Secant varieties to osculating varieties of Veronese embeddings of $\mathbb{P}^n$.

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ABSTRACT: A well known theorem by Alexander-Hirschowitz states that all the higher secant varieties of $V_{n,d}$ (the d-uple embedding of $\mathbb{P}^n$) have the expected dimension, with few known exceptions. We study here the same problem for $T_{n,d}$, the tangential variety to $V_{n,d}$, and prove a conjecture, which is the analogous of Alexander-Hirschowitz theorem, for $n \leq 9$. Moreover. we prove that it holds for any n, d if it holds for $d = 3$. Then we generalize to the case of $O_{k,n,d}$, the k-osculating variety to $V_{n,d}$, proving, for $n = 2$, a conjecture that relates the defectivity of $\sigma_s(O_{k,n,d})$ to the Hilbert function of certain sets of fat points in $\mathbb{P}^n$.

Introduction.

The well known Alexander-Hirschowitz theorem (see [AH1]) states:

Theorem 0.1. *(Alexander-Hirschowitz) Let X be a generic collection of s 2-fat points in $\mathbb{P}^n_\kappa$. If $(I_X)_d \subset \kappa[x_0, ..., x_n]$ is the vector space of forms of degree d which are singular at the points of X, then* $\dim(I_X)_d = \min\{(n+1)d, \binom{n+d}{n}\}$, *as expected, unless:*

- $d = 2, 2 \leq s \leq n$;
- $n = 2,\ d = 4,\ s = 5$;
- $n = 3,\ d = 4,\ s = 9$;
- $n = 4,\ d = 3,\ s = 7$;
- $n = 4,\ d = 4,\ s = 14$.

Notice that with "m-fat point at $P \in \mathbb{P}^n$" we mean the scheme defined by the ideal $I_P^m \subset \kappa[x_0, ..., x_n]$.

An equivalent reformulation of the theorem is in the language of higher secant varieties; let $V_{n,d} \subset \mathbb{P}^N$, with $N = \binom{n+d}{n} - 1$, be the d-ple (Veronese) embedding of $\mathbb{P}^n$, and let $\sigma_s(V_{n,d})$ be its $(s-1)^{th}$ higher secant variety , that is, the closure of the union of the $\mathbb{P}^{s-1}$'s which are s-secant to $V_{n,d}$. Then Theorem 0.1 is equivalent to:

Theorem 0.2. *All the higher secant varieties $\sigma_s(V_{n,d})$ have the expected dimension* $\min\{s(n+1)-1, \binom{n+d}{n} - 1\}$, *except when s, n, d are as in the exceptions of Theorem 0.1.*

An application of the theorem is in terms of the Waring problem for forms (or of the decomposition of a supersymmetric tensor), i.e. that the general form of degree d in $n+1$ variables can be written as the sum of $\lceil \frac{1}{n+1}\binom{n+d}{d} \rceil$ dth powers of linear forms, with the same list of exceptions (e.g. see [Ge] or [IK]).

In [CGG] a similar problem has been studied, namely when the dimension of $\sigma_s(T_{n,d})$ is as expected, where $T_{n,d}$ is the tangential variety of the Veronese variety $V_{n,d}$. This too translates into a problem of representation of forms: the generic form parameterized by $\sigma_s(T_{n,d})$ is a form F of degree d which can be written as $F = L_1^{d-1}M_1 + ... + L_s^{d-1}M_s$, where the L_i, M_i's are linear forms.

The following conjecture was stated in [CGG]:

Conjecture 1: *The secant variety $\sigma_s(T_{n,d})$ has the expected dimension,* $\min\{2sn+s-1, \binom{n+d}{n}-1\}$, *except when:*

i) $d = 2$, $2 \le 2s < n$;

ii) $d = 3$, $s = n = 2, 3, 4$.

In the same paper the conjecture was proved for $d = 2$ (any s, n) and for $s \le 5$ (any d, n), while in [B] it is proved for $n = 2, 3$ (any s, d).

In [CGG](via inverse systems) it is shown that $\sigma_s(T_{n,d})$ is defective if and only if a certain 0-dimensional scheme $Y \subset \mathbb{P}^n$ does not impose independent conditions to forms of degree d in $R := \kappa[x_0, ..., x_n]$. The scheme $Y = Z_1 \cup ... \cup Z_s$ is supported at s generic points $P_1, ..., P_s \in \mathbb{P}^n$, and at each of them the scheme Z_i lies between the 2-fat point and the 3-fat point on P_i (we will call Z_i a $(2, 3, n)$-scheme, for details see section 1 below).

Hence Conjecture 1 can be reformulated in term of $(I_Y)_d$ having the expected dimension, with the same exceptions, in analogy with the statement of Theorem 0.1.

Theorem 0.1 has been proved thanks to the Horace differential Lemma, (AH2, Proposition 9.1; see also here Proposition 1.5) and an induction procedure which has a delicate beginning step for $d = 3$; different proofs for this case are in [Ch1], [Ch2] and in the more recent [BO], where an excellent history of the question can be found.

Also the proof of Conjecture 1 presents the case of $d = 3$ as a crucial one; the first main result in this paper (Corollary 2.5) is to prove that if Conjecture 1 holds for $d = 3$, then it holds also for $d \ge 4$ (and any n, s). The procedure we use is based on Horace differential Lemma too.

We also prove Conjecture 1 for all $n \le 9$, since with that hypothesis we can check the case $d = 3$ by making use of COCOA (see Corollary 2.4).

A more general problem can be considered (see also [BCGI]): let $O_{k,n,d}$ be the k-osculating variety to $V_{n,d} \subset \mathbb{P}^N$, and study its $(s-1)^{th}$ higher secant variety $\sigma_s(O_{k,n,d})$. Again, we are interested in the problem of determining all s for which $\sigma_s(O_{k,n,d})$ is defective, i.e. for which its dimension is strictly less than its expected dimension (for precise definitions and setting of the problem, see Section 1 of the present paper and in particular Question Q(k,n,d)).

Also in this general case we found in [BCGI] (via inverse systems) that $\sigma_s(O_{k,n,d})$ is defective if and only if a certain 0-dimensional scheme $Y \subset \mathbb{P}^n$ does not impose independent conditions to forms of degree d in $R := \kappa[x_0, ..., x_n]$. The scheme $Y = Z_1 \cup ... \cup Z_s$ is supported at s generic points $P_1, ..., P_s \in \mathbb{P}^n$, and at each of them the scheme Z_i lies between the $(k+1)$ and the $(k+2)$-fat point at P_i (for details see Lemma 1.2 below).

The following (quite immediate) lemma ([BCGI] 3.1) describes what can be deduced about the postulation of the scheme Y from information on fat points:

Lemma 0.3. *Let $P_1, ..., P_s$ be generic points in $\mathbb{P}^n$, and set $X := (k+1)P_1 \cup ... \cup (k+1)P_s$, $T := (k+2)P_1 \cup ... \cup (k+2)P_s$. Now let Z_i be a 0-dimensional scheme supported at P_i, $(k+1)P_i \subset Z_i \subset (k+2)P_i$, and set $Y := Z_1 \cup ... \cup Z_s$. Then, Y is regular in degree d if $h^1(\mathcal{I}_T(d)) = 0$ or if $h^0(\mathcal{I}_X(d)) = 0$.*
Moreover, Y is not regular in degree d if
(i) $h^1(\mathcal{I}_X(d)) > max\{0, \deg(Y) - \binom{d+n}{n}\}$
or if
(ii) $h^0(\mathcal{I}_T(d)) > max\{0, \binom{d+n}{n} - \deg(Y)\}$.

All cases studied in [BCGI] lead us to state the following:

Conjecture 2a. *The secant variety $\sigma_s(O_{k,n,d})$ is defective if and only if Y is as in case (i) or (ii) of the Lemma above.*

The conjecture amounts to saying that I_Y does not have the expected Hilbert function in degree d only when "forced" by the Hilbert function of one of the fat point schemes X, T.

Notice that (i), respectively (ii), obviously implies that X, respectively T, is defective. Hence, if Conjecture 2a holds and Y is defective in degree d, then either T or X are defective in degree d too, and the defectivity of Y is either given by the defectivity of X or forced by the high defectivity of T.

Thus if the conjecture holds, we have another occurrence of the "ubiquity" of fat points: the problem of $\sigma_s(O_{k,n,d})$ having the right dimension reduces to a problem of computing the Hilbert function in degree d of two schemes of s generic fat points in $\mathbb{P}^n$, all of them having multiplicity $k+1$, respectively $k+2$.

In [BC]and [BF] the conjecture is proved in $\mathbb{P}^2$ for $s \le 9$.

Notice that the Conjecture 2a implies the following one, more geometric, which relates the defectivity of $\sigma_s(O_{k,n,d})$ to the dimensions of the k^{th} and the $(k+1)^{th}$ osculating space at a generic point of the $(s-1)^{th}$ higher secant variety of the Veronese variety $\sigma_s(V_{n,d})$:

Conjecture 2b. *If the secant variety $\sigma_s(O_{k,n,d})$ is defective then at a generic point $P \in \sigma_s(V_{n,d})$, either the k^{th} osculating space $O_{k,\sigma_s(V_{n,d}),P}$ does not have dimension* $\min\{s\binom{k+n}{n} - 1, \binom{d+n}{n} - 1\}$*, or the $(k+1)^{th}$ osculating space $O_{k+1,\sigma_s(V_{n,d}),P}$ does not have dimension* $\min\{s\binom{k+n+1}{n} - 1, \binom{d+n}{n} - 1\}$.

The implication follows from the fact that (see [BBCF]) for $P \in < P_1, ..., P_s >$:

$$O_{k,\sigma_s(V_{n,d}),P} =< O_{k,V_{n,d},P_1}, O_{k,V_{n,d},P_2}, \dots, O_{k,V_{n,d},P_s} > .$$

The other main result in this paper is in section 3, where we will prove Conjecture 2a for $n = 2$.

Section 1: Preliminaries and Notations.

In this paper we will always work over a field κ such that $\kappa = \overline{\kappa}$ and $\mathrm{char}\kappa = 0$.

1.1 Notations.

(i) If $P \in \mathbb{P}^n$ is a point and I_P is the ideal of P in $\mathbb{P}^n$, we denote by mP the fat point of multiplicity m supported at P, i.e. the scheme defined by the ideal I_P^m.

(ii) Let $X \subseteq \mathbb{P}^N$ be a closed irreducible projective variety; the $(s-1)^{th}$ *higher secant variety* of X is the closure of the union of all linear spaces spanned by s points of X, and it will be denoted by $\sigma_s(X)$.

(iii) Let $X \subset \mathbb{P}^N$ be a variety, and let $P \in X$ be a smooth point; we define *the k^{th} osculating space to X at P* as the linear space generated by $(k+1)P \cap X$ (i.e. by the k^{th} infinitesimal neighbourhood of P in X) and we denote it by $O_{k,X,P}$; hence $O_{0,X,P} = \{P\}$, and $O_{1,X,P} = T_{X,P}$, the projectivised tangent space to X at P.

Let $U \subset X$ be the dense set of the smooth points where $O_{k,X,P}$ has maximal dimension. The *k^{th} osculating variety to X* is defined as:

$$O_{k,X} = \overline{\bigcup_{P \in U} O_{k,X,P}}.$$

(iv) We denote by $V_{n,d}$ the d-uple Veronese embedding of $\mathbb{P}^n$, i.e. the image of the map defined by the linear system of all forms of degree d on $\mathbb{P}^n$: $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$, where $N = \binom{n+d}{n} - 1$.

(v) We denote the k^{th} osculating variety to the Veronese variety by $O_{k,n,d} := O_{k,V_{n,d}}$. When $k = 1$, the osculating variety is called *tangential variety* and it is denoted by $T_{n,d}$.

Hence, the $(s-1)^{th}$ higher secant variety of the k^{th} osculating variety to the Veronese variety $V_{n,d}$ will be denoted by $\sigma_s(O_{k,n,d})$.

Since the case $d \leq k$ is trivial, and the description for $k = 1$ given in [CGG], together with [BCGI, Proposition 4.4] describe the case $d = k+1$ completely, from now on we make the general assumption, which will be implicit in the rest of the paper, that $d \geq k+2$.

It is easy to see ([BCGI] 2.3) that the dimension of $O_{k,n,d}$ is always the expected one, that is, $\dim O_{k,n,d} = \min\{N,\ n + \binom{k+n}{n} - 1\}$. The expected dimension for $\sigma_s(O_{k,n,d})$ is:

$$\operatorname{expdim} \sigma_s(O_{k,n,d}) = \min\{\, N,\ s(n + \binom{k+n}{n} - 1) + s - 1\}$$

(there are $\infty^{s(\dim O_{k,n,d})}$ choices of s points on $O_{k,n,d}$, plus ∞^{s-1} choices of a point on the $\mathbb{P}^{s-1}$ spanned by the s points; when this number is too big, we expect that $\sigma_s(O_{k,n,d}) = \mathbb{P}^N$).

