Ideals of varieties parameterized by certain symmetric tensors

Alessandra Bernardi

ABSTRACT. The ideal of a Segre variety $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t} \hookrightarrow \mathbb{P}^{(n_1+1)\cdots(n_t+1)-1}$ is generated by the 2-minors of a generic hypermatrix of indeterminates (see [**TH**] and [**Gr**]). We extend this result to the case of Segre-Veronese varieties. The main tool is the concept of "weak generic hypermatrix" which allows us to treat also the case of projection of Veronese surfaces from a set of generic points and of Veronese varieties from a Cohen-Macaulay subvariety of codimension 2.

1 Introduction

In this paper we study the generators of the ideal of Segre-Veronese varieties and the ideal of projections of Veronese surfaces from a set of generic points and, more generally, of Veronese varieties from a Cohen-Macaulay subvariety of codimension 2.

A Segre variety parameterizes completely decomposable tensors (Definition 2.1). In [**TH**] (Theorem 1.5) it is proved that the ideal of a Segre variety is generated by all 2-minors of a generic hypermatrix of indeterminates.

In this paper we prove an analogous statement for Segre-Veronese varieties (see [**CGG**]). Segre-Veronese varieties parameterizes certain symmetric decomposable tensors (see Section 3); we prove (in Theorem 3.11) that their ideal is generated by 2-minors of a generic symmetric hypermatrix (Definition 3.5).

The idea we use is the following. We define "weak generic hypermatrices" (see Definition 3.8) and we prove that the ideal generated by 2-minors of a weak generic hypermatrix is a prime ideal (Proposition 3.10). Then we show that a symmetric hypermatrix of indeterminates is weak generic and we can conclude, since the ideal generated by its 2-minors defines, set-theoretically, a Segre-Veronese variety.

An analogous idea is used in Sections 4 and 5 in order to find the generators of projections of Veronese varieties from a subvariety of codimension 2.

Denote with $Y_{n,d}$ the Veronese variety obtained as the d-uple embedding of $\mathbb{P}^n$ into $\mathbb{P}^{\binom{n+d}{d}-1}$. We construct a hypermatrix in such a way that its 2-minors together with some linear equations generate an ideal I that defines set-theoretically a projection of the surface $Y_{2,d}$ from a finite set of s generic points, with $s \leq \binom{d}{2}$. Then we prove that such hypermatrix is weak generic. Finally in Theorem 4.7 we prove that I is actually the ideal of the projected surface.

This construction can be generalized to projections of Veronese varieties $Y_{n,d}$, for all $n, d > 0$, from a subvariety of codimension 2 and of degree $s = \binom{t+1}{2} + k \leq \binom{d}{2}$ for some non negative integers t, k, d such that $0 < t < d-1$ and $0 \leq k \leq t$ (see Section 5).

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2 Preliminaries

Let $K = \overline{K}$ be an algebraically closed field of characteristic zero, and let $V_1, \dots, V_t$ be vector spaces over K of dimensions $n_1, \dots, n_t$ respectively. We will call en element $T \in V_1 \otimes \cdots \otimes V_t$ a tensor of size $n_1 \times \cdots \times n_t$.

Let $E_j = \{\underline{e}_{j,1}, \dots, \underline{e}_{j,n_j}\}$ be a basis for the vector space V_j, $j = 1, \dots, t$. We define a basis E for $V_1 \otimes \cdots \otimes V_t$ as follows:

$$E := \{\underline{e}_{i_1,\dots,i_t} = \underline{e}_{1,i_1} \otimes \cdots \otimes \underline{e}_{t,i_t} \mid 1 \leq i_j \leq n_j, \forall j = 1, \dots, t\}. \qquad (1)$$

A tensor $T \in V_1 \otimes \cdots \otimes V_t$ can be represented via a so called "hypermatrix" (or "array")

$$\mathcal{A} = (a_{i_1,\dots,i_t})_{1 \leq i_j \leq n_j\,,\, j=1,\dots,t}$$

with respect to the basis E defined in (1), i.e.:

$$T = \sum_{1 \leq i_j \leq n_j,\, j=1,\dots,t} a_{i_1,\dots,i_t} \underline{e}_{i_1,\dots,i_t}.$$

Definition 2.1. *A tensor $T \in V_1 \otimes \cdots \otimes V_t$ is called "decomposable" if, for all $j = 1, \dots, t$, there exist $\underline{v}_j \in V_j$ such that $T = \underline{v}_1 \otimes \cdots \otimes \underline{v}_t$.*

Definition 2.2. *Let $E_j = \{\underline{e}_{j,1}, \dots, \underline{e}_{j,n_j}\}$ be a basis for the vector space V_j for $j = 1, \dots, t$. Let also $\underline{v}_j = \sum_{i=1}^{n_j} a_{j,i} \underline{e}_{j,i} \in V_j$ for $j = 1, \dots, t$. The image of the following embedding*

$$\begin{array}{ccl} \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_t) & \hookrightarrow & \mathbb{P}(V_1 \otimes \cdots \otimes V_t) \\ ([\underline{v}_1] \ , \ \dots \ , \ [\underline{v}_t]) & \mapsto & [\underline{v}_1 \otimes \cdots \otimes \underline{v}_t] = \\ & & = \sum_{1 \leq i_j \leq n_j,\, j=1,\dots,t} [(a_{1,i_1} \cdots a_{t,i_t}) \underline{e}_{i_1,\dots,i_t}] \end{array}$$

is well defined and it is known as "Segre Variety". We denote it by $Seg(V_1 \otimes \cdots \otimes V_t)$.

Remark: A Segre variety $Seg(V_1 \otimes \cdots \otimes V_t)$ parameterizes the decomposable tensors of $V_1 \otimes \cdots \otimes V_t$.

A set of equations defining $Seg(V_1 \otimes \cdots \otimes V_t)$ is well known (one of the first reference for a set-theoretical description of the equations of Segre varieties is [**Gr**]). Before introducing that result we need the notion of d-minor of a hypermatrix.

Notation:

- The hypermatrix $\mathcal{A} = (x_{i_1,\dots,i_t})_{1 \leq i_j \leq n_j\,,\, j=1,\dots,t}$ is said to be a *generic hypermatrix of indeterminates* (or more simply *generic hypermatrix*) of $S := K[x_{i_1,\dots,i_t}]_{1 \leq i_j \leq n_j\,,\, j=1,\dots,t}$, if the entries of $\mathcal{A}$ are the independent variables of S.
- We denote by S_t the homogeneous degree t part of the polynomial ring S.
- We will always suppose that we have fixed a basis E_i for each V_i and the basis E for $V_1 \otimes \cdots \otimes V_t$ as in (1).
- When we will write "$\mathcal{A}$ is the hypermatrix associated to the tensor T" (or vice versa) we will always assume that the association is via the fixed basis E. Moreover if the size of T is $n_1 \times \cdots \times n_t$, then $\mathcal{A}$ is of the same size.

