Bounds on short cylinders and uniqueness results for degenerate Kolmogorov equation

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Abstract

We consider the Cauchy problem for hypoelliptic Kolmogorov equations in the form

$$\partial_t u = \sum_{i,j=1}^{m} a_{i,j}(z)\partial_{x_i x_j} u + \sum_{j=1}^{m} a_j(z)\partial_{x_j} u + \sum_{i,j=1}^{N} b_{i,j} x_i \partial_{x_j} u,$$

$(x,t) \in \mathbb{R}^N \times]0,T[, 1 \leq m \leq N$, as well as in its divergence form. We prove that, if $|u(x,t)| \leq M \exp\left(a(t^{-\beta} + |x|^2)\right)$, for some positive constants a, M and $\beta \in]0,1[$ and $u(\cdot,0) \equiv 0$, then $u \equiv 0$. The proof of the main result is based on some previous uniqueness result and on the application of some "estimates in short cylinders", first introduced by Safonov in the study of uniformly parabolic operators.

1 Introduction

We consider second order operators in non-divergence form

$$Lu := \sum_{i,j=1}^{m} a_{i,j}(z)\partial_{x_i x_j} u + \sum_{j=1}^{m} a_j(z)\partial_{x_j} u + \sum_{i,j=1}^{N} b_{i,j} x_i \partial_{x_j} u - \partial_t u, \tag{1.1}$$

as well as in divergence form

$$Lu := \sum_{i,j=1}^{m} \partial_{x_i}\left(a_{i,j}(z)\partial_{x_j} u\right) + \sum_{i,j=1}^{N} b_{i,j} x_i \partial_{x_j} u - \partial_t u, \tag{1.2}$$

where $z = (x,t) \in \mathbb{R}^N \times \mathbb{R}$, $1 \leq m \leq N$ and the coefficients $a_{i,j}$ and a_j are bounded continuous functions. When considering divergence form operators (1.2), we assume that the $\partial_{x_i} a_{i,j}$'s are continuous for $i,j = 1,\dots,m$. The matrix $B = (b_{i,j})_{i,j=1,\dots,N}$ has real, constant entries, $A_0(z) = (a_{i,j}(z))_{i,j=1,\dots,m}$ is symmetric and positive, for every $z \in \mathbb{R}^{N+1}$. Our assumptions are:

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H1 the operator

$$K := \sum_{j=1}^{m} \partial_{x_j}^2 + \sum_{i,j=1}^{N} b_{i,j} x_i \partial_{x_j} - \partial_t, \tag{1.3}$$

is hypoelliptic, *i.e.*, every distributional solution to $Ku = f$ is a smooth classical solution, whenever f is smooth.

H2 There exists a positive constant Λ such that

$$\Lambda^{-1} |\zeta|^2 \leq \langle A_0(z)\zeta, \zeta \rangle \leq \Lambda |\zeta|^2, \quad \forall\, \zeta \in \mathbb{R}^m, \forall\, z \in \mathbb{R}^{N+1}. \tag{1.4}$$

H3 The coefficients $a_{i,j}$, a_j (if L is in its non-divergence form (1.1)) and $\partial_{x_i} a_{i,j}$ (if L is in its divergence form (1.2)) are bounded and Hölder continuous of exponent $\alpha \leq 1$, for $i, j = 1, \ldots, m$ (in the sense of the Definition 2.2 below).

In order to explain our assumptions, we first note that conditions **[H1]**-**[H2]**-**[H3]** are satisfied by every uniformly parabolic operator in non-divergence form, with Hölder continuous coefficients. In that case, K is the heat operator, $m = N$, $B = 0$. On the other hand, it is known that degenerate Kolmogorov operators (with $m < N$) naturally arise in in stochastic theory (see [18], [16] [29], and [17]). Moreover, degenerate Kolmogorov equations also appear in many research fields. For instance, the Kolmogorov equation [18]

$$\partial_{x_1}^2 u + x_1 \partial_{x_2} u = \partial_t u, \qquad (x,t) \in \mathbb{R}^3,$$

occurs in the financial problem of the evaluation of the path dependent options (see [30], [4], [10], [12]), in kinetic theory (see [5], [26] [9], [21]), as well as in visual perception (see [23], [31]). We also quote the papers [1], [2] and their bibliography for other applications.

In Section 2 we recall the main issues of the general theory for degenerate Kolmogorov operators (1.1) that will be needed in this paper. Here we point out that the well known Hörmarder condition can be used to check the hypoellipticity of K. Indeed, K can be written as

$$K = \sum_{j=1}^{m} X_j^2 + Y, \tag{1.5}$$

where

$$X_j = \partial_{x_j}, \ i = 1, \ldots, m, \qquad\qquad Y = \langle x, B\nabla \rangle - \partial_t. \tag{1.6}$$

$\nabla = (\partial_{x_1}, \ldots, \partial_{x_N})$ and $\langle \cdot, \cdot \rangle$ are, respectively, the gradient and the inner product in $\mathbb{R}^N$. K is hypoelliptic if, and only if, it satisfies the Hörmarder condition:

$$\text{rank Lie}\,(X_1, \ldots, X_m, Y) = N + 1, \quad \text{at every point of } \mathbb{R}^{N+1}. \tag{1.7}$$

In this paper we are concerned with the classical solutions of the Cauchy problem

$$\begin{cases} Lu = 0 & \text{in } \mathbb{R}^N \times]0, T[, \\ u(\cdot, 0) = 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{1.8}$$

If Ω is an open subset of $\mathbb{R}^{N+1}$, and $f \in C(\Omega)$, a classical solution of the equation $Lu = f$ is a function $u \in C(\Omega)$ that has continuous derivatives $\partial_{x_i} u$ and $\partial_{x_i x_j} u$ (for $i, j = 1, \dots, m$) and Yu, and satisfies the equation $Lu = f$ at every point of Ω. A classical solution of (1.8) is a function $u \in C(\mathbb{R}^N \times [0, T[)$ that is a solution of $Lu = 0$ in $\mathbb{R}^N \times]0, T[)$ and satisfies $u(x, 0) = 0$ for every $x \in \mathbb{R}^N$.