When $\dim \sigma_s(O_{k,n,d}) < \operatorname{expdim} \sigma_s(O_{k,n,d})$, the osculating variety is said to be *defective*.

In [BCGI], taking into account that the cases with $n = 1$ can be easily described, while if $n \geq 2$ and $d = k$ one has $\dim \sigma_s(O_{k,n,d}) = N$, we raised the following question:

Question Q(k,n,d): *For all k, n, d such that $d \geq k+1$, $n \geq 2$, describe all s for which $\sigma_s(O_{k,n,d})$ is defective, i.e.*

$$\dim \sigma_s(O_{k,n,d}) < \min\{\, N,\ s(n + \binom{k+n}{n} - 1) + s - 1\} = \min\{ \binom{d+n}{n} - 1,\ s\binom{k+n}{n} + sn - 1\}.$$

We were able to answer the question for s, n, d, k in several ranges, thanks to the following lemma (see [BCGI] 2.11 and results of Section 2):

Lemma 1.2 *For any $k, n, d \in \mathbb{N}$ such that $n \geq 2$, $d \geq k+1$, there exists a 0-dimensional subscheme $Z = Z(k,n) \in \mathbb{P}^n$ depending only from k and n and not from d, such that:*

(a) Z is supported on a point P, and one has:

$$(k+1)P \subset Z(k,n) \subset (k+2)P, \qquad \textit{with} \qquad l(Z) = \binom{k+n}{n} + n;$$

(b) denoting by $Y = Y(k, n, s)$ the generic union in $\mathbb{P}^n$ of $Z_1, \ldots, Z_s$ where $Z_i \cong Z$ for $i = 1, \ldots, s$, then

$$\dim \sigma_s(O_{k,n,d}) = \operatorname{expdim} \sigma_s(O_{k,n,d}) - h^0(\mathcal{I}_Y(d)) + \max\{0, \binom{d+n}{n} - l(Y)\}$$

In particular, $\sigma_s(O_{k,n,d})$ is not defective if and only if Y is regular in degree d, i.e. $h^0(\mathcal{I}_Y(d)) \cdot h^1(\mathcal{I}_Y(d)) = 0$.

The homogeneous ideal of this 0-dimensional scheme Z is defined in [BCGI] 2.5 through inverse systems, so we don't have an explicit geometric description of it in the general case. Anyway, for $k = 1$ it is possible to describe it geometrically as follows (see [CGG] Section 2):

Definition 1.3. Let P be a point in $\mathbb{P}^n$, and L a line through P; we say that a 0-dimensional scheme $X \subset \mathbb{P}^n$ is a $(2, 3, n)$-scheme supported on P with direction L if $I_X = I_P^3 + I_L^2$. Hence, the length of a $(2, 3, n)$-point is $2n + 1$. The scheme $Z(1, n)$ of Lemma 1.2 is a $(2, 3, n)$-scheme.

We say that a subscheme of $\mathbb{P}^n$ is a generic union of s $(2, 3, n)$-schemes if it is the union of $X_1, \ldots, X_s$ where X_i is a $(2, 3, n)$-scheme supported on P_i with direction L_i, with $P_1, \ldots, P_s$ generic points and $L_1, \ldots, L_s$ generic lines through $P_1, \ldots, P_s$.

We are going to use these schemes in Section 2, so we need to know more about them; but first we recall the Differential Horace Lemma of [**AH2**], writing it in the context where we shall use it.

Definition 1.4. In the algebra of formal functions $\kappa[[\mathbf{x}, y]]$, where $\mathbf{x} = (x_1, ..., x_{n-1})$, a *vertically graded* (with respect to y) ideal is an ideal of the form:

$$I = I_0 \oplus I_1 y \oplus ... \oplus I_{m-1} y^{m-1} \oplus (y^m)$$

where for $i = 0, ..., m-1$, $I_i \subset \kappa[[\mathbf{x}]]$ is an ideal.

Let Q be a smooth n-dimensional integral scheme, let K be a smooth irreducible divisor on Q. We say that $Z \subset Q$ is a *vertically graded subscheme* of Q with base K and support $z \in K$, if Z is a 0-dimensional scheme with support at the point z such that there is a regular system of parameters $(\mathbf{x}, y)$ at z such that $y = 0$ is a local equation for K and the ideal of Z in $\widehat{\mathcal{O}}_{Q,z} \cong \kappa[[\mathbf{x}, y]]$ is vertically graded.

Let $Z \subset Q$ be a vertically graded subscheme with base K, and $p \geq 0$ be a fixed integer; we denote by $Res_K^p(Z) \subset Q$ and $Tr_K^p(Z) \subset K$ the closed subschemes defined, respectively, by the ideals:

$$\mathcal{I}_{Res_K^p(Z)} := \mathcal{I}_Z + (\mathcal{I}_Z : \mathcal{I}_K^{p+1})\mathcal{I}_K^p, \qquad \mathcal{I}_{Tr_K^p(Z),K} := (\mathcal{I}_Z : \mathcal{I}_K^p) \otimes \mathcal{O}_K.$$

In $Res_K^p(Z)$ we take away from Z the $(p+1)^{th}$ "slice"; in $Tr_K^p(Z)$ we consider only the $(p+1)^{th}$ "slice". Notice that for $p = 0$ we get the usual trace and residual schemes: $Tr_K(Z)$ and $Res_K(Z)$.

Finally, let $Z_1, ..., Z_r \subset Q$ be vertically graded subschemes with base K and support z_i, $Z = Z_1 \cup ... \cup Z_r$, and $\mathbf{p} = (p_1, ..., p_r) \in \mathbb{N}^r$.

We set:

$$Tr_K^{\mathbf{p}}(Z) := Tr_K^{p_1}(Z_1) \cup ... \cup Tr_K^{p_r}(Z_r), \quad Res_K^{\mathbf{p}}(Z) := Res_K^{p_1}(Z_1) \cup ... \cup Res_K^{p_r}(Z_r).$$

Proposition 1.5. (Horace differential Lemma, [**AH2**] Proposition 9.1) *Let H be a hyperplane in $\mathbb{P}^n$ and let $W \subset \mathbb{P}^n$ be a 0-dimensional closed subscheme.*

Let $S_1, ..., S_r$, $Z_1, ..., Z_r$ be $0-$dimensional irreducible subschemes of $\mathbb{P}^n$ such that $S_i \cong Z_i$, $i = 1, ..., r$, Z_i has support on H and is vertically graded with base H, and the supports of $S = S_1 \cup .... \cup S_r$ and $Z = Z_1 \cup ... \cup Z_r$ are generic in their respective Hilbert schemes. Let $\mathbf{p} = (p_1, ..., p_r) \in \mathbb{N}^r$. Assume:
a) $H^0(\mathcal{I}_{Tr_H W \cup Tr_H^{\mathbf{p}}(Z),H}(n)) = 0$ and
b) $H^0(\mathcal{I}_{Res_H W \cup Res_H^{\mathbf{p}}(Z)}(n-1)) = 0$,
then

$$H^0(\mathcal{I}_{W \cup S}(n)) = 0.$$

Definition 1.6. A 2-*jet* is a 0-dimensional scheme $J \subset \mathbb{P}^n$ with support at a point $P \in \mathbb{P}^n$ and degree 2; namely the ideal of J is of type: $I_P^2 + I_L$, where $L \subset \mathbb{P}^n$ is a line containing P. We will say that $J_1, ..., J_s$ are *generic* in $\mathbb{P}^n$, if the points $P_1, ..., P_s$ are generic in $\mathbb{P}^n$ and $L_1, \dots, L_s$ are generic lines through $P_1, \dots, P_s$.

Remark 1.7. Let $X \subset \mathbb{P}^n$ be a $(2,3,n)$-scheme supported at P with direction L and $(y_1, \dots, y_n)$ be local coordinates around P, such that L becomes the y_n-axis; then, $I_X = (y_1 y_n^2, \dots, y_{n-1} y_n^2, y_n^3, y_1^2, y_1 y_2, \dots, y_{n-1}^2)$ (y_n appears only in the first n generators). Let H, respectively K, be a hyperplane through L, respectively transversal to L; then, we can assume $I_H = (y_{n-1})$, respectively $I_K = (y_n)$. We now compute $Res_H^p(X)$ and $Tr_H^p(X)$. One has:

a) $Res_H X = Res_H^0(X)$, $I_{Res_H(X)} = (I_X : y_{n-1}) = (y_1, \dots, y_{n-1}, y_n^2)$, hence $Res_H X$ is a 2-jet lying on L;

b) $Tr_H(X) = Tr_H^0(X)$, $I_{Tr_H(X)} = I_X + (y_{n-1}) = (y_1 y_n^2, \dots, y_{n-2} y_n^2, y_n^3, y_1^2, y_1 y_2, \dots, y_{n-2}^2)$, hence $Tr_H(X)$ is a $(2,3,n-1)$-scheme of H.

Hence the scheme X as a vertically graded scheme with base H has only two layers (strata); in other words, $Tr_H^p(X)$ is empty for $p > 1$, and $Res_H^1(X)$ is a (2,3,n-1)-scheme of H, while $Tr_H^1(X)$ is a 2-jet lying on L.

Now we want to compute $Res_K^p(X)$ and $Tr_K^p(X)$. Consider first:

b) $I_{Tr_K(X)} = I_X + (y_n) = (y_n, y_1^2, y_1 y_2, \dots, y_{n-1}^2)$, hence $Tr_H(X)$ is a 2-fat point of $K \cong \mathbb{P}^{n-1}$,

a) $I_{Res_K X} = (I_X : y_n) = (y_1 y_n, \dots, y_{n-1} y_n, y_n^2, y_1^2, y_1 y_2, \dots, y_{n-1}^2)$, hence $Res_K X$ is a 2-fat point of $\mathbb{P}^n$.

So the scheme X, as a vertically graded scheme with base K, has only three layers (strata); the 0-layer is $Tr_K(X) = Tr_K^0(X)$, the 1-layer is the 0-layer of $Res_K X = Res_K^0(X)$, hence it is again a 2-fat point of $K \cong \mathbb{P}^{n-1}$, and the 2-layer is the 1-layer of $Res_K X$, hence it is a point of $\mathbb{P}^n$. In other words, $Tr_H^p(X)$ is empty for $p > 2$, $Res_K^1(X)$ is a a 2-fat point of $\mathbb{P}^n$, while $Tr_K^1(X)$ is a 2-fat point of K; $Res_K^2(X)$ is a 2-fat point of K doubled in a direction transversal to K (i.e., $I_{Res_K^2(X)} = (y_n^2, y_1^2, y_1 y_2, \dots, y_{n-1}^2)$), while $Tr_K^2(X)$ is a point of $\mathbb{P}^n$.

We will use in the sequel the fact that by adding s generic 2-jets to any 0-dimensional scheme $Z \subset \mathbb{P}^n$ we impose a maximal number of independent conditions to forms in $I_Z(d)$, for all d. This is probably classically known, but we write a proof here for lack of a reference:

Lemma 1.8 *Let $Z \subseteq \mathbb{P}^n$ be a scheme, and let $J \subset \mathbb{P}^n$ be a generic 2-jet. Then:*

$$h^0(\mathcal{I}_{Z\cup J}(d)) = \max\{h^0(\mathcal{I}_Z(d)) - 2, 0\}.$$

Proof: Let P be the support of J; then we know that $h^0(\mathcal{I}_{Z\cup P}(d)) = \max\{h^0(\mathcal{I}_Z(d))-1, 0\}$, so if $h^0(\mathcal{I}_Z(d)) \leq 1$ there is nothing to prove. Let $h^0(\mathcal{I}_Z(d)) \geq 2$, then $h^0(\mathcal{I}_{Z\cup P}(d)) = h^0(\mathcal{I}_Z(d)) - 1 \geq 1$. Since J is generic, if $h^0(\mathcal{I}_{Z\cup J}(d)) = h^0(\mathcal{I}_{Z\cup P}(d))$, then every form of degree d containing $Z \cup P$ should have double intersection with almost every line containing P, hence it should be singular at P. This means that when we force a form in the linear system $|H^0(\mathcal{I}_Z(d))|$ to vanish at P, then we are automatically imposing to the form to be singular at P, and this holds for P in a dense open set of $\mathbb{P}^n$, say U. If the form f is generic in $|H^0(\mathcal{I}_Z(d))|$, its zero set V meets U in a non empty subset of V, so f is singular at whatever point P' we choose in $V \cap U$, and this means that the hypersurface V is not reduced. Since the dimension of the linear system $|H^0(\mathcal{I}_Z(d))|$ is at least 2, this is impossible by Bertini Theorem (e.g. see [**J**], Theorem 6.3).

