OPTIMAL HOMEOMORPHISMS BETWEEN CLOSED CURVES

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Abstract. The concept of natural pseudo-distance has proven to be a powerful tool for measuring the dissimilarity between topological spaces endowed with continuous real-valued functions. Roughly speaking, the natural pseudo-distance is defined as the infimum of the change of the functions' values, when moving from one space to the other through homeomorphisms, if possible. In this paper, we prove the first available result about the existence of optimal homeomorphisms between closed curves, i.e. inducing a change of the function that equals the natural pseudo-distance.

Introduction

Formalizing the concept of shape for topological spaces and manifolds, as well as providing an efficient comparison of shapes, has been a widely researched topic in the last decade. As such, a class of methods has been developed with the purpose of performing a topological exploration of the shape, according to some quantitative geometric properties provided by a real function chosen to extract shape features [1, 3, 18, 20, 25].

In this context, Size Theory was introduced at the beginning of the 1990s [12, 13, 15], supported by the adoption of a suitable mathematical tool: the *natural pseudo-distance* [7, 8, 10].

In the formalism of Size Theory, a shape is modelled as a pair (X, φ), where X is a topological space and $\varphi : X \to \mathbb{R}$ is a continuous function [1, 15]. Such a pair is called a *size pair* and φ is called a *measuring function*. The role of φ is to take into account only the properties considered relevant for the shape comparison problem at hand, while disregarding the irrelevant ones, as well as to impose the desired invariance properties.

The natural pseudo-distance is a measure of the dissimilarity between two size pairs (X, φ), (Y, ψ). Roughly speaking, it is defined as the infimum of the variation of the values of φ and ψ, when we move from X to Y through homeomorphisms, if possible (see Definition 1.2). Therefore, two objects have the same shape if they share the same shape properties, expressed by the measuring functions' values, that is, their natural pseudo-distance vanishes.

Earlier results about the natural pseudo-distance can be divided into two classes. One class provides constraints on the possible values taken by the natural pseudo-distance between two size pairs (X, φ), (Y, ψ). For example, if the considered topological spaces X and Y are smooth closed manifolds and the measuring functions are also smooth, then the natural pseudo-distance is an integer sub-multiple of the

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Euclidean distance between two suitable critical values of the measuring functions [8]. In particular, this integer can only be either 1 or 2 in the case of curves [7], while it cannot be greater than 3 in the case of surfaces [10].

The other class of results furnishes lower bounds for the natural pseudo-distance [9, 17]. In particular it is possible to estimate the natural pseudo-distance by using the concept of *size function* [5, 9]. Size functions are shape descriptors able to reduce the comparison of shapes to the comparison of certain countable sets of points in the real plane [11, 14, 16]. This reduction allows us to study the space of all homeomorphisms between the considered topological spaces, without actually computing them.

The research on size functions has led to a formal setting, which has turned out to be useful, not only from a theoretical point of view, but also on the applicative side (see, e.g., [2, 4, 6, 23, 24]).

Besides being a useful theoretical tool for applications in shape comparison, the natural pseudo-distance is challenging from the mathematical point of view, and several questions about its properties need for further investigation. One among them consists in establishing the hypotheses ensuring the existence of *optimal homeomorphisms* between size pairs, i.e. homeomorphisms realizing the natural pseudo-distance. It is possible to show that, in general, such homeomorphisms do not exist (see, e.g., Section 2).

In this paper, we provide the first available result about the existence of optimal homeomorphisms. To be more precise, we prove that, under appropriate conditions, it is always possible to construct a homeomorphism between two closed curves (i.e. compact and without boundary 1-manifolds), satisfying the property of optimality (Theorem 3.5). This result can be seen as a necessary first step towards the study of this problem in a more general setting, e.g. when manifolds of arbitrary dimensions are involved.

The subject of our work fits in the current mathematical research and interest in simple closed curves, motivated by problems concerning shape comparison in Computer Vision (cf. e.g., [21, 22]).

The paper is divided into three sections. Section 1 deals with some of the standard facts on the comparison of size pairs via the natural pseudo-distance. In particular, the definition of the natural pseudo-distance δ and its main properties are given, focusing on the concept of d-approximating sequence. Section 2 is devoted to the description of some simple and meaningful examples showing that none of the conditions we require in stating our main result can be dropped. In Section 3 we prove our main result concerning the existence and the construction of an optimal homeomorphism between two smooth closed curves endowed with Morse measuring functions (Theorem 3.5).