Before stating our main result, we recall that the classical uniqueness results due to Krzyzanski in [19] apply to our non divergence form operator (1.1):

$$|u(x,t)| \le M e^{c|x|^2}, \ (x,t) \in \mathbb{R}^N \times]0,T], \qquad \Rightarrow \qquad u \equiv 0, \tag{1.9}$$

(see [19], and Theorem B in [3]). Concerning the divergence form operator (1.2), the uniqueness of the solution of (1.8) has been proved by Di Francesco and Pascucci in [11]:

$$\int_0^T \int_{\mathbb{R}^N} e^{-c|x|^2} |u(x,t)| dx dt < \infty \qquad \Rightarrow \qquad u \equiv 0. \tag{1.10}$$

(Theorem 1.6 in [11], see also Theorem 3.1 [24]). It is remarkable that the also the assumption $u \ge 0$ in $\mathbb{R}^N \times]0, T]$ implies $u \equiv 0$ (see [24] and [13]). The main achievements of this paper are the following uniqueness result:

Theorem 1.1 *Let L be in non divergence form* (1.1)*, satisfying conditions* **[H1]-[H2]-[H3]***. Let $u \in C(\mathbb{R}^N \times [0, T])$ be a solution of the Cauchy problem* (1.8)*. If there exist three constants $a, M > 0$ and $\beta \in]0, 1[$, such that*

$$|u(x,t)| \le M \exp\big(a(t^{-\beta} + |x|^2)\big), \tag{1.11}$$

for every $(x,t) \in \mathbb{R}^N \times]0, T[$, then $u \equiv 0$.

Theorem 1.2 *Let L be in divergence form* (1.2)*, satisfying conditions* **[H1]-[H2]-[H3]***. Let $u \in C(\mathbb{R}^N \times [0, T])$ be a solution of the Cauchy problem* (1.8)*. If there exist two constants $a > 0$ and $\beta \in]0, 1[$, such that*

$$\int_0^T \int_{\mathbb{R}^N} \exp\big(-a(t^{-\beta} + |x|^2)\big) |u(x,t)| dx dt < \infty, \tag{1.12}$$

then $u \equiv 0$.

Our Theorems 1.1 and 1.2 extend some uniqueness results for parabolic operators, where growth conditions, analogous to (1.10) and (1.9), are replaced by some non-uniform in time conditions. We first quote the paper by Shapiro [28], where the uniqueness of the solution is proved under the assumption that $\|u(\cdot,t)\|_{L^\infty} = o(t^{-1})$ as $t \to 0$. More recently, Chung [6] show that the Cauchy problem for the heat equation has a unique solution satisfying (1.11). Chung and Kim [7] prove that the growth condition (1.11) is optimal in the sense that the uniqueness result fails when assuming (1.11) with $\beta = 1$. Indeed, in [7] it is proved that the function

$$u(x,t) = \frac{1}{2\pi i} \int_{\partial D_M} \frac{1}{(2\pi t)^{N/2}} \exp\left(-\frac{(x-\zeta)^2}{4t}\right) \exp\left(e^{\zeta^2}\right) d\zeta,$$

$$D_M = \left\{ \xi + i\eta \in \mathbb{C} \mid \xi \ge M, |\eta| \le \frac{\pi}{2\xi} \right\}$$

where the integral is taken counterclockwise, is a non trivial solution of the Cauchy problem for the heat equation and satisfies $|u(x,t)| \le C_\varepsilon \exp\left(\frac{\varepsilon}{t}\right)$, for every positive ε. Let us also recall that the following famous example due to Tychonoff

$$u(x,t) = \sum_{k=0}^{\infty} \varphi^{(k)}(t) \frac{x^2 k}{(2k)!}, \quad \varphi(t) = \begin{cases} \exp\left(-\frac{1}{t^2}\right) & \text{if } t > 0, \\ 0 & \text{if } t \le 0, \end{cases}$$

show that also the growth condition in x (1.11) is optimal. Since the class of Kolmogorov operators (1.1) contains the parabolic ones, the above examples show that the growth condition (1.11) is sharp also for the operators considered here.

The uniqueness result proved by Chung in [6] has been extended by Ferretti in [14] to uniformly parabolic operators with measurable coefficients, both in divergence form and in non-divergence form. The main tools used in [14] are the "estimates in short cylinders", first introduced by Safonov in the study of uniformly parabolic operators [27]. The method of [14] relies on a geometric construction based on the invariance of the heat equation with respect to the usual Euclidean change of variable and on the caloric rescaling $\delta_\lambda(x,t) = \big(\lambda x, \lambda^2 t\big)$.

In this paper we prove Theorems 1.1 and 1.2 by following the method due to Ferretti. To that aim, we prove in Section 3 some estimates on short cylinders analogous to the ones in [27], by using a simpler approach based on the a priori bounds of the fundamental solution of L. We adapt the ideas of [14] to the non-Euclidean geometry of the Lie group related to the Kolmogorov equations (1.3) and to its pseudo-distance. Some differences with respect to the proof of [14] are due to the fact that only a pseudo-triangular inequality holds in the Lie group, instead of the usual triangle inequality. Moreover, we cannot rely on the estimate on short cylinders with arbitrarily large basis.

This paper is organized as follows. In Section 2 we recall the main issues of the general theory of the hypoelliptic Kolmogorov equations and of the related Lie group. In Section 3 we prove the estimates in short cylinders, for both divergence and non divergence form operators. In Section 4 we give the proof of Theorem 1.1, and in Section 5 we prove Theorem 1.2.

2 Lie group structure

In this section we recall the definition of the Lie group related to Kolmogorov operators and some known results concerning the fundamental solution of the operator (1.1) (or in its divergence form (1.2)) useful in the sequel.

First of all, we recall that the following property is equivalent to the hypoellipticity of the operator K defined in (1.3): *there exists a basis of* $\mathbb{R}^N$ *such that* B *has the form*

$$\begin{pmatrix} * & B_1 & 0 & \dots & 0 \\ * & * & B_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \dots & B_r \\ * & * & * & \dots & * \end{pmatrix} \tag{2.1}$$

where B_j is a matrix $m_{j-1} \times m_j$ of rank m_j, with

$$m_0 := m \geq m_1 \geq \ldots \geq m_r \geq 1, \qquad\qquad m_0 + m_1 + \ldots + m_r = N,$$

and $*$ are constant and arbitrary blocks. We refer to [20] for the proof of the equivalence of the two conditions. The hypoellipticity of K is also equivalent to the following condition: if we set

$$E(s) = \exp(-sB^T), \quad A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{C}(t) = \int_0^t E(s)AE^T(s)ds, \tag{2.2}$$

then: *K is hypoelliptic if, and only if, $\mathcal{C}(t)$ is positive, for every $t > 0$.* In the sequel, we assume that the basis of $\mathbb{R}^N$ is such that B has the form (2.1).