□

Let $Z \subseteq \mathbb{P}^n$ be a zero-dimensional scheme; the following simple Lemma gives a criterion for adding to Z a scheme D which lies on a smooth hypersurface $\mathcal{F} \subseteq \mathbb{P}^n$ and is made of s generic 2-jets on $\mathcal{F}$, in such a way that D imposes independent conditions to forms of a given degree in the ideal of Z (see Lemma 4 in [**Ch1**] and Lemma 1.9 in [**CGG2**] for the case of simple points on a hypersurface).

Lemma 1.9 *Let $Z \subseteq \mathbb{P}^n$ be a zero dimensional scheme. Let $\mathcal{F} \subseteq \mathbb{P}^n$ be a smooth hypersurface of degree d and let $Z' = Res_{\mathcal{F}} Z$. Let $P_1, \ldots, P_s$ be generic points on $\mathcal{F}$, let $L_1, \ldots, L_s$ lines with $P_i \in L_i$, and such that each line L_i is generic in $T_{P_i}(\mathcal{F})$; let J_i be the 2-jet with support at P_i and contained in L_i. We denote by $D_s = J_1 \cup \ldots \cup J_s$ the union of these s 2-jets generic in $\mathcal{F}$.*

i) If $\dim(I_{Z+D_{s-1}})_t \geq \dim(I_{Z'})_{t-d} + 2$, then $\dim(I_{Z+D_s})_t = \dim(I_Z)_t - 2s$;

ii) if $\dim(I_{Z'})_{t-d} = 0$ and $\dim(I_Z)_t \leq 2s$, then $\dim(I_{Z+D_s})_t = 0$.

Proof: *i)* By induction on s. If $s = 1$, by assumption $\dim(I_Z)_t \geq \dim(I_{Z'})_{t-d} + 2$, hence in the exact sequence $0 \to H^0(\mathcal{I}_{Z'}(t-d)) \stackrel{\phi}{\to} H^0(\mathcal{I}_Z(t-d)) \to H^0(\mathcal{I}_{Z\cap\mathcal{F},\mathcal{F}}(t)) \to \ldots$ the cokernel of the map ϕ has dimension at least 2 and so $(I_Z)_t$ cuts on $\mathcal{F}$ a linear system (i.e. $|H^0(\mathcal{I}_{Z\cap\mathcal{F},\mathcal{F}}(t))|$) of (projective) dimension ≥ 1. We have $\dim(I_{Z+P_1})_t = \dim(I_Z)_t - 1$, since otherwise each hypersurface in $|(I_Z)_t|$ would contain the generic point P_1 of $\mathcal{F}$, that is, would contain $\mathcal{F}$.

Assume $\dim(I_{Z+J_1})_t = \dim(I_{Z+P_1})_t = \dim(I_Z)_t - 1$; this means that if we impose to $S \in |(I_Z)_t|$ the passage through P_1 automatically we impose to S to be tangent to L_1 at P_1, and L_1 being generic in $T_{P_1}(\mathcal{F})$, this means that each S passing through P_1 is tangent to $\mathcal{F}$ at P_1. Let's say that this holds for P_1 in the open not empty subset U of $\mathcal{F}$; for S generic in $|(I_Z)_t|$, $U' = S \cap \mathcal{F} \cap U$ is not empty, hence the generic S is tangent to $\mathcal{F}$ at each $P \in U'$. This means that $|(I_Z)_t|$ cuts on $\mathcal{F}$ a linear system of positive dimension whose generic element is generically non reduced, and this is impossible, by Bertini Theorem (e.g. see [**J**], Theorem 6.3).

Now let $s > 1$. Since $\dim(I_{Z+D_{s-2}})_t \geq \dim(I_{Z+D_{s-1}})_t > \dim(I_{Z'})_{t-d}$ by assumption, and $Res_{\mathcal{F}}(Z + D_{s-1}) = Z'$, the case $s = 1$ gives $\dim(I_{Z+D_s})_t = \dim(I_{Z+D_{s-1}})_t - 2$. So, by the induction hypothesis, we get

$$\dim(I_{Z+D_s})_t = (\dim(I_Z)_t - 2(s-1)) - 2 = \dim(I_Z)_t - 2s.$$

ii) Assume first $\dim(I_Z)_t \leq 2$; it is enough to prove $\dim(I_{Z+J_1})_t = 0$ since then also $\dim(I_{Z+D_s})_t = 0$. If $\dim(I_Z)_t = 2$ this follows by *i)* and if $\dim(I_Z)_t = 0$ this is trivial. If $\dim(I_Z)_t = 1$, then if $\dim(I_{Z+P_1})_t = 0$ we are done. If $\dim(I_{Z+P_1})_t = 1$, then by the genericity of P_1 we have that the unique S in the system contains $\mathcal{F}$, i.e. $S = \mathcal{F} \cup G$, but then $Z' \subseteq G$, which contradicts $\dim(I_{Z'})_{t-d} = 0$.

Otherwise, let $\dim(I_Z)_t = 2v + \delta \geq 3$, $\delta = 0, 1$. If $\delta = 0$, then $\dim(I_{Z+D_{v-1}})_t \geq 2 = \dim(I_{Z'})_{t-d} + 2$, and by *i)* we get $\dim(I_{Z+D_v})_t = \dim(I_Z)_t - 2v = 0$, and, since $s \geq v$, it follows that $\dim(I_{Z+D_s})_t = 0$.

If $\delta = 1$, then $\dim(I_{Z+D_{v-1}})_t \geq 3 \geq \dim(I_{Z'})_{t-d} + 2$, and, by *i)*, $\dim(I_{Z+D_{v-1}})_t = 3$ and $\dim(I_{Z+D_v})_t = \dim(I_Z)_t - 2v = 1$. Notice that the only element in $(I_{Z+D_v})_t$ cannot have $\mathcal{F}$ as a fixed component, otherwise we would have $\dim(I_{Z'})_{t-d} = 1$ and not $= 0$; hence $\dim(I_{Z+D_v+P_{v+1}})_t = 0$ and so, since $2s \geq 2v + 1$ and $D_v \cup P_{v+1} \subset D_s$, $\dim(I_{D_s})_t = 0$.

$\square$

Now we give a Lemma which will be of use in the proof of Theorem 2.2.

Lemma 1.10 *Let $R \subseteq \mathbb{P}^n$ be a zero dimensional scheme contained in a $(2,3,n)$-scheme with $r = \deg Y \leq 2n$; assume moreover that, if $r \geq n+1$, then R is a flat limit of the union of a 2-fat point of $\mathbb{P}^n$ and of a scheme (eventually empty) contained in a 2-fat point of a $\mathbb{P}^{n-1}$, and that, if $r \leq n$, then R is contained in a 2-fat point of a $\mathbb{P}^{n-1}$. Then, there exists a flat family for which R is a special fiber and the generic fiber is the generic union in $\mathbb{P}^n$ of δ 2-fat points, h 2-jets and ϵ simple points, where $r = (n+1)\delta + 2h + \epsilon$, $0 \leq \delta \leq 1$, $0 \leq \epsilon \leq 1$, and $2h + \epsilon \leq n$.*

Proof: In the following we denote by 2_tP a 2-fat point of a linear variety $K \subseteq \mathbb{P}^n$, $K \cong \mathbb{P}^t$. We first notice that if A is a subscheme of 2_nP with $\deg A = n$ then A is a scheme of type $2_{n-1}P$. The proof is by induction on n: if $n = 2$, the statement is trivial since the only scheme of degree 2 in $\mathbb{P}^2$ is a 2-jet, i.e. a 2_1P. Now assume the assertion true for $n-1$, let A be a subscheme of 2_nP with $\deg A = n$ and let H be a hyperplane through the support of A. Since $\deg 2_nP \cap H = n$, we have $n - 1 \leq \deg A \cap H \leq n$. If $\deg A \cap H = n$ then $A = 2_{n-1}P$ and we are done. If $\deg A \cap H = n-1$ then $Res_H A$ is a simple point, and by induction $A \cap H = 2_{n-2}P$. Hence there is a hyperplane K such that $A \cap H$ is a 2-fat point of $H \cap K$, and working for example in affine coordinates, it is easy to see that A is a 2-fat point of the $\mathbb{P}^{n-1}$ generated by $H \cap K$ and a normal direction to H.

In order to prove the Lemma, it is enough to prove that the generic union in $\mathbb{P}^n$ of h 2-jets and ϵ simple points, with $0 \leq \epsilon \leq 1$ and $2h + \epsilon \leq n$, specializes to any possible subscheme M of a scheme of type $2_{n-1}P$: in fact, if $r \leq n$ we are done, if $r \geq n+1$, the collision of a 2_nP with M gives R.

By induction on n: if $n = 2$, the statement is trivial. Let us now consider the generic union of h 2-jets and ϵ simple points in $\mathbb{P}^n$, with $0 \leq \epsilon \leq 1$ and $2h + \epsilon \leq n$. We have two cases.

Case 1: if $2h + \epsilon \leq n-1$, we specialize everything inside a hyperplane H where, by induction assumption, this scheme specializes to any possible subscheme of a scheme of type $2_{n-2}P$, i.e., to any possible subscheme of degree $\leq n-1$ of a scheme of type $2_{n-1}P$.

Case 2: If $2h + \epsilon = n$, we have to show that the generic union of h 2-jets and ϵ simple points specializes to a scheme $2_{n-1}P$.

If n is odd, then $h = \frac{n-1}{2}$ and $\epsilon = 1$; by induction assumption, $\frac{n-1}{2}$ 2-jets specialize to a scheme of type $2_{n-2}P$, and the generic union of the last one with a simple point specializes to a scheme of type $2_{n-1}P$.

If n is even, then $h = \frac{n}{2}$ and $\epsilon = 0$; by induction assumption, $\frac{n}{2} - 1$ 2-jets specialize to a scheme of degree $n-2$ contained in a scheme of type $2_{n-2}P$, which is a $2_{n-3}P$, so it is enough to prove that the generic union of the last one with a 2-jet specializes to a scheme of type $2_{n-1}P$.

In affine coordinates $x_1, \dots, x_n$, let $x_{n-2} = x_{n-1} = x_n = 0$ be the linear subspace containing $2_{n-3}P$, so that $I_{2_{n-3}P} = (x_1, \dots, x_{n-3})^2 \cap (x_{n-2}, x_{n-1}, x_n)$, and let $(x_1, \dots, x_{n-3}, x_{n-2} - a, x_{n-1}^2, x_n)$ be the ideal of a 2-jet moving along the x_{n-2}-axis; then it is immediate to see that the limit for $a \to 0$ of $(x_1, \dots, x_{n-3})^2 \cap (x_{n-2}, x_{n-1}, x_n) \cap (x_1, \dots, x_{n-3}, x_{n-2} - a, x_{n-1}^2, x_n)$ is $(x_1, \dots, x_{n-1})^2 \cap (x_n)$, which is the ideal of a $2_{n-1}P$.

$\square$

2. On Conjecture 1.

We want to study $\sigma_s(T_{n,d})$, and we have seen that its dimension is given by the Hilbert function of s generic $(2,3,n)$-points in $\mathbb{P}^n$.

Definition 2.0 For each n and d we define $s_{n,d}$, $r_{n,d} \in \mathbb{N}$ as the two positive integers such that

$$\binom{d+n}{n} = (2n+1)s_{n,d} + r_{n,d}, \qquad 0 \le r_{n,d} < 2n+1.$$

In the following we denote by $X_{s,n} \subset \mathbb{P}^n$ the zero dimensional scheme union of s generic $(2,3,n)$-schemes $A_1, \dots, A_s$. We also denote by $X_{s_{n,d}}$ the scheme $X_{s,n}$, with $s = s_{n,d}$. Hence $X_{s_{n,d}}$ is the union of the maximum number of generic $(2,3,n)$-points that we expect to impose independent conditions to forms od degree d. We will also use $X_{s_{n,d}+1}$ to indicate $X_{s+1,n}$ when $s = s_{n,d}$.

With $Y_{n,d} \subset \mathbb{P}^n$ we denote a scheme generic union of $X_{s_{n,d}}$ and $R_{n,d}$, where $R_{n,d}$ is a zero dimensional scheme contained in a $(2,3,n)$-point, with $\deg(R_{n,d}) = r_{n,d}$.

A 0-dimensional subscheme A of $\mathbb{P}^n$ is said to be "$\mathcal{O}_{\mathbb{P}^n}(d)$-numerically settled" if $degA = h^0(\mathcal{O}_{\mathbb{P}^n}(d))$; in this case, $h^0(\mathcal{I}_A(d)) = 0$ if and only if $h^1(\mathcal{I}_A(d)) = 0$. The scheme $Y_{n,d}$ is $\mathcal{O}_{\mathbb{P}^n}(d)$-numerically settled for all n, d.