1. Preliminaries

In Size Theory, a *size pair* is a pair (X, φ), where X is a non-empty, compact, locally connected Hausdorff space and $\varphi : X \to \mathbb{R}$ is a continuous function called a *measuring function*. Let *Size* be the collection of all the size pairs, and let $(X, \varphi), (Y, \psi)$ be two size pairs. We denote by $H(X, Y)$ the set of all homeomorphisms from X to Y.

Definition 1.1. *If $H(X,Y) \neq \emptyset$, the function $\Theta : H(X,Y) \to \mathbb{R}$ given by*

$$\Theta(f) = \max_{x \in X} |\varphi(x) - \psi(f(x))|$$

is called the natural size measure *with respect to the measuring functions φ and ψ.*

Roughly speaking, $\Theta(f)$ measures how much f changes the values taken by the measuring functions, at corresponding points.

Definition 1.2. *We shall call* natural pseudo-distance *the pseudo-distance δ : $Size \times Size \to \mathbb{R} \cup \{+\infty\}$ defined as*

$$\delta\left((X,\varphi),(Y,\psi)\right) = \begin{cases} \inf\limits_{f \in H(X,Y)} \Theta(f), & \text{if } H(X,Y) \neq \emptyset \\ +\infty, & \text{otherwise.} \end{cases}$$

Note that δ is not a distance, since two different size pairs $(X,\varphi),(Y,\psi)$ can have a vanishing pseudo-distance. In that case, X and Y are only sharing the same shape properties with respect to the chosen functions φ and ψ, respectively. Moreover, we observe that the infimum of $\Theta(f)$ for f varying in $H(X,Y)$ is not always attained. When it is, we shall say that each homeomorphism $f \in H(X,Y)$ with $\Theta(f) = \delta\left((X,\varphi),(Y,\psi)\right)$ is an *optimal homeomorphism*. On the other hand, Definition 1.2 implies that, if $H(X,Y) \neq \emptyset$, we can always find a sequence (f_k) of homeomorphisms from X to Y, such that $\lim\limits_{k\to\infty} \Theta(f_k) = \delta\left((X,\varphi),(Y,\psi)\right)$.

Definition 1.3. *Let $(X,\varphi),(Y,\psi)$ be two size pairs, with X,Y homeomorphic and $\delta\left((X,\varphi),(Y,\psi)\right) = d$. Every sequence (f_k) of homeomorphisms $f_k : X \to Y$ such that $\lim\limits_{k\to\infty} \Theta(f_k) = d$ is said to be a d-*approximating sequence *from (X,φ) to (Y,ψ).*

Remark 1.4. *We observe that (f_k) is a d-approximating sequence from (X,φ) to (Y,ψ) if and only if (f_k^{-1}) is a d-approximating sequence from (Y,ψ) to (X,φ).*

The main goal of this paper is to show that an optimal homeomorphism exists between two size pairs (X,φ) and (Y,ψ), under the following conditions:

- (a) (X,φ) and (Y,ψ) have vanishing natural pseudo-distance, i.e. it holds that $\delta((X,\varphi),(Y,\psi)) = 0$;
- (b) X and Y are two curves of class C^2;
- (c) φ and ψ are Morse (i.e., smooth and having invertible Hessian at each critical point) measuring functions.

This result will be formally given and proved later (Theorem 3.5), in the case of closed curves. However, we remark that the hypothesis of closed curves will be assumed only for the sake of simplicity. Indeed, it can be weakened to compact 1-manifolds having non-empty boundary, without much affecting the following reasonings.

Remark 1.5. *The reader may wonder why we are defining the natural pseudo-distance δ in terms of homeomorphisms instead of diffeomorphisms, since the above assumption (c) requires that the measuring functions are Morse. The answer is that, under condition (b), Definition 1.2 is invariant with respect to such a choice. Indeed, it is well known that each homeomorphism between compact differentiable 1-manifolds can be approximated arbitrarily well by diffeomorphisms. On the other hand, dealing with homeomorphisms allows us to slim down examples and proves from useless technical steps.*

2. Meaningful examples

We provide here three meaningful examples showing that the assumptions (a), (b), (c) introduced in Section 1 are the less restrictive we can consider in order to ensure the existence of an optimal homeomorphism between two size pairs (X, φ) and (Y, ψ). Indeed, if one among them is dropped, then Theorem 3.5 does not hold.

First of all, let us observe that for every size pair (Z, ω) with Z a closed curve of class C^k, an embedding $h : Z \to \mathbb{R}^3$ of class C^k exists such that $z(p) = \omega(h^{-1}(p))$ for each point $p = (x(p), y(p), z(p)) \in h(Z)$. Moreover, if ω is Morse, we can assume that z is Morse on $h(Z)$, too. In other words, there is no lack of generality in assuming that the measuring function associated with Z is obtained by restriction of the z-coordinate in $\mathbb{R}^3$.