Under the above equivalent conditions, Hörmarder constructed in [15] the fundamental solution of K:

$$\begin{aligned}\Gamma(x,t,\xi,\tau) = & \frac{(4\pi)^{-\frac{N}{2}}}{\sqrt{\det \mathcal{C}(t-\tau)}} \cdot \\ & \cdot \exp\Big(-\frac{1}{4}\langle \mathcal{C}^{-1}(t-\tau)(x - E(t-\tau)\xi), x - E(t-\tau)\xi\rangle - (t-\tau)\mathrm{tr}B\Big),\end{aligned} \tag{2.3}$$

if $t > \tau$, and $\Gamma(x,t,\xi,\tau) = 0$ if $t \leq \tau$.

The operator K has the remarkable property to being invariant with respect to a Lie group structure $\mathcal{G} = \big(\mathbb{R}^{N+1}, \circ\big)$ first studied in [20]:

$$(x,t) \circ (\xi,\tau) = (\xi + E(\tau)x, t+\tau), \quad (x,t),(\xi,\tau) \in \mathbb{R}^{N+1}, \tag{2.4}$$

where $E(\tau)$ is the matrix in (2.2). The Lie group invariance is stated as follows: if we set $w(z) = u\,(\zeta \circ z)$, for some $\zeta \in \mathbb{R}^{N+1}$, then

$$Kw(z) = (Ku)\,(\zeta \circ z)\,. \tag{2.5}$$

Accordingly, the fundamental solution Γ is invariant with respect to group $\mathcal{G}$:

$$\Gamma(x,t,\xi,\tau) = \Gamma\big((\xi,\tau)^{-1} \circ (x,t)\big) := \Gamma\big((\xi,\tau)^{-1} \circ (x,t), 0, 0\big). \tag{2.6}$$

Moreover, if (and only if) all the $*$-block in (2.1) are null, then K is homogeneous of degree two with respect the family of following dilations,

$$\delta(\lambda) := \big(D(\lambda), \lambda^2\big) = \mathrm{diag}\left(\lambda I_{m_0}, \lambda^3 I_{m_1}, \ldots, \lambda^{2r+1} I_{m_r}, \lambda^2\right), \tag{2.7}$$

(I_{m_j} denotes the $m_j \times m_j$ identity matrix), *i.e.* if we set $w(z) = u\,(\delta(\lambda)z)$, for some $\lambda > 0$, then

$$Kw(z) = \lambda^2 (Ku)\,(\delta(\lambda)z)\,, \tag{2.8}$$

(see Proposition 2.2 in [20]). If K is homogeneous, then the following identities hold:

$$E(\lambda^2 t)D(\lambda) = D(\lambda)E(t), \quad D(\lambda)\mathcal{C}(t)D(\lambda) = \mathcal{C}(\lambda^2 t), \quad \mathcal{C}^{-1}(t) = D(\lambda)\mathcal{C}^{-1}(\lambda^2 t)D(\lambda), \tag{2.9}$$

for every positive λ and t (see Remark 2.1 and Proposition 2.3 in [20]). As a consequence, Γ is a $\delta(\lambda)$-homogeneous function:

$$\Gamma(\delta(\lambda)z) = \lambda^{-Q}\Gamma(z), \quad \forall\, z \in \mathbb{R}^{N+1} \setminus \{0\},\ \lambda > 0,$$

where

$$Q = m_0 + 3m_1 + \ldots, (2r+1)m_r.$$

Since

$$\det\bigl(\delta(\lambda)\bigr) = \det\left(\mathrm{diag}(\lambda I_{m_0}, \lambda^3 I_{m_3}, \ldots, \lambda^{2r+1} I_{m_r}, \lambda^2)\right) = \lambda^{Q+2}, \tag{2.10}$$

the number $Q+2$ is said *homogeneous dimension* of $\mathbb{R}^{N+1}$ with respect to the dilation group $(\delta(\lambda))_{\lambda>0}$ and Q is said *spatial homogeneous dimension* of $\mathbb{R}^N$ with respect to $(\delta(\lambda))_{\lambda>0}$.

We next define a norm which is homogeneous with respect to $(\delta(\lambda))_{\lambda>0}$.

Definition 2.1 *For every $z = (x,t) \in \mathbb{R}^{N+1}$, $z \neq (0,0)$, we set $\|z\|_{\mathcal{G}} = \rho$, where ρ is the unique positive solution to the equation*

$$\frac{x_1^2}{\rho^{2q_1}} + \cdots + \frac{x_N^2}{\rho^{2q_N}} + \frac{t^2}{\rho^4} = 1,$$

and q_j are the positive integers such that $\delta(\lambda) = \mathrm{diag}(\lambda^{q_1}, \ldots, \lambda^{q_N}, \lambda^2)$. We put $\|(0,0)\|_{\mathcal{G}} = 0$, and we denote $|x|_{\mathcal{G}} = \|(x,0)\|_{\mathcal{G}}$.

It is easy to check that $\|\cdot\|_{\mathcal{G}}$ is a homogeneous function of degree 1 with respect the dilation $\delta(\lambda)$, *i.e.*

$$\|\delta(\lambda)z\|_{\mathcal{G}} = \lambda\|z\|_{\mathcal{G}}, \qquad \text{for every} \ \ \lambda > 0, \ \text{and} \ \ z \in \mathbb{R}^{N+1}. \tag{2.11}$$

We explicitly remark that $|x|_{\mathcal{G}} = 1$ if, and only if, the usual Euclidean norm $|x|$ equals 1. Moreover, from (2.9), it follows that

$$\delta(\lambda)\,(z \circ \zeta) = (\delta(\lambda)z) \circ (\delta(\lambda)\zeta)\,, \qquad \text{for every} \ \ \lambda > 0, \ \text{and} \ \ z, \zeta \in \mathbb{R}^{N+1}. \tag{2.12}$$

Note that there exists a constant $\mathbf{c} \geq 1$ such that

$$\|z \circ \zeta\|_{\mathcal{G}} \leq \mathbf{c}\,(\|z\|_{\mathcal{G}} + \|\zeta\|_{\mathcal{G}})\,, \qquad \text{for all } z, \zeta \in \mathbb{R}^{N+1}. \tag{2.13}$$

The homogeneity properties of the operator K whose all the $*$-block in (2.1) are null, somehow extend to any operator K as follows. For K as in (1.3), we define the homogeneous operator K_0 by setting

$$K_0 u := \sum_{j=1}^{m} \partial_{x_j}^2 u + Y_0 u, \qquad Y_0 = \langle\, B_0, \nabla\rangle - \partial_t \tag{2.14}$$

where

$$B_0 = \begin{pmatrix} 0 & B_1 & 0 & \cdots & 0 \\ 0 & 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B_r \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{2.15}$$