Remark 2.1 Let A be a 0-dimensional $\mathcal{O}_{\mathbb{P}^n}(d)$-numerically settled subscheme of $\mathbb{P}^n$, and assume $h^0(\mathcal{I}_A(d)) = 0$. Let $B \subseteq A$ and $C \supseteq A$ be 0-dimensional subschemes of $\mathbb{P}^n$; then, $h^0(\mathcal{I}_C(d)) = 0$, and $h^1(\mathcal{I}_B(d)) = 0$, or equivalently $h^0(\mathcal{I}_B(d)) = degA - degB$.

Hence if we prove $h^0(\mathcal{I}_{Y_{n,d}}(d)) = 0$ then we know that $h^1(\mathcal{I}_{Y_{n,d}}(d)) = 0$, and

$h^0(\mathcal{I}_{X_{s,n}}(d)) = 0$ for all $s > s_{n,d}$,

$h^1(\mathcal{I}_{X_{s,n}}(d)) = 0$ for all $s \le s_{n,d}$.

Moreover, if $h^0(\mathcal{I}_{Y_{n,d}}(d)) = 0$ then also $h^0(\mathcal{I}_D(d)) = 0$, where D denotes a generic union of $X_{s_{n,d}}$, of $\lfloor \frac{r_{n,d}}{2} \rfloor$ 2-jets and of $r_{n,d} - 2\lfloor \frac{r_{n,d}}{2} \rfloor$ simple points. In fact, we have $h^0(\mathcal{I}_{X_{s_{n,d}}}(d)) = \deg(R_{n,d}) = r_{n,d}$ and we conclude by Lemma 1.8.

The same conclusion (i.e. $h^0(\mathcal{I}_D(d)) = 0$) holds in the weaker assumption that $h^1(\mathcal{I}_{X_{s_{n,d}}}(d)) = 0$, since in this case $h^0(\mathcal{I}_{X_{s_{n,d}}}(d)) = \binom{d+n}{n} - deg(X_{s_{n,d}}) = r_{n,d}$ and we get $h^0(\mathcal{I}_D(d)) = 0$ by Lemma 1.8.

Theorem 2.2 *Suppose that for all $n \ge 5$, we have $h^1(\mathcal{I}_{X_{s_{n,3}}}(3)) = 0$ and $h^0(\mathcal{I}_{X_{s_{n,3}+1}}(3)) = 0$; then $h^0(\mathcal{I}_{Y_{n,d}}(d)) = h^1(\mathcal{I}_{Y_{n,d}}(d)) = 0$, for all $d \ge 4$, $n \ge 4$.*

Proof: Let us consider a hyperplane $H \subset \mathbb{P}^n$; we want a scheme Z with support on H, made of $(2,3,n)$-schemes, and an integer vector $\mathbf{p}$, such that the "differential trace" $Tr^{\mathbf{p}}_H(Z) \subset H$ is $\mathcal{O}_{\mathbb{P}^{n-1}}(d)$-numerically settled.

Let us consider $n \geq 5$ first. Since $0 \leq r_{n-1,d} \leq 2n-2$, we write $r_{n-1,d} = n\delta + 2h + \epsilon$, with $0 \leq \epsilon \leq 1$, $0 \leq \delta \leq 1$ and $2h + \epsilon \leq n$.

We denote by Z the zero dimensional scheme union of $s_{n-1,d} + h + \epsilon + \delta$ (hence $\delta = 0$ if $0 \leq r_{n-1,d} \leq n$, while $\delta = 1$ if $n+1 \leq r_{n-1,d} \leq 2n-2$), $(2,3,n)$-schemes $Z_1, \dots, Z_{s_{n-1,d}+h+\epsilon+\delta}$, where each Z_i is supported at P_i with direction L_i, and:

- the P_i's are generic on H, $i = 1, \dots, s_{n-1,d} + h + \epsilon + \delta$;

- $L_i \subset H$ for $i = 1, \dots, s_{n-1,d} + h$;

- if $(\epsilon, \delta) \neq (0,0)$, the corresponding lines $L_{s_{n-1,d}+h+1}$, $L_{s_{n-1,d}+h+2}$ have generic directions in $\mathbb{P}^n$ (hence not contained in H.

In case $n = 4$, instead, we write $r_{3,d} = 2h + \epsilon$, with $0 \leq \epsilon \leq 1$, and Z is given as before. Notice that in this case $0 \leq h \leq 3$, and it can appear only one line $L_{s_{3,d}+h+1}$, not contained in H.
We want to use the Horace differential Lemma 1.5, where the role of the schemes H and Z appearing in the statement of the Lemma are played by our hyperplane H and the scheme Z just defined, and with:
$W = A_{s_{n-1,d}+h+\epsilon+1} \cup \dots \cup A_{s_{n,d}} \cup R_{n,d}$,
$S = A_1 \cup \ldots \cup A_{s_{n-1,d}+h+\epsilon+\delta}$,
$\mathbf{p} = (\underbrace{0, \dots, 0}_{s_{n-1,d}}, \underbrace{1, \dots, 1}_{h}, \underbrace{2}_{\epsilon}, \underbrace{0}_{\delta})$.
so that $Tr_H W = \emptyset$ and $Res_H W = W$, and $Y_{n,d} = W \cup S$.
Notice that this construction is possible, since $s_{n-1,d} + h + 2 \leq s_{n,d}$ (and even more than that): see Appendix A, A.1.
In order to simplify notations, we set:

$$T^j_i := Tr^j_H(Z_i), \quad R^j_i := Res^j_H(Z_i), \quad j = 0,1,2, \quad i = 1, \dots, s_{n-1,d} + h + \epsilon + \delta,$$

$$T := Tr_H W \cup Tr^{\mathbf{p}}_H(Z) = T^0_1 \cup \ldots \cup T^0_{s_{n-1,d}} \cup T^1_{s_{n-1,d}+1} \cup \ldots \cup T^1_{s_{n-1,d}+h} \cup T^2_{s_{n-1,d}+h+\epsilon} \cup T^0_{s_{n-1,d}+h+\epsilon+\delta},$$

$$R := Res_H W \cup Res^{\mathbf{p}}_H(Z) = W \cup R^0_1 \cup \ldots \cup R^0_{s_{n-1,d}} \cup R^1_{s_{n-1,d}+1} \cup \ldots \cup R^1_{s_{n-1,d}+h} \cup R^2_{s_{n-1,d}+h+\epsilon} \cup R^0_{s_{n-1,d}+h+\epsilon+\delta}.$$

Observe that, by Remark 1.7 :

$T^0_1, \dots, T^0_{s_{n-1,d}}$ are $(2,3,n-1)$-points in $H \cong \mathbb{P}^{n-1}$, and $R^0_1, \dots, R^0_{s_{n-1,d}}$ are 2-jets in H;
$T^1_{s_{n-1,d}+1}, \dots, T^1_{s_{n-1,d}+h}$ are 2-jets in H and $R^1_{s_{n-1,d}+1}, \dots, R^1_{s_{n-1,d}+h}$ are $(2,3,n-1)$-points in H;
$T^2_{s_{n-1,d}+h+\epsilon}$ is, when appearing, a simple point of H, and $R^2_{s_{n-1,d}+h+\epsilon+\delta}$ is a 2-fat point of H doubled in a direction transversal to H;
$T^0_{s_{n-1,d}+h+\epsilon+\delta}$ is, when appearing, a 2-fat point on H, and $R^0_{s_{n-1,d}+h+\epsilon}$ is a 2-fat point in $\mathbb{P}^n$ with support on H.
We will also make use of the scheme:

$$B := W \cup R^1_{s_{n-1,d}+1} \cup \ldots \cup R^1_{s_{n-1,d}+h} \cup R^2_{s_{n-1,d}+h+\epsilon}.$$

Let us consider the following four statements:

$$\mathbf{Prop}(n,d) : h^0(\mathcal{I}_{Y_{n,d}}(d)) = 0; \qquad \mathbf{Reg}(n,d) : h^1(\mathcal{I}_{X_{s,n}}(d)) = 0 \text{ and } h^0(\mathcal{I}_{X_{s,n+1}}(d)) = 0,$$

$$\mathbf{Degue}(n,d) : h^0(\mathcal{I}_R(d-1)) = 0; \qquad \mathbf{Dime}(n,d) : h^0(\mathcal{I}_{T,H}(d)) = 0.$$

If **Degue**(n,d) and **Dime**(n,d) are true, we know that **Prop**(n,d) is true too, by Proposition 1.5.

For the first values of n, d, we will need an "ad hoc" construction, which is given by the following:

Lemma 2.3 *Let $d = 4$ and $n \in \{4,5,6\}$, then* **Prop**(n,d) *holds.*

Proof of the Lemma.

Case $n = 4$. Here we use the construction of R and T described above, hence we need to show that **Degue**$(4,4)$ and **Dime**$(4,4)$ hold. Since $s_{3,4} = 5$, and $r_{3,4} = 0$, T is made of five generic $(2,3,3)$-points in $H \cong \mathbb{P}^3$, so **Dime**$(4,4)$ holds (i.e. $h^0(\mathbb{P}^3, I_{T,H}(4)) = h^0(\mathbb{P}^3, I_{X_{5,3}}(4) = 0)$, e.g. see [**CGG1**].

In order to prove **Degue**$(4,4)$ we want to apply Lemma 1.2, with R made of five 2-jets plus the scheme $B = W$; hence we need to show that $h^0(\mathcal{I}_B(3)) \le 10$, while $h^0(\mathcal{I}_{Res_H(B)}(2)) = 0$. Since here $s_{4,4} = 7 = r_{4,4}$, while $r_{3,4} = 0$, we have that $B = W = Res_H(B)$ and it is given by A_6 and A_7, plus $R_{4,4}$. Hence we have $h^1(\mathcal{I}_B(3)) = 0$, since B is contained in the scheme made of 3 generic $(2,3,4)$-points (which is known to have maximal Hilbert function, by [**CGG1**] or [**B**]); $h^1(\mathcal{I}_B(3)) = 0$ is equivalent to saying that $h^0(\mathcal{I}_B(3)) = 2s_{3,4} = 10$, as required. Moreover $h^0(\mathcal{I}_B(2)) = 0$, since there is one only form of degree two passing through two generic $(2,3,4)$-points in $\mathbb{P}^4$, given by the hyperplane containing the two double lines, doubled. Since the support of $R_{4,4}$ is generic, we get $h^0(\mathcal{I}_B(2)) = 0$. So we have that **Degue**$(4,4)$ holds, and **Prop**$(4,4)$ holds too.

Case $n = 5$. Here we need to use a different construction. We have $s_{5,4} = 11$, $r_{5,4} = 5$, $s_{4,4} = 7 = r_{4,4}$. We want to use the Horace differential Lemma 1.5 with $Z = Z_1 \cup \ldots \cup Z_8 \cup R_{5,4}$, where $Z_1, \ldots, Z_8$ are $(2,3,5)$ schemes supported at generic points of H with direction $L_1, \ldots, L_8 \subset H$, and we specialize $R_{5,4}$ so that $R_{5,4} \subset H$, contained in a generic $(2,3,4)$-scheme of H; with $W = A_9 \cup A_{10} \cup A_{11}$, and with $\mathbf{p} = (\underbrace{0,\ldots,0}_{7},1,0)$.

Hence $T = Tr_H W \cup Tr_H^{\mathbf{p}}(Z) = T_1^0 \cup T_2^0 \cup \ldots \cup T_7^0 \cup T_8^1 \cup R_{5,4}$ and $R = Res_H W \cup Res_H^{\mathbf{p}}(Z) = W \cup R_1^0 \cup R_2^0 \cup \ldots \cup R_7^0 \cup R_8^1$.

We have that the ideal sheaf of $T_1^0 \cup T_2^0 \cup \ldots \cup T_7^0 \cup R_{5,4}$ has $h^1 = 0$ and $h^0 = 2$ in degree 4, by using the previous case and the fact that $R_{5,4}$ is contained in a $(2,3,4)$-point, so $h^0(\mathcal{I}_{T,H}(4)) = 0$ by Lemma 1.8, since T_8^1 is a 2-jet in $H \cong \mathbb{P}^4$. We also have $h^0(\mathcal{I}_R(3)) = 0$. In fact, let us denote by U the scheme $U = R_8^1 \cup W$. In order to apply Lemma 1.9 (the R_i^0's are 2-jets) to get $h^0(\mathcal{I}_R(3)) = 0$, we need to show that $h^0(\mathcal{I}_{Res_H U}(2)) = 0$ and $h^1(\mathcal{I}_U(3)) = 0$. Since U is included in the union of four $(2,3,5)$-points, which impose independent conditions in degree three (e.g. see [**CGG1**]), $h^1(\mathcal{I}_U(3)) = 0$ follows. Moreover, $Res_H(U)$ is made by three $(2,3,5)$-points, and again $h^0(\mathcal{I}_{Res_H U}(2)) = 0$ is known by [**CGG1**].

Now, $h^0(\mathcal{I}_{T,H}(4)) = 0 = h^0(\mathcal{I}_R(3))$ imply **Prop**$(5,4)$ by Lemma 1.5, and we are done.

