A short course on American options

Andrea Pascucci
Dipartimento di Matematica, Università di Bologna
Piazza di Porta S. Donato 5
40126 Bologna (Italy)
email: pascucci@dm.unibo.it
Web: `http://www.dm.unibo.it/~pascucci/`

February 2009
Métodos Matemáticos y Simulación Numérica en Ingeniería y Ciencias Aplicadas
PhD program of the Universities of A Coruña, Santiago de Compostela and Vigo (Spain)

Contents

1 American options in discrete time

1.1 Discrete markets

We briefly recall the main results for market models in discrete time: we refer for instance to [38] and [32] for a comprehensive description and complete proofs. We consider a discrete market model, built on a probability space $(\Omega, \mathcal{F}, P)$ with Ω having a *finite* number of elements and where we assume that $P(\{\omega\}) > 0$ for every $\omega \in \Omega$. Given a time interval[1] $[0, T]$, we suppose that all transactions take place only at fixed dates

$$0 = t_0 < t_1 < \cdots < t_N = T,$$

and that the market consists of $d+1$ securities: one bond B with dynamics

$$\begin{cases} B_0 = 1, \\ B_n = B_{n-1}(1+r), \qquad n = 1, \dots, N, \end{cases} \tag{1.1}$$

where r is the risk-free rate, and d stocks $S = (S^1, \dots, S^d)$ with dynamics

$$\begin{cases} S_0^i \in \mathbb{R}_+, \\ S_n^i = S_{n-1}^i \left(1 + \mu_n^i\right), \qquad n = 1, \dots, N, \end{cases} \tag{1.2}$$

where μ_n^i is a real random variable that represents the yield rate of the i-th asset over the n-th period $[t_{n-1}, t_n]$. We set

$$\mu_n = (\mu_n^1, \dots, \mu_n^d)$$

and define the filtration

$$\begin{cases} \mathcal{F}_0 = \{\emptyset, \Omega\}, \\ \mathcal{F}_n = \sigma\left(\mu_k \mid k \leq n\right), \qquad n = 1, \dots, N. \end{cases}$$

As usual $\mathcal{F}_n$ represents the amount of information available in the market *at time* t_n. We assume that μ_n is independent of $\mathcal{F}_{n-1}$ for any $n = 1, \dots, N$, and $\mathcal{F}_N = \mathcal{F}$.

Definition 1.1. *A portfolio (or strategy) is a stochastic process in* $\mathbb{R}^{d+1}$

$$(\alpha, \beta) = (\alpha_n^1, \dots, \alpha_n^d, \beta_n)_{n=1,\dots,N}.$$

In the preceding definition α_n^i (respectively β_n) represents the amount of the asset S^i (resp. of the bond) held in the portfolio *during the n-th period,* i.e. from t_{n-1} to t_n. Therefore we define the value of the strategy (α, β) at time t_n as

$$V_n^{(\alpha,\beta)} := \alpha_n S_n + \beta_n B_n = \sum_{i=1}^{d} \alpha_n^i S_n^i + \beta_n B_n, \qquad n = 1, \dots, N, \tag{1.3}$$

and

$$V_0^{(\alpha,\beta)} = \alpha_1 S_0 + \beta_1 B_0.$$

Usually, when (α, β) is fixed, we omit the superscript and simply write V instead of $V^{(\alpha,\beta)}$.

Definition 1.2. *We denote by $\mathcal{A}$ the family of the strategies (α, β) that are*

[1] Let us recall that the unit of time is the year: to fix the ideas, $t = 0$ denotes today's date and T the expiration date of a derivative.

i) self-financing, that is the relation

$$V_{n-1} = \alpha_n S_{n-1} + \beta_n B_{n-1} \tag{1.4}$$

holds for every $n = 1, \dots, N$;

ii) predictable, that is (α_n, β_n) *is* $\mathcal{F}_{n-1}$*-measurable for every* $n = 1, \dots, N$.

Note that if $(\alpha, \beta) \in \mathcal{A}$ then $V^{(\alpha,\beta)}$ is a real stochastic process adapted to the filtration $(\mathcal{F}_n)$ and we have

$$V_n - V_{n-1} = \alpha_n (S_n - S_{n-1}) + \beta_n (B_n - B_{n-1}),$$

that is, the variation of the value of the portfolio only depends on the variation of the prices of the assets. The discounted price of the i-th asset is the defined by

$$\widetilde{S}_n^i = \frac{S_n^i}{B_n}, \qquad n = 0, \dots, N.$$

Note that, since $B_0 = 1$, then $\widetilde{S}_0^i = S_0^i$. The discounted value of the strategy (α, β) is

$$\widetilde{V}_n = \alpha_n \widetilde{S}_n + \beta_n.$$

Then the self-financing condition reads $\widetilde{V}_{n-1} = \alpha_n \widetilde{S}_{n-1} + \beta_n$ or equivalently

$$\widetilde{V}_n = \widetilde{V}_{n-1} + \alpha_n (\widetilde{S}_n - \widetilde{S}_{n-1}). \tag{1.5}$$

In particular we remark that a strategy is identified (recursively by (1.5)) by the initial wealth V_0 and the predictable process α.

1.2 Martingale measure and arbitrage price

Definition 1.3. *Given a discrete market* (B, S) *on the probability space* $(\Omega, \mathcal{F}, P)$*, a martingale measure is a probability* Q *on* $(\Omega, \mathcal{F})$ *such that:*

i) Q *is equivalent to* P*;*

ii) for every $n = 1, \dots, N$ *we have*

$$E^Q \left[\widetilde{S}_n \mid \mathcal{F}_{n-1} \right] = \widetilde{S}_{n-1}, \tag{1.6}$$

i.e $\widetilde{S}$ *is a* Q*-martingale.*

By the martingale property, we have

$$S_0 = \widetilde{S}_0 = E^Q \left[\widetilde{S}_n \right] \tag{1.7}$$

that has an important economic interpretation: it says that the expected value of the future normalized prices is equal to the current price. Therefore (1.7) is a *risk-neutral pricing formula*: the expected value of $\widetilde{S}_n$ with respect to the measure Q corresponds to the value given by an investor who reckons that the current market price of the asset is correct (and so he/she is neither disposed nor averse to buy the asset).

A key property of any strategy $(\alpha, \beta) \in \mathcal{A}$ is that its discounted value is a Q-martingale:

Proposition 1.4. *If Q is a martingale measure and $(\alpha,\beta)\in\mathcal{A}$, then $\widetilde{V}^{(\alpha,\beta)}$ is a Q-martingale: in particular the following risk-neutral pricing formula holds:*

$$V_0^{(\alpha,\beta)} = E^Q\left[\widetilde{V}_n^{(\alpha,\beta)}\right], \qquad n \le N. \tag{1.8}$$

Proof. Taking the expectation conditional to $\mathcal{F}_{n-1}$ in the second formula in (1.5), we get

$$E^Q\left[\widetilde{V}_n \mid \mathcal{F}_{n-1}\right] = \widetilde{V}_{n-1} + E^Q\left[\alpha_n(\widetilde{S}_n - \widetilde{S}_{n-1}) \mid \mathcal{F}_{n-1}\right] =$$

(since α is predictable)

$$= \widetilde{V}_{n-1} + \alpha_n E^Q\left[\widetilde{S}_n - \widetilde{S}_{n-1} \mid \mathcal{F}_{n-1}\right] = \widetilde{V}_{n-1}$$

since $\widetilde{S}$ is a Q-martingale. □

Definition 1.5. *A strategy $(\alpha,\beta)\in\mathcal{A}$ is called an arbitrage if its value V is such that*

i) $V_0 = 0$;

ii) $V_N \ge 0$;

iii) $P(V_N > 0) > 0$.

A discrete market is arbitrage-free if and only if there exist no arbitrage strategies.

Definition 1.6. *A European-style derivative (or simply, a claim) is a random variable X on $(\Omega,\mathcal{F},P)$. X is called replicable if $(\alpha,\beta)\in\mathcal{A}$ exists such that $X = V_N^{(\alpha,\beta)}$: in that case (α,β) is called replicating strategy. A market is complete if any European derivative is replicable.*

Theorem 1.7. [Fundamental theorem of asset pricing]

i) A discrete market is arbitrage-free if and only if there exists at least one martingale measure.

ii) In an arbitrage-free market, X is replicable if and only if $E^Q[X]$ is constant, independent of the martingale measure Q. In that case, for every replicating strategy $(\alpha,\beta)\in\mathcal{A}$ and for every martingale measure Q, it holds that

$$E^Q\left[\widetilde{X} \mid \mathcal{F}_n\right] = \widetilde{V}_n^{(\alpha,\beta)} =: \widetilde{H}_n, \qquad n = 0,\dots,N. \tag{1.9}$$

The process $\widetilde{H}$ is called discounted arbitrage price (or risk-neutral price) of X.

iii) An arbitrage-free market is complete if and only if there exists a unique martingale measure Q.

Note that, by (1.9), the arbitrage price of an option is independent of the martingale measure Q and of the replicating strategy (α,β). Actually one can show that, in an arbitrage free market, a claim X is replicable if and only if $E^Q[X]$ is independent of the martingale measure Q.

1.3 Binomial model

In the binomial model there is only one risky asset defined by

$$S_n = S_{n-1}(1+\mu_n), \qquad n = 1, \dots, N$$

where μ_n are i.i.d. random variables such that

$$1+\mu_n = \begin{cases} u & \text{with probability } p, \\ d & \text{with probability } 1-p, \end{cases}$$

for $p \in]0,1[$ and $0 < d < u$. In other terms, the law of μ_n is a linear combination of Dirac deltas: $p\delta_{u-1} + (1-p)\delta_{d-1}$. Consequently we have

$$P(S_n = u^k d^{n-k} S_0) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad 0 \le k \le n \le N.$$

The binomial model is the simplest example of a discrete market that is arbitrage-free and complete.

Theorem 1.8. *In the binomial model the condition*

$$d < 1 + r < u, \tag{1.10}$$

is equivalent to the existence and uniqueness of the martingale measure Q, under which $\mu_1, \dots, \mu_N$ are i.i.d. random variables and

$$Q(1+\mu_n = u) = q := \frac{1+r-d}{u-d}. \tag{1.11}$$

1.4 American options

From now on we assume that the market S is arbitrage-free, that is there exists at least a martingale measure Q.

An American derivative is characterized by the possibility of early exercise at every time t_n, $0 \le n \le N$, during the life span of the contract. To describe an American derivative it is therefore necessary to specify the premium (i.e. the payoff) that has to be paid to the owner in case he/she exercises the option at time t_n with $n \le N$. For example, in the case of an American Call option with underlying asset S and strike K, the payoff at time t_n is $X_n = (S_n - K)^+$.

Definition 1.9. *An American derivative is a non-negative discrete stochastic process $X = (X_n)$, adapted to the filtration $(\mathcal{F}_n)$.*

Since the choice of the best time to exercise an American option must depend only on the information available at that moment, the following definition of *exercise strategy* seems natural.

Definition 1.10. *A stopping time*

$$\nu : \Omega \longrightarrow \{0, 1, \dots, N\},$$

i.e. a random variable such that

$$\{\nu = n\} \in \mathcal{F}_n, \qquad n = 0, \dots, N, \tag{1.12}$$

is called exercise strategy (or exercise time). We denote the set of all exercise strategies by $\mathcal{T}_0$.

Intuitively, given a trajectory $\omega \in \Omega$ of the underlying market, the natural number $\nu(\omega)$ represents the moment when one decides to exercise the American derivative. The condition (1.12) merely means that the decision to exercise at time n depends on $\mathcal{F}_n$, i.e. on the information available at time t_n.

Definition 1.11. *Given an American option X and an exercise time $\nu \in \mathcal{T}_0$, the random variable X_ν defined by*

$$(X_\nu)(\omega) = X_{\nu(\omega)}(\omega), \qquad \omega \in \Omega,$$

is called payoff of X relative to the strategy ν. An exercise time ν_0 is called optimal in Q if

$$E^Q\left[\widetilde{X}_{\nu_0}\right] = \sup_{\nu \in \mathcal{T}_0} E^Q\left[\widetilde{X}_\nu\right]. \tag{1.13}$$

We observe that the random variable $\widetilde{X}_\nu$ can be interpreted as the discounted payoff of an *European option*: so $E^Q\left[\widetilde{X}_\nu\right]$ gives the risk-neutral price of the option (this depending of course on the martingale measure Q), when the option is exercised following the strategy ν.

In an arbitrage-free and complete market, the price of a *European* option with payoff X_N is by definition equal to the value of a replicating strategy: in particular, *the discounted arbitrage price is a Q-martingale.* Pricing an American option is a slightly more delicate matter since it is clear that it is generally not possible to determine a replicating strategy i.e. a strategy $(\alpha, \beta) \in \mathcal{A}$ such that $V_n^{(\alpha,\beta)} = X_n$ for any n: this is due to the fact that $\widetilde{V}^{(\alpha,\beta)}$ is a Q-martingale while $\widetilde{X}$ is simply an adapted process.

Let us begin by observing that, by arbitrage arguments, it is possible to find upper and lower bounds for an initial price of X that will be denoted by H_0. We set

$$\mathcal{A}_X^+ = \{(\alpha, \beta) \in \mathcal{A} \mid V_n^{(\alpha,\beta)} \geq X_n, \; n = 0, \dots, N\},$$

the family of those strategies in $\mathcal{A}$ that super-replicate X. To avoid introducing arbitrage opportunities, the price H_0 must be less or equal to the initial value $V_0^{(\alpha,\beta)}$ for every $(\alpha, \beta) \in \mathcal{A}_X^+$ and so

$$H_0 \leq \inf_{(\alpha,\beta) \in \mathcal{A}_X^+} V_0^{(\alpha,\beta)}.$$

On the other hand we put

$$\mathcal{A}_X^- = \{(\alpha, \beta) \in \mathcal{A} \mid \text{ there exists } \nu \in \mathcal{T}_0 \text{ s.t. } X_\nu \geq V_\nu^{(\alpha,\beta)}\}.$$

Intuitively, an element (α, β) of $\mathcal{A}_X^-$ represents a strategy on which a short position is assumed to borrow money and buy the American option, knowing that there exists an exercise strategy ν yielding a payoff X_ν greater or equal to $V_\nu^{(\alpha,\beta)}$, corresponding to the amount necessary to close the short position in the strategy (α, β). The initial price H_0 of X must necessarily be greater or equal to $V_0^{(\alpha,\beta)}$ for every $(\alpha, \beta) \in \mathcal{A}_X^-$: if this were not true, one could easily build an arbitrage strategy. Then we have

$$\sup_{(\alpha,\beta) \in \mathcal{A}_X^-} V_0^{(\alpha,\beta)} \leq H_0.$$

Therefore we determined an interval which the initial price H_0 must belong to, in order to avoid introducing arbitrage opportunities. Let us show now that risk-neutral pricing relative to an optimal exercise strategy respects such conditions.

Proposition 1.12. *In an arbitrage-free market, for every martingale measure Q it holds that*

$$\sup_{(\alpha,\beta) \in \mathcal{A}_X^-} V_0^{(\alpha,\beta)} \leq \sup_{\nu \in \mathcal{T}_0} E^Q\left[\widetilde{X}_\nu\right] \leq \inf_{(\alpha,\beta) \in \mathcal{A}_X^+} V_0^{(\alpha,\beta)}. \tag{1.14}$$

Proof. For any $(\alpha,\beta)\in\mathcal{A}_X^-$, there exists $\nu_0\in\mathcal{T}_0$ such that $V_{\nu_0}^{(\alpha,\beta)}\le X_{\nu_0}$. Further, $\widetilde{V}^{(\alpha,\beta)}$ is a Q-martingale and so by the Optional sampling theorem, we have

$$V_0^{(\alpha,\beta)}=\widetilde{V}_0^{(\alpha,\beta)}=E^Q\left[\widetilde{V}_{\nu_0}^{(\alpha,\beta)}\right]\le E^Q\left[\widetilde{X}_{\nu_0}\right]\le\sup_{\nu\in\mathcal{T}_0}E^Q\left[\widetilde{X}_\nu\right],$$

hence we obtain the first inequality in (1.14), by the arbitrariness of $(\alpha,\beta)\in\mathcal{A}_X^-$.

On the other hand, if $(\alpha,\beta)\in\mathcal{A}_X^+$ then, again by the Optional sampling theorem, for every $\nu\in\mathcal{T}_0$ we have

$$V_0^{(\alpha,\beta)}=E^Q\left[\widetilde{V}_\nu^{(\alpha,\beta)}\right]\ge E^Q\left[\widetilde{X}_\nu\right],$$

hence we get the second inequality in (1.14), by the arbitrariness of $(\alpha,\beta)\in\mathcal{A}_X^+$ and $\nu\in\mathcal{T}_0$. □

The definition of arbitrage price of an American option is based on the Doob's decomposition theorem which we first present.