As noticed above, the matrices E_0, $\mathcal{C}_0$ and $\mathcal{C}_0^{-1}$, defined as in (2.2) with B_0 instead of B, satisfy (2.9). Moreover, for every given $T > 0$, there exists a positive constant c_T such that

$$\begin{aligned} \langle \mathcal{C}_0(t)x, x\rangle (1 - c_T t) \le \langle \mathcal{C}(t)x, x\rangle \le \langle \mathcal{C}_0(t)x, x\rangle (1 + c_T t), \\ \left\langle \mathcal{C}_0^{-1}(t)y, y\right\rangle (1 - c_T t) \le \left\langle \mathcal{C}^{-1}(t)y, y\right\rangle \le \left\langle \mathcal{C}_0^{-1}(t)y, y\right\rangle (1 + c_T t); \end{aligned} \tag{2.16}$$

for every $x, y \in \mathbb{R}^N, t \in [-T, T], t \neq 0$ (see Lemma 3.3 in [20]). Moreover, there exist two positive constants k'_T, k''_T such that

$$k'_T t^Q (1 - c_T t) \le \det \mathcal{C}(t) \le k''_T t^Q (1 + c_T t),$$

for every $(x, t) \in \mathbb{R}^N \times]0, T]$, with $t < \frac{1}{c_T}$ (see formula (3.14) in [20]). As a direct consequence, of the above bounds and of the continuity of $\mathcal{C}$ we get

$$\det \mathcal{C}(t) \ge c'_T t^Q, \quad t \in]0, T], \tag{2.17}$$

for some positive constant c'_T. Analogously, in the sequel we will also use the inequality

$$\left\langle \mathcal{C}^{-1}(t)E(t)y, E(t)y\right\rangle \ge c''_T \left\langle E_0(1)^T \mathcal{C}_0^{-1}(1)E_0(1)D\left(\tfrac{1}{\sqrt{t}}\right)y, D\left(\tfrac{1}{\sqrt{t}}\right)y\right\rangle, \tag{2.18}$$

$(y, t) \in \mathbb{R}^N \times]0, T]$, which plainly follows from (2.16), (2.9) and from the following identity $E(t)^T \mathcal{C}^{-1}(t)E(t) = \mathcal{C}^{-1}(-t)$.

For any operator $K = \sum_{j=1}^m \partial_{x_j}^2, +\langle x, B\nabla\rangle - \partial_t$ we consider the matrix B_0 in (2.15) related to B and the norm $\| \cdot \|_{\mathcal{G}}$ in Definition 2.1. Note that, even if K is non homogeneous, the norm is defined in terms of the dilation group related to K_0, which actually only depends on the matrix B of K.

Definition 2.2 *Let $\alpha \in]0, 1]$. We say that a function $f : \mathbb{R}^{N+1} \to \mathbb{R}$ is Hölder continuous of exponent α, in short $f \in C^\alpha$, if there exists a positive constant c such that*

$$|f(z) - f(\zeta)| \le c \, \|\zeta^{-1} \circ z\|_{\mathcal{G}}^\alpha, \qquad \text{for every } \ z, \zeta \in \mathbb{R}^{N+1}. \tag{2.19}$$

We remark that a triangle inequality for non-homogeneous operators has been proved in Lemma 2.1 in [13], but in this case the constant $\mathbf{c}$ depends on the compact set where z and ζ are assumed to belong. On the other hand, if $z = (x, 0)$ and $\zeta = (\xi, 0)$, then formula (2.4) reads $(x, 0) \circ (\xi, 0) = (\xi + x, 0)$, and since in this case the operation "$\circ$" does not depend on the matrix B, we have

$$\|(x, 0) \circ (\xi, 0)\|_{\mathcal{G}} \le \mathbf{c} \left(\|(x, 0)\|_{\mathcal{G}} + \|(\xi, 0)\|_{\mathcal{G}}\right) = \mathbf{c} \left(|x|_{\mathcal{G}} + |\xi|_{\mathcal{G}}\right), \qquad \text{for all } x, \xi \in \mathbb{R}^N, \tag{2.20}$$

also for non homogeneous operators K.

We finally recall some known results about the fundamental solution of the operator L in its non-divergence form (1.1), as well as in its divergence form (1.2). If L satisfies hypotheses **[H1]-[H2]-[H3]**, then the Levi's parametrix method provides the existence of a fundamental solution Γ of L, which satisfies the following upper and lower bounds: for every positive T

there exist two constant coefficients operators K^-, K^+ and two positive constants c^-, c^+, such that,

$$c^- \Gamma^-(x,t,y,s) \le \Gamma(x,t,y,s) \le c^+ \Gamma^+(x,t,y,s), \tag{2.21}$$

for every $(x,t),(y,s) \in \mathbb{R}^{N+1}$, with $0 < t-s \le T$, where Γ^- and Γ^+ denote the fundamental solutions of the operators

$$K^- = \Lambda^- \sum_{i=1}^m \partial_{x_i}^2 + Y, \qquad K^+ = \Lambda^+ \sum_{i=1}^m \partial_{x_i}^2 + Y.$$

We point out that the constants Λ^-, Λ^+ and c^-, c^+ in (2.21) only depend on T, on Λ in **[H2]**, on the matrix B and on the Hölder constant of the coefficients $a_{i,j}$'s. We recall that the functions Γ^- and Γ^+ are explicitly written in the form (2.3) with the matrix A in (2.2) replaced by the matrices $\Lambda^- \mathrm{diag}\,(I_m, 0, \dots, 0)$ and $\Lambda^+ \mathrm{diag}\,(I_m, 0, \dots, 0)$ respectively (see Theorem 1.4 in [10] and Theorem 1.5 in [13]).

We end this section by quoting a local pointwise estimate of the solutions of $Lu = 0$ in its divergence form (see Corollary 1 in [8]). We define the cylinder of center at $(\xi,\tau) \in \mathbb{R}^{N+1}$ and radius r as:

$$\widetilde{H}_R(\xi,\tau) = \left\{(x,t) \in \mathbb{R}^{N+1} : \tau - R^2 < t < \tau + R^2;\ |x - E(t-\tau)\xi|_{\mathcal{G}} < R\right\}. \tag{2.22}$$

Theorem 2.3 *Let L be a divergence form operator satisfying* **[H1]-[H2]***, and let u be a weak solution of $Lu = 0$ in $\widetilde{H}_r(x_0,t_0)$, with $0 < r \le 1$. Then there exists a positive constant c which only depends on the matrix B, on the constants appearing in* **[H1]-[H2]** *and on the homogeneous dimension Q such that, for every $p \ge 1$, it holds*

$$\sup_{\widetilde{H}_\varrho(x_0,t_0)} |u|^p \le \frac{c}{(r-\varrho)^{Q+2}} \int_{\widetilde{H}_r(x_0,t_0)} |u(y,s)|^p dy\, ds,$$

for every $\varrho \in \left[\frac{r}{2}, r\right]$.

