Harnack inequality and no-arbitrage bounds for self-financing portfolios

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Abstract

We give a direct proof of the Harnack inequality for a class of Kolmogorov operators associated with a linear SDE and we find the explicit expression of the optimal Harnack constant. We discuss some possible implication of the Harnack inequality in finance: specifically we infer no-arbitrage bounds for the value of self-financing portfolios in terms of the initial wealth.

1 Introduction

Consider the linear SDE in $\mathbb{R}^N$

$$dX_t = \big(B(t)X_t + b(t)\big)dt + \sigma(t)dW_t, \tag{1.1}$$

where W is a d-dimensional Brownian motion with $d \leq N$ and $\sigma(t)$, $B(t)$, $b(t)$ are $L^\infty_{\text{loc}}(\mathbb{R})$-functions with values respectively in the matrix spaces of dimension $N \times d$, $N \times N$, $N \times 1$.

Equations of the form (1.1) naturally arise in several classical models in physics and in mathematical finance (see Section 2). It is well-known that (1.1), associated with the initial condition $X_{t_0} = x_0$, has the unique solution

$$X_t = E_{t_0}(t)\left(x_0 + \int_{t_0}^t E_{t_0}^{-1}(s)b(s)ds + \int_{t_0}^t E_{t_0}^{-1}(s)\sigma(s)dW_s\right),$$

where

$$E_{t_0}(t) = \exp\left(\int_{t_0}^t B(s)ds\right). \tag{1.2}$$

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Moreover X_t has multinormal distribution with mean

$$m_{t_0,x_0}(t) = E_{t_0}(t)\left(x_0 + \int_{t_0}^{t} E_{t_0}^{-1}(s)b(s)ds\right), \tag{1.3}$$

and covariance matrix

$$\mathcal{C}_{t_0}(t) = E_{t_0}(t)\left(\int_{t_0}^{t} E_{t_0}^{-1}(s)\sigma(s)\left(E_{t_0}^{-1}(s)\sigma(s)\right)^* ds\right) E_{t_0}^*(t). \tag{1.4}$$

Note that the matrix $E_{t_0}^{-1}(s)\sigma(s)\left(E_{t_0}^{-1}(s)\sigma(s)\right)^*$ appearing in the previous integral has rank d. Nevertheless, it is remarkable that even when $d < N$ the $N \times N$ matrix $\mathcal{C}_{t_0}(t)$ can be strictly positive definite, as the following example shows.

Example 1.1 KOLMOGOROV [11]. *Consider, the following SDE in* $\mathbb{R}^2$

$$\begin{cases} dX_t^1 = \mu dt + \sigma_0 dW_t, \\ dX_t^2 = X_t^1 dt, \end{cases} \tag{1.5}$$

with μ *and* σ_0 *positive constants. It is a SDE of the form* (1.1) *with*

$$B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} \sigma_0 \\ 0 \end{pmatrix}, \qquad b = \begin{pmatrix} \mu \\ 0 \end{pmatrix},$$

so that $1 = d < N = 2$*, and a direct computation gives*

$$\mathcal{C}_{t_0}(t_0 + t) = \sigma_0^2 \begin{pmatrix} t & \frac{t^2}{2} \\ \frac{t^2}{2} & \frac{t^3}{3} \end{pmatrix} > 0 \qquad \text{for any } t > 0.$$

In Section 2 we discuss some applications of (1.5) in finance. Here we recall that the (1.5) is a simplified version of the Langevin equation which describes the motion of a particle into a viscous fluid: in this case the coefficient σ_0 is the magnitude of the stochastic force and X_t^1, X_t^2 respectively represent the velocity and the position of the particle. We refer to the paper by Bossy, Jabir and Talay [5] for some recent results about more general Lagrangian stochastic models.

In this paper we consider the SDE (1.1) under the assumption:

[H.1] *the matrix* $\mathcal{C}_t(T)$ *is positive definite for every* $t < T$.