It is possible to extend the notion of "d-minor of a matrix" to that one of "d-minor of a hypermatrix".

Definition 2.3. *Let $V_1, \dots, V_t$ be vector spaces of dimensions $n_1, \dots, n_t$, respectively, and let (J_1, J_2) be a partition of the set $\{1, \dots, t\}$. If $J_1 = \{h_1, \dots, h_s\}$ and $J_2 = \{1, \dots, t\} \backslash J_1 = \{k_1, \dots, k_{t-s}\}$, the (J_1, J_2)-Flattening of $V_1 \otimes \cdots \otimes V_t$ is the following:*

$$V_{J_1} \otimes V_{J_2} = (V_{h_1} \otimes \cdots \otimes V_{h_s}) \otimes (V_{k_1} \otimes \cdots \otimes V_{k_{t-s}}).$$

Definition 2.4. *Let $V_{J_1} \otimes V_{J_2}$ be any flattening of $V_1 \otimes \cdots \otimes V_t$ and let $f_{J_1,J_2} : \mathbb{P}(V_1 \otimes \cdots \otimes V_t) \xrightarrow{\sim} \mathbb{P}(V_{J_1} \otimes V_{J_2})$ be the obvious isomorphism. Let $\mathcal{A}$ be a hypermatrix associated to a tensor $T \in V_1 \otimes \cdots \otimes V_t$; let $[T'] = f_{J_1,J_2}([T]) \in \mathbb{P}(V_{J_1} \otimes V_{J_2})$ and let A_{J_1,J_2} be the matrix associated to T'. Then the d-minors of the matrix A_{J_1,J_2} are said to be "d-minors of $\mathcal{A}$".*

Sometimes we will improperly write "a d-minor of a tensor T", meaning that it is a d-minor of the hypermatrix associated to such a tensor via the fixed basis E of $V_1 \otimes \cdots \otimes V_t$.

Example: d-minors of a decomposable tensor.

Let $V_1, \dots, V_t$ and $(J_1, J_2) = (\{h_1, \dots, h_s\}, \{k_1, \dots, k_{t-s}\})$ as before. Consider the following composition of maps:

$$\mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_t) \overset{s_1 \times s_2}{\rightarrow} \mathbb{P}(V_{J_1}) \times \mathbb{P}(V_{J_2}) \overset{s}{\rightarrow} \mathbb{P}(V_{J_1} \otimes V_{J_2})$$

where $Im(s_1 \times s_2) = Seg(V_{J_1}) \times Seg(V_{J_2})$ and $Im(s)$ is the Segre variety of two factors.

Consider the basis (made as E above) E_{J_1} for V_{J_1} and E_{J_2} for V_{J_2}. In terms of coordinates, the composition $s \circ (s_1 \times s_2)$ is described as follows.

Let $\underline{v}_i = (a_{i,1}, \ldots, a_{i,n_i}) \in V_i$ for each $i = 1, \ldots, t$ and $T = \underline{v}_1 \otimes \cdots \otimes \underline{v}_t \in V_1 \otimes \cdots \otimes V_t$; then:

$$s_1 \times s_2([(a_{1,1}, \ldots, a_{1,n_1})], \ldots, [(a_{t,1}, \ldots, a_{t,n_t})]) = ([(y_{1,\ldots,1}, \ldots, y_{n_{h_1},\ldots,n_{h_s}})], [(z_{1,\ldots,1}, \ldots, z_{n_{k_1},\ldots,n_{k_{t-s}}})])$$

where $y_{l_1,\ldots,l_s} = a_{h_1,l_1} \cdots a_{h_s,l_s}$, for $l_m = 1, \ldots, n_m$ and $m = 1, \ldots, s$;
and $z_{l_1,\ldots,l_{t-s}} = a_{k_1,l_1} \cdots a_{k_{t-s},l_{t-s}}$ for $l_m = 1, \ldots, n_m$ and $m = 1, \ldots, t-s$.

If we rename the variables in V_{J_1} and in V_{J_2} as: $(y_{1,\ldots,1}, \ldots, y_{n_{h_1},\ldots,n_{h_s}}) = (y_1, \ldots, y_{N_1})$, with $N_1 = n_{h_1} \cdots n_{h_s}$, and $(z_{1,\ldots,1}, \ldots, z_{n_{k_1},\ldots,n_{k_{t-s}}}) = (z_1, \ldots, z_{N_2})$, with $N_2 = n_{k_1} \cdots n_{k_{t-s}}$, then:

$$s([(y_1, \ldots, y_{N_1})], [(z_1, \ldots, z_{N_2})]) = [(q_{1,1}, q_{1,2}, \ldots, q_{N_1,N_2})] = ((s_1 \times s_2) \circ s)([T]),$$

where $q_{i,j} = y_i z_j$ for $i = 1, \ldots, N_1$ and $j = 1, \ldots, N_2$. We can easily rearrange coordinates and write $((s_1 \times s_2) \circ s)([T])$ as a matrix:

$$((s_1 \times s_2) \circ s)([T]) = \begin{pmatrix} q_{1,1} & \cdots & q_{1,N_2} \\ \vdots & & \vdots \\ q_{N_1,1} & \cdots & q_{N_1,N_2} \end{pmatrix}. \qquad (2)$$

A d-minor of the tensor T is a d-minor of the matrix $((s_1 \times s_2) \circ s)([T])$ defined in (2).

Example: The 2-minors of a hypermatrix $\mathcal{A} = (a_{i_1,\ldots,i_t})_{1 \leq i_j \leq n_j, j=1,\ldots,t}$ are all of the form:

$$a_{i_1,\ldots,i_l,\ldots,i_t} a_{l_1,\ldots,l_m,\ldots,l_t} - a_{i_1,\ldots,l_m,\ldots,i_t} a_{l_1,\ldots,i_m,\ldots,l_t}$$

for $1 \leq i_j, l_j \leq n_j$, $j = 1, \ldots, t$ and $1 \leq m \leq t$.