Accordingly, in the examples and figures we describe here, we shall always assume that the spaces X and Y are endowed with the z-coordinate function, and use the symbol z to denote both $z_{|X}$ and $z_{|Y}$.

Example 1 (Hypothesis (a) fails)**.** We report an example introduced in [8]. It shows that if two size pairs satisfy hypotheses (b) and (c), but have non-vanishing natural pseudo-distance, then an optimal homeomorphism does not always exist.

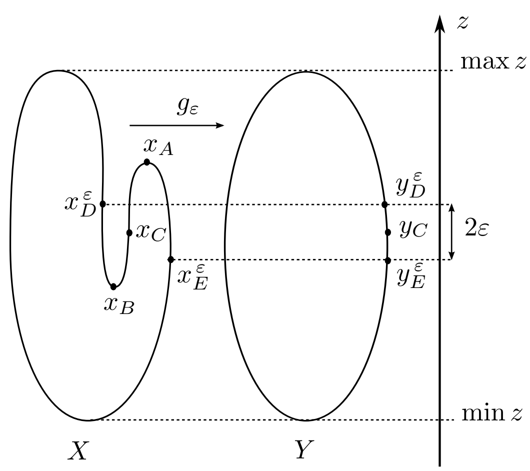

Figure 1. An example of two closed curves X, Y endowed with the Morse function z. No optimal homeomorphism exist between (X, z), (Y, z) because their natural pseudo-distance is non-zero.

Let us consider the two size pairs (X, z), (Y, z) depicted in Figure 1, where X and Y are smooth closed curves in $\mathbb{R}^3$, embedded in the real plane. The functions $z_{|X}$ and $z_{|Y}$ are Morse.

As can be seen in Figure 1, the points x_A, $x_B \in X$ are critical points of the function z and $z(x_C) = \frac{1}{2}(z(x_A) + z(x_B)) = z(y_C)$. In [7] it has been proved that the natural pseudo-distance between homeomorphic smooth closed curves, endowed with Morse measuring functions, is always obtainable in terms of some critical values of the measuring functions. Actually, in this example it is possible to show that the natural pseudo-distance between (X, z) and (Y, z) takes the value $d = \frac{1}{2}(z(x_A) - z(x_B))$. On the other hand, it will also be proved that no optimal homeomorphism exists. Indeed, we can construct a sequence of homeomorphisms (f_k), such that

$\lim_{k\to\infty} \Theta(f_k) = \frac{1}{2}(z(x_A) - z(x_B))$, and show that $\Theta(f) > \frac{1}{2}(z(x_A) - z(x_B))$ for every homeomorphism $f \in H(X,Y)$. The first step consists in proving that, for every $\varepsilon > 0$, a homeomorphism $g_\varepsilon : X \to Y$ exists, such that $\Theta(g_\varepsilon) \leq \frac{1}{2}(z(x_A) - z(x_B)) + 2\varepsilon$. Accordingly, consider the points x_D^ε, x_E^ε, y_D^ε and y_E^ε in Figure 1, verifying $z(x_D^\varepsilon) = z(y_D^\varepsilon) = z(x_C) + \varepsilon$ and $z(x_E^\varepsilon) = z(y_E^\varepsilon) = z(x_C) - \varepsilon$. Choose a homeomorphism g_ε, taking the arc $x_D^\varepsilon x_C x_E^\varepsilon$ to the arc $y_D^\varepsilon y_C y_E^\varepsilon$ in such a way that $g_\varepsilon(x_D^\varepsilon) = y_D^\varepsilon$ and $g_\varepsilon(x_E^\varepsilon) = y_E^\varepsilon$. Outside the arc $x_D^\varepsilon x_C x_E^\varepsilon$ in X define g_ε by mapping, in the unique possible way, every point x to a point $g_\varepsilon(x)$ satisfying $z(x) = z(g_\varepsilon(x))$. For every $k \in \mathbb{N}\backslash\{0\}$ set $f_k = g_{\frac{1}{k}}$. It can be easily verified that $\lim_{k\to\infty} \Theta(f_k) = \frac{1}{2}(z(x_A) - z(x_B))$. It only remains to prove that $\Theta(f) \leq \frac{1}{2}(z(x_A) - z(x_B))$ for no homeomorphism $f \in H(X,Y)$. If such a homeomorphism existed, for every $x \in X$ we would have $|z(x) - z(f(x))| \leq \frac{1}{2}(z(x_A) - z(x_B))$, and hence $z(f(x_A)) \geq z(y_C) \geq z(f(x_B))$. Therefore, points $x \in X$ such that $|z(x) - z(f(x))| > \frac{1}{2}(z(x_A) - z(x_B))$ could be easily found, contradicting our assumption.