In that case, for any $t > t_0$, X_t has a density $x \longmapsto \Gamma(t_0, x_0; t, x)$, where

$$\Gamma(t_0, x_0; t, x) = \frac{1}{\sqrt{(2\pi)^N \det \mathcal{C}_{t_0}(t)}} e^{-\frac{1}{2}\left\langle \mathcal{C}_{t_0}^{-1}(t)\left(x - m_{t_0,x_0}(t)\right), \left(x - m_{t_0,x_0}(t)\right)\right\rangle}. \tag{1.6}$$

Moreover Γ is the fundamental solution to the Kolmogorov differential operator

$$L := \frac{1}{2}\sum_{i,j=1}^{N} a_{ij}(t)\partial_{x_i x_j} + \langle b(t) + B(t)x, \nabla\rangle + \partial_t, \qquad (t,x) \in \mathbb{R}^{N+1}, \quad (1.7)$$

where $A := (a_{ij}) = \sigma\sigma^*$ and $\nabla = (\partial_{x_1}, \cdots, \partial_{x_N})$. Specifically, this means that the function

$$u(t,x) = \int_{\mathbb{R}^N} \Gamma(t,x;T,y)\varphi(y)dy, \qquad t < T, \; x \in \mathbb{R}^N,$$

is a classical solution to the Cauchy problem

$$\begin{cases} Lu = 0 & \text{in }]-\infty, T[\times\mathbb{R}^N, \\ u(T,x) = \varphi(x), & x \in \mathbb{R}^N. \end{cases}$$

The Kolmogorov equation related to the system (1.5) is

$$\frac{\sigma_0^2}{2}\partial_{x_1}^2 u(t,x) + \mu\partial_{x_1}u(t,x) + x_1\partial_{x_2}u(t,x) + \partial_t u(t,x) = 0, \qquad (1.8)$$

and its fundamental solution is

$$\Gamma(s,y;t,x) = \frac{\sqrt{3}}{\pi\sigma_0^2(t-s)^2}\exp\Bigg(- \frac{(x_1 - y_1 - \mu(t-s))^2}{2\sigma_0^2(t-s)} - 3\frac{(2x_2 - 2y_2 - (t-s)(x_1+y_1))^2}{2\sigma_0^2(t-s)^3}\Bigg). \qquad (1.9)$$

It is interesting to remark that assumption [H.1] can be also expressed in geometric-differential terms. In fact, it is equivalent to the following condition due to Hörmander [8]:

[H.2] $\qquad \operatorname{rank} \mathcal{L}(Y_1, \cdots, Y_d, Y)(t,x) = N + 1, \qquad (t,x) \in \mathbb{R}^{N+1},$

where $\mathcal{L}(Y_1, \cdots, Y_d, Y)$ denotes the Lie algebra generated by the vector fields in $\mathbb{R}^{N+1}$

$$Y_i = \sum_{j=1}^{N} \sigma_{ji}\partial_{x_j}, \qquad i = 1, \ldots, d$$

and

$$Y = \langle b(t) + B(t)x, \nabla\rangle + \partial_t.$$

Recall that $\mathcal{L}(Y_1, \cdots, Y_d, Y)(t,x)$ is the vector space generated by $Y_1, \cdots, Y_d$, Y, by their first order commutators $[Y_k, Y]$, $k = 1, \ldots, d$, where $[Y_k, Y]u := Y_k Y u - Y Y_k u$ and by their higher order commutators $[Y_j, ..., [Y_k, Y]...]$, evaluated at the point (t,x).

A third condition, that is equivalent to the Hörmander condition and [H.1], arises in control theory. Given a positive T, a curve $\gamma : [0,T] \longmapsto \mathbb{R}^N$ is L-admissible if it is absolutely continuous and satisfies

$$\gamma'(s) = B(s)\gamma(s) + b(s) + \sigma(s)w(s), \qquad \text{a.e. in } [0,T], \tag{1.10}$$

for a suitable function w with values in $\mathbb{R}^d$. The components $w_1, \cdots, w_d$ of w are called *controls* of the path γ. Then [H.1] and [H.2] are equivalent to the following condition:

[H.3] *for every $x, y \in \mathbb{R}^N$ and $T > 0$, there exists an L-admissible path such that $\gamma(0) = x$ and $\gamma(T) = y$,*

(see, for instance, Karatzas and Shreve [10]). We finally recall that, when B and σ are constant matrices, there is a simple algebraic condition, due to Kalman, that is equivalent to [H.3]. Consider the $N \times dN$ matrix $C(\sigma, B) := \big(\sigma \ B\sigma \ \ldots \ B^{N-1}\sigma\big)$. Then [H.3] is satisfied if, and only if, rank $C(\sigma, B) = N$ (see Jurdjevic [9], Agrachev and Sachkov [1]).