Definition 2.5. *Let $\mathcal{A}$ be a hypermatrix whose entries are in $K[u_1, \ldots, u_r]$. The ideal $I_d(\mathcal{A})$ is the ideal generated by all d-minors of $\mathcal{A}$.*

Example: The ideal of the 2-minors of a generic hypermatrix $\mathcal{A} = (x_{i_1,\ldots,i_t})_{1 \leq i_j \leq n_j, j=1,\ldots,t}$ is

$$I_2(\mathcal{A}) := (x_{i_1,\ldots,i_l,\ldots,i_t} x_{j_1,\ldots,j_l,\ldots,j_t} - x_{i_1,\ldots,j_l,\ldots,i_t} x_{j_1,\ldots,i_l,\ldots,j_t})_{l=1,\ldots,t;\ 1 \leq i_k, j_k \leq n_j,\ k=1,\ldots,t}.$$

It is a classical result (see [**Gr**]) that a set of equations for a Segre Variety is given by all the 2-minors of a generic hypermatrix. In fact, as previously obseved, a Segre variety parameterizes decomposable tensors, i.e. all the "rank one" tensors.

In [**TH**] (Theorem 1.5) it is proved that, if $\mathcal{A}$ is a generic hypermatrix of a polynomial ring S of size $n_1 \times \cdots \times n_t$, then $I_2(\mathcal{A})$ is a prime ideal in S, therefore:

$$I(Seg(V_1 \otimes \cdots \otimes V_t)) = I_2(\mathcal{A}) \subset S.$$

Now we generalize this result to another class of decomposable tensors: those defining "Segre-Veronese varieties".

3 Segre-Veronese varieties

3.1 Definitions and Remarks

Before defining a Segre-Veronese variety we recall that a Veronese variety $Y_{n,d}$ is the d-uple embedding of $\mathbb{P}^n$ into $\mathbb{P}^{\binom{n+d}{d}-1}$, via the linear system associated to the sheaf $\mathcal{O}(d)$, with $d > 0$.

Definition 3.1. *A hypermatrix* $\mathcal{A} = (a_{i_1,\dots,i_d})_{1\le i_j\le n,\, j=1,\dots,d}$ *is said to be "supersymmetric" if* $a_{i_1,\dots,i_d} = a_{i_{\sigma(1)},\dots,i_{\sigma(d)}}$ *for all* $\sigma \in \mathfrak{S}_d$ *where* $\mathfrak{S}_d$ *is the permutation group of* $\{1,\dots,d\}$.

With an abuse of notation we will say that a tensor $T \in V^{\otimes d}$ is supersymmetric if it can be represented by a supersymmetric hypermatrix.

Definition 3.2. *Let* $\tilde{S}$ *be a ring of coordinates on* $\mathbb{P}^{\binom{n+d-1}{d}-1}$ *obtained as the quotient* $\tilde{S} = S/I$ *where* $S = K[x_{i_1,\dots,i_d}]_{1\le i_j\le n,\, j=1,\dots,d}$ *and* I *is the ideal generated by all* $x_{i_1,\dots,i_d} - x_{i_{\sigma(1)},\dots,i_{\sigma(d)}}, \forall\, \sigma \in \mathfrak{S}_d$.
The hypermatrix $(\overline{x}_{i_1,\dots,i_d})_{1\le i_j\le n,\, j=1,\dots,d}$ *whose entries are the indeterminates of* $\tilde{S}$, *is said to be a "generic supersymmetric hypermatrix".*

Remark: Let $H \subset V^{\otimes d}$ be the $\binom{n+d-1}{d}$-dimensional subspace of the supersymmetric tensors of $V^{\otimes d}$, i.e. H is isomorphic to the symmetric algebra $Sym_d(V)$. The ring $\tilde{S}$ above is a ring of coordinates for $\mathbb{P}(H)$. The Veronese variety $Y_{n-1,d} \subset \mathbb{P}^{\binom{n+d-1}{d}-1}$ can be viewed as $Seg(V^{\otimes d}) \cap \mathbb{P}(H) \subset \mathbb{P}(H)$.
Let $\mathcal{A} = (x_{i_1,\dots,i_d})_{1\le i_j\le n,\, j=1,\dots,d}$ be a generic supersymmetric hypermatrix, then it is a known result that:

$$I(Y_{n-1,d}) = I_2(\mathcal{A}) \subset \tilde{S}. \tag{3}$$

See [**Wa**] for set theoretical point of view. In [**Pu**] the author proved that $I(Y_{n-1,d})$ is generated by the 2-minors of a particular catalecticant matrix (for a definition of "Catalecticant matrices" see e.g. either [**Pu**] or [**Ge**]). A. Parolin, in his PhD thesis ([**Pa**]), proved that the ideal generated by the 2-minors of that catalecticant matrix is actually $I_2(\mathcal{A})$ with $\mathcal{A}$ a generic supersymmetric hypermatrix.

In this way we have recalled two very related facts:

- if $\mathcal{A}$ is a generic $n_1 \times \cdots \times n_t$ hypermatrix, then the ideal of the 2-minors of $\mathcal{A}$ is the ideal of the Segre variety $Seg(V_1 \otimes \cdots \otimes V_t)$;
- if $\mathcal{A}$ is a generic supersymmetric $\underbrace{n \times \cdots \times n}_{d}$ hypermatrix, then the ideal of the 2-minors of $\mathcal{A}$ is the ideal of the Veronese variety $Y_{n-1,d}$, with $\dim(V) = n$.

Now we want to prove that a similar result holds also for other kinds of hypermatrices strictly related with those representing tensors parameterized by Segre varieties and Veronese varieties.

Definition 3.3. *Let* $V_1,\dots,V_t$ *be vector spaces of dimensions* $n_1,\dots,n_t$ *respectively. The Segre-Veronese variety* $\mathcal{S}_{d_1,\dots,d_t}(V_1 \otimes \cdots \otimes V_t)$ *is the embedding of* $\mathbb{P}(V_1) \otimes \cdots \otimes \mathbb{P}(V_t)$ *into* $\mathbb{P}^{N-1}$, *where* $N = \left(\Pi_{i=1}^t \binom{n_i+d_i-1}{d_i}\right)$, *given by sections of the sheaf* $\mathcal{O}(d_1,\dots,d_t)$.
I.e. $\mathcal{S}_{d_1,\dots,d_t}(V_1 \otimes \cdots \otimes V_t)$ *is the image of the composition of the following two maps:*

$$\mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_t) \overset{\nu_{d_1}\times\cdots\times\nu_{d_t}}{\longrightarrow} \mathbb{P}^{\binom{n_1+d_1-1}{d_1}-1} \times \cdots \times \mathbb{P}^{\binom{n_t+d_t-1}{d_t}-1}$$

and

$$\mathbb{P}^{\binom{n_1+d_1-1}{d_1}-1} \times \cdots \times \mathbb{P}^{\binom{n_t+d_t-1}{d_t}-1} \overset{s}{\longrightarrow} \mathbb{P}^{N-1}$$

where $Im(\nu_1 \times \cdots \times \nu_t) = Y_{n_1-1,d_1} \times \cdots \times Y_{n_t-1,d_t}$ *and* $Im(s)$ *is the Segre variety with* t *factors.*

Example: If $(d_1,\dots,d_t) = (1,\dots,1)$ then $\mathcal{S}_{1,\dots,1}(V_1 \otimes \cdots \otimes V_t) = Seg(V_1 \otimes \cdots \otimes V_t)$.