Note that the above theorem applies to weak solutions to $Lu = 0$, however the classical solutions considered in this paper are also solutions in the weak sense.

3 Estimates on short cylinders

We prove some pointwise estimates of the solution of the equation $Lu = 0$. Since our proof only relies on the bounds (2.21) of the fundamental solution Γ, the result hold for both non divergence form and divergence form operators L. We next recall the definition of "cylindrical open set" previously used in [22] and in [25]. Let $(\xi,\tau) \in \mathbb{R}^{n+1}$, and let R, h be two positive constants. Then

$$\begin{aligned}
H_R(\xi,\tau,h) &= \left\{(x,t) \in \mathbb{R}^{N+1} : \tau < t < \tau + h;\ |x - E(t-\tau)\xi|_{\mathcal{G}} < R\right\},\\
B_R(\xi,\tau) &= \left\{(x,t) \in \mathbb{R}^{N+1} : t = \tau;\ |(x-\xi)|_{\mathcal{G}} < R\right\},\\
B_R(\xi,\tau,h) &= \left\{(x,t) \in \mathbb{R}^{N+1} \quad t = \tau + h;\ |x - E(t-\tau)\xi|_{\mathcal{G}} < R\right\},\\
\Sigma_R(\xi,\tau,h) &= \left\{(x,t) \in \mathbb{R}^{N+1} \quad \tau < t < \tau + h;\ |x - E(t-\tau)\xi|_{\mathcal{G}} = R\right\},\\
\partial_P H_R(\xi,\tau,h) &= B_R(\xi,\tau) \cup \Sigma_R(\xi,\tau,h)
\end{aligned}$$

denote the open cylinder, its lower and upper basis, its lateral boundary, and its parabolic boundary, respectively. Note that, if $(x,\tau) \in \partial B_R(\xi,\tau)$, then $(E(s)x, \tau+s) \in \Sigma_R(\xi,\tau,h)$, for every $s \in]0,h[$. In [22] Proposition A.1 it is proved the existence of a barrier function for every point of $\partial_P H_R(\xi,\tau,h)$, then the solution of the Cauchy-Dirichlet problem

$$\begin{cases} Lu = 0 & \text{in } H_R(\xi,\tau,h), \\ u = \varphi & \text{in } \partial_P H_R(\xi,\tau,h), \end{cases} \tag{3.1}$$

with $\varphi \in C\left(\partial_P H_R(\xi,\tau,h)\right)$, attains the boundary data at every point of $\partial_P H_R(\xi,\tau,h)$. The main result of this section is the following

Theorem 3.1 *Let L be either in non divergence form* (1.1)*, or in divergence form* (1.2)*. For every $R_0 > 0$ there exist two positive constants $\mathbf{C}$ and $\varepsilon_0 \in]0,1]$, such that, if u is a solution of $Lu = 0$ in $H_R(\xi,\tau,\varepsilon R^2)$, $u = 0$ in $B_R(\xi,\tau)$, for some $(\xi,\tau) \in \mathbb{R}^{N+1}$, $R \in]0,R_0]$ and $\varepsilon \in]0,\varepsilon_0]$, then*

$$|u(E(t)\xi, t+\tau))| \le e^{-\mathbf{C}\frac{R^2}{t}} \sup_{\Sigma_R(\xi,\tau,\varepsilon R^2)} |u|,$$

for every $t \in [0,\varepsilon R^2]$.

Proof. We first note that it is not restrictive to assume $(\xi,\tau) = (0,0)$, since $w(z) := u\left((\xi,\tau)\circ z\right)$ is a solution of $Lw = 0$ in the cylinder $H_R(0,0,\varepsilon R^2)$, where of course the coefficients a_{ij} and a_j of the operator L are computed at $(\xi,\tau)\circ z$ instead of z. We define

$$v(x,t) := \frac{2}{c^-} \int_{\mathbb{R}^N} \Gamma(x,t,y,0) \varphi\left(D\left(\tfrac{1}{R}\right) y\right) dy,$$

with

$$\varphi \in C^\infty(\mathbb{R}^N), \qquad \varphi(x) = 1 \ \text{ if } \ |x|_{\mathcal{G}} \ge 5/6, \quad \varphi(x) = 0 \ \text{ if } \ |x|_{\mathcal{G}} \le 2/3,$$

and c^- the constant in (2.21), related to $T = R_0^2$. We observe that, if $|x|_{\mathcal{G}} = R$,

$$\begin{aligned} v(E(t)x,t) &= \frac{2}{c^-} \int_{\mathbb{R}^N} \Gamma(E(t)x,t,y,0) \varphi\left(D\left(\tfrac{1}{R}\right) y\right) dy \\ &\ge 2 \int_{\mathbb{R}^N} \Gamma^-(E(t)x,t,y,0) \varphi\left(D\left(\tfrac{1}{R}\right) y\right) dy \longrightarrow 2\,\varphi\left(D\left(\tfrac{1}{R}\right) x\right) = 2 \end{aligned}$$

as $t \to 0^+$. Since the convergence is uniform on the compact sets, there exists a positive ε_0 (it is not restrictive to assume $0 < \varepsilon_0 < 1/4$), such that $v(E(t)x,t) \ge 1$ for every $x \in \mathbb{R}^N$, $|x|_{\mathcal{G}} = R$ and $t \in]0,\varepsilon_0 R^2]$. In other words,

$$v \ge 1 \quad \text{in } \ \Sigma_R(0,0,\varepsilon_0 R^2).$$

If u is a solution of $Lu = 0$ in $H_R(0,0,\varepsilon_0 R^2)$, $u = 0$ in $B_R(0,0)$, then

$$u \le v \sup_{\Sigma_R(0,0,\varepsilon_0 R^2)} |u| \quad \text{in } \partial_P H_R(0,0,\varepsilon_0 R^2), \qquad Lu = Lv = 0 \ \text{ in } H_R(0,0,\varepsilon_0 R^2).$$

The maximum principle then gives

$$u(0,t) \le v(0,t) \sup_{\Sigma_R(0,0,\varepsilon_0 R^2)} |u| \qquad \text{for every } \; t \in [0, \varepsilon_0 R^2]. \tag{3.2}$$

In order to conclude the proof, we need an upper bound for $v(0,t)$. First, by using (2.21) and the definition of φ, we obtain for any $t > 0$

$$v(0,t) \le 2\,\frac{c^+}{c^-} \int_{\mathbb{R}^N} \Gamma^+(0,t,y,0)\varphi\left(D\left(\tfrac{1}{R}\right)y\right)dy \le 2\,\frac{c^+}{c^-} \int_{|y|_{\mathcal{G}} \ge R/2} \Gamma^+(0,t,y,0)dy. \tag{3.3}$$