Example 2 (Hypothesis (b) fails)**.** This example, introduced in [8], shows that there does not always exist an optimal homeomorphism between two size pairs satisfying hypotheses (a) and (c), but missing hypothesis (b).

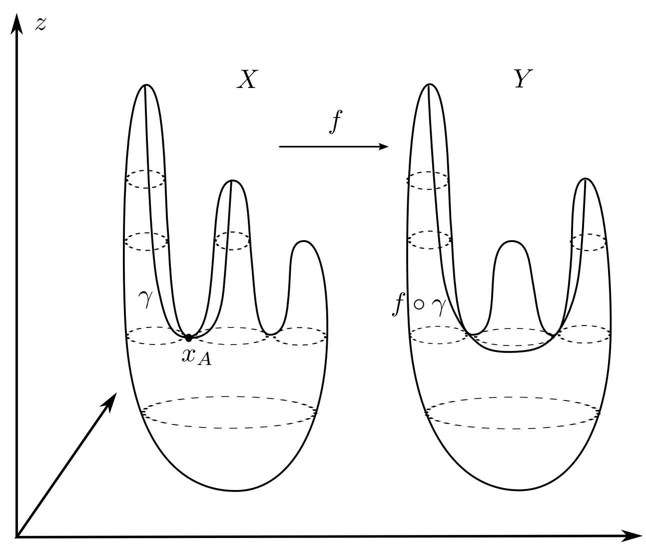

FIGURE 2. An example of two size pairs (X,z) and (Y,z), whose natural pseudo-distance is zero. No optimal homeomorphism exist between (X,z), (Y,z) because X an Y are not closed curves.

Consider the smooth surfaces X and Y displayed in Figure 2 and the corresponding measuring function z. The dotted lines are level curves for the measuring function z. It is easy to show that the natural pseudo-distance between the two size pairs is zero. Indeed, it is possible to isotopically deform the left surface to the right one by "torsion", exchanging the positions of the two smallest humps. This deformation can be performed by an arbitrarily small change in the values of the height z. Therefore, a sequence of homeomorphisms (f_k) from X to Y can be constructed, such that $\lim_{k\to\infty} \Theta(f_k) = 0$.

However, no optimal homeomorphism exists between the two size pairs. Suppose indeed there exists a homeomorphism f such that $\Theta(f) = 0$. Consider a path γ as in Figure 2, chosen in such a way that, in the image of the path, $z(x) = z(x_A)$ for no point $x \in X$ different from x_A. It can be easily verified that the image of the path $f \circ \gamma$ has to contain more than one point at which z takes the value $z(x_A)$. This contradicts the assumptions, since $\Theta(f) = 0$ implies $z(f(x)) = z(x)$ for every x in the image of γ.

Example 3 (Hypothesis (c) fails)**.** This last example shows that there does not always exist an optimal homeomorphism between two closed curves having vanishing natural pseudo-distance, if such curves are endowed with measuring functions missing hypothesis (c).

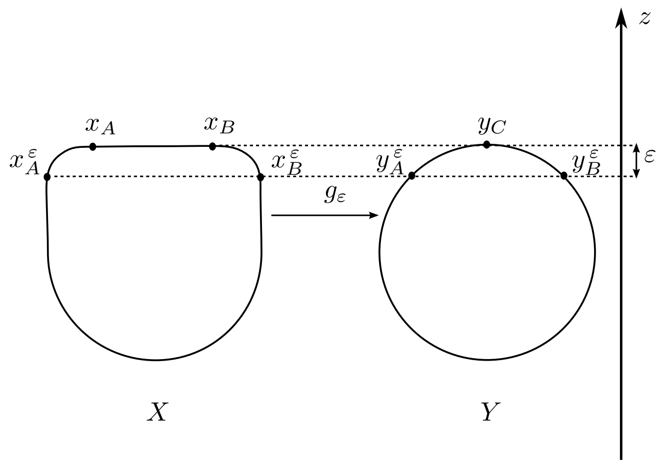

FIGURE 3. An example of two size pairs (X, z) and (Y, z), whose natural pseudo-distance is zero. No optimal homeomorphism exist between (X, z), (Y, z) because $z_{|X}$ is not Morse.

Let us consider the two size pairs (X, z) and (Y, z) in Figure 3, where X and Y are smooth closed curves. As can be seen, the measuring function z is not Morse on X.