In this paper we are concerned with the Harnack inequality for the Kolmogorov operator (1.7) and its applications in finance, in particular no-arbitrage bounds for the value of self-financing portfolios. Harnack inequalities are fundamental tools in the PDEs theory. They provide, for instance, regularity results of the weak solutions of $Lu = 0$ and uniqueness results for the related Cauchy problem. The Harnack inequalities for backward Kolmogorov operators available in literature read as follows: *under the assumption* [H.1]*, consider $(t,x), (T,y) \in \mathbb{R}^{N+1}$ with $t < T$. Then there exists a constant $H = H(t,x,T,y)$, only dependent on L and $(t,x), (T,y)$, such that*

$$u(T,y) \leq H\, u(t,x), \tag{1.11}$$

for every positive solution u to $Lu = 0$.

We next recall the Harnack inequality for Kolmogorov operators proved by Kupcov [12], Garofalo and Lanconelli [7] and Lanconelli and Polidoro [13] by using mean value formulas. It reads

$$u(T,y) \leq H\, u(t,x), \qquad (t,x) \in Q_r(T,y) \quad t = T - cr^2. \tag{1.12}$$

Here $Q_r(T,y)$ is a suitable cylinder of radius r centered at (T,y) (for the precise notation we refer to [13]) and the constants c and H only depend on L and on $Q_r(T,y)$.

In this paper we give a direct proof of the Harnack inequality (1.11) by using a variational argument due to Li and Yau [15], that allows to find explicitly the Harnack constant H. Our main result is the following:

Theorem 1.2 *Assume that L in* (1.7) *verifies hypothesis [H.1] and let u be a positive solution to $Lu = 0$ in $[t_0, t_1[\times\mathbb{R}^N$. Then the Harnack inequality $u(T, y) \le H\, u(t, x)$ holds with*

$$H = H(t, x, T, y) = \sqrt{\frac{\det \mathcal{C}_t(t_1)}{\det \mathcal{C}_T(t_1)}}\, e^{\frac{1}{2}\left\langle \mathcal{C}_t^{-1}(T)(y - m_{t,x}(T)), (y - m_{t,x}(T))\right\rangle}, \qquad (1.13)$$

for any $(t, x), (T, y) \in [t_0, t_1[\times\mathbb{R}^N$ with $t < T$.

We next give two examples that point out the optimality of the constant $H(t, x, T, y)$ in (1.13).

Corollary 1.3 *Let $u :]-\infty, \varepsilon[\times\mathbb{R}^N \to \mathbb{R}$ be a positive solution of the heat equation $\partial_t u + \frac{1}{2}\Delta u = 0$. Then $u(0, y) \le Hu(t, x)$, for any $x, y, \in \mathbb{R}^N, t < 0$, with*

$$H = \sqrt{\frac{(\varepsilon - t)^N}{\varepsilon^N}} \exp\left(\frac{|x - y|^2}{-2t}\right).$$

Remark 1.4 *The optimality of the constant H becomes apparent when applying the above inequality to the fundamental solution of the heat equation:*

$$u(t, x) = (2\pi(\varepsilon - t))^{-N/2} \exp\left(-\frac{|x - y|^2}{2(\varepsilon - t)}\right), \qquad (t, x) \in]-\infty, \varepsilon[\times\mathbb{R}^N.$$

Note that, in this case, we have $u(0, y) = (2\pi\varepsilon)^{-N/2}$.

Corollary 1.5 *Let $u :]-\infty, \varepsilon[\times\mathbb{R}^N \to \mathbb{R}$ be a positive solution of the Kolmogorov equation* (1.8)*. Then $u(0, y) \le Hu(t, x)$, for any $x, y, \in \mathbb{R}^2, t < 0$, with*

$$H = \frac{(\varepsilon - t)^2}{\varepsilon^2} \exp\left(\frac{(x_1 - y_1 - \mu t)^2}{-2\sigma_0^2 t} + 3\frac{(2(y_2 - x_2) + t(x_1 + y_1))^2}{-2\sigma_0^2 t^3}\right).$$