Example: If $t = 1$ and $\dim(V) = n$, then $\mathcal{S}_d(V)$ is the Veronese variety $Y_{n-1,d}$.

Below we describe how to associate to each element of $\mathcal{S}_{d_1,\dots,d_t}(V_1 \otimes \cdots \otimes V_t)$ a decomposable tensor $T \in V_1^{\otimes d_1} \otimes \cdots \otimes V_t^{\otimes d_t}$.

Definition 3.4. *Let* $\underline{n} = (n_1,\dots,n_t)$ *and* $\underline{d} = (d_1,\dots,d_t)$. *If* V_i *are vector spaces of dimension* n_i *for* $i = 1,\dots,t$, *an "*$(\underline{n},\underline{d})$*-tensor" is defined to be a tensor* T *belonging to* $V_1^{\otimes d_1} \otimes \cdots \otimes V_t^{\otimes d_t}$.

Definition 3.5. *Let $\underline{n}$ and $\underline{d}$ as above. A hypermatrix $\mathcal{A} = (a_{i_{1,1},\dots,i_{1,d_1};\dots;i_{t,1},\dots,i_{t,d_t}})_{1\leq i_{j,k}\leq n_j,\, k=1,\dots,d_j,\, j=1,\dots,t}$ is said to be "$(\underline{n},\underline{d})$-symmetric" if $a_{i_{1,1},\dots,i_{1,d_1};\dots;i_{t,1},\dots,i_{t,d_t}} = a_{i_{\sigma_1(1,1)},\dots,i_{\sigma_1(1,d_1)};\dots;i_{\sigma_t(t,1)},\dots,i_{\sigma_t(t,d_t)}}$ for all permutations $\sigma_j \in \mathfrak{S}_{d_j}$ where $\mathfrak{S}_{d_j}$ is the permutation group on $\{(j,1),\dots,(j,d_j)\}$ for all $j = 1,\dots,t$.*

An $(\underline{n},\underline{d})$-tensor $T \in V_1^{\otimes d_1} \otimes \cdots \otimes V_t^{\otimes d_t}$ is said to be an "$(\underline{n},\underline{d})$-symmetric tensor" if it can be represented by an $(\underline{n},\underline{d})$-symmetric hypermatrix.

Definition 3.6. *Let $\underline{n} = (n_1,\dots,n_t)$ and $\underline{d} = (d_1,\dots,d_t)$ and let $R_{[\underline{n},\underline{d}]}$ be the ring of coordinates on $\mathbb{P}^{N-1}$, with $N = \left(\Pi_{i=1}^t \binom{n_i+d_i-1}{d_i}\right)$, obtained from $S = K[x_{i_{1,1},\dots,i_{1,d_1};\dots;i_{t,1},\dots,i_{t,d_t}}]_{1\leq i_{j,k}\leq n_j,\, k=1,\dots,d_j,\, j=1,\dots,t}$ via the quotient modulo $x_{i_{1,1},\dots,i_{1,d_1};\dots;i_{t,1},\dots,i_{t,d_t}} - x_{i_{\sigma_1(1,1)},\dots,i_{\sigma_1(1,d_1)};\dots;i_{\sigma_t(t,1)},\dots,i_{\sigma_t(t,d_t)}}$, for all $\sigma_j \in \mathfrak{S}_{d_j}$ and $j = 1,\dots,t$.*
The hypermatrix $(\overline{x}_{i_{1,1},\dots,i_{1,d_1};\dots;i_{t,1},\dots,i_{t,d_t}})_{1\leq i_{j,k}\leq n_j,\, k=1,\dots,d_j,\, j=1,\dots,t}$ of indeterminates of $R_{[\underline{n},\underline{d}]}$, is said to be a "generic $(\underline{n},\underline{d})$-symmetric hypermatrix".

Remark: Let $H_i \subset V_i^{\otimes d_i}$ be the subspace of supersymmetric tensors of $V_i^{\otimes d_i}$ for each $i = 1,\dots,t$, then $H_1 \otimes \cdots \otimes H_t \subset V_1^{\otimes d_1} \otimes \cdots \otimes V_t^{\otimes d_t}$ is the subspace of the $(\underline{n},\underline{d})$-symmetric tensors of $V_1^{\otimes d_1} \otimes \cdots \otimes V_t^{\otimes d_t}$. The ring $R_{[\underline{n},\underline{d}]}$ above is a ring of coordinates on $\mathbb{P}(H_1 \otimes \cdots \otimes H_t)$. It is not difficult to check that, as sets:

$$\mathbb{P}(H_1 \otimes \cdots \otimes H_t) \cap Seg(V_1^{\otimes d_1} \otimes \cdots \otimes V_t^{\otimes d_t}) = \mathcal{S}_{d_1,\dots,d_t}(V_1 \otimes \cdots \otimes V_t); \tag{4}$$

i.e. $\mathcal{S}_{d_1,\dots,d_t}(V_1 \otimes \cdots \otimes V_t)$ parameterizes the $(\underline{n},\underline{d})$-symmetric decomposable $(\underline{n},\underline{d})$-tensors of $V_1^{\otimes d_1} \otimes \cdots \otimes V_t^{\otimes d_t}$. A consequence of this fact is that a Segre-Veronese variety is set-theoretically given by the 2-minors of an $(\underline{n},\underline{d})$-symmetric hypermatrix of indeterminates.

In Section 3.3 we will prove that the ideal of the 2-minors of the generic $(\underline{n},\underline{d})$-symmetric hypermatrix in $R_{[\underline{n},\underline{d}]}$ is the ideal of a Segre-Veronese variety. We will need the notion of "weak generic hypermatrices" that we are going to introduce.

3.2 Weak Generic Hypermatrices

The aim of this section is Proposition 3.10 which asserts that the ideal generated by 2-minors of a weak generic hypermatrix (Definition 3.8) is prime.