We see that the natural pseudo-distance between (X, z) and (Y, z) is vanishing, but an optimal homeomorphism does not exist. Indeed, it is possible to give a sequence of homeomorphism (f_k), such that $\lim_{k\to\infty} \Theta(f_k) = 0$, and verify that $\Theta(f) > 0$ for every homeomorphism $f \in H(X, Y)$. Similarly to the previous example, for every $\varepsilon > 0$ we first construct a homeomorphism $g_\varepsilon : X \to Y$, moving each point $x \in X$ less then or equal to ε with respect to the measuring function z. This can be done by considering a homeomorphism g_ε taking the arc $x_A^\varepsilon x_B^\varepsilon$, containing the segment $\overline{x_A x_B}$, to the arc $y_A^\varepsilon y_C y_B^\varepsilon$. Observe that $z(x) = z(y_C)$ for every $x \in \overline{x_A x_B}$, and $z(x_A^\varepsilon) = z(x_B^\varepsilon) = z(y_A^\varepsilon) = z(y_B^\varepsilon) = z(y_C) - \varepsilon$. Outside the arc $x_A^\varepsilon x_B^\varepsilon$ in X we define g_ε by mapping every point x to a point $g_\varepsilon(x)$ satisfying $z(x) = z(g_\varepsilon(x))$. For every $k \in \mathbb{N} \setminus \{0\}$ set $f_k = g_{\frac{1}{k}}$. It can be easily verified that $\lim_{k\to\infty} \Theta(f_k) = 0$. However, an optimal homeomorphism $f : X \to Y$ does not exist. Indeed, such a map should verify $\max_{x\in X} |z(x) - z(f(x))| = 0$, and therefore it should take each point of the segment $\overline{x_A x_B}$ to the point y_C, against the injectivity.

3. Main theorem

In this section we prove the main theorem of this paper which states that an optimal homeomorphism exists between two closed curves of class C^2, endowed with Morse measuring functions, and whose natural pseudo-distance is zero (see Theorem 3.5). Roughly speaking, the proof involves the idea to construct such a homeomorphism between the two curves as a continuous extension of a uniformly continuous, bijective map existing between dense subsets of the curves (see Proposition 3.3 and Remark 3.4). The optimality is finally showed in Theorem 3.5.

Let us now introduce some notations and assumptions we shall adopt in the rest of this section.

Let (X, φ), (Y, ψ) be two size pairs, with X, Y two C^2 closed curves, and φ, ψ Morse measuring functions, and suppose that $\delta\left((X, \varphi), (Y, \psi)\right) = 0$.

It is not restrictive to assume that X and Y are metric spaces, endowed with two metrics d_X and d_Y, respectively. Moreover, for the sake of simplicity, from now on we shall assume that the considered curves are connected. However, note that this last hypothesis can be weakened to any finite number of connected components, without much affecting the following reasonings.

Let us now consider two parameterizations $h_X : S^1 \to X$, $h_Y : S^1 \to Y$. The clockwise orientation on $S^1 \subset \mathbb{R}^2$, and the homeomorphisms h_X, h_Y allow us to induce an orientation on X and Y, respectively. For every $x, x' \in X$ (respectively $y, y' \in Y$), we shall denote by $\widehat{xx'}$ (resp. $\widehat{yy'}$) the oriented path on X (resp. Y), induced by h_X (resp. h_Y), from the point x (resp. y) to the point x' (resp. y'), going clockwise along S^1, and including both x and x' (resp. y and y').

Consider the sets $X_{\mathbb{Q}} = \{x = h_X((\cos\theta, \sin\theta)) : \theta \in \mathbb{Q}\}$ and $Y_{\mathbb{Q}} = \{y = h_Y((\cos\theta, \sin\theta)) : \theta \in \mathbb{Q}\}$, and a sequence (f_k) of 0-approximating homeomorphisms from (X, φ) to (Y, ψ), i.e. such that $\lim_{k\to\infty} \Theta(f_k) = 0$. By using the Cantor's diagonalization argument, and from the compactness of X and Y, we can assume (possibly by considering a subsequence) that there exist $\lim_{k\to\infty} f_k(x)$ for every $x \in X_{\mathbb{Q}}$, and $\lim_{k\to\infty} f_k^{-1}(y)$ for every $y \in Y_{\mathbb{Q}}$. We shall set $\lim_{k\to\infty} f_k(x) = y_x \in Y$ for every $x \in X_{\mathbb{Q}}$, and $\lim_{k\to\infty} f_k^{-1}(y) = x_y \in X$ for every $y \in Y_{\mathbb{Q}}$.

Furthermore, since homeomorphisms between closed curves can be orientation-preserving or not, for the sake of simplicity we shall assume (possibly by considering a subsequence) that the orientation is maintained by each f_k. Indeed, if this is not the case, we can consider a new parametrization $\hat{h}_Y$ having opposite orientation with respect to h_Y.