Recalling the explicit expression (2.3) of Γ^+, with $A = \Lambda^+ \mathrm{diag}\,(I_m, 0, \dots, 0)$, we have

$$\begin{aligned}\Gamma^+(0,t,y,0) &= \frac{(4\pi)^{-N/2}}{\sqrt{\det \mathcal{C}(t)}} \exp\left(-\frac{1}{4}\langle \mathcal{C}^{-1}(t)E(t)y, E(t)y\rangle\right)\\ &\le \frac{(4\pi)^{-N/2}}{\sqrt{c'_T t^Q}} \exp\left(-\frac{c''_T}{4}\Big\langle E_0(1)^T \mathcal{C}_0^{-1}(1)E_0(1)D\Big(\tfrac{1}{\sqrt{t}}\Big)y, D\Big(\tfrac{1}{\sqrt{t}}\Big)y\Big\rangle\right),\end{aligned}$$

by (2.17) and (2.18). Then, by the change of variable $\eta := D\left(\frac{1}{\sqrt{t}}\right)y$, we get

$$\int_{|y|_{\mathcal{G}} \ge R/2} \Gamma^+(0,t,y,0)dy \le \frac{(4\pi)^{-N/2}}{\sqrt{c'_T}} \int_{|\eta|_{\mathcal{G}} \ge \frac{R}{2\sqrt{t}}} \exp\left(-\frac{c''_T}{4}\left\langle E_0(1)^T \mathcal{C}_0^{-1}(1)E_0(1)\eta, \eta\right\rangle\right)d\eta.$$

Furthermore, if $t \in]0, \varepsilon_0 R^2]$,

$$c''_T \langle E_0(1)^T \mathcal{C}_0^{-1}(1)E_0(1)\eta, \eta\rangle \ge C_0 \langle \eta, \eta\rangle = C_0 \sum_{j=1}^N \frac{\eta_j^2}{|\eta|_{\mathcal{G}}^{2q_j}} |\eta|_{\mathcal{G}}^{2q_j} \ge C_0 |\eta|_{\mathcal{G}}^2$$

for some positive constant C_0, since $|\eta|_{\mathcal{G}} \ge \frac{R}{2\sqrt{t}} > 1$. Thus,

$$\int_{|y|_{\mathcal{G}} \ge R/2} \Gamma^+(0,t,y,0)dy \le C_T \int_{|\eta|_{\mathcal{G}} \ge \frac{R}{2\sqrt{t}}} e^{-\frac{C_0}{4}|\eta|_{\mathcal{G}}^2} d\eta. \tag{3.4}$$

On the other hand, we have

$$\begin{aligned}\int_{|\eta|_{\mathcal{G}} \ge \frac{R}{2\sqrt{t}}} e^{-\frac{C_0}{4}|\eta|_{\mathcal{G}}^2} d\eta &\le \int_{|\eta|_{\mathcal{G}} \ge \frac{R}{2\sqrt{t}}} e^{-\frac{C_0}{6}|\eta|_{\mathcal{G}}^2} \left(\max_{|\eta|_{\mathcal{G}} \ge \frac{R}{2\sqrt{t}}} e^{-\frac{C_0}{12}|\eta|_{\mathcal{G}}^2}\right) d\eta\\ &\le e^{-\frac{C_0}{48}\frac{R^2}{t}} \int_{|\eta|_{\mathcal{G}} \ge \frac{1}{2\sqrt{\varepsilon_0}}} e^{-\frac{C_0}{12}|\eta|_{\mathcal{G}}^2} \left(\max_{|\eta|_{\mathcal{G}} \ge \frac{1}{2\sqrt{\varepsilon_0}}} e^{-\frac{C_0}{12}|\eta|_{\mathcal{G}}^2}\right) d\eta\\ &\le e^{-\frac{C_0}{48}\frac{R^2}{t}} e^{-\frac{C_0}{48}\frac{1}{\varepsilon_0}} \int_{\mathbb{R}^N} e^{-\frac{C_0}{12}|\eta|_{\mathcal{G}}^2} d\eta.\end{aligned} \tag{3.5}$$

Hence, if we set $\mathbf{C} := \frac{C_0}{48}$, from (3.3), (3.4) and (3.5) it follows that

$$v(0,t) \le 2\,\frac{c^+}{c^-}\, C_T e^{-\mathbf{C}\frac{R^2}{t}} e^{-\mathbf{C}\frac{1}{\varepsilon_0}} \int_{\mathbb{R}^N} e^{-\frac{C_0}{12}|\eta|_{\mathcal{G}}^2} d\eta \le e^{-\mathbf{C}\frac{R^2}{t}},$$

if ε_0 is suitably small, for every $t \in]0, \varepsilon_0 R^2]$. As a consequence, (3.2) yields

$$u(0,t) \le e^{-\mathbf{C}\frac{R^2}{t}} \sup_{\Sigma_R(0,0,\varepsilon_0 R^2)} |u| \qquad \text{for every} \quad t \in]0, \varepsilon_0 R^2],$$

and the proof is accomplished. $\square$

4 Pointwise conditions

In this section we prove that the pointwise condition (1.11) yields the uniqueness of the solution of the Cauchy problem (1.8) for both non divergence and divergence form operators L.

Proposition 4.1 *Let L be either in non divergence form* (1.1)*, or in divergence form* (1.2)*, and let $u \in C(\mathbb{R}^N \times [0,T], \mathbb{R})$ be a solution of the Cauchy problem* (1.8)*. Assume that for some constants $a > 0$, and $\beta \in]0,1[$ we have*

$$|u(x,t)| \le \exp\left(a(t^{-\beta} + |x|^2)\right) \qquad \text{in} \ \ \mathbb{R}^N \times]0,T].$$

Then there exist two positive constants $h_0 \in]0,T]$ and M_0, depending on the operator L and on the constants a and β, such that

$$|u(x,t)| \le M_0 \exp\left(2a|x|^2\right) \qquad \text{in} \ \ \mathbb{R}^N \times]0,h_0].$$

Lemma 4.2 *Consider any $R_0 > \mathbf{c}^{\frac{1}{1-\beta}}$, where $\mathbf{c}$ is the constant in* (2.20)*. Set $\overline{R} = \frac{\mathbf{c} R_0^{2-\beta}}{R_0^{1-\beta} - \mathbf{c}}$, and let $u \in C\left(\overline{H_{\overline{R}}(0,0,T)}, \mathbb{R}\right)$ be a solution of*

$$\begin{cases} Lu = 0 & \text{in } H_{\overline{R}}(0,0,T), \\ u = 0, & \text{in } B_{\overline{R}}(0,0). \end{cases}$$

