let $S \subset X$ be a subset of points of X computing $R_X(L)$ (i.e. S is a minimal set of points of X such that $L \subset< S >$). Clearly Q has to belong to S. Set $S' := \ell_L(S\backslash\{Q\})$ and take $S'' \subseteq S\backslash\{Q\}$ such that $S' = \ell_L(S'')$ and $\sharp(S'') = \sharp(S')$. Since $L \subseteq< S >$, we get that $\dim(< S' >) = \dim(< S >) - 2$. Moreover $L \cap X = \{Q\}$ implies that the minimality of S gives $S'' = S\backslash\{Q\}$. Now, since $\dim(< S' >) = \dim(< S >) - 2$, while $\sharp(S') = \sharp(S'') = \sharp(S) - 1$, the set $\ell_L(S')$ is linearly dependent in $\mathbb{P}^{n-2}$. Since C_L is a rational normal curve of $\mathbb{P}^{n-2}$ and $S' \subset C_L$, the linear dependence of S' implies $\sharp(S') \geq n$. Hence $R_X(P) \geq n$ for any $P \in X$ and then $R_X(L) \geq n + 1$. □

Proposition 4. *Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate m-dimensional subvariety and let $V \subset \mathbb{P}^n$ be a linear subspace. Then*

$$R_X(V) \leq n + 2 - m + \dim(< (X \cap V)_{red} >). \tag{4.4}$$

Proof. Let $A \subset (X \cap V)_{red}$ be a finite set of points such that $\sharp(A) = s + 1$ and $< A >=< (X\cap V)_{red} >$. Let $N \subset V$ be a complementary subspace of $< (X\cap V)_{red} >$, i.e. a linear subspacesuch that $N\cap < (X \cap V)_{red} >= \emptyset$ and $< N, < (X \cap V)_{red} >= V$. Proposition 2 assure the existence of a finite subset of points $B \subset X$ such that $\sharp(B) \leq n + 1 - m$ and $N \subseteq< B >$. Hence we have $V \subseteq< A \cup B >$. □

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