Let us now set $\widetilde{Y} = \{y_x : x \in X_{\mathbb{Q}}\}$ and $\widetilde{X} = \{x_y : y \in Y_{\mathbb{Q}}\}$ and denote by X_*, Y_* the sets $X_{\mathbb{Q}} \cup \widetilde{X}$ and $Y_{\mathbb{Q}} \cup \widetilde{Y}$, respectively. We can define a relation $\rho \subseteq X_* \times Y_*$ by setting

$$(1) \quad (x, y) \in \rho \Leftrightarrow (x \in X_{\mathbb{Q}} \text{ and } y = y_x \in \widetilde{Y}) \text{ or } (y \in Y_{\mathbb{Q}} \text{ and } x = x_y \in \widetilde{X}).$$

Remark 3.1. *Note that the equality $\varphi(x) = \psi(y)$ holds for every $(x, y) \in \rho$. Indeed, since (f_k) is a 0-approximating sequence, if $x \in X_{\mathbb{Q}}$ and $y = y_x \in \widetilde{Y}$, then $|\varphi(x) - \psi(y_x)| = |\varphi(x) - \psi(\lim_{k\to\infty} f_k(x))| = \lim_{k\to\infty} |\varphi(x) - \psi(f_k(x))| = 0$; if $y \in Y_{\mathbb{Q}}$ and $x = x_y \in \widetilde{X}$, by* Remark 1.4 *it follows that $|\varphi(x_y) - \psi(y)| = |\varphi(\lim_{k\to\infty} f_k^{-1}(y)) - \psi(y)| = \lim_{k\to\infty} |\varphi(f_k^{-1}(y)) - \psi(y)| = 0$.*

Following the rough outline exposed at the beginning of this section, we shall first construct a suitable function between X_* and Y_*, proving that it is bijective and uniformly continuous. Such a function can be obtained directly from the relation ρ, by virtue of the following technical lemma.

Lemma 3.2. *The following statements hold:*

(i) *For every real number $\varepsilon > 0$, there exists $\eta > 0$ such that, for every $(x,y),(x',y') \in \rho$ with $d_X(x,x') < \eta$ (respectively $d_Y(y,y') < \eta$), the inequality $d_Y(y,y') < \varepsilon$ (resp. $d_X(x,x') < \varepsilon$) holds.*

(ii) *For every $x \in X_*$ (respectively $y \in Y_*$), there exists $y \in Y_*$ (resp. $x \in X_*$) such that $(x,y) \in \rho$;*

(iii) *For every $(x,y),(x',y') \in \rho$, $x = x'$ if and only if $y = y'$.*

Proof. Let us start by proving assertion (i). We shall confine ourselves to prove that for every real number $\varepsilon > 0$, there exists $\eta > 0$ such that, for every $(x,y),(x',y') \in \rho$ with $d_X(x,x') < \eta$, the inequality $d_Y(y,y') < \varepsilon$ holds. Indeed, the proof of the other case is analogous.

We shall prove the statement by contradiction, i.e. by supposing the existence of a real number $\bar{\varepsilon} > 0$ such that, for every $\eta > 0$, two pairs $(x_\eta, y_\eta),(x'_\eta, y'_\eta) \in \rho$ exist with $d_X\left(x_\eta, x'_\eta\right) < \eta$ and $d_Y\left(y_\eta, y'_\eta\right) \geq \bar{\varepsilon}$. Let us consider two sequences $((x_n,y_n))$, $((x'_n,y'_n))$ of elements in ρ, with $d_X(x_n,x'_n) < \frac{1}{n}$ and $d_Y(y_n,y'_n) \geq \bar{\varepsilon}$ for every $n \in \mathbb{N}$.

Since $X_* = X_{\mathbb{Q}} \cup \widetilde{X}$, it can be assumed (possibly by considering two subsequences) that $x_n, x'_n \in X_{\mathbb{Q}}$ for every index n. Indeed, if this is not the case, we can alternatively assume that $x_n, x'_n \in \widetilde{X}$ for every index n, or that x_n (respectively x'_n) $\in X_{\mathbb{Q}}$ and x'_n (resp. x_n) $\in \widetilde{X}$ for every index n, without much affecting the following reasonings. Observe that our assumption implies that $y_n, y'_n \in \widetilde{Y}$ for every index n.

By the compactness of X, we can hypothesize (possibly by extracting a subsequence) that the sequence (x_n) converges to a point $\bar{x} \in X$. Obviously, it holds that $\lim_{n\to\infty} d_X(x_n,x'_n) = 0$ and hence $d_X(\bar{x},x'_n) \to 0$ for $n \to \infty$, that is also (x'_n) converges to $\bar{x}$.