Case $n = 6$. Here we have $s_{6,4} = 16$, $r_{6,4} = 2$, while $s_{5,4} = 11$, $r_{5,4} = 5$. We want to use the Horace differential Lemma 1.5 with $Z = Z_1 \cup \ldots \cup Z_{13} \cup R_{6,4}$, where $Z_1, \ldots, Z_{13}$ are $(2,3,6)$ schemes supported at generic points of H with direction $L_1, \ldots, L_{12} \subset H$, while L_{13} is not in H, and we specialize $R_{6,4} \subset H$, as a generic 2-jet in H; with $W = A_{14} \cup A_{15} \cup A_{16}$, and with $\mathbf{p} = (\underbrace{0,\ldots,0}_{11},1,2,0)$.

Hence $T = Tr_H W \cup Tr^{\mathbf{p}}_H(Z) = T^0_1 \cup T^0_2 \cup \ldots \cup T^0_{11} \cup T^1_{12} \cup T^2_{13} \cup R_{6,4}$ and $R = Res_H W \cup Res^{\mathbf{p}}_H(Z) = W \cup R^0_1 \cup R^0_2 \cup \ldots \cup R^0_{11} \cup R^1_{12} \cup R^2_{13}$.

We have that $h^0(\mathcal{I}_{T,H}(4)) = 0$ by applying Lemma 1.1 and the previous case.

We also have $h^0(\mathcal{I}_R(3)) = 0$. In fact, let us denote by U the scheme $U = R^1_{12} \cup R^2_{13} \cup W$. In order to apply Lemma 1.9 (the R^0_i's are 2-jets) to get $h^0(\mathcal{I}_R(3)) = 0$, we need to show that $h^0(\mathcal{I}_{Res_H U}(2)) = 0$ and $h^1(\mathcal{I}_U(3)) = 0$.

Since U is included in the union of five $(2,3,6)$-points, which impose independent conditions in degree three (e.g. see [**CGG1**]), $h^1(\mathcal{I}_U(3)) = 0$ follows. Moreover, $Res_H(U)$ is made by three $(2,3,6)$-points plus a 2-fat point inside $H \cong \mathbb{P}^5$. Since there is only one form of degree two passing through three generic $(2,3,6)$-points in $\mathbb{P}^6$, given by the hyperplane containing the three double lines, doubled, we get $h^0(\mathcal{I}_{Res_H U}(2)) = 0$.

Now, $h^0(\mathcal{I}_{T,H}(4)) = 0 = h^0(\mathcal{I}_R(3))$ imply $\mathbf{Prop}(6,4)$ by Lemma 1.5, and we are done. □

Now we come back to the proof of the Theorem for the remaining values of n, d; we will work by induction on both n, d in order to prove statement $\mathbf{Prop}(n,d)$ for $n \geq 4$, $d \geq 5$ and for $n \geq 7$, $d = 4$. We divide the proof in 7 steps.

Step 1. The induction is as follows: we suppose that $\mathbf{Prop}(\nu,\delta)$ is known for all (ν,δ) such that $4 \leq \nu < n$ and $4 \leq \delta \leq d$ or $4 \leq \nu \leq n$ and $4 \leq \delta < d$ and we prove that $\mathbf{Prop}(n,d)$ holds.

The initial cases for the induction are given by Lemma 2.2, and we will also make use of the fact that $\mathbf{Reg}(n,3)$ with $n \geq 4$ and $\mathbf{Reg}(3,d)$ with $d \geq 4$ hold respectively by assumption and by [B], while, by [CGG], we know everything about the Hilbert function of generic $(2,3,n)$-schemes when $d = 2$.

We will be done if we prove that $\mathbf{Degue}(n,d)$ and $\mathbf{Dime}(n,d)$ hold for $n \geq 4$, $d \geq 5$ and for $n \geq 7$, $d = 4$.

Step 2. Let us prove $\mathbf{Dime}(n,d)$. Notice that T is $\mathcal{O}_{\mathbb{P}^{n-1}}(d)$-numerically settled in $H \cong \mathbb{P}^{n-1}$, hence $\mathbf{Dime}(n,d)$ is equivalent to $h^1(\mathcal{I}_{T,H}(d)) = 0$.

The scheme T is the generic union of $X_{s_{n-1,d}}$ with h 2-jets, of ϵ simple points and of δ 2-fat points, where $2h + \epsilon + n\delta = r_{n-1,d}$. Then $\mathbf{Dime}(n,d)$ holds for $n \geq 5$ and $d \geq 4$ since we are assuming that $\mathbf{Prop}(n-1,d)$ is true and the union of h 2-jets, ϵ simple points and of δ 2-fat points can specialize to $R_{n-1,d}$ (see Lemma 1.10).

For $n = 4$ and $d \geq 5$, $\mathbf{Dime}(4,d)$ holds, since we know that $h^1(\mathcal{I}_{X_{s_{3,d}}}(d)) = 0$ by [B] and in this case T is the generic union of $X_{s_{3,d}}$ with h 2-jets and ϵ simple points so we can apply Lemma 1.8.

Step 3. We are now going to prove $\mathbf{Degue}(n,d)$. Since the scheme R is the union of the scheme B and of $s_{n-1,d}$ 2-jets lying on H (see definitions of R and B above), we can use Lemma 1.9 *ii*). Hence, in order to prove that $\dim(I_R)_{d-1} = 0$, i.e. that $\mathbf{Degue}(n,d)$ holds, it is enough to prove that $(I_{Res_H(B)})_{d-2} = 0$ and that $\dim(I_B)_{d-1} \leq 2s_{n-1,d}$.

Step 4. Let us show that$(I_{Res_H(B)})_{d-2} = 0$. We set $t_{n,d} := s_{n,d} - s_{n-1,d} - h - \epsilon - \delta$. The scheme $Res_H(B)$ is given by W plus, if $\epsilon = 1$, one 2-fat point contained in H, plus, if $\delta = 1$, one simple point in H. W is the generic union of $R_{n,d}$ with $t_{n,d}$ $(2,3,n)$-points. Let I denote the ideal of these $t_{n,d}$ $(2,3,n)$-points; if we show that $I_{d-2} = 0$, then also $(I_{Res_H(B)})_{d-2} = 0$.

The idea is to prove that our $(2,3,n)$-points are "too many" to have $I_{d-2} \neq 0$ since they are more than $s_{n,d-2} + 1$; the only problem with this procedure is that there are cases (when $d - 2 = 2$ or 3) where I_{d-2} may not have the expected dimension, so those cases have to be treated in advance.

First let $d = 4$ (and $n \geq 7$); if we show that $t_{n,4} > \frac{n}{2}$, then we are done, since $(I_{X_{s,n}})_2 = 0$ for $s > \frac{n}{2}$, by [**CGG**], Prop 3.3. The inequality $t_{n,4} > \frac{n}{2}$ is treated in Appendix A, A.2, and proved for $n \geq 7$, as required.

Now let $d = 5$ and $n = 4$; here we have that $s_{4,3} + 1 = 4$, but actually there is one cubic hypersuface through four $(2,3,4)$-points in $\mathbb{P}^4$; nevertheless, since $t_{4,5} = 14 - 8 - 0 - 0 = 6$, and it is known (see [**CGG**]or [**B**]) that $(I_{X_{6,4}})_3 = 0$, we are done also in this case.

Eventually, for $d = 5$, $n \geq 5$, or in the general case $d \geq 6$, $n \geq 4$, if we show that $t_{n,d} \geq s_{n,d-2} + 1$, the problem reduces to the fact that $(I_{X_{s_{n,d-2}+1}})_{d-2} = 0$. If $d = 5$, we know that $(I_{X_{s_{n,3}+1}})_3 = 0$ by hypothesis, while for $d \geq 6$ we can suppose that $(I_{X_{s_{n,d-2}+1}})_{d-2} = 0$ by induction on d.

The inequality $t_{n,d} \geq s_{n,d-2} + 1$ is discussed in Appendix A, A.1, and proved for all the required values of n, d.

Thus the condition $(I_{Res_H(B)})_{d-2} = 0$ holds.

Step 5. Now we have to check that $\dim(I_B)_{d-1} \leq 2s_{n-1,d}$. Since $\deg Y_{n,d} = h^0(\mathcal{O}_{\mathbb{P}^n}(d))$ and $\deg T = h^0(\mathcal{O}_{\mathbb{P}^{n-1}}(d))$, then $\deg R = h^0(\mathcal{O}_{\mathbb{P}^n}(d-1))$. The scheme R is the union of the scheme B and of $s_{n-1,d}$ 2-jets lying on H, so $\deg R = \deg B + 2s_{n-1,d}$. Hence $\dim(I_B)_{d-1} \leq 2s_{n-1,d}$ is equivalent to $h^1(\mathcal{I}_B(d-1)) = 0$ (and to $\dim(I_B)_{d-1} = 2s_{n-1,d}$).

Let us consider the case $n \geq 5$ first. Let Q be the scheme $Q = Z_{s_{n-1,d}+1} \cup \ldots \cup Z_{s_{n,d}+h+\epsilon+\delta} \cup A_{s_{n,d}+h+\epsilon+\delta+1} \cup \ldots \cup A_{s_{n,d}} \cup A_{s_{n,d}+1}$, where $A_{s_{n,d}+1}$ is a $(2,3,n)$ scheme containing $R_{n,d}$. We have that B is contained in the scheme Q, which is composed by $s_{n,d} - s_{n-1,d} + 1$ generic $(2,3,n)$-points (notice that $2h + \epsilon + \delta \leq n + 1$, so $Z_{s_{n-1,d}+1}, \ldots, Z_{s_{n,d}+h+\epsilon+\delta}$ are generic, since only the first h of the lines L_i are in H).

The generic union of $s_{n,d-1}$ generic $(2,3,n)$-points in $\mathbb{P}^n$ is the scheme $X_{s_{n,d-1}}$; by induction, or by hypothesis if $d-1 = 3$, we have $h^1(\mathcal{I}_{X_{s_{n,d-1}}}(d-1)) = 0$. Since $s_{n,d} - s_{n-1,d} + 1 \leq s_{n,d-1}$ (see *Step 6*), then $B \subset Q \subset X_{s_{n,d-1}}$ and we conclude by Remark 2.1 that $h^1(\mathcal{I}_B(d-1)) = 0$.

Step 6. We now prove the inequality: $s_{n,d} - s_{n-1,d} + 1 \leq s_{n,d-1}$ $(n \geq 5)$.
We have $\deg Q = \deg B + 2h + \epsilon + n\delta + (2n + 1 - r_{n,d})$, in fact in order to "go from B to Q", we have to add a 2-jet to each of the R_i^1 (h in number), a simple point to $R^2_{s_{n-1,d}+h+\epsilon}$ if $\epsilon = 1$, a 2-fat point of H if $\delta = 1$ and something of degree $(2n + 1 - r_{n,d})$ to $R_{n,d}$.

Since $r_{n,d} \geq 0$ and $2h + \epsilon + n\delta = r_{n-1,d} \leq 2n - 2$, we have: $\deg Q = (2n + 1)(s_{n,d} - s_{n-1,d} + 2) \leq \deg(B) + 2n - 2 + 2n + 1 = \deg(B) + 4n - 1$.

Notice that $\deg(Y_{n,d-1}) = \deg(B) + 2s_{n-1,d}$, so we have: $(2n + 1)(s_{n,d} - s_{n-1,d} + 1) \leq \deg(Y_{n,d-1}) - 2s_{n-1,d} + 4n - 1$.

If we prove that $4n-1-2s_{n-1,d} \leq 0$, we obtain: $(2n+1)(s_{n,d}-s_{n-1,d}+1) \leq \deg(Y_{n,d-1}) = (2n+1)s_{n,d-1}$, and we are done.

The computations to get $4n - 1 - 2s_{n-1,d} \leq 0$ can be found in Appendix A.3.

Step 7. We are only left to prove that $h^1(\mathcal{I}_B(d-1)) = 0$ in case $n = 4$ ($d \geq 5$).

Recall that now $r_{3,d} = 2h + \epsilon \leq 6$, with $0 \leq h \leq 3$, $0 \leq \epsilon \leq 1$. If $r_{3,d} \leq 4$, we can apply the same procedure as in step 5, since the part of the scheme Q with support on H is generic in $\mathbb{P}^4$. Hence we only have to deal with $r_{3,d} = 5, 6$.

The case $r_{3,d} = 5$ does not actually present itself; this can be checked by considering that

$$\binom{d+3}{3} = \frac{(d+3)(d+2)(d+1)}{6} = 7s_{3,d} + r_{3,d} \Rightarrow (d+3)(d+2)(d+1) = 42s_{3,d} + 6r_{3,d}.$$

Hence if $r_{3,d} = 5$, we get $42s_{3,d} + 30 = 7(6s_{3,d} + 4) + 2$, but it is easy to check that $(d + 3)(d + 2)(d + 1)$ never gives a remainder of 2, modulo 7.