Remark 1.6 *Also in the case of the degenerate equation* (1.8)*, the fundamental solution shows the optimality of the constant H. Let us consider the function*

$$u(t, x) = \Gamma(t, x; \varepsilon, 0), \qquad (t, x) \in]-\infty, \varepsilon[\times\mathbb{R}^2.$$

where Γ is defined in (1.9)*. Then $\frac{\sqrt{3}}{\pi\sigma_0^2\varepsilon^2} \exp\left(-\frac{\mu^2\epsilon}{2\sigma_0^2}\right) = u(0, 0) \le Hu(t, x)$, with*

$$u(t, x) = \frac{\sqrt{3}}{\pi\sigma_0^2(\varepsilon - t)^2} \exp\Bigg(-\frac{\big(x_1 + \mu(\epsilon - t)\big)^2}{2\sigma_0^2(\varepsilon - t)} - 3\frac{(2x_2 + (\varepsilon - t)x_1)^2}{2\sigma_0^2(\varepsilon - t)^3}\Bigg).$$

Theorem 1.2 provides a generalization of the result contained in the Corollary 1.2 of [16], that applies to a restricted class of Kolmogorov equations. The proof of Theorem 1.2 is based on the solution of an optimal control problem with quadratic cost. More specifically, we first prove (cf. Proposition 3.1) the following gradient estimate valid for positive solutions to L in $[t_0, t_1[\times\mathbb{R}^N$:

$$\frac{\langle A(t)\nabla u(t,x), \nabla u(t,x)\rangle}{2u(t,x)} \leq -Yu(t,x) - \varphi'(t)u(t,x), \tag{1.14}$$

where $\varphi(t) = \log\sqrt{\det\mathcal{C}_t(t_1)}$. Then the Harnack inequality follows by integrating inequality (1.14) along an L-admissible path of the form (1.10) and the optimal constant in (1.13) is obtained by minimizing the quadratic cost

$$\psi(w) := \int_t^T |w(s)|^2 ds. \tag{1.15}$$

We emphasize that our approach is quite general and applies to many different problems: parabolic equations on manifolds (Li and Yau [15]), porous media and p-diffusion equations (Auchmuty and Bao [2]), and sum of squares of vector fields (Cao and Yau [6]).

We also recall that the same method based on the optimal control theory, where the gradient bound (1.14) is replaced by the local Harnack inequality (1.12) gives similar lower bounds for positive solutions (see [4]).

In Section 2 we apply the Harnack inequality in the framework of the financial no-arbitrage theory and show how it yields an interesting a priori upper bound for the value of a self-financing portfolio in terms of the initial wealth (see Proposition 2.1).

The rest of the paper is organized as follows: in Section 3 we prove the gradient estimate (1.14) for positive solutions of $Lu = 0$; Section 4 contains the proof of Theorem 1.2.

2 Harnack inequality and no-arbitrage bounds

In a standard multi-dimensional Black&Scholes model the risk neutral dynamics of N financial assets is given by

$$dS_t^i = rS_t^i dt + S_t^i \sum_{j=1}^N \sigma_{ij} dW_t^j = rS_t^i dt + S_t^i \sigma^i \cdot dW_t,$$

where $W = \left(W^1, \dots, W^N\right)$ is a standard N-dimensional Brownian motion, σ is a non-singular $N \times N$ matrix with constant real entries and r is the constant risk free rate. We also denote as usual by B_t the bank account defined by

$$dB_t = rB_t dt.$$

We consider a Markovian portfolio that is a process $(\alpha,\beta)=\left(\alpha^1,\dots,\alpha^N,\beta\right)$ of the form

$$\alpha_t=\alpha(t,S_t),\qquad \beta_t=\beta(t,S_t),$$

where (α,β) are smooth functions. The value of (α,β) is defined as the process

$$V_t=\sum_{i=1}^N \alpha_t^i S_t^i+\beta_t B_t=\alpha_t\cdot S_t+\beta_t B_t.$$

We say that (α,β) has the self-financing property if

$$dV_t=\alpha_t\cdot dS_t+\beta_t dB_t. \tag{2.16}$$

It is well-known that condition (2.16) is equivalent to the fact that $V_t=f(t,S_t)$, $\alpha_t=\nabla_S f(t,S_t)$ and $\beta_t=e^{-rt}\left(f(t,S_t)-S_t\cdot\nabla_S f(t,S_t)\right)$ where $f=f(t,S)$ is a solution to the PDE

$$\frac{1}{2}\sum_{i,j=1}^N (\sigma\sigma^*)_{ij}\, S^iS^j\partial_{S^iS^j}f+rS\cdot\nabla_S f+\partial_t f-rf=0. \tag{2.17}$$