Let us now consider the sequences (y_n), (y'_n) in $\widetilde{Y}$. By the compactness of Y, we can assume that they converge to $\bar{y}, \bar{y}' \in Y$, respectively. Moreover, by the hypothesis $d_Y(y_n,y'_n) \geq \bar{\varepsilon}$ for every index n, it means that $\bar{y} \neq \bar{y}'$. On the other hand, by the continuity of φ and ψ, we have $\lim_{n\to\infty} \varphi(x_n) = \varphi(\bar{x})$, $\lim_{n\to\infty} \varphi(x'_n) = \varphi(\bar{x})$, $\lim_{n\to\infty} \psi(y_n) = \psi(\bar{y})$, $\lim_{n\to\infty} \psi(y'_n) = \psi(\bar{y}')$, and by Remark 3.1 we can write $\varphi(\bar{x}) = \psi(\bar{y}) = \psi(\bar{y}') = c \in \mathbb{R}$. In other words, we have $\bar{y} \neq \bar{y}'$ with $\psi(\bar{y}) = \psi(\bar{y}')$. Since ψ is a Morse function, it is necessarily non-constant on the path $\widehat{\bar{y}\bar{y}'}$, therefore there exists a point $y'' \in \widehat{\bar{y}\bar{y}'}$ verifying $|\psi(y'') - c| = C > 0$ (obviously, $y'' \neq \bar{y}, \bar{y}'$). Furthermore, by recalling that $(y_n) = (\lim_{k\to\infty} f_k(x_n))$ converges to $\bar{y}$ and $(y'_n) = (\lim_{k\to\infty} f_k(x'_n))$ converges to $\bar{y}'$, we can find an index N such that, for every $n > N$, an index $K = K(n)$ exists with $y'' \in \widehat{f_k(x_n)f_k(x'_n)}$ for every $k > K$, implying that $f_k^{-1}(y'') \in \widehat{x_n x'_n}$ for large enough indices n and k.

Since $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x'_n = \bar{x}$, it can be assumed that $f_k^{-1}(y'')$ converges to $\bar{x}$. Indeed, if $f_k^{-1}(y'')$ did not converge to $\bar{x}$, then we could consider another point $y''' \in \widehat{\bar{y}'\bar{y}}$ (i.e. the clockwise oriented path from $\bar{y}'$ to $\bar{y}$), verifying $|\psi(y''') - c| = C' > 0$ (obviously, $y''' \neq \bar{y}, \bar{y}'$), and such that $f_k^{-1}(y''')$ converges to $\bar{x}$.

Therefore, by Definition 1.3 and Remark 1.4, both of them in the case $d = 0$, we have $0 = \lim_{k\to\infty} |\varphi(f_k^{-1}(y'')) - \psi(y'')| = |\varphi(\bar{x}) - \psi(y'')| = |c - \psi(y'')|$, i.e. $C = 0$, thus getting a contradiction. This concludes the proof of (i).

The proof of statement (ii) is trivial, and directly follows by the definition of the relation ρ in (1).

Let us now prove (iii). Let $(x, y), (x', y') \in \rho$, with $x = x'$. This means that $d_X(x, x') < \eta$ for every real value $\eta > 0$. Since $(x, y), (x', y') \in \rho$, assertion (i) implies that $d_Y(y, y') < \varepsilon$ for every real value $\varepsilon > 0$, i.e. $y = y'$. Conversely, a similar proof can be given, by exchanging the roles of x, x' and y, y' and applying once more assertion (i). □

In plain words, assertions (ii) and (iii) of Lemma 3.2 tell us that it is possible to define a bijective function $f_* : X_* \to Y_*$ directly from the relation ρ, by setting

$$f_*(x) = y \Leftrightarrow (x, y) \in \rho.$$

Moreover, statement (i) of Lemma 3.2 implies that f_* is uniformly continuous together with its inverse. Finally, we observe that Remark 3.1 implies $\varphi(x) = \psi(f_*(x))$ for every $x \in X_*$. In other words, we have just proved the following proposition.

Proposition 3.3. *$f_* : X_* \to Y_*$ is a bijective, uniformly continuous map with uniformly continuous inverse, and such that $\varphi(x) = \psi(f_*(x))$ for every $x \in X_*$.*

Remark 3.4. *Proposition 3.3 allows us to easily obtain a continuous map from X to Y. Indeed, it is well known that a uniformly continuous function between two metric spaces can be* univocally *extended to a continuous map between two given completions of the spaces themselves.*

In our context, X and Y are compact metric spaces, and hence complete. This means that every Cauchy sequence of elements in X (respectively Y) has a limit in X (resp. Y). Moreover, X and Y are completions of their dense subsets X_* and Y_*, respectively. Finally, we observe that a uniformly continuous function takes Cauchy sequences into Cauchy sequences. Following these considerations, it is easy to show that $f : X \to Y$ with $f_{|X_*} = f_*$ and $f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f_*(x_n)$ for every Cauchy sequence (x_n) of elements in X_*, is a well-defined function, continuously extending the map $f_* : X_* \to Y_*$.