Thus we are only left with the case $r_{3,d} = 6$, when $h = 3$ and $\epsilon = 0$. In this case we have $d \equiv 3 \pmod 7$, hence $d \geq 10$; it is also easy to check that $r_{3,d-1} = 3$ in this case.

We can add $2s_{3,d}$ generic simple points to B, in order to get a scheme B' which is $\mathcal{O}_{\mathbb{P}^4}(d-1)$-numerically settled, so now $h^1(\mathcal{I}_B(d-1)) = 0$ is equivalent to $h^0(\mathcal{I}_{B'}(d-1)) = 0$ (by Remark 2.1).

We want to apply Horace differential Lemma again in order to prove $h^0(\mathcal{I}_{B'}(d-1)) = 0$; so we will define appropriate schemes Z_B, W_B and an integer vector $\mathbf{q}$, such that conditions $a)$ and $b)$ of Proposition 1.5 apply to them, yielding $h^0(\mathcal{I}_{B'}(d-1)) = 0$.

Consider the scheme $Z_B \subset \mathbb{P}^4$, given by $s_{3,d-1} - 1$ $(2,3,4)$-schemes in $\mathbb{P}^4$, such that their support is at generic points of H, and only for the last one of them the line L_i is not in H. Let $W_B \subset \mathbb{P}^4$ be given by $2s_{3,d}$ generic simple points, $s_{4,d} - s_{3,d} - s_{3,d-1} - 2$ generic $(2,3,4)$-schemes, three generic $(2,3,3)$-schemes in $H \cong \mathbb{P}^3$, and the scheme $R_{4,d}$. Let also $\mathbf{q} = (\underbrace{0,\dots,0}_{s_{3,d-1}-3}, \underbrace{1}_{1}, \underbrace{2}_{1})$.

Let $T_B = Tr_H(W_B) \cup Tr_H^{\mathbf{p}}(Z_B) = X_{s_{3,d-1}} \cup E \cup F$, and $R_B = Res_H(W_B) \cup Res_H^{\mathbf{q}}(Z_B)$.

We have that E and F are, respectively, a 2-jet and a simple point in H (they give the "remainder scheme" of degree 3, to get that T_B is $\mathcal{O}_{\mathbb{P}^3}(d-1)$-numerically settled).

The scheme R_B is the union of $2s_{3,d}$ generic simple points, $s_{4,d} - s_{3,d} - s_{3,d-1} - 2$ generic $(2,3,4)$-schemes, the scheme $R_{4,d}$, $s_{3,d-1}$ 2-jets in H, a $(2,3,3)$-scheme in H and a 2-fat point of H doubled in a direction transversal to H.

If we show that $h^0(\mathcal{I}_{R_B}(d-2)) = 0 = h^0(\mathcal{I}_{T_B,H}(d-1))$, then we are done by Proposition 1.5.

We have $h^0(\mathcal{I}_{T_B,H}(d-1)) = 0$, since T_B is $\mathcal{O}_{\mathbb{P}^3}(d-1)$-numerically settled, and is given by $X_{s_{3,d-1}}$, whose ideal sheaf has $h^1 = 0$ in degree $d-1$ by [B], union with a 2-jet and a simple point, so we can apply Lemma 1.8.

In order to show that $h^0(\mathcal{I}_{R_B}(d-2)) = 0$ we want to proceed as in Step 5, i.e by applying Lemma 1.9, since R_B, is made of $s_{3,d-1} - 3$ 2-jets union the $2s_{3,d}$ generic simple points and a scheme that we denote by R'_B. We will be done if we show that $h^0(\mathcal{I}_{Res_H(R_B)}(d-3)) = 0$ and $h^1(\mathcal{I}_{R'_B}(d-2)) = 0$.

The first condition will follow if $s_{4,d} - s_{3,d} - s_{3,d-1} - 2 \geq s_{4,d-3}$, the second condition (since R'_B is contained in the union of $s_{4,d} - s_{3,d} - s_{3,d-1} + 1$ generic $(2,3,4)$-schemes) if $s_{4,d} - s_{3,d} - s_{3,d-1} + 1 \leq s_{4,d-2}$.

Both inequalities are proved in Appendix, A.4.

□

Thanks to some "brute force" computation by **COCOA**, we are able to prove:

Corollary 2.4 *For $4 \leq n \leq 9$, we have:*

i) $h^1(\mathcal{I}_{X_{s_{n,3}}}(3)) = 0$ and $h^0(\mathcal{I}_{X_{s_{n,3}+1}}(3)) = 0$, except for $n = 4$, in which case we have $h^0(\mathcal{I}_{X_{s,4}}(3)) = 0$ for $s \geq 5$.

ii) $h^0(\mathcal{I}_{Y_{n,d}}(d)) = h^1(\mathcal{I}_{Y_{n,d}}(d)) = 0$, for $d \geq 4$.

Proof: Part i) comes from direct computations using CoCoA ([**CO**]). Note that $s_{4,3} = 3$ and that $h^0(\mathcal{I}_{X_{4,4}}(3)) = h^1(\mathcal{I}_{X_{4,4}}(3)) = 1$, see [**CGG1**].
Part ii) comes by applying the Theorem and part i).

Coming back to the language of secant varieties, Theorem 2.2 and Corollary 2.4 give:

Corollary 2.5 *If Conjecture 1 is true for $d = 3$, then it is true for all $d \geq 4$. Moreover, for $n \leq 9$, Conjecture 1 holds.*

3. On Conjecture 2a. The case $n = 2$.

In this section we prove Conjecture 2a for $n = 2$.

We want to use the fact that $\sigma_s(O_{k,n,d})$ is defective if at a generic point its tangent space does not have the expected dimension; actually (see [**BCGI**]) this is equivalent to the fact that for generic $L_i \in R_1$, $F_i \in R_k$, $R = \kappa[x_0, ..., x_n]$, $i = 1, ..., s$ the vector space $< L_1^{d-k} R_k, L_1^{d-k-1} F_1 R_1, ..., L_s^{d-k} R_k, L_s^{d-k-1} F_s R_1 >$ does not have the expected dimension.

Via inverse systems this reduces to the study of $(I_Y)_d$, where $Y = Z_1 \cup ... \cup Z_s$ is a certain 0-dimensional scheme in $\mathbb{P}^n$. Namely, the scheme Y is supported at s generic points $P_1, ..., P_s \in \mathbb{P}^n$, at each of them $l(Z_i) = \binom{k+n}{n} + n$, and $I_{P_i}^{k+2} \subset I_{Z_i} \subset I_{P_i}^{k+1}$ (see Lemma 1.2).

When working in $\mathbb{P}^2$, we can specialize the F_i's to be of the form Π_i^k, where Π_i is a generic linear form through P_i. In this way we get a scheme $\overline{Y} = \overline{Z}_1 \cup ... \cup \overline{Z}_s$, and the structure of each $\overline{Z}_i$ is $((k+2)P_i \cap L_i^2) \cup (k+1)P_i$, where the line L_i is "orthogonal" to $\Pi_i = 0$, i.e. if we put $P_i = (1, 0, 0)$, $\Pi_i = x_1$ and $L_i = \{x_2 = 0\}$, the ideal is of the form: $((x_1, x_2)^{k+2} + (x_2)^2) \cap (x_1, x_2)^{k+1} = (x_1^{k+2}, x_1^{k+1} x_2, x_1^{k-1} x_2^2, ..., x_2^{k+1})$.

Notice that the forms in $I_{\overline{Z}_i}$ have multiplicity at least $k+1$ at P_i and they meet L_i with multiplicity at least $k+2$; moreover the generic form in $I_{\overline{Z}_i}$ has L_i at least as a double component of its tangent cone at P_i.

When $F \in I_{\overline{Z}_i}$ and we speak of its "tangent cone" at P_i, we mean (with the choice of coordinates above) either the form in $\kappa[x_1, x_2]$ obtained by putting $x_0 = 1$ in F and considering the (homogeneous) part of minimum degree thus obtained, or also the scheme (in $\mathbb{P}^2$) defined by such a form.

We will say that L_i is a "simple tangent" for F meaning that for the tangent cone to F at P_i, L_i is a reduced component.

The strategy we adopt to prove Conjecture 2a is the following: if $(I_Y)_d$ does not have the expected dimension, i.e. $h^0(\mathcal{I}_Y(d)) h^1(\mathcal{I}_Y(d)) \neq 0$, then the same happens for $\mathcal{I}_{\overline{Y}}(d)$; hence Conjecture 2a would be proved if we show that whenever $\dim(I_{\overline{Y}})_d$ is more than expected, then $h^1(\mathcal{I}_X(d)) > max\{0, \deg(Y) - \binom{d+n}{n}\}$ or $h^0(\mathcal{I}_T(d)) > max\{0, \binom{d+n}{n} - \deg(Y)\}$, where

$$X := (k+1)P_1 \cup ... \cup (k+1)P_s \subset \mathbb{P}^2; \quad T := (k+2)P_1 \cup \ldots \cup (k+2)P_s \subset \mathbb{P}^2.$$

The following easy technical Bertini-type lemma and its corollary will be of use in the sequel.

Lemma 3.1 *Let F, G be linearly independent polynomials in $\kappa[x]$. Then for almost any $a \in \kappa$, $F + aG$ has at least one simple root.*

Proof. Let M be the greatest common divisor of F and G with $F = MP$, $G = MQ$. Let us consider $PQ' - QP'$, where P' and Q' are the derivatives of P and Q, respectively . Since P and Q have no common roots, it easily follows that $PQ' - QP'$ cannot be identically zero.

For any $\beta \in \kappa$ which is neither a root for $PQ' - QP'$, nor for M, nor for Q, let

$$a = a(\beta) := -\frac{P(\beta)}{Q(\beta)},$$

so $(F + aG)(\beta) = M(\beta)(P + aQ)(\beta) = 0$, and $(F + aG)'(\beta) = (M'(P + aQ) + M(P' + aQ'))(\beta) = (M(P' + aQ'))(\beta) = (M(P' - \frac{P(\beta)}{Q(\beta)}Q'))(\beta) = (\frac{M}{Q})(\beta)(QP' - PQ')(\beta) \neq 0$, hence β is a simple root for $F + aG$. Since β assumes almost every value in κ, so does $a(\beta)$. □

Corollary 3.2 *Let $P = (1, 0, 0) \in \mathbb{P}^2$. Let $f, g \in (I_P^{k+1})_d$, and $f, g \notin (I_P^{k+2})_d$. Assume that f, g, have different tangent cones at P. Then for almost any $a \in \kappa$, $f + ag$ has at least one simple tangent at P.*

Proof. The Corollary follows immediately from Lemma 3.1 by de-homogenising the tangent cones to f, g at P to get two non-zero and non-proportional polynomials $F, G \in \kappa[x]$. □

It will be handy to introduce the following definitions.

Definition 3.3 Let $P \in \mathbb{P}^2$ and L be a line L through P. We say that a scheme supported at one point is of type Z' if its structure is $(k + 1)P \cup ((k + 2)P \cap L)$, and that it is of type $\overline{Z}$ if its structure is $(k + 1)P \cup ((k + 2)P \cap L^2)$.

We will say that a union of schemes of types Z' and/or $\overline{Z}$ is generic, if the points of their support and the relative lines are generic.

The following lemma is the key to prove Conjecture 2a:

Lemma 3.4 *Let $\overline{Y} = \overline{Z}_1 \cup ... \cup \overline{Z}_s \subset \mathbb{P}^2$ be a union of s generic schemes of type $\overline{Z}$, then either:*
(i) $(I_{\overline{Y}})_d = (I_T)_d$;
or
(ii) $\dim(I_{\overline{Y}})_d = \dim(I_X)_d - 2s$.

Proof. Notice that by the genericity of the points and of the lines, the Hilbert function of a scheme with support on $P_1, \dots, P_s$, formed by t schemes of type $\overline{Z}$, by t' schemes of type Z' and by $s - t - t'$ fat points of multiplicity $(k + 1)$ depends only on s, t and t'.

Let W_t be a scheme formed by t schemes of type $\overline{Z}$ and by $s - t$ fat points of multiplicity $(k + 1)$. Let

$$\tau = \max\{t \in \mathbb{N} | \dim(I_{W_t})_d = \dim(I_X)_d - 2t\}.$$

If $\tau = s$, we have $W_s = \overline{Y}$ and $\dim(I_{W_s})_d = \dim(I_X)_d - 2s$, hence (ii) holds.