Theorem 1.13. [Doob's decomposition theorem]
Every discrete adapted process H can be decomposed in

$$H=M+A \tag{1.15}$$

where M is a martingale such that $M_0=H_0$ and A is predictable and such that $A_0=0$. Moreover H is a super-martingale if and only if A is decreasing.

Proof. We set $M_0=H_0$, $A_0=0$ and define recursively

$$\begin{aligned}M_{n+1}&=M_n+H_{n+1}-E\left[H_{n+1}\mid\mathcal{F}_n\right]\\&=H_{n+1}+\sum_{k=0}^{n}\left(H_k-E\left[H_{k+1}\mid\mathcal{F}_k\right]\right),\end{aligned}\tag{1.16}$$

and

$$\begin{aligned}A_{n+1}&=A_n-\left(H_n-E\left[H_{n+1}\mid\mathcal{F}_n\right]\right)\\&=-\sum_{k=0}^{n}\left(H_k-E\left[H_{k+1}\mid\mathcal{F}_k\right]\right).\end{aligned}\tag{1.17}$$

The thesis follows straightforwardly. □

Under the hypothesis that the market is arbitrage-free and complete, the following theorem contains the definition of the initial arbitrage price of an American derivative X.

Theorem 1.14. *Suppose that there exists a unique martingale measure Q. Then there exists $(\alpha,\beta)\in\mathcal{A}_X^+\cap\mathcal{A}_X^-$ and so we have:*

i) $V_n^{(\alpha,\beta)}\ge X_n,\ n=0,\dots,N;$

ii) there exists $\nu_0\in\mathcal{T}_0$ such that $X_{\nu_0}=V_{\nu_0}^{(\alpha,\beta)}$.

Consequently

$$E^Q\left[\widetilde{X}_{\nu_0}\right]=V_0^{(\alpha,\beta)}=\sup_{\nu\in\mathcal{T}_0}E^Q\left[\widetilde{X}_\nu\right],\tag{1.18}$$

defines the initial arbitrage price of X.

Proof. The proof is constructive and is made up of two main steps:

1) construct the smallest super-martingale $\widetilde{H}$ greater than $\widetilde{X}$, usually called *Snell's envelope of the process* $\widetilde{X}$;

2) use Doob's decomposition theorem to isolate the martingale part of the process $\widetilde{H}$ and by this determine a strategy $(\alpha,\beta) \in \mathcal{A}_X^+ \cap \mathcal{A}_X^-$.

Then we can we conclude the proof of the theorem by showing that $\widetilde{H}_0 = \widetilde{V}_0^{(\alpha,\beta)} = V_0^{(\alpha,\beta)}$ and (1.18) holds.

First step: we define iteratively the stochastic process $\widetilde{H}$ by putting

$$\widetilde{H}_n = \begin{cases} \widetilde{X}_N, & n = N, \\ \max\left\{\widetilde{X}_n, E^Q\left[\widetilde{H}_{n+1} \mid \mathcal{F}_n\right]\right\}, & n = 0, \dots, N-1. \end{cases} \tag{1.19}$$

Below we will see that the process $\widetilde{H}$ defines the (discounted) arbitrage price of X. It is indeed an intuitive notion of price that gives rise to the definition above: in fact the option X is worth $H_N = X_N$ at maturity and, at time t_{N-1}, is worth

- X_{N-1} if one decides to exercise it;
- the arbitrage price of a European derivative with payoff H_N and maturity N, in case one decides not to exercise it, and we know that this equals $\frac{1}{1+r}E^Q\left[H_N \mid \mathcal{F}_{N-1}\right]$.

Then it seems reasonable to define

$$H_{N-1} = \max\left\{X_{N-1}, \frac{1}{1+r}E^Q\left[H_N \mid \mathcal{F}_{N-1}\right]\right\},$$

and by repeating this argument backwards, we get definition (1.19).

Evidently, $\widetilde{H}$ is an adapted non-negative stochastic process; further, for every n, we have

$$\widetilde{H}_n \geq E^Q\left[\widetilde{H}_{n+1} \mid \mathcal{F}_n\right], \tag{1.20}$$

that is $\widetilde{H}$ is a *Q-super-martingale*. This means that $\widetilde{H}$ "decreases in mean" and intuitively this corresponds to the fact that, moving forward in time, the advantage of the possibility of early exercise decreases.

Actually $\widetilde{H}$ *is the smallest super-martingale that dominates* $\widetilde{X}$*:* in fact, if M is a Q-super-martingale such that $M_n \geq \widetilde{X}_n$ then

$$M_n \geq \max\{\widetilde{X}_n, E^Q\left[M_{n+1} \mid \mathcal{F}_n\right]\}$$

for every n. Since

$$M_N \geq \widetilde{X}_N = \widetilde{H}_N,$$

the thesis follows by induction. We recall that, in probability theory, the smallest super-martingale that dominates a generic adapted process $\widetilde{X}$ is usually called *Snell's envelope of* $\widetilde{X}$.

Second step: we now prove that there exists $(\alpha,\beta) \in \mathcal{A}_X^+ \cap \mathcal{A}_X^-$. Since $\widetilde{H}$ is a Q-super-martingale, by Doob's decomposition Theorem 1.13 we get

$$\widetilde{H} = M + A$$

where M is a Q-martingale such that $M_0 = \widetilde{H}_0$ and A is a predictable decreasing process with null initial value.

By hypothesis the market is complete and so there exists a strategy $(\alpha, \beta) \in \mathcal{A}$ that replicates the European derivative M_N in the sense that $\widetilde{V}_N(\alpha, \beta) = M_N$. Further, since M and $\widetilde{V} := \widetilde{V}^{(\alpha,\beta)}$ are Q-martingales with the same terminal value, they are equal:

$$\widetilde{V}_n = E^Q\left[\widetilde{V}_N \mid \mathcal{F}_n\right] = E^Q\left[M_N \mid \mathcal{F}_n\right] = M_n. \tag{1.21}$$

Consequently, $(\alpha, \beta) \in \mathcal{A}_X^+$ since $A_n \leq 0$. Moreover we have

$$V_0 = M_0 = \widetilde{H}_0,$$

so that (α, β) is a super-replicating strategy for X that has an initial cost equal to the price of the option, as defined in (1.18).

In order to verify that $(\alpha, \beta) \in \mathcal{A}_X^-$, we put:

$$\nu_0(\omega) = \min\{n \mid \widetilde{H}_n(\omega) = \widetilde{X}_n(\omega)\}, \qquad \omega \in \Omega. \tag{1.22}$$

Since

$$\{\nu_0 = n\} = \{\widetilde{H}_0 > \widetilde{X}_0\} \cap \cdots \cap \{\widetilde{H}_{n-1} > \widetilde{X}_{n-1}\} \cap \{\widetilde{H}_n = \widetilde{X}_n\} \in \mathcal{F}_n$$

for every n, then ν_0 is an stopping time, i.e. an exercise strategy. Further, ν_0 is the first time that $\widetilde{X}_n \geq E^Q\left[\widetilde{H}_{n+1} \mid \mathcal{F}_n\right]$ and so intuitively it represents the first time that it is profitable to exercise the option.

According to Doob's decomposition Theorem (see in particular formula (1.16)), for $n = 1, \dots, N$, we have

$$M_n = \widetilde{H}_n + \sum_{k=0}^{n-1}\left(\widetilde{H}_k - E^Q\left[\widetilde{H}_{k+1} \mid \mathcal{F}_k\right]\right),$$

and consequently

$$M_{\nu_0} = \widetilde{H}_{\nu_0} \tag{1.23}$$

since

$$\widetilde{H}_k = E^Q\left[\widetilde{H}_{k+1} \mid \mathcal{F}_k\right] \quad \text{over} \quad \{k < \nu_0\}.$$

Then, by (1.21), we have

$$\widetilde{V}_{\nu_0} = M_{\nu_0} =$$

(by (1.23))

$$= \widetilde{H}_{\nu_0} =$$

(by the definition of ν_0)

$$= \widetilde{X}_{\nu_0}, \tag{1.24}$$

and this proves that $(\alpha, \beta) \in \mathcal{A}_X^-$.

Conclusion: let us show now that ν_0 is an optimal exercise time. Since $(\alpha, \beta) \in \mathcal{A}_X^+ \cap \mathcal{A}_X^-$, by (1.14) in Proposition 1.12 we get

$$V_0 = \sup_{\nu \in \mathcal{T}_0} E^Q\left[\widetilde{X}_\nu\right].$$

On the other hand, by (1.24) and the Optional sampling theorem, it holds that

$$V_0 = E^Q\left[\widetilde{X}_{\nu_0}\right]$$

and this concludes the proof. □

Remark 1.15. *The preceding theorem is significant from both a theoretical and practical point of view: on one hand it proves that there exists a unique initial price of X that does not give rise to arbitrage opportunities. On the other hand it provides us with a constructive way to determine the main features of X:*

i) the arbitrage price by the recursive formula (1.19)*;*

ii) a hedging strategy $(\alpha, \beta) \in \mathcal{A}_X^+ \cap \mathcal{A}_X^-$;

iii) an optimal exercise strategy ν_0.

□

Formula (1.19) readily gives a recursive algorithm for determining the arbitrage price of an American derivative: this is a particular case of a much more general methodology to solve stochastic optimal control problems, that is called **dynamic programming**. As an example, in Paragraph 1.8 we will use the dynamic programming to study the problem of pricing in an incomplete market.

Remark 1.16. *Fixed $n \le N$, we denote by*

$$\mathcal{T}_n = \{\nu \in \mathcal{T}_0 \mid \nu \ge n\}$$

the family of exercise strategies of an American derivative bought at time t_n. A strategy $\nu_n \in \mathcal{T}_n$ is optimal if it holds that

$$E^Q\left[\widetilde{X}_{\nu_n} \mid \mathcal{F}_n\right] = \sup_{\nu\in\mathcal{T}_n} E^Q\left[\widetilde{X}_{\nu} \mid \mathcal{F}_n\right].$$

If $\widetilde{H}$ is the process in (1.19)*, we denote the first time that it is profitable to exercise the American derivative bought at time n by*

$$\nu_n(\omega) = \min\{k \ge n \mid \widetilde{H}_k(\omega) = \widetilde{X}_k(\omega)\}, \qquad \omega \in \Omega.$$

We can extend Theorem 1.14 and prove that ν_n is the first optimal exercise time following n. To be more precise it holds that

$$\widetilde{H}_n = E^Q\left[\widetilde{X}_{\nu_n} \mid \mathcal{F}_n\right] = \sup_{\nu\in\mathcal{T}_n} E^Q\left[\widetilde{X}_{\nu} \mid \mathcal{F}_n\right]. \tag{1.25}$$

The process $\widetilde{H}$ in (1.19) *is called discounted arbitrage price of X.* □

1.5 Asymptotics: the free-boundary problem

In the binomial model we now fix T, set $\delta = \frac{T}{N}$ and let N go to infinity: a well-known consistency result states that, under natural assumptions, the binomial model approximates the standard Black & Scholes model. More precisely, let us assume the following specific form of the parameters of the model:

$$u = e^{\sigma\sqrt{\delta}}, \qquad d = e^{-\sigma\sqrt{\delta}}, \qquad 1 + r = e^{r_0\delta}, \tag{1.26}$$

for some positive constants σ and r_0.

Given a function $f = f(t, S)$ defined on $[0, T] \times \mathbb{R}_+$ (here f plays the role of the arbitrage price of an American option with payoff $\varphi(t, S)$), the recursive pricing formula (1.19) becomes

$$\begin{cases} f(T, S) = \varphi(T, S), \\ f(t, S) = \max\left\{\varphi(t, S), \frac{1}{1+r}\left(qf(t+\delta, uS) + (1-q)f(t+\delta, dS)\right)\right\}. \end{cases} \tag{1.27}$$

If we set

$$f = f(t,S), \qquad f^u = f(t+\delta, uS), \qquad f^d = f(t+\delta, dS),$$

and define the discrete operator

$$J_\delta f(t,S) = \frac{qf^u + (1-q)f^d}{1+r} - f \tag{1.28}$$

the second equation in (1.27) is equivalent to

$$\max\{J_\delta f(t,S), \varphi(t,S) - f(t,S)\} = 0.$$

By using the standard Taylor expansion, it is not difficult to prove the following

Proposition 1.17. *For every $f \in C^{1,2}([0,T]\times\mathbb{R}_+)$ we have*

$$\lim_{\delta\to 0^+} \frac{J_\delta f(t,S)}{\delta} = L_{\mathrm{BS}} f(t,S),$$

for $(t,S) \in]0,T[\times\mathbb{R}_+$, where

$$L_{\mathrm{BS}} f(t,S) := \partial_t f(t,S) + \frac{\sigma^2 S^2}{2}\partial_{SS} f(t,S) + r_0 S \partial_S f(t,S) - r_0 f(t,S) \tag{1.29}$$

is the Black&Scholes differential operator.

Proof. We first note that by (1.26)

$$q = \frac{1+\varrho-d}{u-d} = \frac{1}{2} + \frac{1}{2\sigma}\left(r - \frac{\sigma^2}{2}\right)\sqrt{\delta} + \mathrm{o}(\sqrt{\delta}) \tag{1.30}$$

as $\delta \to 0$. By the second order Taylor expansion we get (by simplicity, we simply write f instead of $f(t,S)$)

$$f^u - f = \partial_t f \delta + \partial_S f S(u-1) + \frac{1}{2}\partial_{SS} f S^2 (u-1)^2 + \mathrm{o}(\delta) + \mathrm{o}((u-1)^2) =$$

(by (1.26))

$$= \sigma S \partial_S f \sqrt{\delta} + Lf\delta + \mathrm{o}(\delta), \qquad \delta \to 0, \tag{1.31}$$

where

$$Lf = \partial_t f + \frac{\sigma^2}{2} S\partial_S f + \frac{\sigma^2 S^2}{2}\partial_{SS} f,$$

and analogously

$$f^d - f = -\sigma S \partial_S f \sqrt{\delta} + Lf\delta + \mathrm{o}(\delta), \qquad \delta \to 0. \tag{1.32}$$

Then we have

$$\begin{aligned} J_\delta f(t,S) &= -(1+\varrho)f + qf^u + (1-q)f^d \\ &= -r\delta f + q(f^u - f - (f^d - f)) + (f^d - f) + \mathrm{o}(\delta) = \end{aligned}$$

(by (1.31) and (1.32))

$$= -\delta r f + \delta L f + \sqrt{\delta}(2q-1)\sigma S \partial_S f + \mathrm{o}(\delta) =$$

(by (1.30))

$$\begin{aligned} &= -\delta r f + \delta L f + \sqrt{\delta}\left(\left(r - \frac{\sigma^2}{2}\right)\sqrt{\delta} + \mathrm{o}(\sqrt{\delta})\right)\sigma S \partial_S f + \mathrm{o}(\delta) \\ &= \delta L_{\mathrm{BS}} f + \mathrm{o}(\delta), \end{aligned}$$

as $\delta \to 0$ and this concludes the proof. □ □

By Proposition 1.17, the asymptotic version, as $\delta \to 0$, of the discrete problem (1.27) is given by

$$\begin{cases} \max\{L_{\rm BS} f, \varphi - f\} = 0, & \text{in } [0,T[\times\mathbb{R}_+, \\ f(T,S) = \varphi(T,S), & S \in \mathbb{R}_+. \end{cases} \tag{1.33}$$

The convergence of binomial to Black&Scholes prices of American options was proved in [31], [2] and [33].

Problem (1.33) is called a free boundary problem: it contains a *differential inequality* that, from the theoretical point of view, is much more complex to study than the usual parabolic Cauchy problem arising in the analysis of European options. Existence and uniqueness of the solution to problem (1.33) will be proved in the next chapters.

1.6 Binomial algorithm for American options

In this paragraph we consider an American options with payoff of the form $X_n = \varphi(n, S_n)$: this includes the American put as a particular case. We use the notation

$$S_{n,k} := u^k d^{n-k} S_0, \qquad n = 0, \dots, N \text{ and } k = 0, \dots, n, \tag{1.34}$$

and denote the payoff by

$$X_{n,k} = \varphi(n, S_{n,k}).$$

The recursive definition (1.19) gives the following iterative formula for the arbitrage price $H = (H_{n,k})$ of the derivative:

$$\begin{cases} H_{N,k} = X_{N,k}, & 0 \le k \le N, \\ H_{n-1,k} = \max\left\{X_{n-1,k}, \frac{1}{1+r}(qH_{n,k+1} + (1-q)H_{n,k})\right\}, & 0 \le k \le n-1, \end{cases} \tag{1.35}$$

for $n = 1, \dots, N$ and $q = \frac{1+r-d}{u-d}$.