Assume that

$$|u(x,t)| \le \exp\left(a t^{-\beta}\right) \qquad \text{in} \ \ H_{\overline{R}}(0,0,T) \tag{4.1}$$

for some positive constants a and $\beta \in]0,1[$. Then there exists a constant $h_0 \in]0,T]$, only depending on a, β, R_0 and on the operator L, such that

$$|u(x,t)| \le 1 \qquad \text{in} \ \ H_{R_0}(0,0,h_0).$$

Proof. Let $\varepsilon_0, \mathbf{C}$ be the constants of Theorem 3.1. Consider the cylinder $H_{R_0}(0,0,h_0)$, where $h_0 \in]0, \varepsilon_0]$ will be suitably chosen later. For every $j \in \mathbb{N} \cup \{0\}$, we set

$$h_j := \frac{h_0}{R_0^{2j}}, \quad r_j := R_0^{(\beta-1)j+1}, \quad R_j := \sum_{i=0}^{j} \mathbf{c}^{i+1} r_i.$$

Note that, for every $j \in \mathbb{N}$,

$$\varepsilon_j := \frac{h_{j-1}}{r_j^2} = \frac{h_0}{R_0^{2\beta j}} \leq h_0 \leq \varepsilon_0, \qquad R_j < \mathbf{c} R_0 \sum_{i=0}^{\infty} \mathbf{c}^i k^{(\beta-1)i} = \overline{R}. \tag{4.2}$$

We next define, for every $j \in \mathbb{N} \cup \{0\}$,

$$D_j := \{x \in \mathbb{R}^N \ \mid \ (x,0) = (x_0,0) \circ (x_1,0) \circ \ldots \circ (x_j,0), \ |x_i|_{\mathcal{G}} \leq r_i, \ i = 0, \ldots, j\}$$

By repeatedly using the pseudo-triangular inequality (2.20), we easily see that

$$\begin{aligned} |x|_{\mathcal{G}} &\leq \mathbf{c}\big(\|(x_0,0)\|_{\mathcal{G}} + \|(x_1,0) \circ \ldots \circ (x_j,0)\|_{\mathcal{G}}\big) \\ &\leq \mathbf{c}|x_0|_{\mathcal{G}} + \mathbf{c}^2|x_1|_{\mathcal{G}} + \ldots + \mathbf{c}^{j-1}|x_{j-2}|_{\mathcal{G}} + \mathbf{c}^j|x_{j-1}|_{\mathcal{G}} + \mathbf{c}^j|x_j|_{\mathcal{G}} \leq \sum_{i=0}^{j} \mathbf{c}^{i+1} r_i = R_j, \end{aligned}$$

for every $x \in D_j$, hence

$$D_j \subseteq \{x \in \mathbb{R}^N \ \mid \ |x|_{\mathcal{G}} \leq R_j\} \subseteq \{x \in \mathbb{R}^N \ \mid \ |x|_{\mathcal{G}} \leq \overline{R}\}. \tag{4.3}$$

We are now in position to prove the lemma. Since $u(E(t)x,t) \to 0$ as $t \to 0$, uniformly on every compact subset of $\mathbb{R}^N$, then there exists $J \in \mathbb{N}$ such that $|u| \leq 1$ in the set $\big\{(E(t)x,t) \ \mid \ |x|_{\mathcal{G}} \leq \overline{R}, \ 0 \leq t \leq h_J\big\}$ so that, by (4.3), we have

$$|u| \leq 1 \qquad \text{in} \ \ \{(E(t)x,t) \ \mid \ x \in D_J, \ 0 \leq t \leq h_J\}. \tag{B_J}$$

We next note that $t^{-\beta} \leq h_J^{-\beta}$, as $t \in [h_J, h_0]$, then (B_J) and the growth condition (4.1) yield

$$|u| \leq \exp\left(a h_J^{-\beta}\right) \qquad \text{in} \ \ \{(E(t)x,t) \ \mid \ x \in D_J, \ 0 \leq t \leq h_0\}. \tag{A_J}$$

We next claim that (A_J) implies

$$|u| \leq 1 \qquad \text{in} \ \ \{(E(t)x,t) \ \mid \ x \in D_{J-1}, \ 0 \leq t \leq h_{J-1}\}. \tag{B_{J-1}}$$

Let x be any point in D_{J-1} and $t \in [0, h_{J-1}]$. By Theorem 3.1 we have

$$|u(E(t)x,t)| \leq \exp\left(-\mathbf{C}\frac{r_J^2}{t}\right) \sup_{\Sigma_{r_J}\left(x,0,\varepsilon_J r_J^2\right)} |u| \leq \exp\left(-\mathbf{C}\frac{r_J^2}{h_{J-1}}\right) \sup_{\Sigma_{r_J}\left(x,0,\varepsilon_J r_J^2\right)} |u|.$$

On the other hand, if $(y,s) \in \Sigma_{r_J}\left(x,0,\varepsilon_J r_J^2\right)$, then there is a point $x_J \in \mathbb{R}^N$ such that $|x_J|_{\mathcal{G}} = r_J$ and $(y,s) = (E(s)y_J, s)$, where $(y_J,0) = (x,0) \circ (x_J,0)$. Since $y_J \in D_J$ and $0 \leq s \leq h_0$, estimate (A_J) then gives $|u(y,s)| \leq \exp\left(a h_J^{-\beta}\right)$. Thus

$$|u(E(t)x,t)| \leq \exp\left(a h_J^{-\beta} - \mathbf{C}\frac{r_J^2}{h_{J-1}}\right) = \exp\left(-\frac{\mathbf{C}R_0^{2\beta J}}{2h_0} + \frac{R_0^{2\beta J}}{h_0^{\beta}}\left(a - \frac{\mathbf{C}}{2h_0^{1-\beta}}\right)\right) \leq 1,$$

provided that we choose $h_0 \le \min\left\{\varepsilon_0, T, \left(\frac{\mathbf{C}}{2a}\right)^{\frac{1}{1-\beta}}\right\}$. This proves (B_{J-1}). Now (B_{J-1}) gives (A_{J-1}), and so on. Following this "backward induction" argument, we finally obtain

$$|u| \le 1 \quad \text{in} \quad \{(E(t)x, t) \mid x \in D_0,\ 0 \le t \le h_0\}. \tag{B_0}$$

This inequality accomplishes the proof of the Lemma, since the above set is $H_{R_0}(0, 0, h_0)$. $\square$