We are now ready to give the main result of this paper.

Theorem 3.5. *Let (X, φ), (Y, ψ) be two size pairs, with X, Y closed curves of class C^2, and $\varphi : X \to \mathbb{R}$, $\psi : Y \to \mathbb{R}$ Morse measuring functions. If $\delta((X, \varphi), (Y, \psi)) = 0$, then there exists an optimal homeomorphism $f : X \to Y$.*

Proof. We shall prove the statement by showing that the continuous function f defined in Remark 3.4 is an optimal homeomorphism from X to Y.

Let us start by proving that f is a homeomorphism. To do so, we only need to show that f is injective, since every continuous injection between compact Hausdorff spaces is a homeomorphism [19, Thm. 2-103].

Let $x, x' \in X$ and suppose $x \neq x'$. Then $x = \lim_{n\to\infty} x_n$, $x' = \lim_{n\to\infty} x'_n$ for two suitable sequences (x_n) and (x'_n) of elements in X_*, with $\lim_{n\to\infty} x_n \neq \lim_{n\to\infty} x'_n$. This means that we can choose a real number $\bar{\varepsilon} > 0$ such that, for sufficiently large indices n, we have $d_X(x_n, x'_n) > \bar{\varepsilon}$, allowing us to claim that $\lim_{n\to\infty} f_*(x_n) \neq \lim_{n\to\infty} f_*(x'_n)$. Indeed, the equality $\lim_{n\to\infty} f_*(x_n) = \lim_{n\to\infty} f_*(x'_n)$ would imply the existence of a real number, that is $\bar{\varepsilon}$, such that, for every real value $\eta > 0$, two points $f_*(x_n), f_*(x'_n) \in Y_*$ would exist, with $d_Y(f_*(x_n), f_*(x'_n)) < \eta$ and $d_X(x_n, x'_n) > \bar{\varepsilon}$, thus contradicting the uniform continuity of the inverse of f_* (see Proposition 3.3). Hence, the assumption $x \neq x'$ implies that $\lim_{n\to\infty} f_*(x_n) \neq \lim_{n\to\infty} f_*(x'_n)$, i.e. $f(x) \neq f(x')$, thus proving the injectivity of f.

To conclude the proof, we still need to show the optimality of f, i.e. that the equality $\varphi(x) = \psi(f(x))$ holds for every $x \in X$. By Proposition 3.3, this is true when $x \in X_*$. On the other hand, if $x \in X \setminus X_*$, then there exists a sequence (x_n) in X_* converging to x. So, by the continuity of φ, ψ and f, we can write $\varphi(x) = \lim_{n\to\infty} \varphi(x_n)$ and $\psi(f(x)) = \lim_{n\to\infty} \psi(f(x_n))$. Moreover, by recalling once more that the restriction of f to X_* coincides with f_*, and $\psi \circ f_*$ coincides with $\varphi_{|X_*}$, we have $|\varphi(x) - \psi(f(x))| = |\lim_{n\to\infty} \varphi(x_n) - \lim_{n\to\infty} \psi(f(x_n))| = \lim_{n\to\infty} |\varphi(x_n) - \psi(f(x_n))| = \lim_{n\to\infty} |\varphi(x_n) - \psi(f_*(x_n))| = 0$. □

4. Conclusions and future works

In this paper we have proved that there always exists an optimal homeomorphism between two size pairs (X, φ), (Y, ψ) having vanishing natural pseudo-distance, under the assumptions that X, Y are closed curves of class C^2, and φ, ψ are Morse measuring functions. We point out that this result is the first available one concerning the existence of optimal homeomorphisms between size pairs. Indeed, until now the research has been developed mainly focusing on the relations between the natural pseudo-distance and the critical values of the measuring functions, as well as on the estimation of natural pseudo-distance via lower bounds provided by size functions. Our result opens the way to further investigations, in order to obtain a generalization to the case of k-dimensional manifolds endowed with $\mathbb{R}^k$-valued measuring functions, with $k > 1$. In this context, an interesting research line appears to be, for example, to consider measuring functions having finite preimage for each point in the range, or characterized by a behavior analogous to that of Morse functions in the 1-dimensional case.

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