Example 1.17. *Let us consider an American Put option with strike $K = 20$ and price of the underlying asset $S_0 = 20$ in a three-period binomial model with parameters*

$$u = 1.1, \qquad d = 0.9, \qquad r = 0.05.$$

The martingale measure is defined by

$$q = \frac{1+r-d}{u-d} = 0.75.$$

By using the algorithm (1.35), at every step we compare the risk-neutral price to the value in case of early exercise:

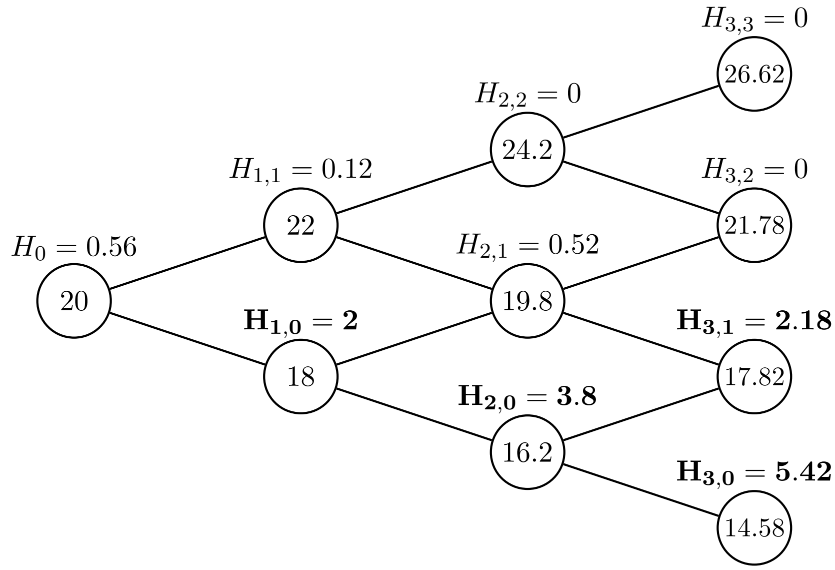
$$\begin{aligned} H_{n-1,k} &= \max\left\{X_{n-1,k}, \frac{1}{1+r}(qH_{n,k+1} + (1-q)H_{n,k})\right\} \\ &= \max\left\{X_{n-1,k}, \frac{1}{1.05}(0.75 * H_{n,k+1} + 0.25 * H_{n,k})\right\}. \end{aligned}$$

In Figure 1 we put the price of the underlying asset and of the derivative respectively inside and outside of the circle. The prices in boldface correspond to early exercise. For example, at the beginning we have that $X_0 = 0$ while

$$E^Q\left[\widetilde{H}_1\right] = \frac{1}{1.05}(0.75 * 0.12 + 0.25 * 2) = 0.56$$

and so it is not profitable to exercise immediately. □

Figure 1 Arbitrage prices of an American Put option with strike 20 and $S_0 = 20$ in a three-period binomial model with parameters $u = 1.1$, $d = 0.9$ and $r = 0.05$



Let us now dwell on the hedging problem, first recalling how it can be solved in the European case.

Remark 1.18. *We use notation* (1.34) *and denote by* $H_{n,k}$ *the price, at the node* $S_{n,k}$ *of the binomial tree, of an European option with payoff* $H_N = \varphi(S_N)$. *The replication condition* $V_n^{(\alpha,\beta)} = H_n$ *is equivalent to*

$$\begin{cases} \alpha_n u S_{n-1,k} + \beta_n B_n = H_{n,k+1}, \\ \alpha_n d S_{n-1,k} + \beta_n B_n = H_{n,k}. \end{cases}$$

The solution of the system gives the replicating strategy for the n-th period $[t_{n-1}, t_n]$*:*

$$\alpha_{n,k} = \frac{H_{n,k+1} - H_{n,k}}{(u-d)S_{n-1,k}}, \qquad \beta_{n,k} = \frac{uH_{n,k} - dH_{n,k+1}}{(u-d)(1+r)^n}. \tag{1.36}$$

Coming back to the American case, theoretically the proof of Theorem 1.14 is constructive (since it is based upon Doob's decomposition) and identifies the hedging strategy with the replicating strategy of the European derivative M_N. However, *M_N is a path-dependent derivative even if X is path-independent.* So the computation of the replicating strategy by the binomial algorithm can be burdensome, since M_N depends on the whole path of the underlying asset and not just on its final value. As a matter of fact, this approach is not used in practice.

Instead, it is worthwhile noting that the process M_n depends on the path of the underlying asset just because it has to keep track of the possible early exercises: but in the moment that the derivative is exercised, hedging is no longer necessary and the problem gets definitely easier.

To fix the ideas, in the preceding example we consider the time $n = 1$ and so we have two cases:

- if $S_1 = S_{1,1} = 22$ then
$$0.12 = H_{1,1} > X_{1,1} = 0,$$
so the option is not exercised, $M_2 = \widetilde{H}_2$ and we can use the usual replication argument (cf. formulas (1.36) below) to determine the strategy
$$\alpha_2 = \frac{H_{2,2} - H_{2,1}}{(u-d)S_{1,1}}, \qquad \beta_2 = \frac{uH_{2,1} - dH_{2,2}}{(u-d)(1+r)},$$
that, with an initial wealth $H_{1,1}$, hedges the American derivative at the subsequent time;

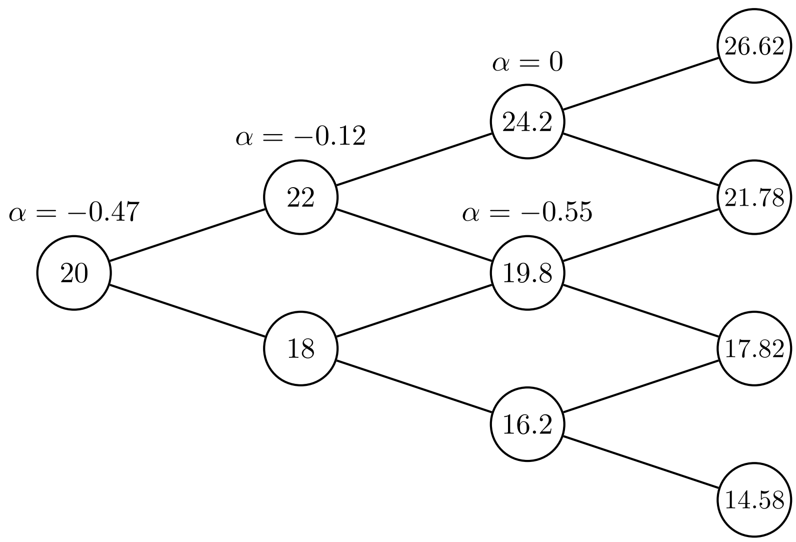
- if otherwise $S_1 = dS_{1,0} = 18$, then

$$1.28 = \frac{1}{1+r} E^Q [H_2] = \frac{qH_{2,1} + (1-q)H_{2,0}}{1+r} < X_{1,0} = 2,$$

  and so the option is exercised. Therefore the position is closed and it is not necessary to determine the hedging strategy[2].

By using standard binomial formulas (1.36) we determine the whole hedging strategy. In Figure 2 we depict the strategy of the preceding example.

Figure 2 Hedging strategy for an American Put option with strike 20 and $S_0 = 20$ in a three-period binomial model with parameters $u = 1.1$, $d = 0.9$ and $r = 0.05$



1.7 Trinomial model

In the trinomial model we assume that there exists only one risky asset whose dynamics is given by

$$S_n = S_{n-1}(1 + \mu_n), \qquad n = 1, \dots, N$$

where μ_n are i.i.d. random variables such that

$$1 + \mu_n = \begin{cases} u & \text{with probability } p_1, \\ m & \text{with probability } p_2, \\ d & \text{with probability } p_3 = 1 - p_1 - p_2, \end{cases} \tag{1.37}$$

for $p_1, p_2 \in]0,1[$ and $0 < d < m < u$.

The trinomial model is the simplest example of an arbitrage-free and incomplete market. Trying to determine a martingale measure by imposing condition (1.6) we infer

$$S_{n-1}(1+r) = E^Q [S_{n-1} (1 + \mu_n) \mid \mathcal{F}_{n-1}], \tag{1.38}$$

[2] Anyway $H_{1,1} = 1.28$ is enough to super-replicate $H_{2,1}$ and $H_{2,0}$ at the subsequent time. In general, the hedging formulas (1.36) provide *a self-financing strategy that super-replicates the payoff of the American option.*

and by setting, for $n = 1, \dots, N$,

$$q_1^n = Q(1+\mu_n = u \mid \mathcal{F}_{n-1}), \quad q_2^n = Q(1+\mu_n = m \mid \mathcal{F}_{n-1}), \quad q_3^n = Q(1+\mu_n = d \mid \mathcal{F}_{n-1}),$$

we get the linear system

$$\begin{cases} uq_1^n + mq_2^n + dq_3^n = 1+r, \\ q_1^n + q_2^n + q_3^n = 1, \end{cases} \tag{1.39}$$

that generally admits infinite solutions, thus proving the non uniqueness of the martingale measure. On the other hand, the condition $V_N = X$ of replicability for a claim X in the last period, gives the linear system (dual of (1.39))

$$\begin{cases} \alpha_N u S_{N-1} + \beta_N B_N = X^u, \\ \alpha_N m S_{N-1} + \beta_N B_N = X^m, \\ \alpha_N d S_{N-1} + \beta_N B_N = X^d, \end{cases}$$

that generally does not admit a solution (α_N, β_N) thus proving the incompleteness of the market.

1.8 Pricing in an incomplete market by Dynamic Programming

We consider a standard trinomial market model, with $N = 2$, where the dynamics of the risky asset is given by

$$S_0 = 1, \qquad S_n = S_{n-1}(1+\mu_n), \qquad n = 1, 2$$

where μ_n are i.i.d. random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, such that

$$P(\mu_n = -1/2) = P(\mu_n = 0) = P(\mu_n = 1) = \frac{1}{3}, \qquad n = 1, 2.$$

We assume that the short rate is null, $r = 0$.

We consider the problem of pricing and hedging an European Call option with payoff

$$\varphi(S_2) = (S_2 - 1)^+,$$

by minimization of the "shortfall" risk criterion. More precisely, by means of the Dynamic Programming (DP) algorithm, we aim to determine a self-financing strategy with non-negative value V (that is, such that $V_n \geq 0$ for any n) that minimizes

$$E^P \left[\mathcal{U}(V_2, S_2) \right],$$

where

$$\mathcal{U}(V, S) = (\varphi(S) - V)^+$$

is the shortfall risk function.

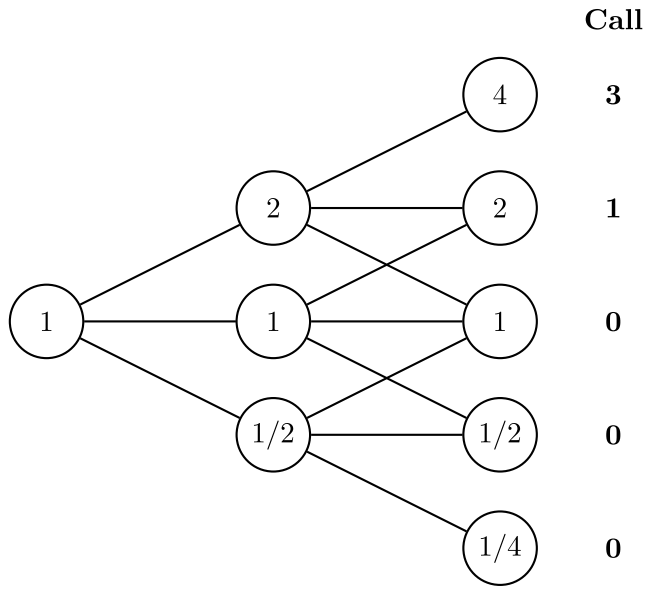
We first represent the binomial tree with the prices of the underlying asset. By (1.5), the value V of a self-financing strategy (α, β) satisfies

$$V_n = V_{n-1} + \alpha_n S_{n-1}\mu_n = V_{n-1} + \begin{cases} \alpha_n S_{n-1}, \\ 0, \\ -\frac{\alpha_n S_{n-1}}{2}. \end{cases} \tag{1.40}$$

Then $V_n \geq 0$ for any n if and only if $V_0 \geq 0$ and

$$-\frac{V_{n-1}}{S_{n-1}} \leq \alpha_n \leq \frac{2V_{n-1}}{S_{n-1}}, \qquad n = 1, 2.$$

In the general framework of a model with N periods, the DP algorithm consists of two steps:



i) we compute

$$R_{N-1}(V,S) := \min_{\alpha\in\left[-\frac{V}{S},\frac{2V}{S}\right]} E^P\left[\mathcal{U}\left(V+S\alpha\mu_N, S\left(1+\mu_N\right)\right)\right]$$

for S varying among the possible values of S_{N-1}. Recalling that we are considering predictable strategies, we denote by $\alpha_N = \alpha_N(V)$ the minimum point for V varying among the possible values of V_{N-1};

ii) for $n \in \{N-1, N-2, \dots, 1\}$, we compute

$$R_{n-1}(V,S) := \min_{\alpha\in\left[-\frac{V}{S},\frac{2V}{S}\right]} E^P\left[R_n\left(V+S\alpha\mu_n, S\left(1+\mu_n\right)\right)\right]$$

for S varying among the possible values of S_{n-1}. We denote by $\alpha_n = \alpha_n(V)$ the minimum point for V varying among the possible values of V_{n-1}.

In our setting, as a first step of the DP algorithm we compute $R_1(V,S)$ for $S \in \left\{2, 1, \frac{1}{2}\right\}$. We have

$$\begin{aligned} R_1(V,2) &= \min_{\alpha\in[-V/2,V]} E^P\left[\mathcal{U}\left(V+2\alpha\mu_2, 2(1+\mu_2)\right)\right] \\ &= \min_{\alpha\in[-V/2,V]} E^P\left[\left((2(1+\mu_2)-1)^+ - (V+2\alpha\mu_2)\right)^+\right] \\ &= \min_{\alpha\in[-V/2,V]} \frac{1}{3}\left((3-V-2\alpha)^+ + (1-V)^+\right) = \frac{4}{3}(1-V)^+, \end{aligned}$$

and the minimum is attained in

$$\alpha_2 = V. \tag{1.41}$$

Next we have

$$\begin{aligned} R_1(V,1) &= \min_{\alpha\in[-V,2V]} E^P\left[\mathcal{U}\left(V+\alpha\mu_2, 1+\mu_2\right)\right] \\ &= \min_{\alpha\in[-V,2V]} E^P\left[\left(\mu_2^+ - (V+\alpha\mu_2)\right)^+\right] \\ &= \min_{\alpha\in[-V,2V]} \frac{1}{3}(1-V-\alpha)^+ = \frac{1}{3}(1-3V)^+, \end{aligned}$$