Let $\tau < s$: we will prove that $(I_{\overline{Y}})_d = (I_T)_d$. Let W be the scheme

$$W = W_\tau = \overline{Z}_1 \cup \dots \cup \overline{Z}_\tau \cup (k + 1)P_{\tau+1} \cup \dots \cup (k + 1)P_s.$$

and let

$$W'_{(j)} = \overline{Z}_1 \cup \dots \cup \overline{Z}_\tau \cup (k + 1)P_{\tau+1} \cup \dots \cup Z'_j \cup \dots\dots \cup (k + 1)P_s, \qquad \tau + 1 \le j \le s,$$

$$W''_{(j)} = \overline{Z}_1 \cup \dots \cup \overline{Z}_\tau \cup (k + 1)P_{\tau+1} \cup \dots \cup \overline{Z}_j \cup \dots\dots \cup (k + 1)P_s, \qquad \tau + 1 \le j \le s,$$

that is $W'_{(j)}$, respectively $W''_{(j)}$, is the scheme obtained from W by substituting the fat point $(k+1)P_j$ with a scheme of type Z', respectively $\overline{Z}$, so

$$W \subset W'_{(j)} \subset W''_{(j)},$$

and $\deg W'_{(j)} = \deg W + 1$, $\deg W''_{(j)} = \deg W + 2$ (for $\tau = s-1$, $W''_{(s)} = \overline{Y}$).

If $(I_{W''_{(j)}})_d = 0$, then trivially $(I_{\overline{Y}})_d = (I_T)_d = 0$ and we are done. So assume that $(I_{W''_{(j)}})_d \neq 0$.

By the definition of τ we have that $\dim(I_{W''_{(j)}})_d > \dim(I_X)_d - 2(\tau+1) = \dim(I_W)_d - 2$, hence we get

$$0 \leq \dim(I_W)_d - \dim(I_{W''_{(j)}})_d \leq 1.$$

Let us consider the two possible cases.

Case 1: $\dim(I_W)_d - \dim(I_{W'_{(j)}})_d = 0$, $\tau+1 \leq j \leq s$.

In this case we have $(I_W)_d = (I_{W'_{(j)}})_d$. This means that every form $F \in (I_W)_d$ meets the line L_j with multiplicity at least $k+2$; but since the line L_j is generic through P_j, this yields that every line through P_j is met with multiplicity at least $k+2$, hence

$$(I_W)_d \subset (I_{P_j}^{k+2})_d, \quad \text{for} \quad \tau+1 \leq j \leq s. \tag{1}$$

In particular, we have that

$$(I_W)_d = (I_{W''_{(s)}})_d. \tag{2}$$

Now consider the schemes

$$W_{(i,s)} = \overline{Z}_1 \cup \ldots \cup \overline{Z}_{i-1} \cup (k+1)P_i \cup \overline{Z}_{i+1} \cup \ldots \cup \overline{Z}_\tau \cup (k+1)P_{\tau+1} \cup \ldots \cup (k+1)P_{s-1} \cup \overline{Z}_s \ , \qquad 1 \leq i \leq \tau,$$

$$W'_{(i,s)} = \overline{Z}_1 \cup \ldots \cup \overline{Z}_{i-1} \cup Z'_i \cup \overline{Z}_{i+1} \cup \ldots \cup \overline{Z}_\tau \cup (k+1)P_{\tau+1} \cup \ldots \cup (k+1)P_{s-1} \cup \overline{Z}_s \ , \qquad 1 \leq i \leq \tau,$$

i.e. $W_{(i,s)}$ is the scheme obtained from W by substituting the fat point $(k+1)P_i$ to the scheme $\overline{Z}_i$ and a scheme $\overline{Z}_s$, of type $\overline{Z}$, to the fat point $(k+1)P_s$, while $W'_{(i,s)}$ is the scheme obtained from $W_{(i,s)}$ by substituting a scheme Z'_i, of type Z', to the fat point $(k+1)P_i$.

The schemes $W_{(i,s)}$ and W are made of τ schemes of type $\overline{Z}$ and $s-\tau$ $(k+1)$-fat points; the schemes $W'_{(i,s)}$ and $W'_{(s)}$ are made of τ schemes of type $\overline{Z}$, $s-\tau-1$ $(k+1)$-fat points and one scheme of type Z'. This yields that:

$$\dim(I_{W_{(i,s)}})_d = \dim(I_W)_d = \dim(I_{W'_{(s)}})_d = \dim(I_{W'_{(i,s)}})_d.$$

Hence every form $F \in (I_{W_{(i,s)}})_d$ meets the generic line L_i with multiplicity at least $k+2$, thus we get

$$(I_{W_{(i,s)}})_d \subset (I_{P_i}^{k+2})_d, \quad \text{for} \quad 1 \leq i \leq \tau. \tag{3}$$

and from this and (2) we have

$$(I_{W_{(i,s)}})_d = (I_{W''_{(s)}})_d = (I_W)_d. \tag{4}$$

By (1), (3) and (4) it follows that $(I_W)_d = (I_T)_d$, hence, since $W \subset \overline{Y} \subset T$, we get (i).

Case 2 : $\dim(I_W)_d - \dim(I_{W'_{(j)}})_d = 1$, $\tau+1 \leq j \leq s$.

In this case we have

$$\dim(I_{W'_{(j)}})_d = \dim(I_{W''_{(j)}})_d.$$

Let $F \in (I_{W'_{(j)}})_d = (I_{W''_{(j)}})_d$; hence L_j appears with multiplicity two in the tangent cone of F. If $F \notin (I_{P_j}^{k+2})_d$, then let L'_j be a generic line not in the tangent cone of F at P_j. By substituting the line L'_j to L_j in the construction of $W'_{(j)}$, we get another form $G \in (I_W)_d$, $G \notin (I_{P_j}^{k+2})_d$, with the double line L'_j in its tangent cone. Then, by Corollary 3.2, the generic form $F + aG$ has a simple tangent at P_j, and this is a contradiction since a generic choice of the line L_j should yield $(I_{W'_{(j)}})_d = (I_{W''_{(j)}})_d$. Hence $F \in (I_{P_j}^{k+2})_d$, for $\tau + 1 \leq j \leq s$.

With an argument like the one we used in *Case 1*, we also get that $F \in (I_{P_j}^{k+2})_d$ for $1 \leq j \leq \tau$, and (i) easily follows. □

Now we are ready to prove Conjecture 2a.

Theorem 3.5 *The secant variety $\sigma_s(O_{k,2,d})$ is defective if and only if one of the following holds:*
(i) $h^1(\mathcal{I}_X(d)) > max\{0, \deg(Y) - \binom{d+n}{n}\}$, or
(ii) $h^0(\mathcal{I}_T(d)) > max\{0, \binom{d+n}{n} - \deg(Y)\}$.

Proof. Since if Y is defective in degree d, then $\overline{Y}$ is, but, by Lemma 3.4, either $\dim(I_{\overline{Y}})_d = \dim(I_X)_d - 2s$, hence

$$h^1(\mathcal{I}_X(d)) = h^1(\mathcal{I}_{\overline{Y}}(d)) - 2s > max\{0, \deg(\overline{Y}) - \binom{d+n}{n}\} = max\{0, \deg(Y) - \binom{d+n}{n}\},$$

or $(I_{\overline{Y}})_d = (I_T)_d$, hence

$$h^0(\mathcal{I}_T(d)) = h^0(\mathcal{I}_{\overline{Y}}(d)) > max\{0, \binom{d+n}{n} - \deg(\overline{Y})\} = max\{0, \binom{d+n}{n} - \deg(Y)\}.$$

□

APPENDIX: Calculations

A.1 We want to prove that (for $n \geq 4$ and $d \geq 6$ or for $n \geq 5$ and $d = 5$):

$$s_{n,d} - s_{n-1,d} - h - \epsilon - \delta - 1 \geq s_{n,d-2}$$

Recall:
$s_{n,d}(2n+1) + r_{n,d} = \binom{n+d}{d}$; $s_{n-1,d}(2n-1) + r_{n-1,d} = \binom{n+d-1}{d}$; $s_{n,d-2}(2n+1) + r_{n,d-2} = \binom{n+d-2}{d-2}$.

Hence our inequality becomes:

$\frac{1}{2n+1}\left[\binom{n+d}{d} - r_{n,d}\right] - \frac{1}{2n-1}\left[\binom{n+d-1}{d} - r_{n-1,d}\right] - h - \epsilon - \delta - 1 - \frac{1}{2n+1}\left[\binom{n+d-2}{d-2} - r_{n,d-2}\right] \geq 0$

By using binomial equalities and reordering this is:

$\frac{1}{2n+1}\left[\binom{n+d-1}{d} + \binom{n+d-2}{d-1} + \binom{n+d-2}{d-2}\right] - \frac{1}{2n-1}\binom{n+d-1}{d} + \frac{r_{n-1,d}}{2n-1} - h - \epsilon - \delta - 1 - \frac{1}{2n+1}\binom{n+d-2}{d-2} + \frac{1}{2n+1}(r_{n,d-2} - r_{n,d}) \geq 0$

i.e.

$\frac{1}{2n+1}\binom{n+d-2}{d-1} - \frac{2}{(2n+1)(2n-1)}\binom{n+d-1}{d} + \frac{r_{n-1,d}}{2n-1} - h - \epsilon - \delta - 1 + \frac{1}{2n+1}(r_{n,d-2} - r_{n,d}) \geq 0$

By using binomial equalities again:

$\frac{1}{2n+1}\binom{n+d-2}{d-1} - \frac{2}{(2n+1)(2n-1)}\left[\binom{n+d-2}{d} + \binom{n+d-2}{d-1}\right] + \frac{r_{n-1,d}}{2n-1} - h - \epsilon - \delta - 1 + \frac{1}{2n+1}(r_{n,d-2} - r_{n,d}) \geq 0$

i.e.

$\frac{1}{2n+1}\binom{n+d-2}{d-1}(1 - \frac{2}{2n-1}) - \frac{2}{(2n+1)(2n-1)}\binom{n+d-2}{d} + \frac{r_{n-1,d}}{2n-1} - h - \epsilon - \delta - 1 + \frac{1}{2n+1}(r_{n,d-2} - r_{n,d}) \geq 0$

i.e.

$\binom{n+d-2}{d-1}\frac{[2n(d-1)-3d+2]}{d(4n^2-1)} + \frac{r_{n-1,d}}{2n-1} - h - \epsilon - \delta - 1 + \frac{1}{2n+1}(r_{n,d-2} - r_{n,d}) \geq 0$

Now, $\frac{r_{n-1,d}}{2n-1} \geq 0$, while $h + \epsilon + \delta \leq \frac{n}{2}$, and $r_{n,d-2} - r_{n,d} \geq -2n$, i.e. $\frac{1}{2n+1}(r_{n,d-2} - r_{n,d}) \geq -\frac{2n}{2n+1} \geq -1$, so our inequality holds if:

$\binom{n+d-2}{d-1}\frac{[2n(d-1)-3d+2]}{d(4n^2-1)} - \frac{n}{2} - 2 \geq 0$

It is quite immediate to check that the right hand side is an increasing function in d, e.g. by writing it as follows:

$\binom{n+d-2}{n-1}[2n - 3 - \frac{2n+2}{d}] - (\frac{n}{2} + 2)(4n^2 - 1) \geq 0.$

i.e.

$\binom{n+d-2}{n-1}[2n - 3 - \frac{2n+2}{d}] - 2n^3 - 8n^2 + \frac{n}{2} + 2 \geq 0.$

Let us consider the case $d = 6$ first; our inequality becomes:

$\binom{n+4}{5}\frac{(10n-16)}{6} - 2n^3 - 8n^2 + \frac{n}{2} + 2 \geq 0.$

i.e.

$\frac{(n+4)(n+3)(n+2)(n+1)n(5n-8)}{360} - 2n^3 - 8n^2 + \frac{n}{2} + 2 \geq 0.$

i.e.

$\frac{(n+4)(n+3)(n+2)(n+1)n(5n-8)-20n^2(n+2)}{360} + \frac{n}{2} + 2 \geq 0.$

i.e.

$\frac{n(n+2)}{360}[(n+4)(n+3)(n+1)(5n-8) - 720n] + \frac{n}{2} + 2 \geq 0.$

Which, for $n \geq 4$, is easily checked to be true. Hence we are done for $n \geq 4$, $d \geq 6$.

Now let us consider the case $d = 5$; our inequality becomes:

$\binom{n+3}{4}\frac{(8n-13)}{5} - 2n^3 - 8n^2 + \frac{n}{2} + 2 \geq 0.$

i.e.

$\frac{(n+3)(n+2)(n+1)n(8n-13)}{120} - 2n^3 - 8n^2 + \frac{n}{2} + 2 \geq 0.$

i.e.

$(n^4 + 6n^3 + 11n^2 + 6n)(8n - 13) - 240n^3 - 960n^2 + 60n + 240 \geq 0.$

i.e.

$8n^5 + 35n^4 - 230n^3 - 1015n^2 - 18n + 240 \geq 0.$

i.e.