Definition 3.7. *A k-th section of a hypermatrix $\mathcal{A} = (x_{i_1,\dots,i_t})_{1\leq i_j\leq n_j, j=1,\dots,t}$ is a hypermatrix of the form*

$$\mathcal{A}_{i_k=l} = (x_{i_1,\dots,i_t})_{1\leq i_j\leq n_j, j=1,\dots,\hat{k},\dots,t, i_k=l}.$$

Remark: If a hypermatrix $\mathcal{A}$ represents a tensor $T \in V_1 \otimes \cdots \otimes V_t$, then a k-th section of $\mathcal{A}$ is a hypermatrix representing a tensor $T' \in V_1 \otimes \cdots \otimes \hat{V}_k \otimes \cdots \otimes V_t$.

We introduce now the notion of "weak generic hypermatrices"; this is, in some sense, a generalization of the one of "weak generic box" in [**TH**].

Definition 3.8. *Let $K[u_1,\dots,u_r]$ be a ring of polynomials. A hypermatrix $\mathcal{A} = (f_{i_1,\dots,i_t})_{1\leq i_j\leq n_j, j=1,\dots,t}$, where all $f_{i_1,\dots,i_t} \in K[u_1,\dots,u_r]_1$, is called a "weak generic hypermatrix of indeterminates" (or briefly "weak generic hypermatrix") if:*

1. *all the entries of $\mathcal{A}$ belong to $\{u_1,\dots,u_r\}$;*
2. *there exists an entry $f_{i_1,\dots,i_t}$ such that $f_{i_1,\dots,i_t} \neq f_{k_1,\dots,k_t}$ for all $(k_1,\dots,k_t) \neq (i_1,\dots,i_t)$, $1 \leq k_j \leq n_j$, $j = 1,\dots,t$;*
3. *the ideals of 2-minors of all sections of $\mathcal{A}$ are prime ideals.*

Lemma 3.9. *Let $I, J \subset R = K[u_1,\dots,u_r]$ be ideals such that $J = (I, u_1,\dots,u_q)$ with $q < r$. Let $f \in R$ be a polynomial independent of $u_1,\dots,u_q$ and such that $I : f = I$. Then $J : f = J$.*

Proof. We need to prove that if $g \in R$ is such that $fg \in J$, then $g \in J$.

Any polynomial $g \in R$ can be written as $g = g_1 + g_2$ where $g_1 \in (u_1, \ldots, u_q)$ and g_2 is independent of $u_1, \ldots, u_q$. Clearly $g_1 \in J$. Now $fg_2 = fg - fg_1 \in J$ and fg_2 is independent of $u_1, \ldots, u_q$. This implies that $fg_2 \in I$, then $g_2 \in I \subset J$ because $I : f = I$ by hypothesis. Therefore $g = g_1 + g_2 \in J$. $\square$

Now we can state the main proposition of this section. The proof that we are going to exhibit follows the ideas the proof of Theorem 1.5 in [**TH**], where the author proves that the ideal generated by 2-minors of a generic hypermatrix of indeterminates is prime. In the same proposition (Proposition 1.12) it is proved that also the ideal generated by 2-minors of a "weak generic box" is prime. We give here an independent proof for weak generic hypermatrix, since it is a more general result; moreover we do not follow exactly the same lines as in [**TH**].

Proposition 3.10. *Let $R = K[u_1, \ldots, u_r]$ be a ring of polynomials and let $\mathcal{A} = (f_{i_1,\ldots,i_t})_{1\leq i_j \leq n_j,\, j=1,\ldots,t}$ be a weak generic hypermatrix as defined in 3.8. Then the ideal $I_2(\mathcal{A})$ is a prime ideal in R.*

Proof. Since $\mathcal{A} = (f_{i_1,\ldots,i_t})_{1\leq i_j \leq n_j,\, j=1,\ldots,t}$ is a weak generic hypermatrix, there exists an entry $f_{i_1,\ldots,i_t}$ that verifies the item 2. in Definition 3.8. It is not restrictive to assume that such $f_{i_1,\ldots,i_t}$ is $f_{1,\ldots,1}$.

Let $F, G \in R$ s.t. $FG \in I_2(\mathcal{A})$. We want to prove that either $F \in I_2(\mathcal{A})$ or $G \in I_2(\mathcal{A})$. Let $Z = \{f^k_{1,\ldots,1} \mid k \geq 0\} \subset R$ and let R_Z be the localization of R at Z. Let also $\varphi : R \to R_Z$ such that

$$\varphi(f_{j_1,\ldots,j_t}) = \frac{f_{j_1,1,\ldots,1} \cdots f_{1,\ldots,\,1,j_t}}{f^{t-1}_{1,\ldots,1}},$$

$\varphi(K) = K$ and $\varphi(u_i) = u_i$ for $u_i \in \{u_1, \ldots, u_r\} \backslash \{f_{i_1,\ldots,i_t}\}$ if not empty. Clearly $\varphi(m) = 0$ for all 2-minors m of $\mathcal{A}$. Hence $\varphi(I_2(\mathcal{A})) = 0$. Since $F(\ldots, f_{j_1,\ldots,j_t}, \ldots)G(\ldots, f_{j_1,\ldots,j_t}, \ldots) \in I_2(\mathcal{A})$ then $F(\ldots, \varphi(f_{j_1,\ldots,j_t}), \ldots) \cdot G(\ldots, \varphi(f_{j_1,\ldots,j_t}), \ldots) = 0_{R_Z}$. The localization R_Z is a domain because R is a domain, thus either $F(\ldots, \varphi(f_{j_1,\ldots,j_t}), \ldots) = 0_{R_Z}$, or $G(\ldots, \varphi(f_{j_1,\ldots,j_t}), \ldots) = 0_{R_Z}$. Suppose that $F\left(\ldots, \frac{f_{j_1,1,\ldots,1} \cdots f_{1,\ldots,\,1,j_t}}{f^{t-1}_{1,\ldots,1}}, \ldots\right) = 0_{R_Z}$. This implies that

$$F(\ldots, f_{j_1,\ldots,f_{j_t}}, \ldots) = F\left(\ldots, \frac{f_{j_1,1,\ldots,1} \cdots f_{1,\ldots,\,1,j_t}}{f^{t-1}_{1,\ldots,1}}, \ldots\right) + H, \tag{5}$$