and the minimum is attained in

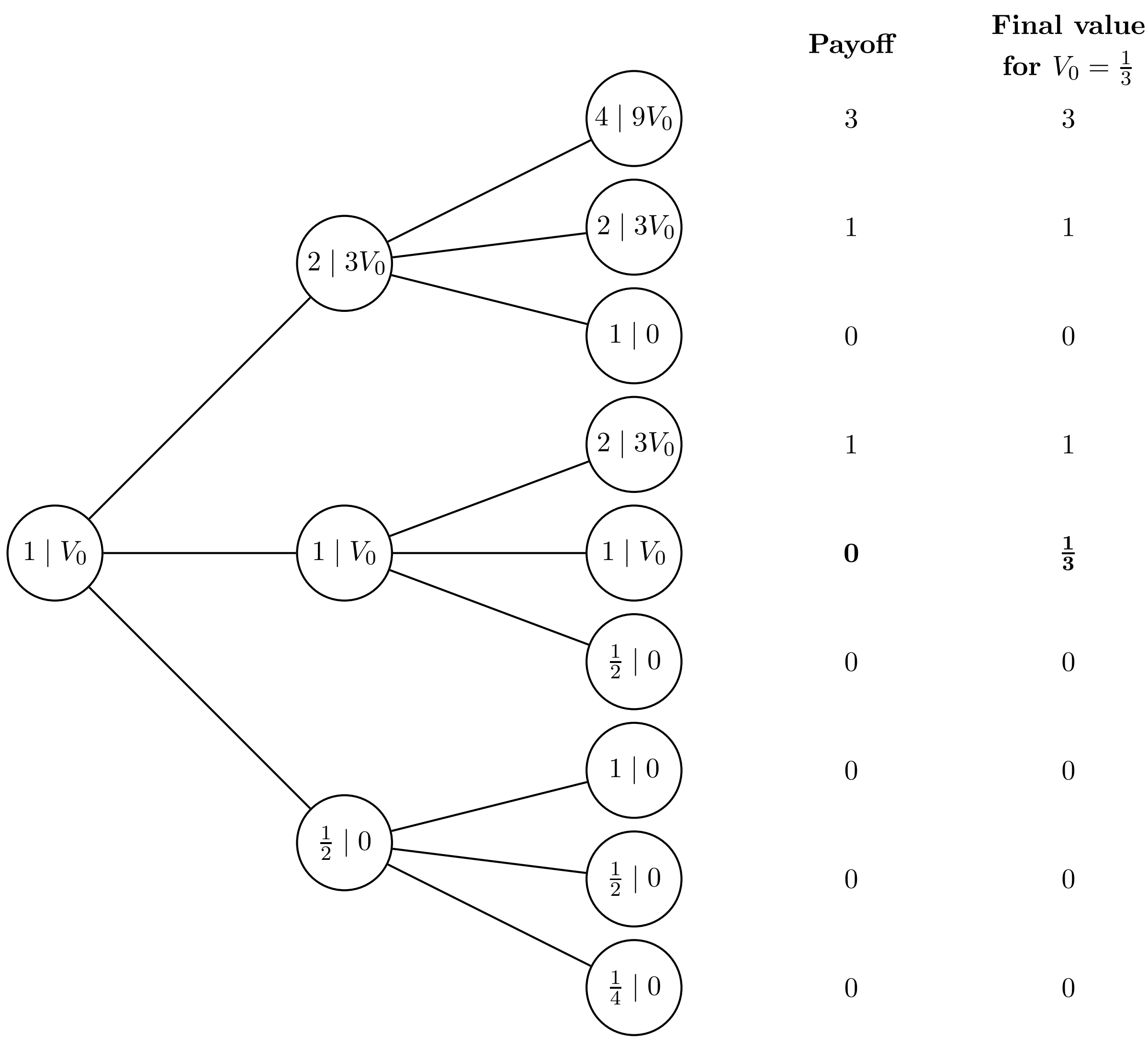
$$\alpha_2 = 2V. \tag{1.42}$$

Moreover we have

$$\begin{aligned}
R_1\left(V, \frac{1}{2}\right) &= \min_{\alpha\in[-2V,4V]} E^P\left[\mathcal{U}\left(V + \frac{\alpha\mu_2}{2}, \frac{1+\mu_2}{2}\right)\right] \\
&= \min_{\alpha\in[-2V,4V]} E^P\left[\left(\underbrace{\left(\frac{1+\mu_2}{2} - 1\right)^+}_{=0} - \underbrace{\left(V + \frac{\alpha\mu_2}{2}\right)}_{\geq 0}\right)^+\right] = 0,
\end{aligned}$$

and the minimum is attained in any

$$\alpha_2 \in [-2V, 4V]. \tag{1.43}$$



The second step consists in computing the risk at the initial time:

$$
\begin{aligned}
R_0(V,1) &= \min_{\alpha\in[-V,2V]} E^P\left[R_1(V+\alpha\mu_1, 1+\mu_1)\right] \\
&= \frac{1}{3}\min_{\alpha\in[-V,2V]}\left(R_1(V,1)+R_1(V+\alpha,2)\right) \\
&= \frac{1}{3}\min_{\alpha\in[-V,2V]}\left(\frac{1}{3}(1-3V)^+ + \frac{4}{3}(1-(V+\alpha))^+\right) \\
&= \frac{5}{9}(1-3V)^+, \qquad (1.44)
\end{aligned}
$$

and the minimum is attained in

$$\alpha_1 = 2V. \qquad (1.45)$$

By formula (1.44) for $R_0(V,1)$, it is clear that an initial wealth $V \geq \frac{1}{3}$ is sufficient to make the shortfall risk null or, in more explicit terms, *to super-replicate the payoff.*

Next we determine the shortfall strategy, that is the self-financing strategy that minimizes the shortfall risk. Let us denote by V_0 the initial wealth: by (1.45) we have

$$\alpha_1 = 2V_0.$$

Consequently, by (1.40) we get

$$
V_1 = V_0 + \begin{cases} 2V_0, & \text{for } \mu_1 = 1, \\ 0, & \text{for } \mu_1 = 0, \\ -V_0, & \text{for } \mu_1 = -\frac{1}{2}. \end{cases}
$$

Then by (1.41)-(1.42)-(1.43) we have

$$
\alpha_2 = \begin{cases} 3V_0, & \text{if } S_1 = 2, \\ 2V_0, & \text{if } S_1 = 1, \\ 0, & \text{if } S_1 = \frac{1}{2}, \end{cases}
$$

and we can easily compute the final value V_2 by means of (1.40). We represent in the figure the trinomial tree with the prices of the underlying asset and the values of the shortfall strategy inside the circles. On the right side we also indicate the final values of the option and of the shortfall strategy corresponding to $V_0 = \frac{1}{3}$. We remark that we have perfect replication in all scenarios except for the trajectory $S_0 = S_1 = S_2 = 1$ for which we have super-replication: the terminal value of the shortfall strategy $V_2 = \frac{1}{3}$ is strictly greater than the payoff of the call option that in this case is null.

2 Obstacle problem for parabolic PDEs

In this chapter we prove the existence of a solution to the free boundary problem

$$\begin{cases} \max\{Lu - ru, \varphi - u\} = 0, & \text{in } \mathcal{S}_T :=]0, T[\times\mathbb{R}^N, \\ u(0,\cdot) = \varphi, & \text{in } \mathbb{R}^N. \end{cases} \tag{2.46}$$

This corresponds to the construction of the Snell envelope (cf. Step 1 in the proof of Theorem 1.14). Note that (2.46) is the continuous-time version of (1.19).

In (2.46) L is a parabolic operator with variable coefficients of the form

$$Lu := \frac{1}{2}\sum_{i,j=1}^{N} c_{ij}\partial_{x_i x_j} u + \sum_{i=1}^{N} b_i \partial_{x_i} u - \partial_t u, \tag{2.47}$$

where (t, x) is an element of $\mathbb{R} \times \mathbb{R}^N$ and (c_{ij}) is a symmetric matrix, under the assumption that the coefficients $c_{ij} = c_{ij}(t, x)$, $b_j = b_j(t, x)$ and $r = r(t, x)$ are bounded Hölder continuous functions (cf. Hypothesis 2). We shall systematically use the notation

$$L_r u = Lu - ru.$$

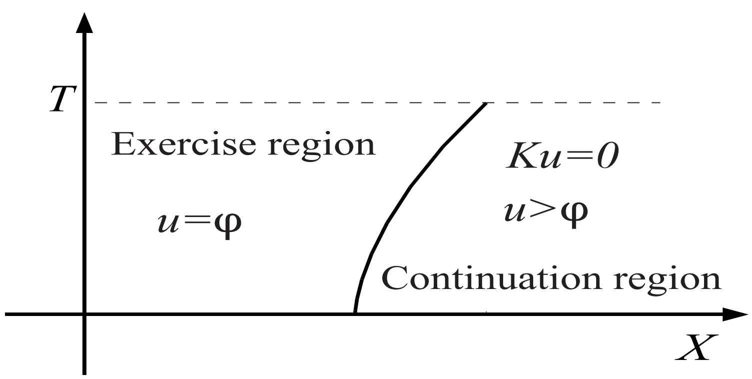
In Chapter 4 we shall prove that the price of an American option can be expressed in terms of the solution u to (2.46). By the first equation in (2.46) we get that $u \geq \varphi$ so the strip

$$\mathcal{S}_T :=]0, T[\times\mathbb{R}^N$$

is divided in two parts:

i) *the exercise region* where $u = \varphi$;

ii) *the continuation region* where $u > \varphi$ and $L_r u = 0$ i.e. the price of the derivative verifies a Black-Scholes' type PDE.



Indeed problem (2.46) is equivalent to:

$$\begin{cases} L_r u \leq 0, & \text{in } \mathcal{S}_T, \\ u \geq \varphi, & \text{in } \mathcal{S}_T, \\ (u - \varphi)\, L_r u = 0, & \text{in } \mathcal{S}_T, \\ u(0, x) = \varphi(0, x), & x \in \mathbb{R}^N. \end{cases} \tag{2.48}$$

This kind of problem is usually called *obstacle problem.* The solution is a function such that:

i) it is super-solution[3] of L_r (i.e. it holds that $L_r u \leq 0$);

ii) it is greater or equal to the obstacle which is represented by the function φ;

iii) it solves the equation $L_r u = 0$ when $u > \varphi$;

iv) it assumes the initial condition.

Actually we can verify that u is the smallest super-solution greater than the obstacle, in analogy with the notion of Snell envelope. The obstacle problem is a particular *free-boundary problem*, since the boundary that separates the continuation and exercise regions is an unknown of the problem.

One of the main features of problem (2.46) is that in general it does not admit a classical solution belonging to C^2 even if φ is a smooth function. Therefore it is necessary to introduce a weak formulation of the problem that may be based upon different notions of a generalized solution. A general theory of existence and regularity has been developed by many authors since the seventies.

The *variational approach* to problem (2.48) consists of looking for the solution as a minimum of a functional within an appropriate functional space whose elements possess *first order square integrable weak derivatives.* In the literature the variational approach has been developed in Bensoussan and Lions [9], Kinderlehrer and Stampacchia [30], Friedman [21] and applied to financial modeling by Bensoussan [7], Jaillet, Lamberton and Lapeyre [24]. More recently, since the introduction of the notion of viscosity solution (cf. Crandall, Ishii and Lions [13]), the pricing of American options in the *viscosity* sense have been studied in Barles [3], Fleming and Soner [17], Varadhan [41]. The notions of variational solution and, above all, of viscosity solution are very weak and allow one to get existence results under very general hypotheses.

Another notion of generalized solution, the so called *solution in strong sense or strong solution,* has been studied (cf. Friedman [20]). Strong solutions have *second order weak derivatives* so that the PDE can be written pointwisely a.e.; even though the theory of strong solutions generally requires more restrictive hypotheses (that are indeed verified practically in all the actual cases), strong solutions should be preferable in financial applications because of their better regularity properties. For this reason, we shall seek the solution to the problem (2.46) in this framework, following the presentation in [16], [38] and [39].

2.1 Fundamental solutions and the Cauchy problem

We suppose that the operator L in (2.47) is uniformly parabolic, i.e. the following holds:

Hypothesis 1. *There exists a positive constant* Λ *such that*

$$\Lambda^{-1}|\xi|^2 \leq \sum_{i,j=1}^{N} c_{ij}(t,x)\xi_i\xi_j \leq \Lambda|\xi|^2, \qquad t \in \mathbb{R}, \ x, \xi \in \mathbb{R}^N. \tag{2.50}$$

The prototype for the class of uniformly parabolic operators is the heat operator with constant coefficients, that has the identity matrix as (c_{ij}) and $b_i \equiv 0$.

[3] The term "super-solution" comes from the well-known fact in the classical theory of differential equations, that under rather general hypotheses, by the maximum principle, it holds that $L_r u \leq 0$ if and only if $u \geq H_u^O$ for every domain O for which the Dirichlet problem for L_r with boundary datum u

$$\begin{cases} L_r H = 0, & \text{in } O, \\ H|_{\partial O} = u, \end{cases} \tag{2.49}$$

is solvable with solution $H = H_u^O$.

In the theory of parabolic equations, it is natural to give to the time variable t "double weight" with respect to the space variables x. In order to introduce the hypothesis below, we define the *parabolic Hölder spaces.*

Definition 2.19. *Let $\alpha \in]0,1[$ and O be an open subset of $\mathbb{R}^{N+1}$. We denote with $C_P^\alpha(O)$, the space of functions u, bounded on O and for which there exists a constant C such that*

$$|u(t,x) - u(s,y)| \le C\left(|t-s|^{\frac{\alpha}{2}} + |x-y|^\alpha\right), \tag{2.51}$$

for every $(t,x),(s,y) \in O$. We define the norm

$$\|u\|_{C_P^\alpha(O)} = \sup_{(t,x)\in O} |u(t,x)| + \sup_{\substack{(t,x),(s,y)\in O\\(t,x)\neq(s,y)}} \frac{|u(t,x)-u(s,y)|}{|t-s|^{\frac{\alpha}{2}} + |x-y|^\alpha}.$$

Let us denote respectively with $C_P^{1+\alpha}(O)$ and $C_P^{2+\alpha}(O)$ the Hölder spaces defined by the following norms:

$$\|u\|_{C_P^{1+\alpha}(O)} = \|u\|_{C_P^\alpha(O)} + \sum_{i=1}^N \|\partial_{x_i} u\|_{C_P^\alpha(O)},$$

$$\|u\|_{C_P^{2+\alpha}(O)} = \|u\|_{C_P^{1+\alpha}(O)} + \sum_{i,j=1}^N \|\partial_{x_i x_j} u\|_{C_P^\alpha(O)} + \|\partial_t u\|_{C_P^\alpha(O)}.$$

We write $u \in C_{P,\mathrm{loc}}^{k+\alpha}(O)$ if $u \in C_P^{k+\alpha}(O_1)$ for every bounded open set O_1 such that $\overline{O}_1 \subseteq O$.

We assume the following regularity hypothesis on the coefficients of the operator:

Hypothesis 2. *The coefficients are bounded and Hölder continuous: $c_{ij}, b_j, r \in C_P^\alpha(\mathbb{R}^{N+1})$ for some $\alpha \in]0,1[$ and for every $1 \le i,j \le N$.*

The following classical results hold (see, for instance, [19] or the more recent exposition [15]):

Theorem 2.20. [Existence of a fundamental solution]
Under the Hypotheses 1 and 2, the operator L_r has a fundamental solution $\Gamma = \Gamma(t,x;s,y)$ that is a positive function, defined for $x,y \in \mathbb{R}^N$ and $t > s$, such that for every bonded and continuous function φ on $\mathbb{R}^N$, the function u defined by

$$u(t,x) = \int_{\mathbb{R}^N} \Gamma(t,x;s,y)\varphi(y)dy, \qquad x \in \mathbb{R}^N,\ t > s, \tag{2.52}$$

and by $u(s,\cdot) = \varphi$, belongs to $C_P^{2+\alpha}(]s,+\infty[\times\mathbb{R}^N) \cap C([s,+\infty[\times\mathbb{R}^N)$ and solves the Cauchy problem

$$\begin{cases} Lu - ru = 0, & \text{in }]s,+\infty[\times\mathbb{R}^N,\\ u(s,\cdot) = \varphi, & \text{in } \mathbb{R}^N. \end{cases} \tag{2.53}$$

2.2 Functional setting and a priori estimates

We now introduce the definition of parabolic Sobolev spaces where we aim to set the obstacle problem and we present some preliminary results to prove the existence of a strong solution. The proof of such results can be found, for example, in Lieberman [34].

Definition 2.21. *Given a domain O in $\mathbb{R} \times \mathbb{R}^N$ and $1 \le p \le \infty$, we denote with $S^p(O)$ the space of the functions $u \in L^p(O)$ for which the weak derivatives*

$$\partial_{x_i} u, \partial_{x_i x_j} u, \partial_t u \in L^p(O)$$

for every $i, j = 1, \dots, N$. We write $u \in S^p_{\mathrm{loc}}(O)$ if $u \in S^p(O_1)$ for every bounded domain O_1 such that $\overline{O}_1 \subseteq O$.

We point out that, as in Definition 2.19 of the parabolic Hölder spaces, the time derivative has double weight.

Definition 2.22. *A strong solution to the problem (2.46) is a function $u \in S^1_{\mathrm{loc}}(\mathcal{S}_T) \cap C(\overline{\mathcal{S}}_T)$ satisfying the equation*

$$\max\{L_r u, \varphi - u\} = 0$$

almost everywhere in $\mathcal{S}_T$ and assuming the initial datum pointwisely. We say that $\bar{u}$ is a strong super-solution to (2.46) if $u \in S^1_{\mathrm{loc}}(\mathcal{S}_T) \cap C(\overline{\mathcal{S}}_T)$ and it verifies

$$\begin{cases} \max\{L_r \bar{u}, \varphi - \bar{u}\} \le 0, & a.e.\ in\ \mathcal{S}_T, \\ \bar{u}(0, \cdot) \ge \varphi, & in\ \mathbb{R}^N, \end{cases} \tag{2.54}$$

The parabolic version of the classical Sobolev-Morrey embedding theorem holds. In the following statements O_1, O_2 denote bounded domains in $\mathbb{R} \times \mathbb{R}^N$ with $\overline{O}_1 \subseteq O_2$.