Putting $\log S=\left(\log S^1,\dots,\log S^N\right)$, by the change of variables

$$f(t,S)=e^{rt}u(t,\log S),$$

equation (2.17) becomes

$$\frac{1}{2}\sum_{i,j=1}^N (\sigma\sigma^*)_{ij}\,\partial_{x_ix_j}u(t,x)+b\cdot\nabla u(t,x)+\partial_t u(t,x)=0, \tag{2.18}$$

where b is the vector defined by

$$b_i=r-\frac{1}{2}\sum_{j=1}^N \sigma_{ij}^2,\qquad i=1,\dots,N. \tag{2.19}$$

By applying the Harnack inequality in Theorem 1.2, we get the following

Proposition 2.1 *Consider any self-financing and admissible (i.e. such that $V_t\geq 0$ for any t) portfolio defined on $[0,T[$. We have*

$$V(t,S_t)\leq e^{rt}H(S_0,S_t,t)V(0,S_0),\qquad 0\leq t<T, \tag{2.20}$$

where

$$H(S_0,S_t,t)=\left(\frac{T}{T-t}\right)^{\frac{N}{2}}e^{\frac{1}{2t}\left|\sigma^{-1}\left(\log\frac{S_t}{S_0}-tb\right)\right|^2}$$
$$=\left(\frac{T}{T-t}\right)^{\frac{N}{2}}\exp\left(\frac{1}{2t}\sum_{j=1}^N\left(\sum_{i=1}^N(\sigma^{-1})_{ij}\left(\log\frac{S_t^i}{S_0^i}-tb_i\right)\right)^2\right),$$

with b as in (2.19).

Remark 2.2 *Formula* (2.20) *provides an a priori estimate of the future value of a self-financing portfolio given in terms of the initial wealth. The estimate is sharp since it is given in terms of the optimal Harnack constant in* (1.13)*, and could be useful for portfolio optimization purposes.*

Formula (2.20) *gives also a proof of the absence of arbitrage opportunities in the market: indeed* (2.20) *implies that* $V(t, S_t)$ *cannot be positive starting from a null initial wealth and therefore the market is arbitrage free.*

We emphasize that (2.20) *gives a pointwise estimate of* $V(t, S_t)$ *and not simply an estimate of its expectation. Note also that the constant* $H(S_0, S_t, t)$ *blows up as* $t \to T$*: even this is a well-known feature of Harnack inequalities in PDEs theory, though it seems less intuitive from the financial point of view.*

We next show an example of a pricing PDE related to (1.5) (so that $d < N$ in (1.1)). Consider a geometric Asian option in the one-dimensional Black-Scholes model (cf. for instance [3]). In this case the risk neutral dynamics is given by

$$\begin{cases} dS_t = rS_t dt + \sigma_0 S_t dW_t, \\ dG_t = \log S_t dt. \end{cases} \tag{2.21}$$

Under the change of variables

$$X_t^1 = \log S_t, \qquad X_t^2 = G_t,$$

we obtain the linear system (1.5) with $\mu = r - \frac{\sigma_0^2}{2}$. As a consequence of Corollary 1.5 we get a pointwise bound for the value of any admissible portfolio related to Asian options (2.21).

Proposition 2.3 *Consider any self-financing and admissible portfolio for* (2.21)*, defined on* $[0, T[$*. We have*

$$V(t, S_t, G_t) \le e^{rt} H(S_0, S_t, G_t, t) V(0, S_0, 0), \qquad 0 < t < T,$$

where

$$H(S_0, S_t, G_t, t) = \left(\frac{T}{T-t}\right)^2 \exp\left(\frac{1}{2\sigma_0^2 t}\left(\log\frac{S_t}{S_0} - t\left(r - \frac{\sigma_0^2}{2}\right)\right)^2 + \frac{3}{2\sigma_0^2 t^3}\Big(2G_t - t\log\left(S_t S_0\right)\Big)^2\right).$$

3 Gradient estimate

The following gradient estimate for positive solutions to $Lu = 0$ holds.