where H belongs to the ideal $(f_{j_1,\ldots,j_t} f^{t-1}_{1,\ldots,1} - f_{j_1,1\ldots,1} \cdots f_{1,\ldots,1,j_t})_{1\leq j_k \leq n_j,\, k=1,\ldots,t} \subset R_Z$.
Now let $H_{t-1} = f_{j_1,\ldots,j_t} f^{t-1}_{1,\ldots,1} - f_{j_1,1\ldots,1} \cdots f_{1,\ldots,1,j_t}$. Then

$$H_{t-1} = f_{1_1,j_2,\ldots,j_t} f_{j_1,1,\ldots,1} f^{t-2}_{j_1,\ldots,j_t} + (f_{1,\ldots,1} f_{j_1,\ldots,j_t} - f_{1,j_2,\ldots,j_t} f_{j_1,1\ldots,1}) f^{t-2}_{j_1,\ldots,j_t} -$$
$$- f_{1,j_2,\ldots,j_t} f_{j_1,1,j_3,\ldots,j_t} \cdots f_{j_1,\ldots,j_{t-1},1} \equiv_{I_2(\mathcal{A})}$$
$$f_{1,j_2,\ldots,j_t} f_{j_1,1,\ldots,1} f^{t-2}_{1,\ldots,1} - f_{1,j_2,\ldots,j_t} f_{j_1,1,j_3,\ldots,j_t} \cdots f_{j_1,\ldots,j_{t-1},1} = H_{t-2}.$$

Proceeding analogously for $H_{t-2}, \ldots, H_1$, it is easy to verify that $H_{t-1} \in I_2(\mathcal{A})$. Hence H belongs to the ideal of R_Z generated by $I_2(\mathcal{A})$. This fact, together with (5), implies that also F belongs to the ideal of R_Z generated by $I_2(\mathcal{A})$. Therefore we obtained that if $\varphi(F) = 0_{R_Z}$, then there exists $\nu > 0$ such that

$$f^{\nu}_{1,\ldots,1} F(\ldots, f_{j_1,\ldots,j_t}, \ldots) \in I_2(\mathcal{A}) \subset R. \tag{6}$$

Clearly $I_2(\mathcal{A}) \subset (I_2(\mathcal{A}), f_{n_1,\ldots,n_t})$, hence, by (6), $f^{\nu}_{1,\ldots,1} F \in (I_2(\mathcal{A}), f_{n_1,\ldots,n_t})$. Now, because of our choice of $f_{1,\ldots,1}$, $f^{\nu}_{1,\ldots,1}$ is independent of $f_{n_1,\ldots,n_t}$, then, by Lemma 3.9, the polynomial F belongs to $(I_2(\mathcal{A}), f_{n_1,\ldots,n_t})$. Hence we can write $F = F_1 + F_2$ where $F_1 \in I_2(\mathcal{A})$ and $F_2 \in (f_{n_1,\ldots,n_t})$, that is to say $F = F_1 + f_{n_1,\ldots,n_t} \tilde{F}_2$ with $\deg(\tilde{F}_2) < \deg(F)$. We want to prove that $F \in I_2(\mathcal{A})$; we proceed by induction on $\deg(F)$. If $\deg(F) = 0$, since $\varphi(F) = 0_{R_Z}$, we have $F = 0 \in I_2(\mathcal{A})$. Now let $\deg(F) > 0$.
Obviously $f^{\nu}_{1,\ldots,1} f_{n_1,\ldots,n_t} \tilde{F}_2 = f^{\nu}_{1,\ldots,1} F - f^{\nu}_{1,\ldots,1} F_1 \in I_2(\mathcal{A})$.
Let's notice that, since $\varphi(G) \neq 0_{R_Z}$, from (6), if $G = f^{\lambda}_{j_1,\ldots,j_t}$ and if $FG \in I_2(\mathcal{A})$ for some $\lambda > 0$ and some entry $f_{j_1,\ldots,j_t}$ of $\mathcal{A}$, then there exists $\nu > 0$ such that $f^{\nu}_{1,\ldots,1} F \in I_2(\mathcal{A})$.
We deduce that there exists $\mu > 0$ s. t. $f^{\nu+\mu}_{1,\ldots,1} \tilde{F}_2 \in I_2(\mathcal{A})$. Now, by induction hypothesis on the degree of F, we have that $\tilde{F}_2 \in I_2(\mathcal{A})$. Therefore $F \in I_2(\mathcal{A})$. $\square$

3.3 Ideals of Segre -Veronese varieties

Since a Segre-Veronese variety is given set-theoretically by the 2-minors of an $(\underline{n}, \underline{d})$-symmetric hypermatrix of indeterminates (see (4)), if we prove that any $(\underline{n}, \underline{d})$-symmetric hypermatrix of indeterminates is weak generic, we will have, as a consequence of Proposition 3.10, that its 2-minors are a set of generators for the ideals of Segre-Veronese varieties.

Remark: If $\mathcal{A} = (a_{i_1,\dots,i_d})_{1 \leq i_j \leq n;\, j=1,\dots,d}$ is a supersimmetric hypermatrix of size $\underbrace{n \times \cdots \times n}_{d}$, then also a k-th section $\mathcal{A}_{i_k=l}$ of $\mathcal{A}$ is a supersymmetric hypermatrix of size $\underbrace{n \times \cdots \times n}_{d-1}$.

In fact, since $\mathcal{A}$ is supersymmetric, then $a_{i_1,\dots,i_d} = a_{i_{\sigma(1)},\dots,i_{\sigma(d)}}$ for all $\sigma \in \mathfrak{S}_d$. The section $\mathcal{A}_{i_k=l}$ is obtained from $\mathcal{A}$ by imposing $i_k = l$. Therefore $\mathcal{A}_{i_k=l} = (a_{i_1,\dots,i_k=l,\dots i_d})$ is such that $a_{i_1,\dots,i_k=l,\dots i_d} = a_{i_{\sigma(1)},\dots,i_{\sigma(k)}=l,\dots,i_{\sigma(d)}}$, for all $\sigma \in \mathfrak{S}_d$ such that $\sigma(k) = k$, hence such σ's can be viewed as elements of the permutation group on the set $\{1, \dots, k-1, k+1, \dots, d\}$ that is precisely $\mathfrak{S}_{d-1}$.

Remark: If $[T] \in Y_{n-1,d}$, then a hypermatrix obtained as a section of the hypermatrix representing T, can be associated to a tensor T' such that $[T'] \in Y_{n-1,d-1}$.