Theorem 2.23 (Sobolev-Morrey embedding theorem)**.** *For every $p > N + 2$ there exists a positive constant C depending only on p, N, O_1 and O_2, such that*

$$\|u\|_{C_P^{1+\alpha}(O_1)} \le C \|u\|_{S^p(O_2)}, \qquad \alpha = 1 - \frac{N+2}{p},$$

for all $u \in S^p(O_2)$.

A second useful result from classical functional analysis is the following a priori interior estimate.

Theorem 2.24 (Interior estimates in S^p)**.** *Assume that L_r satisfies Hypothesis 1 and has bounded continuous coefficients. Then for every $p \in]1, \infty[$ there exists a positive constant C, depending only on p, N, L_r, O_1 and O_2, such that*

$$\|u\|_{S^p(O_1)} \le C \left(\|u\|_{L^p(O_2)} + \|L_r u\|_{L^p(O_2)} \right),$$

for all $u \in S^p(O_2)$.

2.3 Strong solutions

We lay down the hypotheses on the obstacle function:

Hypothesis 3. *The function φ is continuous on $\overline{\mathcal{S}}_T$, locally Lipschitz continuous and for every bounded open set O such that $\overline{O} \subseteq \mathcal{S}_T$ there exists a constant C such that*

$$\sum_{i,j=1}^{N} \xi_i \xi_j \partial_{x_i x_j} \varphi(t, x) \ge C|\xi|^2 \qquad \xi \in \mathbb{R}^N,\ (t, x) \in O, \tag{2.55}$$

in the distributional sense, i.e.

$$\sum_{i,j=1}^{N} \xi_i \xi_j \int_O \varphi \partial_{x_i x_j} \psi \ge C|\xi|^2 \int_O \psi,$$

for all $\xi \in \mathbb{R}^N$ and $\psi \in C_0^\infty(O)$ with $\psi \ge 0$.

Condition (2.55) gives the local lower boundedness of the matrix of the second order spatial (distributional) derivatives. We point out that *all the functions belonging to C^2 verify Hypothesis 3 and also all the locally Lipschitz continuous and convex functions,* including so the payoff functions of the call and put options. On the contrary the function $\varphi(x) = -x^+$ does not satisfy condition (2.55) since its second order distributional derivative is a Dirac delta with negative sign that is "not bounded from below".

Remark 2.25. *It is worth noting, since it will be used in the sequel, that a consequence of the previous hypothesis is that $L_r\varphi$ is locally lower bounded.*

The main result of this chapter is the following existence result.

Theorem 2.26. *Under the Hypotheses 1, 2 and 3, if there exists a strong super-solution $\bar{u}$ to the problem* (2.46)*, then there exists also a strong solution u such that $u \le \bar{u}$ in $\mathcal{S}_T$. Moreover, $u \in S^p_{\text{loc}}(\mathcal{S}_T)$ for every $p \ge 1$ and consequently, by the embedding Theorem 2.23, $u \in C^{1+\alpha}_{P,\text{loc}}(\mathcal{S}_T)$ for all $\alpha \in]0,1[$.*

Remark 2.27. *In typical financial applications, the obstacle is related to the option payoff function ψ: for example, in the case of a call option, $N = 1$ and*

$$\psi(S) = (S - K)^+, \qquad S > 0.$$

In general, if ψ is a Lipschitz continuous function, then there exists a positive constant C such that

$$|\psi(S)| \le C(1 + S), \qquad S > 0,$$

and after the transformation

$$\varphi(t, x) = \psi(t, e^x),$$

we have that

$$|\varphi(t, x)| \le C(1 + e^x), \qquad x \in \mathbb{R}.$$

In this case a super-solution of the obstacle problem is

$$\bar{u}(t, x) = Ce^{\gamma t}\left(1 + e^x\right), \qquad t \in [0, T],\ x \in \mathbb{R},$$

where γ is an appropriate positive constant: in fact it is evident that $\bar{u} \ge \varphi$ and moreover, when $N = 1$,

$$L_r\bar{u} = Ce^{\gamma t}\left(-r - \gamma\right) + Ce^{x+\gamma t}\left(\frac{1}{2}c_{11} + b_1 - r - \gamma\right) \le 0,$$

for γ large enough. □

Remark 2.28. *Concerning the regularity of the solution, we emphasize that on the grounds of Definition 2.19 of the space $C^{1+\alpha}_{P,\text{loc}}$, the strong solution u of Theorem 2.26 is a locally Hölder continuous function, together with its first spatial derivatives $\partial_{x_1}u, \dots, \partial_{x_N}u$ of exponent α for all $\alpha \in]0,1[$.*

2.4 Obstacle problem on bounded cylinders: the penalization method

In this section we prove existence and uniqueness of a strong solution to the obstacle problem

$$\begin{cases} \max\{Lu - ru, \varphi - u\} = 0, & \text{in } B(T) :=]0,T[\times B, \\ u|_{\partial_P B(T)} = g, \end{cases} \tag{2.56}$$

where B is the Euclidean ball with radius R, $R > 0$ being fixed in all this section,

$$B = \{x \in \mathbb{R}^N \mid |x| < R\},$$

and $\partial_P B(T)$ denotes the parabolic boundary of $B(T)$:

$$\partial_P B(T) := \partial B(T) \setminus (\{T\} \times B).$$

The idea is to find a solution to (2.56) as the limit of solutions to a sequence of approximating problems, involving *non-linear PDEs* for which standard existence results are available.

We impose a condition analogous to Hypothesis 3 on the obstacle:

Hypothesis 4. *The function φ is Lipschitz continuous on $\overline{B(T)}$ and the weak convexity condition* (2.55) *holds with $O = B(T)$. Furthermore $g \in C(\partial_P B(T))$ and $g \geq \varphi$.*

We say that $u \in S^1_{\rm loc}(B(T)) \cap C(\overline{B(T)})$ is a strong solution to problem (2.56) if the differential equation is verified a.e. on $B(T)$ and the boundary datum is taken pointwisely. The main result of this section is the following

Theorem 2.29. *Under the Hypotheses 1, 2 and 4 there exists a strong solution u to the problem* (2.56)*. Moreover, for every $p \geq 1$ and O, domain satisfying $\overline{O} \subseteq B(T)$, there exists a positive constant c, depending only on $L_r, O, B(T), p$ and on the L^∞-norms of g and φ, such that*

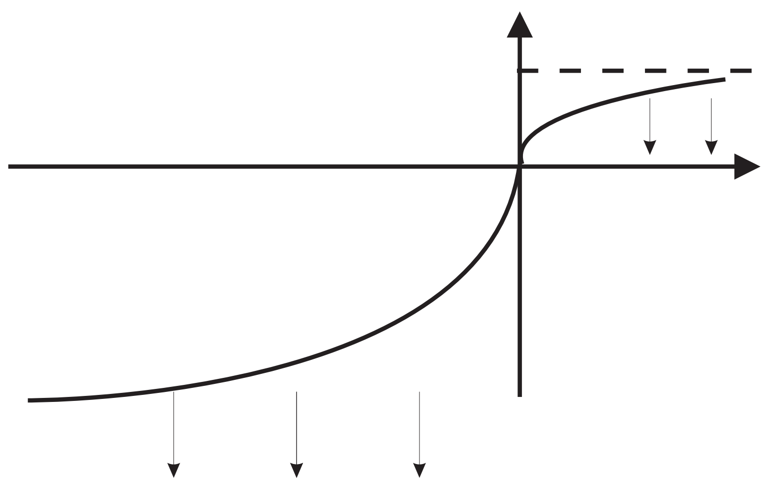
$$\|u\|_{S^p(O)} \leq c. \tag{2.57}$$

We prove Theorem 2.29 by using a penalization technique. We consider a family $(\beta_\varepsilon)_{\varepsilon\in]0,1[}$ of functions in $C^\infty(\mathbb{R})$: for every $\varepsilon > 0$, β_ε is a bounded, increasing function with bounded first order derivative such that

$$\beta_\varepsilon(0) = 0, \quad \beta_\varepsilon(s) \leq \varepsilon, \qquad s > 0.$$

Moreover we require that

$$\lim_{\varepsilon\to 0} \beta_\varepsilon(s) = -\infty, \qquad s < 0.$$



The *penalized problem* is defined as

$$\begin{cases} L_r u = \beta_\varepsilon(u - \varphi), & \text{in } B(T), \\ u|_{\partial_P B(T)} = g. \end{cases} \tag{2.58}$$

The existence of a classical solution to (2.58) is guaranteed by the following result (cf., for instance, [19] or [16]).

Theorem 2.30. *Under Hypotheses 1 and 2, let us take* $g \in C(\partial_P B(T))$ *and* $h = h(z,u) \in \mathrm{Lip}\left(\overline{B(T)} \times \mathbb{R}\right)$. *Then there exists a classical solution* $u \in C_P^{2+\alpha}(B(T)) \cap C(\overline{B(T)})$ *to the problem*

$$\begin{cases} L_r u = h(\cdot, u), & \text{in } B(T), \\ u|_{\partial_P B(T)} = g. \end{cases}$$

Moreover there exists a positive constant c, *depending only on* L_r, h *and* $B(T)$, *such that*

$$\sup_{B(T)} |u| \le e^{cT}(1 + \|g\|_{L^\infty}). \tag{2.59}$$

Proof of Theorem 2.29. We apply Theorem 2.30 with

$$h(\cdot, u) = \beta_\varepsilon(u - \varphi),$$

in order to infer the existence of a classical solution $u_\varepsilon \in C_P^{2+\alpha}(B(T)) \cap C(\overline{B(T)})$ of the penalized problem (2.58). After the simple change of variable

$$v(t,x) = e^{t\|r\|_\infty} u(t,x),$$

we can always assume that $r \ge 0$.

The crucial estimate to be proved is the following

$$|\beta_\varepsilon(u_\varepsilon - \varphi)| \le \widetilde{c} \tag{2.60}$$

with $\widetilde{c}$ constant *not depending on* ε.

Since $\beta_\varepsilon \le \varepsilon$ we have to prove only the estimate from below. We denote with ζ a minimum point of the function $\beta_\varepsilon(u_\varepsilon - \varphi) \in C(\overline{B(T)})$ and we suppose that $\beta_\varepsilon(u_\varepsilon(\zeta) - \varphi(\zeta)) \le 0$, otherwise there is nothing to prove. If $\zeta \in \partial_P B(T)$ then

$$\beta_\varepsilon(g(\zeta) - \varphi(\zeta)) \ge \beta_\varepsilon(0) = 0.$$

Viceversa, if $\zeta \in B(T)$, then, since β_ε is an increasing function, also $u_\varepsilon - \varphi$ assumes the (negative) minimum in ζ and therefore[4]

$$L(u_\varepsilon - \varphi)(\zeta) - r(u - \varphi)(\zeta) \ge 0,$$

i.e.

$$L_r u_\varepsilon(\zeta) \ge L_r \varphi(\zeta). \tag{2.61}$$

[4] We remark that if $v \in C^2$ has a minimum in ζ, then we have

$$\nabla v(\zeta) = 0, \quad \partial_t v(\zeta) = 0, \quad D^2 v(\zeta) \ge 0.$$

Then there exists a symmetric and positive semi-definite matrix M such that

$$D^2 v(\zeta) = M^2 = \left(\sum_{h=1}^N m_{ih} m_{hj}\right)_{i,j} = \left(\sum_{h=1}^N m_{ih} m_{jh}\right)_{i,j}$$

and therefore we have

$$Lv(z) = \frac{1}{2} \sum_{i,j=1}^N c_{ij}(z) \sum_{h=1}^N m_{ih} m_{jh} = \frac{1}{2} \sum_{h=1}^N \left(\sum_{i,j=1}^N c_{ij}(z) m_{ih} m_{jh}\right) \ge 0.$$

In our case $v = u_\varepsilon - \varphi$, and although φ is generally not a C^2 function, the above argument can be made rigorous by a standard regularization technique.

Now, by Hypothesis 4 (see also Remark 2.25), $L_r\varphi(\zeta)$ is bounded from below. So by (2.61) we get

$$\beta_\varepsilon(u_\varepsilon(\zeta)-\varphi(\zeta)) = L_r u_\varepsilon(\zeta) \geq L_r\varphi(\zeta) \geq \widetilde{c},$$

with $\widetilde{c}$ not depending on ε thus proving the estimate (2.60).

By the maximum principle[5] we have

$$\sup_{B(T)} |u_\varepsilon| \leq \sup_{B(T)} |g| + T\widetilde{c}. \tag{2.62}$$

Then by the a priori estimates in S^p, Theorems 2.24, and the estimates (2.60), (2.62) we infer that the norm $\|u_\varepsilon\|_{S^p(O)}$ is bounded uniformly with respect to ε, for every open set O included with its closure in $B(T)$ and for every $p \geq 1$. It follows that there exists a subsequence of (u_ε) weakly convergent for $\varepsilon \to 0$ in S^p (and in $C_P^{1+\alpha}$) on compact subsets of $B(T)$ to a function u (and by (2.60) we also have $u \geq \varphi$). Furthermore

$$L_r u = \lim_{\varepsilon\to 0} L_r u_\varepsilon = \lim_{\varepsilon \to 0} \beta_\varepsilon(u_\varepsilon - \varphi) \leq 0 \qquad \text{in } L^1_{\text{loc}},$$

so that $Lu \leq 0$ a.e. in $B(T)$. Finally, $L_r u = 0$ a.e. on the set $\{u > \varphi\}$.

We next conclude the proof of Theorem 2.29 by showing that $u \in C(\overline{B(T)})$ and $u = g$ on $\partial_P B(T)$. To study the behaviour of the solution at the boundary, we use a standard tool in PDE theory, the *barrier functions*: given a point $(t,x) \in \partial_P B(T)$, a barrier function for L_r in (t,x) is a function $w \in C^2(V \cap \overline{B(T)}; \mathbb{R})$, where V is a neighborhood of (t,x), such that

i) $L_r w \leq -1$ in $V \cap B(T)$;

ii) $w(t,x) = 0$ and $w > 0$ in $V \cap \overline{B(T)} \setminus \{(t,x)\}$.

It is well known that that every point of the parabolic boundary of $B(T)$ admits a barrier function (cf., for instance, [19] p.68 or [38], Lemma 8.25).

Given $\bar{z} = (\bar{t},\bar{x}) \in \partial_P B(T)$ and $\delta > 0$, we consider an open neighborhood V of $\bar{z}$ such that

$$|g(z) - g(\bar{z})| \leq \delta, \qquad \text{for } z = (t,x) \in V \cap \partial_P B(T),$$

and a barrier function w for L_r in $V \cap \partial_P B(T)$ exists. We put

$$v^{\pm}(z) = g(\bar{z}) \pm (\delta + k_\delta w(z))$$

where k_δ is a sufficiently large constant such that

$$L_r(u_\varepsilon - v^+) = \beta_\varepsilon(u_\varepsilon - \varphi) - L_r v^+ \geq \beta_\varepsilon(u_\varepsilon - \varphi) + k_\delta + r(g(\bar{z}) + \delta) \geq 0,$$

and $u_\varepsilon \leq v^+$ on $\partial(V \cap B(T))$. Note that, by (2.60) and (2.62), the constant k_δ can be chosen to be independent of ε. Then, by the maximum principle we have $u_\varepsilon \leq v^+$ on $V \cap B(T)$, and by an analogous argument, we also have $u_\varepsilon \geq v^-$ on $V \cap B(T)$.

Thus as $\varepsilon \to 0^+$, we get

$$g(\bar{z}) - \delta - k_\delta w(z) \leq u(z) \leq g(\bar{z}) + \delta + k_\delta w(z), \qquad z \in V \cap B(T),$$

and consequently

$$g(\bar{z}) - \delta \leq \liminf_{z\to\bar{z}} u(z) \leq \limsup_{z \to \bar{z}} u(z) \leq g(\bar{z}) + \delta, \qquad z \in V \cap B(T).$$

This proves the thesis by the arbitrariness of δ. □

We conclude this section by proving a comparison principle for the obstacle problem.