Proposition 3.1 *Assume hypothesis [H.1] and set*

$$\varphi(t) = \log \sqrt{\det \mathcal{C}_t(t_1)}, \qquad t < t_1.$$

Then for any positive solution u to $Lu = 0$ in the strip $[t_0, t_1[\times\mathbb{R}^N$ we have

$$\frac{\langle A(t)\nabla u(t,x), \nabla u(t,x)\rangle}{2u(t,x)} \le -Yu(t,x) - \varphi'(t)u(t,x). \tag{3.22}$$

Proof. Let $\Gamma(t, x; t_1, y)$ be the fundamental solution of (1.7), defined for $x, y \in \mathbb{R}^N$ and $t < t_1$. We first show that Γ verifies the equation

$$\frac{\langle A(t)\nabla_x\Gamma(t,x;t_1,y), \nabla_x\Gamma(t,x;t_1,y)\rangle}{2\Gamma(t,x;t_1,y)} = -Y\Gamma(t,x;t_1,y) - \varphi'(t)\Gamma(t,x;t_1,y), \tag{3.23}$$

and then prove the gradient estimate (3.22) by means of a representation formula for u. From (1.6) it follows that

$$\begin{aligned}\log\Gamma(t,x;t_1,y) = &-\frac{N}{2}\log 2\pi - \varphi(t)\\ &-\frac{1}{2}\left\langle \mathcal{C}_t^{-1}(t_1)\left(y - m_{t,x}(t_1)\right), \left(y - m_{t,x}(t_1)\right)\right\rangle.\end{aligned}$$

Then we have

$$\nabla_x \log\Gamma(t,x;t_1,y) = -E_t^*(t_1)\mathcal{C}_t^{-1}(t_1)\left(m_{t,x}(t_1) - y\right) \tag{3.24}$$

and

$$\begin{aligned}-Y\log\Gamma(t,x;t_1,y) = &\left\langle b(t) + B(t)x, E_t^*(t_1)\mathcal{C}_t^{-1}(t_1)\left(m_{t,x}(t_1) - y\right)\right\rangle + \varphi'(t)\\ &+\frac{1}{2}\left\langle\left(\frac{d}{dt}\mathcal{C}_t^{-1}(t_1)\right)\left(y - m_{t,x}(t_1)\right), \left(y - m_{t,x}(t_1)\right)\right\rangle\\ &+\left\langle \mathcal{C}_t^{-1}(t_1)\left(m_{t,x}(t_1) - y\right), \frac{d}{dt}\left(m_{t,x}(t_1) - y\right)\right\rangle.\end{aligned} \tag{3.25}$$

Since Γ is the fundamental solution of (1.7), we have

$$L\log\Gamma(t,x;t_1,y) + \frac{1}{2}\left\langle A(t)\nabla_x\log\Gamma(t,x;t_1,y), \nabla_x\log\Gamma(t,x;t_1,y)\right\rangle = 0 \tag{3.26}$$

Therefore if we set

$$(f_{ij}(t)) := E_t^*(t_1)\mathcal{C}_t^{-1}(t_1)E_t(t_1)$$

and we use (3.24), we find

$$-Y\log\Gamma(t,x;t_1,y) = \frac{1}{2}\langle A(t)\nabla_x\log\Gamma(t,x;t_1,y),\nabla_x\log\Gamma(t,x;t_1,y)\rangle$$
$$-\frac{1}{2}\sum_{i,j=1}^{N} a_{ij}(t)f_{ij}(t).$$

Evaluating the above expression and (3.25) at $y = m_{t,x}(t_1)$, we finally obtain

$$\varphi'(t) = -\frac{1}{2}\sum_{i,j=1}^{N} a_{ij}(t)f_{ij}(t).$$

This proves that

$$Y\log\Gamma(t,x;t_1,y) + \frac{1}{2}\langle A(t)\nabla_x\log\Gamma(t,x;t_1,y),\nabla_x\log\Gamma(t,x;t_1,y)\rangle = -\varphi'(t)$$

which is equivalent to (3.23).

In order to conclude the proof, we fix $T < t_1$ and use the representation formula

$$u(t,x) = \int_{\mathbb{R}^N}\Gamma(t,x;T,y)u(T,y)dy, \qquad (t,x)\in[t_0,T]\times\mathbb{R}^N.$$

Then we have

$$-Yu - \varphi'(t)u = \int_{\mathbb{R}^N}\big(-Y\Gamma(\cdot,\cdot;T,y) - \varphi'(t)\Gamma(\cdot,\cdot;T,y)\big)\,u(T,y)dy =$$

(by (3.23))

$$= \frac{1}{2}\int_{\mathbb{R}^N}\frac{\langle A(t)\nabla\Gamma(\cdot,\cdot;T,y),\nabla\Gamma(\cdot,\cdot;T,y)\rangle}{\Gamma(\cdot,\cdot;T,y)}\,u(T,y)dy \geq$$