Proof of Proposition 4.1. Fix $R_0 > \mathbf{c}^{\frac{1}{1-\beta}}$, $x_0 \in \mathbb{R}^N$, and set

$$v(x, t) := u\left((x_0, 0) \circ (x, t)\right) = u(x + E(t)x_0, t).$$

Then v satisfies

$$|v(x, t)| \le \exp\left(at^{-\beta}\right) \exp\left(2a|x|^2\right) \exp\left(2a|E(t)x_0|^2\right) \qquad \text{for every } (x, t) \text{ in } H_{\overline{R}}(0, 0, T),$$

where $\overline{R}$ is as in Lemma 4.2. Then

$$|u(E(t)x_0, t)| = |v(0, t)| \le M_0 \exp\left(2a|E(t)x_0|^2\right) \qquad \text{for every } (x_0, t) \in \mathbb{R}^N \times]0, h_0],$$

with $M_0 = \sup_{H_{\overline{R}}(0,0,T)} \exp\left(2a|x|^2\right)$.

In order to conclude the proof, it is enough to consider any $(y, t) \in \mathbb{R}^N \times]0, h_0]$, and set in the previous inequality $x_0 = E(-t)y$, so that $(E(t)x_0, t) = (y, t)$. $\square$

Proof of Theorem 1.1. If u is a solution of the Cauchy problem (1.8) and satisfies (1.11), then Proposition 4.1 yields

$$|u(x, t)| \le Me^{c|x|^2}, \qquad (x, t) \in \mathbb{R}^N \times]0, T],$$

for some positive constants c and M. Hence $u \equiv 0$, by (1.9). $\square$

5 Integral conditions

In this section we prove that the integral condition (1.12) is equivalent to the pointwise condition (1.11) for divergence form operators L.

Proposition 5.1 *Let $u \in C(\mathbb{R}^N \times [0, T], \mathbb{R})$ be a solution of the Cauchy problem* (1.8)*, with L in divergence form* (1.2)*, and assume that for some constants $a > 0$, and $\beta \in]0, 1[$ we have*

$$\int_0^T \int_{\mathbb{R}^N} \exp\left(-a(t^{-\beta} + |x|^2)\right)|u(x, t)|dxdt < \infty.$$

Then there exist two positive constants b and M, such that

$$|u(x, t)| \le M \exp\left(b(t^{-\beta} + |x|^2)\right) \qquad \text{in } \mathbb{R}^N \times \left]0, \tfrac{T}{2}\right].$$

Proof. Let (x,t) be any point in $\mathbb{R}^N \times \left]0, \frac{T}{2}\right]$. Consider the cylinder $\widetilde{H}_r(x,t)$, with $p=1$, $r=\sqrt{\frac{t}{2}}$, and $\rho = \frac{r}{2}$. Since $\widetilde{H}_{\sqrt{t/2}}(x,t) \subset \mathbb{R}^N \times]0,T[$, by Theorem 2.3 we have

$$|u(x,t)| \le \frac{c}{t^{\frac{Q+2}{2}}} \int\limits_{\widetilde{H}_{\sqrt{t/2}}(x,t)} |u(y,s)| dy\, ds \le \frac{c\, c_H}{t^{\frac{Q+2}{2}}} \int\limits_{\widetilde{H}_{\sqrt{t/2}}(x,t)} e^{-a(s^{-\beta}+|y|^2)} |u(y,s)| dy\, ds,$$

where

$$c_H = \sup_{(\xi,\tau)\in \widetilde{H}_{\sqrt{t/2}}(x,t)} e^{a(\tau^{-\beta}+|\xi|^2)}.$$

Note that $(\xi,\tau) \in \widetilde{H}_{\sqrt{t/2}}(x,t)$ if, and only if $(\xi,\tau) = (x,t)\circ(y,s) = (y+E(s)x, t+s)$, for some $(y,s)\in\mathbb{R}^{N+1}$ such that $|y|_{\mathcal{G}} < \sqrt{\frac{t}{2}}$ and $|s| < \frac{t}{2}$. Since $t \le \frac{T}{2}$, there exists a positive constant c_T such that

$$|\xi| \le |y| + \|E(s)\|\, |x| \le c_T(|x|+1), \qquad \text{and } \tau > \tfrac{t}{2}.$$

Thus

$$\sup_{(\xi,\tau)\in \widetilde{H}_{\sqrt{t/2}}(x,t)} e^{a(\tau^{-\beta}+|\xi|^2)} \le C_T\, e^{a\left(\left(\frac{t}{2}\right)^{-\beta}+|x|^2\right)},$$

for some positive constant C_T. Using again the fact that $\widetilde{H}_{\sqrt{t/2}}(x,t) \subset \mathbb{R}^N\times]0,T[$ we finally find

$$|u(x,t)| \le \frac{c\, C_T}{t^{\frac{Q+2}{2}}}\, e^{a\left(\left(\frac{t}{2}\right)^{-\beta}+|x|^2\right)} \int\limits_{\mathbb{R}^N\times]0,T[} e^{-a(s^{-\beta}+|y|^2)}|u(y,s)| dy\, ds,$$

and the claim easily follows from the fact that

$$t^{-\frac{Q+2}{2}}\, e^{\frac{2^\beta a}{t^\beta}} = e^{\frac{b}{t^\beta}} \left(t^{-\frac{Q+2}{2}} e^{\frac{2^\beta a - b}{t^\beta}} \right),$$

and the last term vanishes, as $t \to 0$, whenever $b < 2^\beta a$. $\square$

Proof of Theorem 1.2. If u is a solution of the Cauchy problem (1.8) and satisfies (1.12), then Propositions 5.1 and 4.1 yield

$$|u(x,t)| \le M e^{c|x|^2}, \qquad (x,t)\in\mathbb{R}^N\times\left]0,\tfrac{T}{2}\right],$$

for some positive constants c and M. Hence $u\equiv 0$ in $\mathbb{R}^N\times\left]0,\frac{T}{2}\right]$, by (1.10). As a consequence, u is a solution of the Cauchy problem

$$\begin{cases} Lu = 0 & \text{in } \mathbb{R}^N\times\left]\frac{T}{2},T\right[, \\ u\left(\cdot,\frac{T}{2}\right) = 0, \end{cases}$$

and, by (1.12), satisfies

$$e^{-\frac{2^\beta a}{T^\beta}} \int_{T/2}^{T}\int_{\mathbb{R}^N} e^{-a|x|^2}|u(x,t)| dxdt < \infty.$$

Then, by applying again (1.10), we get $u\equiv 0$ also in $\mathbb{R}^N\times\left]\frac{T}{2},T\right]$. $\square$

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