[5] For any $v \in C^2$, it holds

$$\sup_{B(T)} |v| \leq \sup_{\partial_p B(T)} |v| + T \sup_{B(T)} |L_r v|.$$

Proposition 2.31. *If u is a strong solution to the problem* (2.56) *and v a super-solution, i.e.* $v \in S^1_{\text{loc}}(B(T)) \cap C(\overline{B(T)})$ *and*

$$\begin{cases} \max\{L_r v, \varphi - v\} \le 0, & \textit{a.e. in } B(T), \\ v|_{\partial_P B(T)} \ge g, \end{cases}$$

then $u \le v$ in $B(T)$. In particular the solution to (2.56) *is unique.*

Proof. By contradiction, we suppose that the open set defined by

$$D := \{z \in B(T) \mid u(z) > v(z)\}$$

is not empty. Then, since $u > v \ge \varphi$ in D, we have that

$$L_r u = 0, \quad L_r v \le 0 \qquad \text{in } D,$$

and $u = v$ on ∂D. Then the maximum principle implies $u \le v$ in D and we get a contradiction. □

2.5 Obstacle problem on the strip

We prove Theorem 2.26 by solving a sequence of obstacle problems on a family of bounded cylinders that cover the strip $\mathcal{S}_T$, namely

$$B_n(T) =]0, T[\times \{|x| < n\}, \qquad n \in \mathbb{N}.$$

For every $n \in \mathbb{N}$, let us consider a function $\chi_n \in C(\mathbb{R}^N; [0,1])$ such that $\chi_n(x) = 1$ if $|x| \le n - \frac{1}{2}$ and $\chi_n(x) = 0$ if $|x| \ge n$, and we set

$$g_n(t,x) = \chi_n(x)\varphi(t,x) + (1 - \chi_n(x))\bar{u}(t,x), \qquad (t,x) \in \mathcal{S}_T.$$

By Theorem 2.29, for every $n \in \mathbb{N}$, there exists a strong solution u_n to the problem

$$\begin{cases} \max\{L_r u, \varphi - u\} = 0, & \text{in } B_n(T), \\ u|_{\partial_P B_n(T)} = g_n, \end{cases}$$

By Proposition 2.31

$$\varphi \le u_{n+1} \le u_n \le \bar{u}, \qquad \text{in } B_n(T),$$

and we can conclude the proof by using again the arguments of Theorem 2.29, based upon the a priori estimates in S^p_{loc} and the barrier functions.

Remark 2.32. *Theorem 2.26 gives an existence result: the uniqueness of the strong solution in the class of non-rapidly increasing functions will be proved in the next chapter as a consequence of the representation formula of Theorem 3.36.*

Remark 2.33. *The strong solution found in Theorem 2.26 is also a solution in the weak and viscosity senses. This means that the other weaker notions on generalized solution gain the stronger regularity properties of the strong solutions, in particular they are in $C^{1+\alpha}_{P,\text{loc}}(\mathcal{S}_T)$ for all $\alpha \in]0,1[$. A proof of this claim can be found, for instance, in [16].*

Remark 2.34. *Regarding the optimal regularity of the solution to the obstacle problem* (2.46)*, using the (quite involved) techniques developed by L. Caffarelli and his collaborators, it is possibile to prove that the solution is, up to S^∞-regularity, as smooth as the obstacle function: in particular, if $\varphi \in C^{2+\alpha}_{P,\text{loc}}$ we not only have that $u \in S^p_{\text{loc}}(\mathcal{S}_T)$ for every $p \ge 1$ as proved in Theorem 2.26, but also that $u \in S^\infty_{\text{loc}}(\mathcal{S}_T)$, that is we have the local Lipschitz continuity of the first order derivatives. For more details we refer to Caffarelli, Petrosyan and Shahgholian [11], Petrosyan and Shahgholian [40], Frentz, Nyström, Pascucci, and Polidoro [18].*

3 Optimal stopping problem

In this chapter we prove a representation formula for the strong solution to the obstacle problem

$$\begin{cases} \max\{Lu - ru, \varphi - u\}, & \text{in } \mathcal{S}_T :=]0, T[\times \mathbb{R}^N, \\ u(T, \cdot) = \varphi, \end{cases} \tag{3.63}$$

where r and φ are given functions and, with the notation $(c_{ij}) = \sigma\sigma^*$,

$$L = \frac{1}{2}\sum_{i,j=1}^{N} c_{ij}\partial_{x_i x_j} + \sum_{j=1}^{N} b_j \partial_{x_j} + \partial_t \tag{3.64}$$

is the Kolmogorov operator associated to the N-dimensional SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t. \tag{3.65}$$

More precisely, we represent u in terms of the solution of the optimal stopping problem related to (3.65) thus proving in particular the existence of a solution to such a problem. Note that L in (3.64) is the backward[6] version of the operator considered in the previous chapter.

In (3.65) we denote by $W = (W^1, \dots, W^N)$ a standard Brownian motion defined on a filtered space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$. As usual we put

$$L_r u = Lu - ru$$

and assume Hypotheses 1 and 2, i.e. L_r is a uniformly parabolic operator with bounded and Hölder continuous coefficients. Under these assumptions, the SDE (3.65) has a solution X and, by Theorem 2.20, the operator L has a fundamental solution Γ that is the transition density of the process X: more precisely, if $X^{t,x}$ denotes the solution to (3.65) starting from x at time t then, for any $T > t$, $\Gamma(t, x, T, \cdot)$ is the density of the random variable $X_T^{t,x}$, i.e.

$$P\left(X_T^{t,x} \in H\right) = \int_H \Gamma(t, x, T, y)dy$$

for any Borel set H.

We also recall the following generalized[7] Ito formula (cf., for instance, Theorem 5.79 in [38]):

Theorem 3.35. *Let $f = f(t, x) \in S^p(\mathbb{R} \times \mathbb{R}^N)$ with $p > 1 + \frac{N+2}{2}$. Then we have*

$$df(t, X_t) = Lf(t, X_t)dt + \sum_{i,j=1}^{N} \sigma_{ij}(t, X_t)\partial_{x_i} f(t, X_t)dW_t^j.$$

3.1 Feynman-Kac representation and uniqueness

The main result of this chapter states that the solution to the obstacle problem (3.63) can be expressed in terms of the solution to the optimal stopping problem related to the diffusion X.

Theorem 3.36 (Feynman-Kač formula)**.** *Under Hypotheses 1 and 2, let u be a strong solution to* (3.63) *such that*

$$|u(t, x)| \le Ce^{\lambda|x|^2}, \qquad (t, x) \in \mathcal{S}_T, \tag{3.66}$$

[6] In the simplest case, $L = \frac{1}{2}\triangle + \partial_t$ is the backward (or adjoint, since it is obtained by integration by parts) version of the standard heat operator $\frac{1}{2}\triangle - \partial_t$.

[7] The standard Ito formula holds true for C^2 functions: recall that its proof is based on Taylor formula.

for some positive constants C and λ, with λ sufficiently small. Then for any $(t,x)\in\mathcal{S}_T$, we have

$$u(t,x)=\sup_{\tau\in\mathcal{T}_{t,T}} E\left[e^{-\int_t^\tau r(s,X_s^{t,x})ds}\varphi(\tau,X_\tau^{t,x})\right],$$

where $\mathcal{T}_{t,T}$ denotes the family of the stopping times with values in $[t,T]$.

Remark 3.37. *Since Theorem 3.36 gives a representation formula, it also implies the uniqueness of the strong solution satisfying estimate* (3.66).

Proof. As for the standard Feynman-Kač formula, the proof is based on the Itô formula: since a strong solution is generally not in C^2, then we have to apply the generalized Itô formula in Theorem 3.35 by means of a localization argument. For more clarity, we only treat the case $r=0$.

We set $B_R=\{x\in\mathbb{R}^N \mid |x|<R\}$, $R>0$, and for a fixed $x\in B_R$ we denote by τ_R the first exit time of $X^{t,x}$ from B_R. Under our assumptions, it is well-known that $E[\tau_R]$ is finite.

We show that for any $(t,x)\in]0,T[\times B_R$ and $\tau\in\mathcal{T}_{t,T}$ such that $\tau\le\tau_R$ a.s., it holds

$$u(t,x)=E\left[u(\tau,X_\tau^{t,x})-\int_t^\tau Lu(s,X_s^{t,x})ds\right]. \tag{3.67}$$

Since $u\in S^p_{\text{loc}}(\mathcal{S}_T)$ for any $p\ge1$ then, for any positive and suitably small ε, there exists a function $u^{\varepsilon,R}$ such that $u^{\varepsilon,R}\in S^p(\mathbb{R}^{N+1})$ for any $p\ge1$ and $u^{\varepsilon,R}=u$ in $]t,T-\varepsilon[\times B_R$.

We next apply Itô formula to $u^{\varepsilon,R}$ and using the fact that $u^{\varepsilon,R}=u$ in $]t,T-\varepsilon[\times B_R$, we get

$$u(\tau,X_\tau^{t,x})=u(t,x)+\int_t^\tau Lu(s,X_s^{t,x})ds+\int_t^\tau \nabla u(s,X_s^{t,x})\sigma(s,X_s^{t,x})dW_s, \tag{3.68}$$

for any $\tau\in\mathcal{T}_{t,T}$ such that $\tau\le\tau_R\wedge(T-\varepsilon)$. Since $u\in C^{1+\alpha}_{P,\text{loc}}$ then $(\nabla u)\sigma$ is a bounded function on $]t,T-\varepsilon[\times B_R$ so that

$$E\left[\int_t^\tau \nabla u(s,X_s^{t,x})\sigma(s,X_s^{t,x})dW_s\right]=0.$$

Thus, taking expectations in (3.68), we conclude the proof of formula (3.67), since $\varepsilon>0$ is arbitrary.

Next we recall that $Lu\le0$ a.e.: since the law of $X^{t,x}$ is absolute continuous with respect to the Lebesgue measure, we have

$$E\left[\int_t^\tau Lu(s,X_s^{t,x})ds\right]\le0,\qquad \tau\in\mathcal{T}_{t,T},$$

so that from (3.67) we deduce

$$u(t,x)\ge E\left[u(\tau\wedge\tau_R,X_{\tau\wedge\tau_R}^{t,x})\right],\qquad \tau\in\mathcal{T}_{t,T}. \tag{3.69}$$

Now we pass to the limit as $R\to+\infty$: it holds

$$\lim_{R\to+\infty}\tau\wedge\tau_R=\tau$$

and, by the growth condition (3.66), we have

$$\left|u(\tau\wedge\tau_R,X_{\tau\wedge\tau_R}^{t,x})\right|\le C\exp\left(\lambda\sup_{t\le s\le T}\left|X_s^{t,x}\right|^2\right).$$

By standard maximal estimates (cf. for instance Theorem 9.32 in [38]) the random variable on the right hand side is integrable, thus by the dominated convergence theorem, passing to the limit in (3.69) as $R \to +\infty$, we infer

$$u(t,x) \geq E\left[u(\tau, X_\tau^{t,x})\right] \geq E\left[\varphi(\tau, X_\tau^{t,x})\right].$$

This proves that

$$u(t,x) \geq \sup_{\tau \in \mathcal{T}_{t,T}} E\left[\varphi(\tau, X_\tau^{t,x})\right].$$

We conclude the proof by setting

$$\tau_0 = \inf\{s \in [t,T] \mid u(s, X_s^{t,x}) = \varphi(s, X_s^{t,x})\}.$$

Since $Lu = 0$ a.e. on $\{u > \varphi\}$, it holds

$$E\left[\int_t^{\tau_0 \wedge \tau_R} Lu(s, X_s^{t,x}) ds\right] = 0,$$

so that by (3.67) we have

$$u(t,x) = E\left[u(\tau_0 \wedge \tau_R, X_{\tau_0 \wedge \tau_R}^{t,x})\right].$$

Using the previous argument to pass to the limit as $R \to +\infty$, we finally deduce

$$u(t,x) = E\left[u(\tau_0, X_{\tau_0}^{t,x})\right] = E\left[\varphi(\tau_0, X_{\tau_0}^{t,x})\right].$$

□

3.2 Gradient estimates

Using the Feynman-Kač representation, it is possible to prove useful properties of the strong solution under additional specific assumptions. For instance, let us assume that the function φ is Lipschitz continuous in x, uniformly in t, that is

$$|\varphi(t,x) - \varphi(t,y)| \leq C|x-y|, \qquad (t,x),(t,y) \in \mathcal{S}_T,$$

for some positive constant C. Then we can prove that the spatial gradient $\nabla u = (\partial_{x_1} u, \dots, \partial_{x_N} u)$ is bounded in $\mathcal{S}_T$. More precisely the following proposition holds.

Proposition 3.38. *Under the hypotheses of Theorem 3.36, let us assume that φ and the coefficients of the SDE* (3.65) *are Lipschitz continuous in x, uniformly in t, on $\mathcal{S}_T$. Moreover let r be constant or φ be a bounded function. Then the strong solution u of the obstacle problem* (3.63) *verifies*

$$\nabla u \in L^\infty(\mathcal{S}_T).$$

Proof. Let us first consider the case of constant r. The thesis follows by the general inequality

$$\left|\sup_\tau F(\tau) - \sup_\tau G(\tau)\right| \leq \sup_\tau |F(\tau) - G(\tau)|$$

valid for any functions F, G. Indeed, by Feynman-Kač formula, we have

$$|u(t,x) - u(t,y)| \leq \sup_{\tau \in \mathcal{T}_{t,T}} E\left[e^{-r(\tau - t)}\left|\varphi(\tau, X_\tau^{t,x}) - \varphi(\tau, X_\tau^{t,y})\right|\right] \leq$$

(by the Lipschitz condition, for some positive constant c)

$$\leq c \sup_{\tau \in \mathcal{T}_{t,T}} E\left[\left| X_\tau^{t,x} - X_\tau^{t,y} \right| \right] \leq$$

(by the well-known continuous dependence on the initial datum of the solution of a SDE with Lipschitz continuous coefficients)

$$\leq c_1 |x - y|,$$

where the constant c_1 depends only on T and on the Lipschitz constants of φ and of the coefficients.

In case φ is bounded, the thesis follows by an analogous argument, using the fact that the product of bounded Lipschitz continuous functions

$$(t, x) \mapsto e^{-\int_t^\tau r(s, X_s^{t,x}) ds} \varphi(\tau, X_\tau^{t,x})$$

is a Lipschitz continuous function. □

4 American options in continuous time

In this chapter we present the main results on pricing and hedging American derivatives by extending to continuous time the ideas introduced in the discrete-market setting. Even in the simplest Black-Scholes market model, the hedging and pricing problems for American options need very refined mathematical tools. In the complete-market setting, Bensoussan [8] and Karatzas [27], [28] developed a probabilistic approach based upon the notion of Snell envelope in continuous time and the Doob-Meyer decomposition. The problem was also studied by Jaillet, Lamberton and Lapeyre [24] who employed variational techniques, and more recently by Oksendal and Reikvam [37], Gatarek e Świech [22] who employed the theory of viscosity solutions. Here we present an analytical Markovian approach, based upon the existence results for the obstacle problem and the Feynman-Kač representation formula previously proved. In order to avoid technicalities and to show clearly the main ideas, we only consider the Black-Scholes market model case: the case of a complete market with N risky assets can be treated in a complete analogous way.

4.1 Pricing and hedging in the Black-Scholes model

Since in the theory of American options, dividends play an essential role, we assume the following risk-neutral dynamics for the underlying asset under the martingale measure Q:

$$dS_t = (r-q)S_t dt + \sigma S_t dW_t. \tag{4.70}$$

In (4.70) r is the risk-free rate, σ is the volatility parameter, $q \geq 0$ is the dividend yield and W is a real Brownian motion on the filtered space $(\Omega, \mathcal{F}, Q, \mathcal{F}_t)$. Then the discounted price $\widetilde{S}_t = e^{-rt}S_t$ has the following dynamics:

$$d\widetilde{S}_t = -q\widetilde{S}_t dt + \sigma \widetilde{S}_t dW_t. \tag{4.71}$$

Definition 4.39. *An American option is a process of the form*

$$(\psi(t, S_t))_{t\in[0,T]}$$

where ψ is a convex Lipschitz continuous function on $[0,T]\times\mathbb{R}_+$: $\psi(t,S_t)$ represents the premium obtained by exercising the option at time t.