Theorem 3.11. *Let $\underline{n} = (n_1, \dots, n_t)$ and $\underline{d} = (d_1, \dots, d_t)$. Let $H_i \subset V_i^{\otimes d_i}$ be the subspace of supersymmetric tensors of $V_i^{\otimes d_i}$ for $i = 1, \dots, t$ and let $R_{[\underline{n},\underline{d}]}$ be the ring of coordinates on $\mathbb{P}(H_1 \otimes \cdots \otimes H_t) \subset \mathbb{P}(V_1^{\otimes d_1} \otimes \cdots \otimes V_t^{\otimes d_t})$ defined in Definition 3.6. If $\mathcal{A}$ is a generic $(\underline{n}, \underline{d})$-symmetric hypermatrix of $R_{[\underline{n},\underline{d}]}$, then $\mathcal{A}$ is a weak generic hypermatrix and the ideal of the Segre-Veronese variety $\mathcal{S}_{d_1,\dots,d_t}(V_1 \otimes \cdots \otimes V_t)$ is*

$$I(\mathcal{S}_{d_1,\dots,d_t}(V_1 \otimes \cdots \otimes V_t)) = I_2(\mathcal{A}) \subset R_{[\underline{n},\underline{d}]}$$

with $d_i > 0$ for $i = 1, \dots, t$.

Proof. The proof is by induction on $\sum_{i=1}^t d_i$.

The case $\sum_{i=1}^t d_i = 1$ is not very significant because if $\dim(V_1) = n_1$, so $\mathcal{S}_1(V_1) = Y_{n_1-1,1} = \mathbb{P}(V_1)$, then $I(\mathcal{S}_1(V_1)) = I(\mathbb{P}(V))$ i.e. the zero ideal (in fact the 2-minors of $\mathcal{A}$ do not exist).

If $\sum_{i=1}^t d_i = 2$ the two possible cases for the Segre-Veronese varieties are either $\mathcal{S}_2(V_1)$ or $\mathcal{S}_{1,1}(V_1, V_2)$. Clearly, if $\dim(V_1) = n_1$, then $\mathcal{S}_2(V_1) = Y_{n_1-1,2}$ is Veronese variety and the theorem holds because of (3). Analogously $\mathcal{S}_{1,1}(V_1, V_2) = Seg(V_1 \otimes V_2)$ and again the theorem is known to be true ([**TH**]).

Assume that the theorem holds for every $(\underline{n}, \underline{d})$-symmetric hypermatrix with $\sum_{i=1}^t d_i \leq r - 1$. Then, by Proposition 3.10, the ideal generated by the 2-minors of such an $(\underline{n}, \underline{d})$-symmetric hypermatrix is a prime ideal.

Now, let $\mathcal{A}$ be an $(\underline{n}, \underline{d})$-symmetric hypermatrix with $\sum_{i=1}^t d_i = r$. The first two properties that characterize a weak generic hypermatrix (see Definition 3.8) are immediately verified for $\mathcal{A}$. For the third one we have to check that the ideals of the 2-minors of all sections $\mathcal{A}_{i_{p,q}=l}$ of $\mathcal{A}$ are prime ideals.
If we prove that $\mathcal{A}_{i_{p,q}=l}$ represents an $(\underline{n}, \underline{d}')$-symmetric hypermatrix (with $\underline{d}' = (d_1, \dots, d_p - 1, \dots, d_t))$) we will have, by induction hypothesis, that $\mathcal{A}_{i_{p,q}=l}$ is a weak generic hypermatrix and hence its 2-minors generate a prime ideal.
The hypermatrix $\mathcal{A} = (a_{i_{1,1},\dots,i_{1,d_1};\dots;i_{t,1},\dots,i_{t,d_t}})_{1 \leq i_{j,k} \leq n_j,\, k=1,\dots,d_j,\, j=1,\dots,t}$ is $(\underline{n}, \underline{d})$-symmetric, hence, by definition, $a_{i_{1,1},\dots,i_{1,d_1};\dots;i_{t,1},\dots,i_{t,d_t}} = a_{i_{\sigma_1(1,1)},\dots,i_{\sigma_1(1,d_1)};\dots;i_{\sigma_t(t,1)},\dots,i_{\sigma_t(t,d_t)}}$ for all permutations $\sigma_j \in \mathfrak{S}_{d_j}$ where $\mathfrak{S}_{d_j}$ is the permutation group on $\{(j,1), \dots, (j, d_j)\}$ for all $j = 1, \dots, t$.
The hypermatrix $\mathcal{A}_{i_{p,q}=l} = (a_{i_{1,1},\dots,i_{1,d_1};\dots,i_{p,q}=l,\dots;i_{t,1},\dots,i_{t,d_t}})$, obtained from $\mathcal{A}$ by imposing $i_{p,q} = l$, is $(\underline{n}, \underline{d}')$-symmetric because

$$a_{i_{1,1},\dots,i_{1,d_1};\dots,i_{p,q}=l,\dots;i_{t,1},\dots,i_{t,d_t}} = a_{i_{\sigma_1(1,1)},\dots,i_{\sigma_1(1,d_1)};\dots;i_{\sigma_p(p,1)},\dots,i_{p,q}=l,\dots i_{\sigma_p(p,d_p)};\dots;i_{\sigma_t(t,1)},\dots,i_{\sigma_t(t,d_t)}}$$

for all $\sigma_j \in \mathfrak{S}_{d_j}$, $j = 1, \dots, \hat{p}, \dots, t$, and for $\sigma_p \in \mathfrak{S}_{d_p-1}$, where $\mathfrak{S}_{d_p-1}$ is the permutation group on the set of indices $\{(p,1), \dots, \widehat{(p,q)}, \dots, (p, d_p)\}$ (this is a consequence of the first Remark of this section). Hence $I_2(\mathcal{A}_{i_{p,q}=l})$ is prime by induction, and $\mathcal{A}$ is weak generic, so also $I_2(\mathcal{A})$ is prime.

Since by definition $\mathcal{S}_{d_1,\dots,d_t}(V_1 \otimes \cdots \otimes V_t) = \mathbb{P}(H_1 \otimes \cdots \otimes H_t) \cap Seg(V_1 \otimes \cdots \otimes V_t)$, we have that $I_2(\mathcal{A})$ is a set of equations for $\mathcal{S}_{d_1,\dots,d_t}(V_1 \otimes \cdots \otimes V_t)$ (see (4)), hence, because of the primeness of $I_2(\mathcal{A})$ that we have just proved, $I_2(\mathcal{A}) \subset R_{[\underline{n},\underline{d}]}$ is the ideal of $\mathcal{S}_{d_1,\dots,d_t}(V_1 \otimes \cdots \otimes V_t)$. $\square$

4 Projections of Veronese surfaces

In this section we want to use the tool of weak generic hypermatrices in order to prove that the ideal of a projection of a Veronese surface $Y_{2,d} \subset \mathbb{P}^{\binom{d+2}{d}-1}$ from a finite number $s \leq \binom{d}{2}$ of generic points on it is the prime ideal defined by the order 2-minors of some particular tensor.