$n^3(8n^2 + 35n - 230 - \frac{1015}{n} - \frac{18}{n^2} + \frac{240}{n^3}) \geq 0.$

Which, for $n \geq 6$, holds. So we are left to prove our inequality for $d = 5 = n$; in this case we have: $s_{5,5} = [\frac{272}{11}] = 24$, $s_{4,5} = [\frac{126}{9}] = 14$ and $r_{4,5} = 0$, hence $h = \epsilon = 0$, while $s_{5,3} = [\frac{56}{11}] = 5$; so: $s_{5,5} - s_{4,5} - 1 \geq s_{5,3}$ becomes: $24 - 14 - 1 \geq 5$, which holds.

$\square$

A.2 We want to prove that, for all $n \geq 7$:

$$s_{n,4} - s_{n-1,4} - h - \epsilon - \delta > \frac{n}{2}$$

i.e.

$\binom{n+4}{4}/(2n+1) - r_{n,4}/(2n+1) - \binom{n-1+4}{4}/(2n-1) + r_{n-1,4}/(2n-1) - h - \epsilon - \delta > \frac{n}{2}$

i.e.

$\frac{(n+4)(n+3)(n+2)(n+1)}{24(2n+1)} - \frac{(n+3)(n+2)(n+1)n}{24(2n-1)} - \frac{n}{2} - \frac{r_{n,4}}{(2n+1)} + \frac{r_{n-1,4}}{(2n-1)} - h - \epsilon - \delta > 0$

Now:

$\frac{r_{n,4}}{(2n+1)} \leq \frac{2n}{(2n+1)} < 1$, hence $-\frac{r_{n,4}}{(2n+1)} > -1$;

$r_{n-1,4} \geq 0$;

and $h + \epsilon + \delta \leq \frac{n}{2}$, i.e. $-h - \epsilon - \delta \geq -\frac{n}{2}$.

Therefore we get:

$\frac{(n+3)(n+2)(n+1)}{24} \cdot [\frac{(n+4)}{(2n+1)} - \frac{n}{(2n-1)}] - \frac{n}{2} - \frac{r_{n,4}}{(2n+1)} + \frac{r_{n-1,4}}{(2n-1)} - h - \epsilon - \delta >$

$\frac{(n+3)(n+2)(n+1)}{24} \cdot [\frac{n+4}{2n+1} - \frac{n}{2n-1}] - \frac{n}{2} - \frac{n}{2} - 1 =$

$= \frac{(n+3)(n+2)(n+1)}{24} \cdot \frac{[(2n-1)(n+4) - n(2n+1)]}{(2n+1)(2n-1)} - n - 1 =$

$(n+1)\left[\frac{(n+3)(n+2)(3n-2)}{12(4n^2-1)} - 1\right] > 0$

i.e.

$(n+3)(n+2)(3n-2) - 12(4n^2-1) > 0$

i.e.

$3n^3 - 35n^2 + 8n > 0$

which is true for $n \geq 12$.

Let us check the cases $n = 7, 8, 9, 10, 11$.

If $n = 7$ we have: $s_{7,4} = [\frac{1}{15}\binom{11}{4}] = 22$ (with $r_{7,4} = 0$); $s_{6,4} = 16$, since $\binom{10}{4} = 210 = 16 \cdot 13 + 2$, so $r_{6,4} = 2$ and $h = 1$, $\epsilon = \delta = 0$.

Our inequality becomes: $22 - 16 - 1 > 7/2$, which holds.

If $n = 8$ we have: $s_{8,4} = [\frac{1}{15}\binom{12}{4}] = 33$ (with $r_{8,4} = 0$); $s_{7,4} = 22$, $r_{7,4} = 0$ and $h = \epsilon = \delta = 0$.

Our inequality becomes: $33 - 22 > 4$, which holds.

If $n = 9$ we have: $s_{9,4} = [\frac{1}{15}\binom{13}{4}] = 47$ (with $r_{9,4} = 10$); $s_{8,4} = 33$, and $h = \epsilon = \delta = 0$.

Our inequality becomes: $47 - 33 > 9/2$, which holds.

If $n = 10$ we have: $s_{10,4} = [\frac{1}{15}\binom{14}{4}] = 66$ (with $r_{10,4} = 11$) ; $s_{9,4} = 47$, and $h = 5$, $\epsilon = \delta = 0$.

Our inequality becomes: $66 - 47 - 5 > 5$, which holds.

If $n = 11$ we have: $s_{10,4} = [\frac{1}{15}\binom{15}{4}] = 91$; $s_{10,4} = 66$, and $h = 5$, $\epsilon = 1$, $\delta = 0$.

Our inequality becomes: $91 - 66 - 5 - 1 > 11/2$, which holds. $\square$

A.3 We want to prove that, for $d \geq 5$, $n \geq 4$ or $d = 4$, $n \geq 7$:

$$4n - 1 \leq 2s_{n-1,d}. \qquad (*)$$

Since $r_{n-1,d} \leq 2n-2$, it is enough to prove that:

$\frac{2}{2n-1}\left[\binom{n-1+d}{n-1} - 2n + 2\right] \geq 4n - 1$ which is:

$\binom{n-1+d}{n-1} \geq \frac{(4n-1)(2n-1)}{2} + 2n - 2$ that is:

$\binom{n-1+d}{n-1} \geq 4n^2 - n - \frac{3}{2} \qquad (**)$

which is surely true if

$\binom{n-1+d}{n-1} \geq 4n^2 - n$ is true.

Notice that the function $\binom{n-1+d}{n-1}$ is an increasing function in d. For $d = 4$, the inequality becomes:

$\frac{n(n^3+6n^2+11n+6)}{24} \geq 4n^2 - n$, which can be written:

$n^3 + 6n^2 + 11n + 6 \geq 96n - 24$, i.e.

$n^3 + 6n^2 - 85n + 30 \geq 0$ which is surely true if the following is true:

$n^2 + 6n - 85 \geq 0$. The last one is verified for $n \geq 8$, so we are done for $d = 4$ and $n \geq 8$.

If $(n, d) = (7, 4)$, $s_{n-1,d} = 16$ since $\binom{10}{4} = 210 = 16 \cdot 13 + 2$, and $(*)$ becomes: $4 \cdot 7 - 1 \leq 2 \cdot 16$ which is true.

Since the function $\binom{n-1+d}{n-1}$ is an increasing function in d, we have proved the initial inequality for $d \geq 4$ and $n \geq 8$.

For $d = 5$ $(**)$ becomes: $n^5 + 10n^4 + 35n^3 - 430n^2 + 144n + 120 \geq 0$ which is true for $n = 5, 6, 7$. We have hence proved the initial inequality for $d \geq 5$ and $n \geq 5$.

If $(n, d) = (4, 5)$, $s_{n-1,d} = 8$ since $\binom{8}{3} = 8 \cdot 7$, and $(*)$ becomes: $4 \cdot 4 - 1 \leq 2 \cdot 8$ which is true.

For $d = 6$ $(**)$ becomes: $n(n+1)(n+2)(n+3)(n+4)(n+5) - 120(6)(4n^2 - n - 1) \geq 0$ which is true for $n = 4$. We conclude that the initial inequality is true for $d \geq 5$ and $n \geq 4$.

A.4 We want to show that (for $d \geq 10$): $s_{4,d} - s_{3,d} - s_{3,d-1} - 2 \geq s_{4,d-3}$ and $s_{4,d} - s_{3,d} - s_{3,d-1} + 1 \leq s_{4,d-2}$.

The first inequality is equivalent to:

$$\left[\frac{1}{9}\binom{d+4}{4}\right] - \frac{1}{7}\binom{d+3}{3} + \frac{6}{7} - \frac{1}{7}\binom{d+2}{3} + \frac{3}{7} - 2 \geq \left[\frac{1}{9}\binom{d+1}{4}\right]$$

which follows if:

$\frac{1}{9}\binom{d+4}{4} - \frac{1}{9}\binom{d+1}{4} \geq \frac{1}{7}\binom{d+3}{3} + \frac{1}{7}\binom{d+2}{3} - \frac{9}{7} + 4$

i.e.

$\frac{d+1}{9}\frac{[(d+4)(d+3)(d+2) - d(d-1)(d-2)]}{24} \geq \frac{1}{7}\left(\frac{(d+1)(d+2)(2d+3)}{6}\right) + \frac{19}{7}$

i.e.

$\frac{d+1}{9}\frac{(12d^2+24d+24)}{24} \geq \frac{1}{42}(d+1)(d+2)(2d+3)+\frac{19}{7}$

i.e.

$\frac{(d^2+2d+2)}{3} \geq \frac{2d^2+7d+6}{7}+\frac{114}{7(d+1)}$

i.e.

$d^2-7d-4 \geq \frac{342}{d+1}$

Which is easily checked to hold for $d \geq 10$.

Now let us consider the second inequality, which is equivalent to:

$$\left[\frac{1}{9}\binom{d+4}{4}\right] - \frac{1}{7}\binom{d+3}{3} + \frac{6}{7} - \frac{1}{7}\binom{d+2}{3} + \frac{3}{7} + 1 \leq \left[\frac{1}{9}\binom{d+2}{4}\right]$$

which follows if:

$\frac{1}{9}\binom{d+4}{4} - \frac{1}{9}\binom{d+2}{4} \leq \frac{1}{7}\binom{d+3}{3} + \frac{1}{7}\binom{d+2}{3} - \frac{9}{7} - 3$

i.e.

$\frac{(d+1)(d+2)}{9}\frac{[(d+4)(d+3)-d(d-1)]}{24} \leq \frac{1}{7}\left(\frac{(d+1)(d+2)(2d+3)}{6}\right) - \frac{30}{7}$

i.e.

$\frac{(d+1)(d+2)}{9}\frac{(8d+12)}{24} \geq \frac{1}{42}(d+1)(d+2)(2d+3) - \frac{30}{7}$

i.e.

$\frac{1}{9} \geq \frac{1}{7} - \frac{180}{7(d+1)(d+2)(2d+3)}$

Which is easily checked to hold for $d \geq 10$.

REFERENCES

[AH1]: J. Alexander, A. Hirschowitz. *Polynomial interpolation in several variables.* J. of Alg. Geom. **4** (1995), 201-222.

[AH2]: J. Alexander, A. Hirschowitz. *An asymptotic vanishing theorem for generic unions of multiple points.* Invent. Math. **140** (2000), 303-325.

[B]: E. Ballico, *On the secant varieties to the tangent developable of a Veronese variety*, J. Algebra **288** (2005), 279–286.

[BBCF] E. Ballico, C.Bocci, E.Carlini, C.Fontanari. *Osculating spaces to secant varieties.* Rend. Circ. Mat. Palermo **53** (2004), 429-436.

[BF]: E. Ballico, C.Fontanari,*On the secant varieties to the osculating variety of a Veronese surface*, Central Europ. J. of Math. **1** (2003), 315-326.

[BC]: A.Bernardi, M.V.Catalisano. *Some defective secant varieties to osculating varieties of Veronese surfaces.* Collect. Math. **57**, n.1, (2006), 43-68.

[BCGI]: A.Bernardi, M.V.Catalisano, A.Gimigliano, M.Idà. *Osculating Varieties of Veronesean and their higher secant varieties.* Canadian J. of Math. **59**, (2007), 488-502.

[BO]: M.C.Brambilla, G.Ottaviani, *On the Alexander-Hirschowitz theorem.* J. Pure and Appl. Algebra **211** (2008), 1229-1251.

[CGG]: M.V.Catalisano, A.V.Geramita, A.Gimigliano. *On the Secant Varieties to the Tangential Varieties of a Veronesean.* Proc. A.M.S. **130** (2001), 975-985.

[Ch1]: K.Chandler. *A brief proof of a maximal rank theorem for generic double points in projective space.* Trans. Amer. Math. Soc. **353** (2001), 1907-1920.

[Ch2]: K.Chandler. *Linear systems of cubics singular at general points of projective space.* Composition Math. **134** (2002), 269-282.

[Ge]: A.V.Geramita. *Inverse Systems of Fat Points*, Queen's Papers in Pure and Applied Math. **102**, *The Curves Seminar at Queens', vol. X* (1998).

[IK]: A.Iarrobino, V.Kanev. *Power Sums, Gorenstein algebras, and determinantal loci.* Lecture Notes in Math. **1721**, Springer, Berlin, (1999).

[I]: A.Iarrobino. *Inverse systems of a symbolic algebra III: Thin algebras and fat points.* Compos. Math. **108** (1997), 319-356.

[J]: J.P.Jouanoulou. *Théoremes de Bertini et applications.* Progress in Math. **42**, Birkauser, Boston, MA, (1983).

[Se]: B. Segre, *Un'estensione delle varietà di Veronese ed un principio di dualità per le forme algebriche I and II.* Rend. Acc. Naz. Lincei (8) **1** (1946), 313-318 and 559-563.

[Te]: A.Terracini. *Sulle* V_k *per cui la varietà degli* S_h $(h+1)$*-seganti ha dimensione minore dell'ordinario.* Rend. Circ. Mat. Palermo **31** (1911), 392-396.

[W]: K.Wakeford. *On canonical forms.* Proc. London Math. Soc. (2) **18** (1919/20). 403-410.

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