An exercise strategy is a stopping time on $(\Omega, \mathcal{F}, Q, \mathcal{F}_t)$ taking values in $[0,T]$: we denote with $\mathcal{T}_T$ the family of all exercise strategies. We say that $\tau_0 \in \mathcal{T}_T$ is an optimal strategy if

$$E^Q\left[e^{-r\tau_0}\psi(\tau_0, S_{\tau_0})\right] = \sup_{\tau\in\mathcal{T}_T} E^Q\left[e^{-r\tau}\psi(\tau, S_\tau)\right].$$

The following result relates the parabolic obstacle problem to the corresponding problem for the Black-Scholes differential operator

$$L_{\text{BS}}f(t,S) := \frac{\sigma^2 S^2}{2}\partial_{SS}f(t,S) + (r-q)S\partial_S f(t,S) + \partial_t f(t,S) - rf(t,S).$$

Theorem 4.40. *There exists a unique strong solution $f \in S^p_{\text{loc}}(]0,T[\times\mathbb{R}_+)$, $p\geq 1$, to the obstacle problem*

$$\begin{cases} \max\{L_{\text{BS}}f, \psi - f\} = 0, & \text{in }]0,T[\times\mathbb{R}_+, \\ f(T,\cdot) = \psi(T,\cdot), & \text{in } \mathbb{R}_+, \end{cases} \tag{4.72}$$

satisfying the following properties:

i) for every $(t,x) \in [0,T[\times\mathbb{R}_+$, *we have*

$$f(t,x) = \sup_{\substack{\tau\in\mathcal{T}_T\\ \tau\in[t,T]}} E^Q\left[e^{-r(\tau-t)}\psi(\tau, S_\tau^{t,x})\right], \tag{4.73}$$

where $S^{t,x}$ *is solution to the SDE* (4.70) *with initial condition* $S_t = x$;

ii) f *admits first partial derivative with respect to* S *in the classical sense and we have*

$$\partial_S f \in C^\alpha_{P,\mathrm{loc}} \cap L^\infty(]0,T[\times\mathbb{R}_+), \qquad \forall \alpha \in]0,1[. \tag{4.74}$$

Proof. With the change of variables

$$u(t,x) = f(t,e^x), \qquad \varphi(t,x) = \psi(t,e^x)$$

the problem (4.72) is equivalent to the obstacle problem

$$\begin{cases} \max\{Lu - ru, \varphi - u\} = 0, & \text{in }]0,T[\times\mathbb{R},\\ u(T,\cdot) = \varphi(T,\cdot), & \text{in } \mathbb{R}, \end{cases}$$

for the parabolic operator with constant coefficients

$$Lu = \frac{\sigma^2}{2}\partial_{xx}u + \left(r - q - \frac{\sigma^2}{2}\right)\partial_x u + \partial_t u.$$

The existence of a strong solution is guaranteed by Theorem 2.26 and by the following Remark 2.27. Furthermore, again by Remark 2.27, u is upper bounded by a super-solution and lower bounded by φ so that an exponential-growth estimate similar to (3.66) holds: then we can apply the Feynman-Kač representation theorem, Theorem 3.36, which justifies formula (4.73) and proves the uniqueness of the solution. Finally, the global boundedness of the gradient can be proved by proceeding as in the proof of Proposition 3.38. □

We now consider a strategy (α_t, β_t) with[8] $\alpha \in \mathbb{L}^2_{\mathrm{loc}}$ and $\beta \in \mathbb{L}^1_{\mathrm{loc}}$, whose value process is defined as

$$V_t = V_t^{(\alpha,\beta)} := \alpha_t S_t + \beta_t B_t.$$

Hereafter, for greater convenience, when (α_t, β_t) is fixed we omit the superscript and simply write V_t instead of $V_t^{(\alpha,\beta)}$. We recall that (α_t, β_t) is self-financing if and only if

$$dV_t = \alpha_t\left(dS_t + qS_t dt\right) + \beta_t dB_t.$$

Substituting (4.70) in the previous formula and using the identity $\beta_t B_t = V_t - \alpha_t S_t$, we obtain the following

Proposition 4.41. *Put* $\widetilde{V}_t = e^{-rt}V_t$. *A strategy* (α,β) *is self-financing if and only if*

$$d\widetilde{V}_t = \alpha_t\left(d\widetilde{S}_t + q\widetilde{S}_t dt\right),$$

i.e.

$$\begin{aligned}\widetilde{V}_t &= V_0 + \int_0^t \alpha_s d\widetilde{S}_s + \int_0^t \alpha_s q\widetilde{S}_s ds\\ &= V_0 + \int_0^t \alpha_s\sigma\widetilde{S}_s dW_s. \end{aligned} \tag{4.75}$$

In particular every self-financing strategy is determined only by its initial value and by its α*-component. Furthermore,* $\widetilde{V}$ *is a* Q*-local martingale.*

[8] $\mathbb{L}^p_{\mathrm{loc}}$ denotes the space of the progressively measurable processes Y such that $\int_0^T |Y_t|^p dt$ is finite a.s.

On the grounds of the previous proposition, the discounted value of all self-financing strategies is a *local* martingale: in the following we are interested in the strategies whose value is a *true* martingale. Therefore we denote by $\mathcal{A}$ the family of the self-financing strategies (α, β) such that[9] $\alpha \in \mathbb{L}^2(P)$: a noteworthy example is represented by the strategies with α bounded process. It is known (cf., for instance, Proposition 10.33 in [38]) that the discounted value of every $(\alpha, \beta) \in \mathcal{A}$ is a Q-martingale. Let us now prove a version of the no-arbitrage principle.

Lemma 4.42. [No-arbitrage principle]
Let V^1, V^2 be the values of two self-financing strategies in $\mathcal{A}$ and assume that

$$V^1_\tau \le V^2_\tau \qquad a.e. \tag{4.76}$$

for some $\tau \in \mathcal{T}_T$. Then it holds that

$$V^1_0 \le V^2_0.$$

Proof. The thesis is immediate consequence of (4.76), of the martingale property of $\widetilde{V}^1, \widetilde{V}^2$ and of Doob's optional sampling theorem. □

Just as in the discrete case, we define the rational price of an American option by comparing it from above and from below to the value of appropriate self-financing strategies. Such an argument is necessary because, differently from the European case, the payoff $\psi(t, S_t)$ of an American option is not replicable in general, this meaning that there does not exist a self-financing strategy assuming the same value of the payoff at every single time. In fact by Proposition 4.41 the discounted value of a self-financing strategy is a martingale (or, in analytical terms, solution to a parabolic PDE) while $\psi(t, S_t)$ is a generic process, not necessarily a martingale.

Let us call

$$\mathcal{A}^+_\psi = \{(\alpha, \beta) \in \mathcal{A} \mid V^{(\alpha,\beta)}_t \ge \psi(t, S_t),\ t \in [0, T] \text{ a.s.}\},$$

the family of self-financing strategies that super-replicate the payoff $\psi(t, S_t)$. Intuitively, in order to avoid arbitrage opportunities, *the initial price of the American option must be less or equal to the initial value $V^{(\alpha,\beta)}_0$ for every $(\alpha, \beta) \in \mathcal{A}^+_\psi$.*

Furthermore, let us set

$$\mathcal{A}^-_\psi = \{(\alpha, \beta) \in \mathcal{A} \mid \text{there exists } \tau \in \mathcal{T}_T \text{ t.c. } \psi(\tau, S_\tau) \ge V^{(\alpha,\beta)}_\tau \text{ a.s.}\}.$$

We can think of $(\alpha, \beta) \in \mathcal{A}^-_\psi$ as a strategy on which we assume a short position to obtain funds to invest in the American option. In other words, $V^{(\alpha,\beta)}_0$ represents the amount that we can initially borrow to buy the option that has to be exercised, exploiting the early-exercise possibility, at time τ to obtain the payoff $\psi(\tau, S_\tau)$ which is greater or equal to $V^{(\alpha,\beta)}_\tau$, amount necessary to close the short position on the strategy (α, β). To avoid arbitrage opportunities, intuitively *the initial price of the American option must be greater or equal to $V^{(\alpha,\beta)}_0$ for all $(\alpha, \beta) \in \mathcal{A}^-_\psi$.*

These remarks are formalized by the following result that can be proved as Proposition 1.12.

Proposition 4.43. *We have*

$$\sup_{(\alpha,\beta)\in\mathcal{A}^-_\psi} V^{(\alpha,\beta)}_0 \le \sup_{\tau\in\mathcal{T}_T} E^Q\left[e^{-r\tau}\psi(\tau, S_\tau)\right] \le \inf_{(\alpha,\beta)\in\mathcal{A}^+_\psi} V^{(\alpha,\beta)}_0.$$

In particular for every $(\alpha, \beta) \in \mathcal{A}^-_\psi \cap \mathcal{A}^+_\psi$, it holds

$$V^{(\alpha,\beta)}_0 = \sup_{\tau\in\mathcal{T}_T} E^Q\left[e^{-r\tau}\psi(\tau, S_\tau)\right].$$

[9] $\mathbb{L}^2(P)$ denotes the space of the progressively measurable processes Y such that $E^P\left[\int_0^T |Y_t|^2 dt\right]$ is finite.

Theorem 4.45 below allows us to say that there exists $(\bar{\alpha},\bar{\beta}) \in \mathcal{A}_\psi^+ \cap \mathcal{A}_\psi^-$ and therefore the following definition is well-posed.

Definition 4.44. *The arbitrage price of the American option $\psi(t,S_t)$ is the initial value of any strategy $(\bar{\alpha},\bar{\beta}) \in \mathcal{A}_\psi^+ \cap \mathcal{A}_\psi^-$: in particular, we have*

$$V_0^{(\bar{\alpha},\bar{\beta})} = \inf_{(\alpha,\beta)\in\mathcal{A}_\psi^+} V_0^{(\alpha,\beta)} = \sup_{(\alpha,\beta)\in\mathcal{A}_\psi^-} V_0^{(\alpha,\beta)} = \sup_{\tau\in\mathcal{T}_T} E^Q\left[e^{-r\tau}\psi(\tau,S_\tau)\right].$$

Theorem 4.45. *Let f be the strong solution to the obstacle problem (4.72). The self-financing strategy (α,β) defined by*

$$V_0^{(\alpha,\beta)} = f(0,S_0), \qquad \alpha_t = \partial_S f(t,S_t),$$

belongs to $\mathcal{A}_\psi^+ \cap \mathcal{A}_\psi^-$. Consequently $f(0,S_0)$ is the arbitrage price of $\psi(t,S_t)$. Furthermore an optimal exercise strategy is defined by

$$\tau_0 = \inf\{t \in [0,T] \mid f(t,S_t) = \psi(t,S_t)\}, \tag{4.77}$$

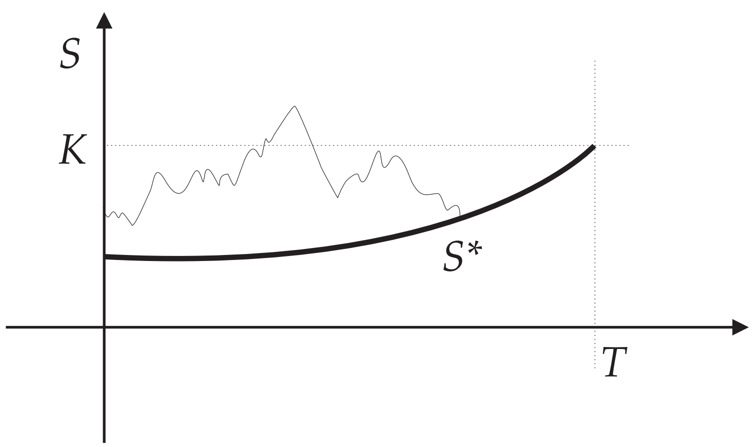
and it holds that

$$V_0^{(\alpha,\beta)} = E^Q\left[e^{-r\tau_0}\psi(\tau_0,S_{\tau_0})\right] = \sup_{\tau\in\mathcal{T}_T} E^Q\left[e^{-r\tau}\psi(\tau,S_\tau)\right],$$

where

$$S_t = S_0 e^{\sigma W_t + \left(r-q-\frac{\sigma^2}{2}\right)t},$$

is the solution to the SDE (4.70) with initial condition S_0.



Proof. The idea is to use the generalized Itô formula, Theorem 3.35, to compute the stochastic differential of $f(t,S_t)$ and to separate the martingale part from the drift part of the process[10]. Since $f \in S^p_{\mathrm{loc}}([0,T]\times\mathbb{R}_+)$, we do not have a global estimate of f and its derivatives (and therefore of $L_{\mathrm{BS}}f$), but only a local one: then we must use a localization argument. Fixed $R>0$, we consider the stopping time

$$\tau_R = T \wedge \inf\{t \mid S_t \in]0,1/R[\cup]R,+\infty[\}.$$

As in the proof of Theorem 3.36, by Itô formula we have that, for all $\tau \in \mathcal{T}_T$,

$$e^{-r(\tau\wedge\tau_R)} f(\tau\wedge\tau_R, S_{\tau\wedge\tau_R}) = f(0,S_0) + \int_0^{\tau\wedge\tau_R} \sigma\widetilde{S}_t\partial_S f(t,S_t)dW_t + \int_0^{\tau\wedge\tau_R} e^{-rt}L_{\mathrm{BS}}f(t,S_t)dt \tag{4.78}$$

[10] This corresponds to Step 2 in the proof of Theorem 1.14. In a general framework, this kind of result is usually called the Doob-Meyer decomposition theorem.

or equivalently, by (4.75),

$$e^{-r(\tau\wedge\tau_R)} f(\tau\wedge\tau_R, S_{\tau\wedge\tau_R}) = \widetilde{V}_{\tau\wedge\tau_R} + \int_0^{\tau\wedge\tau_R} e^{-rt} L_{\mathrm{BS}} f(t, S_t)dt, \tag{4.79}$$

where $\widetilde{V}$ is the discounted value of the self-financing strategy (α,β) defined by the initial value $f(0,S_0)$ and $\alpha_t = \partial_S f(t,S_t)$. Let us point out the analogy with the hedging strategy and the delta of a European option. A crucial remark is that $\widetilde{V}$ is a Q-martingale (not only a local one) since $\partial_S f$ is a bounded function by (4.74), and therefore $(\alpha,\beta)\in\mathcal{A}$.

Let us now prove that, for all $\tau\in\mathcal{T}_T$, it holds that

$$\lim_{R\to\infty} \widetilde{V}_{\tau\wedge\tau_R} = \widetilde{V}_\tau. \tag{4.80}$$

In fact we have

$$\begin{aligned} &E\left[\left(\int_{\tau\wedge\tau_R}^{\tau} \sigma\widetilde{S}_t \partial_S f(t,S_t)dW_t\right)^2\right] \\ =&E\left[\left(\int_0^T \sigma\widetilde{S}_t \partial_S f(t,S_t)\mathbb{1}_{\{\tau\wedge\tau_R\le t\le\tau\}}dW_t\right)^2\right] = \end{aligned}$$

(by the Itô isometry, since the integrand belongs to $\mathbb{L}^2$)

$$= E\left[\int_0^T \left(\sigma\widetilde{S}_t \partial_S f(t,S_t)\mathbb{1}_{\{\tau\wedge\tau_R\le t\le\tau\}}\right)^2 dt\right] \xrightarrow[R\to\infty]{} 0$$

by the dominated convergence Theorem, being $\partial_S f\in L^\infty$.

Now we can prove that $(\alpha,\beta)\in\mathcal{A}^+_\psi\cap\mathcal{A}^-_\psi$. First of all, since $L_{\mathrm{BS}} f\le 0$ a.e. and S_t has positive density, by (4.79), we have

$$V_{t\wedge\tau_R} \ge f(t\wedge\tau_R, S_{t\wedge\tau_R})$$

for all $t\in[0,T]$ and $R>0$. Taking the limit in R, by (4.80) and the continuity of f, we have

$$V_t \ge f(t,S_t) \ge \psi(t,S_t), \qquad t\in[0,T],$$

and this proves that $(\alpha,\beta)\in\mathcal{A}^+_\psi$.

Secondly, since $L_{\mathrm{BS}} f(t,S_t)=0$ a.s. on $\{\tau_0\ge t\}$ with τ_0 defined by (4.77), again by (4.79) we have

$$V_{\tau_0\wedge\tau_R} = f(\tau_0\wedge\tau_R, S_{\tau_0\wedge\tau_R})$$

for all $R>0$. Taking the limit in R as above, we get

$$V_{\tau_0} = f(\tau_0, S_{\tau_0}) = \psi(\tau_0, S_{\tau_0}).$$

This proves that $(\alpha,\beta)\in\mathcal{A}^-_\psi$ and concludes the proof. □

4.2 American call and put options

By Theorem 4.45 we have the following expressions for the prices of call and put American options in the Black-Scholes model, with risk-neutral dynamics (4.70) for the underlying asset:

$$\begin{aligned} C(T,S_0,K,r,q) &= \sup_{\tau\in\mathcal{T}_T} E\left[e^{-r\tau}\left(S_0 e^{\sigma W_\tau + \left(r-q-\frac{\sigma^2}{2}\right)\tau} - K\right)^+\right], \\ P(T,S_0,K,r,q) &= \sup_{\tau\in\mathcal{T}_T} E\left[e^{-r\tau}\left(K - S_0 e^{\sigma W_\tau + \left(r-q-\frac{\sigma^2}{2}\right)\tau}\right)^+\right]. \end{aligned}$$

In the preceding expressions, $C(T, S_0, K, r, q)$ and $P(T, S_0, K, r, q)$ denote respectively the prices at time 0 of call and put American options with terminal time T, initial price of the underlying asset S_0, strike K, interest rate r and dividend yield q. For American options explicit formulas as in the European case are not known, and to compute the prices and the hedging strategies is generally necessary to use numerical methods.