(by Hölder inequality)

$$\geq \frac{1}{2}\left(\int_{\mathbb{R}^N}\Gamma(\cdot,\cdot;T,y)\,u(T,y)dy\right)^{-1}\cdot$$
$$\cdot\langle A\int_{\mathbb{R}^N}\nabla\Gamma(\cdot,\cdot;T,y)u(T,y)dy,\int_{\mathbb{R}^N}\nabla\Gamma(\cdot,\cdot;T,y)u(T,y)dy\rangle$$
$$= \frac{\langle A\nabla u,\nabla u\rangle}{2u}.$$

□

4 Proof of Theorem 1.2

We first prove the following

Lemma 4.1 *The L-admissible path $\bar\gamma$ correspondent to the control*

$$\bar w(s) = \sigma^*(s)E_s(t)\mathcal{C}_t^{-1}(T)\left(y - m_{t,x}(T)\right),$$

minimizes the quadratic cost

$$\psi(w) = \int_t^T |w(s)|^2 ds.$$

Moreover the minimum cost is

$$\psi(\bar w) = \int_t^T |w(s)|^2 ds = \left\langle \mathcal{C}_t^{-1}(T)\left(y - m_{t,x}(T)\right), \left(y - m_{t,x}(T)\right)\right\rangle. \tag{4.27}$$

Proof. Consider the Hamiltonian function

$$\mathcal{H}(x,p,w) = |w|^2 + p\left(Bx + \sigma w + b\right), \qquad p = (p_1, \cdots, p_N),$$

related to the control problem

$$\begin{cases} \gamma'(s) = B(s)\gamma(s) + b(s) + \sigma(s)w(s), \\ \gamma(t) = x, \ \gamma(T) = y. \end{cases}$$

From the classical control theory (see, for instance, Theorem 3, p.180 in [14]), the optimal control is of the form

$$w(s) = \sigma^*(s)p^*(s) \tag{4.28}$$

with p such that $p' = -pB$. The L-admissible path corresponding to (4.28) is

$$\gamma(s) = m_{t,x}(s) + \mathcal{C}_t(s)E_t^{*-1}(s)p^*(t)$$

where $p^*(t)$ is determined by imposing the condition $\gamma(T) = y$: specifically we have

$$p^*(t) = E_t^*(T)\mathcal{C}_t^{-1}(T)\left(y - m_{t,x}(T)\right),$$

and this concludes the proof. □

Proof of Theorem 1.2. Let $\bar\gamma$ be the optimal L-admissible path in Lemma 4.1 and $\bar w$ the corresponding optimal control. By adding the quantity

$$\frac{1}{2}u(s,\bar\gamma(s))|\bar w(s)|^2 - \langle \sigma^*(s)\nabla u(s,\bar\gamma(s)), \bar w(s)\rangle$$

to both sides of (3.22) evaluated at the point $(s,\bar\gamma(s))$, we find

$$Yu(s,\bar\gamma(s)) + \langle \sigma^*(s)\nabla u(s,\bar\gamma(s)), \bar w(s)\rangle \le -\varphi'(s)u(s,\bar\gamma(s)) + \frac{1}{2}u(s,\bar\gamma(s))|\bar w(s)|^2.$$

Then using the fact that $\bar{\gamma}$ is an L-admissible path, we get

$$\frac{d}{ds}u(s,\bar{\gamma}(s)) \leq -\varphi'(s)u(s,\bar{\gamma}(s)) + \frac{1}{2}u(s,\bar{\gamma}(s))|\bar{w}(s)|^2.$$

Dividing by u and integrating in the variable s over the interval $[t,T]$, we finally prove that

$$\log\frac{u(T,y)}{u(t,x)} \leq \log\frac{\sqrt{\det\mathcal{C}_t(t_1)}}{\sqrt{\det\mathcal{C}_T(t_1)}} + \frac{1}{2}\int_s^t |\bar{w}(s)|^2\,ds,$$

or, equivalently,

$$u(T,y) \leq \sqrt{\frac{\det\mathcal{C}_t(t_1)}{\det\mathcal{C}_T(t_1)}}u(t,x)e^{\frac{1}{2}\psi(\bar{w})},$$

with $\psi(\bar{w})$ as in (4.27). □

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