The following result establishes a symmetry relation between the prices of American call and put options.

Proposition 4.46. *We have that*

$$C(T, S_0, K, r, q) = P(T, K, S_0, q, r). \tag{4.81}$$

Proof. If we set

$$Z_t = e^{\sigma W_t - \frac{\sigma^2}{2}t},$$

we recall that Z is a Q-martingale with unitary mean and, with respect to the measure $\widetilde{Q}$ defined by

$$\frac{d\widetilde{Q}}{dQ} = Z_T,$$

the process

$$\widetilde{W}_t = W_t - \sigma t$$

is a Brownian motion.

Let us note that we have

$$\begin{aligned} C(T, S_0, K, r, q) &= \sup_{\tau \in \mathcal{T}_T} E^Q \left[Z_\tau e^{-q\tau} \left(S_0 - K e^{-\sigma W_\tau + \left(q - r + \frac{\sigma^2}{2}\right)\tau} \right)^+ \right] \\ &= \sup_{\tau \in \mathcal{T}_T} E^Q \left[Z_T e^{-q\tau} \left(S_0 - K e^{-\sigma W_\tau + \left(q - r + \frac{\sigma^2}{2}\right)\tau} \right)^+ \right] \\ &= \sup_{\tau \in \mathcal{T}_T} E^{\widetilde{Q}} \left[e^{-q\tau} \left(S_0 - K e^{-\sigma \widetilde{W}_\tau + \left(q - r - \frac{\sigma^2}{2}\right)\tau} \right)^+ \right]. \end{aligned}$$

The thesis follows because, by symmetry, $-\widetilde{W}$ is a $\widetilde{Q}$-Brownian motion. □

We study now some qualitative properties of the prices: on the grounds of Proposition 4.46 it is enough to consider the case of the American put. In the following statement we denote with

$$P(T, S) = \sup_{\tau \in \mathcal{T}_T} E \left[e^{-r\tau} \left(K - S e^{\sigma W_\tau + \left(r - q - \frac{\sigma^2}{2}\right)\tau} \right)^+ \right], \tag{4.82}$$

the price of the American put option.

Proposition 4.47. *The following properties hold:*

i) for all $S \in \mathbb{R}_+$, the function $T \mapsto P(T, S)$ is increasing. In other words, if we fix the parameters of the option, the price of the put option decreases when we get closer to maturity;

ii) for all $T \in [0, T]$, the function $S \mapsto P(T, S)$ is decreasing, convex and

$$\lim_{S \to 0^+} P(T, S) = K; \tag{4.83}$$

iii) for all $(T, S) \in [0, T[\times \mathbb{R}_+$, we have that

$$-1 \le \partial_S P(T, S) \le 0.$$

Proof. i) is trivial. ii) is immediate consequence of (4.82), of the properties of the payoff function and of the fact that the properties of monotony and convexity are preserved by the sup operation, i.e. if (g_τ) is a family of increasing and convex functions then also their sup

$$g := \sup_\tau g_\tau$$

is increasing and convex.

Then $\partial_S P(T, S) \le 0$ since $S \mapsto P(T, S)$ is decreasing. Furthermore, if we set $\psi(S) = (K - S)^+$ we have

$$|\psi(S) - \psi(S')| \le |S - S'|,$$

and so to prove the third property it is enough to proceed just as in the proof of Proposition 3.38 and to observe that

$$\begin{aligned}&\left| E\left[e^{-r\tau}\psi\left(S_0 e^{\sigma W_\tau + \left(r - q - \frac{\sigma^2}{2}\right)\tau}\right) - e^{-r\tau}\psi\left(S_0' e^{\sigma W_\tau + \left(r - q - \frac{\sigma^2}{2}\right)\tau}\right)\right]\right| \\ &\le |S_0 - S_0'| E\left[e^{\sigma W_\tau - \left(q + \frac{\sigma^2}{2}\right)\tau}\right] \le\end{aligned}$$

(since $q \ge 0$)

$$\le |S_0 - S_0'| E\left[e^{\sigma W_\tau - \frac{\sigma^2}{2}\tau}\right] =$$

(since the exponential martingale has unitary mean)

$$= |S_0 - S_0'|.$$

□

4.3 Early exercise premium

In this section, we study the relation between the prices of the European put option and American put option by introducing the concept of *early exercise premium*. In the following we denote by $f = f(t, S)$ the solution to the obstacle problem (4.72) relative to the payoff function of the put option

$$\psi(t, S) = (K - S)^+.$$

For $t \in [0, T]$, we define[11]

$$S^*(t) = \inf\{S > 0 \mid f(t, S) > \psi(t, S)\}.$$

$S^*(t)$ is called *critical price at time* t and corresponds to the point where f "touches" the payoff ψ; the map $t \mapsto S^*(t)$ is called the *free boundary*. Note that $S^*(t) < K$ for $t < T$: in fact if we had $S^*(t) \ge K$ then it should hold that

$$f(t, S^*(t)) = \psi(t, S^*(t)) = 0,$$

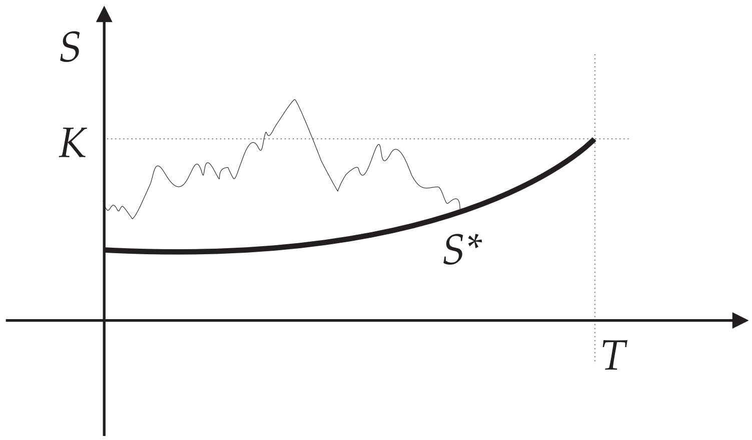
and this is absurd since $f > 0$ by definition (4.82).

[11]Mathematically or by arbitrage arguments, it can be proved that the set $\{S > 0 \mid f(t, S) > \psi(t, S)\}$ is non-empty and $S^*(t) > 0$.

Remark 4.48 (Smooth fit principle). *Since the first order derivative $\partial_S P$ of the American put price is (Hölder) continuous*[12]*, we immediately get the following additional information at the free boundary:*

$$\partial_S P(t, S^*(t)) = \partial_S \psi(t, S^*(t)) = -1. \tag{4.84}$$

Condition (4.84) *is usually known as the smooth fit principle.*



Lemma 4.49. *For all $(t, S) \in [0, T[\times \mathbb{R}_+$, we have that*

$$L_{\mathrm{BS}} f(t, S) = (qS - rK)\mathbb{1}_{\{S<S^*(t)\}}. \tag{4.85}$$

In particular $L_{\mathrm{BS}} f$ is a bounded and discontinuous function.

Proof. Since $S^*(t) \in]0, K]$, by (4.83) and the convexity of $S \mapsto f(t, S)$ (cf. Proposition 4.47-ii)), we infer that $f(t, S) = K - S$ for $S \le S^*(t)$. Moreover, by definition of S^*, we have $f(t, S) > \psi(t, S)$ for $S > S^*(t)$ and therefore we conclude that

$$L_{\mathrm{BS}} f(t, S) = \begin{cases} 0, & \text{for } S > S^*(t), \\ qS - rK, & \text{for } S < S^*(t). \end{cases}$$

□

Remark 4.50. *By* (4.85)*, knowing that $L_{\mathrm{BS}} f \le 0$, we deduce that $S^*(t) \le \frac{rK}{q}$: in particular, if $q > r$ then $\lim\limits_{t \to T^-} S^*(t) < K$.* □

Now we go back to formula (4.78) with $\tau = T$: since $L_{\mathrm{BS}} f$ is bounded, we can take the limit as $R \to +\infty$ and then get

$$e^{-rT} f(T, S_T) = f(0, S_0) + \int_0^T e^{-rt} L_{\mathrm{BS}} f(t, S_t) dt + \int_0^T \sigma \widetilde{S}_t \partial_S f(t, S_t) dW_t,$$

and taking expectation, by (4.85),

$$p(S_0) = P(S_0) + \int_0^T e^{-rt} E^Q \left[(qS_t - rK)\mathbb{1}_{\{S_t \le S^*(t)\}} \right] dt, \tag{4.86}$$

where $p(S_0)$ and $P(S_0)$ denote respectively the price at time 0 of the European and American options with maturity T. The expression (4.86) gives the difference $P(S_0) - p(S_0)$, usually called *early exercise premium:* it quantifies the value of the possibility of exercising before maturity. (4.86) has been proved originally by Kim [29].

[12] This is generally true for any strong solution by Remark 2.28.

4.4 Numerical methods

Standard finite difference schemes can be adapted to the numerical solution of the obstacle problem: Brennan and Schwartz [10] first investigated these methods and the applications to the pricing of options with early exercise. Jaillet, Lamberton and Lapeyre [24] and Han-Wu [23] gave a rigorous justification of the method (see also Zhang [43] for a complete proof of strong convergence of the schemes and an extension to models with jumps). Barraquand and Martineau [5], Barraquand and Pudet [6], Dempster and Hutton [14] propose various refinements of the previous techniques which lead to more accurate approximations of exotic options. Among other numerical methods proposed in literature we quote the finite elements in Achdou and Pironneau [1], the ADI methods in Villeneuve and Zanette [42] and the wavelet methods in Matache, Nitsche and Schwab [36]. In a different spirit, semi-explicit approximation formulas were given by MacMillan [35], Barone-Adesi and Whaley [4], Carr and Faguet [12], Jourdain and Martini [25, 26].

References

[1] Y. Achdou and O. Pironneau, *Computational Methods for Option Pricing*, Society for Industrial and Applied Mathematics, 2005.

[2] K. Amin and A. Khanna, *Convergence of American option values from discrete- to continuous-time financial models*, Math. Finance, 4 (1994), pp. 289–304.

[3] G. Barles, *Convergence of numerical schemes for degenerate parabolic equations arising in finance theory*, in Numerical methods in finance, Cambridge Univ. Press, Cambridge, 1997, pp. 1–21.

[4] G. Barone-Adesi and R. Whaley, *Efficient analytic approximation of American option values*, J. of Finance, 42 (1987), pp. 301–320.

[5] J. Barraquand and D. Martineau, *Numerical valuation of high dimensional multivariate American securities*, J. of Financial and Quantitative Analysis, 30 (1995), pp. 383–405.

[6] J. Barraquand and T. Pudet, *Pricing of American path-dependent contingent claims*, Math. Finance, 6 (1996), pp. 17–51.

[7] A. Bensoussan, *On the theory of option pricing*, Acta Appl. Math., 2 (1984), pp. 139–158.

[8] ——, *On the theory of option pricing*, Acta Appl. Math., 2 (1984), pp. 139–158.

[9] A. Bensoussan and J.-L. Lions, *Applications of variational inequalities in stochastic control*, vol. 12 of Studies in Mathematics and its Applications, North-Holland Publishing Co., Amsterdam, 1982. Translated from the French.

[10] M. J. Brennan and E. S. Schwartz, *The valuation of the American put option*, J. of Finance, 32 (1977), pp. 449–462.

[11] L. Caffarelli, A. Petrosyan, and H. Shahgholian, *Regularity of a free boundary in parabolic potential theory*, J. Amer. Math. Soc., 17 (2004), pp. 827–869 (electronic).

[12] P. Carr and D. Faguet, *Valuing finite lived options as perpetual*, Morgan Stanley working paper, (1996).

[13] M. G. Crandall, H. Ishii, and P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.), 27 (1992), pp. 1–67.

[14] M. A. H. Dempster and J. P. Hutton, *Pricing American stock options by linear programming*, Math. Finance, 9 (1999), pp. 229–254.

[15] M. Di Francesco and A. Pascucci, *On a class of degenerate parabolic equations of Kolmogorov type*, AMRX Appl. Math. Res. Express, (2005), pp. 77–116.

[16] M. Di Francesco, A. Pascucci, and S. Polidoro, *The obstacle problem for a class of hypoelliptic ultraparabolic equations*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 464 (2008), pp. 155–176.

[17] W. H. Fleming and H. M. Soner, *Controlled Markov processes and viscosity solutions*, vol. 25 of Stochastic Modelling and Applied Probability, Springer, New York, second ed., 2006.

[18] M. Frentz, K. Nyström, A. Pascucci, and S. Polidoro, *Optimal regularity in the obstacle problem for Kolmogorov operators related to American Asian options*, preprint AMS Acta, (2008).

[19] A. Friedman, *Partial differential equations of parabolic type*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1964.

[20] ——, *Parabolic variational inequalities in one space dimension and smoothness of the free boundary*, J. Functional Analysis, 18 (1975), pp. 151–176.

[21] ——, *Variational principles and free-boundary problems*, Robert E. Krieger Publishing Co. Inc., Malabar, FL, second ed., 1988.

[22] D. Gatarek and A. Świech, *Optimal stopping in Hilbert spaces and pricing of American options*, Math. Methods Oper. Res., 50 (1999), pp. 135–147.

[23] H. Han and X. Wu, *A fast numerical method for the Black-Scholes equation of American options*, SIAM J. Numer. Anal., 41 (2003), pp. 2081–2095 (electronic).

[24] P. Jaillet, D. Lamberton, and B. Lapeyre, *Variational inequalities and the pricing of American options*, Acta Appl. Math., 21 (1990), pp. 263–289.

[25] B. Jourdain and C. Martini, *American prices embedded in European prices*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 18 (2001), pp. 1–17.

[26] B. Jourdain and C. Martini, *Approximation of American put prices by European prices via an embedding method*, Ann. Appl. Probab., 12 (2002), pp. 196–223.

[27] I. Karatzas, *On the pricing of American options*, Appl. Math. Optim., 17 (1988), pp. 37–60.

[28] ——, *Optimization problems in the theory of continuous trading*, SIAM J. Control Optim., 27 (1989), pp. 1221–1259.

[29] I. J. Kim, *The analytic valuation of American options*, Rev. Financial Studies, 3 (1990), pp. 547–572.

[30] D. Kinderlehrer and G. Stampacchia, *An introduction to variational inequalities and their applications*, vol. 31 of Classics in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000. Reprint of the 1980 original.

[31] H. J. Kushner, *Probability methods for approximations in stochastic control and for elliptic equations*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1977. Mathematics in Science and Engineering, Vol. 129.

[32] D. Lamberton and B. Lapeyre, *Introduction to stochastic calculus applied to finance*, Chapman & Hall, London, 1996. Translated from the 1991 French original by Nicolas Rabeau and Francois Mantion.

[33] D. Lamberton and G. Pagès, *Sur l'approximation des réduites*, Ann. Inst. H. Poincaré Probab. Statist., 26 (1990), pp. 331–355.

[34] G. Lieberman, *Second order parabolic differential equations*, World Scientific, Singapore, 1996.

[35] L. MacMillan, *Analytic approximation for the American put price*, Adv. in Futures and Options Research, 1 (1986), pp. 119–139.

[36] A.-M. Matache, P.-A. Nitsche, and C. Schwab, *Wavelet Galerkin pricing of American options on Lévy driven assets*, Quant. Finance, 5 (2005), pp. 403–424.

[37] B. Oksendal and K. Reikvam, *Viscosity solutions of optimal stopping problems*, Stochastics Stochastics Rep., 62 (1998), pp. 285–301.

[38] A. Pascucci, *Calcolo stocastico per la finanza (Italian)*, English edition forthcoming, Springer Finance, 2008.

[39] A. Pascucci, *Free boundary and optimal stopping problems for American Asian options*, Finance Stoch., 12 (2008), pp. 21–41.

[40] A. Petrosyan and H. Shahgholian, *Parabolic obstacle problems applied to finance*, Contemp. Math., 439 (2007), pp. 117–133.

[41] S. Varadhan, *PDE in finance*, `http://math.nyu.edu/faculty/varadhan/`, (2004).

[42] S. Villeneuve and A. Zanette, *Parabolic ADI methods for pricing American options on two stocks*, Math. Oper. Res., 27 (2002), pp. 121–149.

[43] X. L. Zhang, *Numerical analysis of American option pricing in a jump-diffusion model*, Math. Oper. Res., 22 (1997), pp. 668–690.