In [**TH**] the case in which s is a binomial number (i.e. $s = \binom{t+1}{2}$ for some positive integer $t \leq d-1$) is done.

In this section we try to extend that result to a projection of a Veronese surface from any number $s \leq \binom{d}{2}$ of generic points.

Notice that in [**Gi**] and in [**GL1**] the authors study the projection of Veronese surfaces $Y_{n-1,d}$ from $s = \binom{d}{2}+k$, $0 \leq k \leq d$, for some non negative integer k, (this corresponds to the case of a number of points between the two consecutive binomial numbers $\binom{d}{2}$ and $\binom{d+1}{2}$).

Let $Z = \{P_1, \dots, P_s\} \subset \mathbb{P}^2$ be a set of generic points in $\mathbb{P}^2$, where $s = \binom{t+1}{2} + k \leq \binom{d}{2}$ with $0 < t \leq d-1$ and $0 \leq k \leq t$ (actually we may assume $t \leq d-2$ because the case $t = d-1$ and $k = 0$ corresponds to the known case of the "Room Surfaces" - see [**GG**]). Let $J \subset S = K[w_1, w_2, w_3]$ be the ideal $J = I(Z)$, i.e. $J = \wp_1 \cap \dots \cap \wp_s$ with $\wp_i = I(P_i) \subset S$ prime ideals for $i = 1, \dots, s$.

Let J_t be the degree t part of the ideal J and let $Bl_Z(\mathbb{P}^2)$ be the blow up of $\mathbb{P}^2$ at Z. We indicate with $\tilde{J}_t$ the very ample linear system of the strict transforms of the curves defined by J_t. If $\varphi_{J_d} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^{\binom{d+2}{2}-s-1}$ is the rational morphism associated to J_d and if $\varphi_{\tilde{J}_d} : Bl_Z(\mathbb{P}^2) \to \mathbb{P}^{\binom{d+2}{2}-s-1}$ is the morphism associated to $\tilde{J}_d$, the variety $X_{Z,d}$ we want to study is $\overline{Im(\varphi_{J_d})} = Im(\varphi_{\tilde{J}_d})$. This variety can also be viewed as the projection of the Veronese surface $Y_{2,d} \subset \mathbb{P}^{\binom{d+2}{2}-1}$ from s generic points on it.

The first thing to do is to describe J_d as vector space.

4.1 The ideal of generic points in the projective plane

There is a classical result, Hilbert-Burch Theorem (see, for instance, [**CGO**]), that gives a description of the generators of J. I.e. the ideal $J \subset S = K[w_1, w_2, w_3]$ is generated by $t-k+1$ forms $F_1, \dots, F_{t-k+1} \in S_t$ and by h forms $G_1, \dots, G_h \in S_{t+1}$ where $h = 0$ if $0 \leq k < t/2$ and $h = 2k - d$ if $t/2 \leq k \leq t$. What follows now is the constructions of the F_j's and the G_i's (the same description is presented in [**GL1**]).

If $\mathbf{t/2 \leq k \leq t}$, for a generic choice of points $P_1, \dots, P_s$, the generators of J can be chosen to be the maximal minors of:

$$\mathcal{L} := \begin{pmatrix} L_{1,1} & \cdots & L_{1,2k-t} & Q_{11} & \cdots & Q_{1,t-k+1} \\ \vdots & & \vdots & \vdots & & \vdots \\ L_{k,1} & \cdots & L_{k,2k-t} & Q_{k,1} & \cdots & Q_{k,t-k+1} \end{pmatrix} \in M_{k,k+1}(S) \tag{7}$$

where $L_{i,j} \in S_1$ and $Q_{h,l} \in S_2$ for all $i, h = 1, \dots, k$, $j = 1, \dots, 2k-t$ and $l = 1, \dots, t-k+1$.
The forms $F_j \in S_t$ are the minors of $\mathcal{L}$ obtained by deleting the $2k-t+j$-th column, for $j = 1, \dots, t-k+1$; the forms $G_i \in S_{t+1}$ are the minors of $\mathcal{L}$ obtained by deleting the i-th column, for $i = 1, \dots, 2k-t$.
The degree $(t+1)$ part of the ideal J is clearly $J_{t+1} =< w_1F_1, \dots, w_3F_{t-k+1}, G_1, \dots, G_{2k-t} >$. If we set $\tilde{G}_{i,j} = w_iF_j$ for $i = 1, 2, 3$, $j = 1, \dots, t-k+1$ we can write:

$$J_{t+1} =< \tilde{G}_{1,1}, \dots, \tilde{G}_{3,t-k+1}, G_1, \dots, G_{2k-t} > .$$

Notice that $w_1F_1 = \tilde{G}_{1,1}, \dots, w_3F_{t-k+1} = \tilde{G}_{3,t-k+1}$ are linearly independent (see, for example, [**CGO**]).

If $\mathbf{0 \leq k < t/2}$, then J is generated by maximal minors of:

$$\mathcal{L} := \begin{pmatrix} Q_{1,1} & \cdots & \cdots & \cdots & Q_{1,t-k+1} \\ \vdots & & & & \vdots \\ Q_{k,1} & \cdots & \cdots & \cdots & Q_{k,t-k+1} \\ L_{11} & \cdots & \cdots & \cdots & L_{1,t-k+1} \\ \vdots & & & & \vdots \\ L_{t-2k,1} & \cdots & \cdots & \cdots & L_{t-2k,t-k+1} \end{pmatrix} \in M_{t-k,t-k+1}(S) \tag{8}$$

where $L_{i,j} \in S_1$ and $Q_{h,l} \in S_2$ for all $i = 1, \dots, t-2k$, $j, l = 1, \dots, t-k+1$ and $h = 1, \dots, k$.
The forms $F_j \in S_t$ are the minors of $\mathcal{L}$ obtained by deleting the j-th column for $j = 1, \dots, t-k+1$.
Again $J_{t+1} =< w_1F_1, \dots, w_3F_{t-k+1} >$ but now those generators are not necessarily linearly independent.
Using the same notation of the previous case one can write:

$$J_{t+1} =< \tilde{G}_{1,1}, \dots, \tilde{G}_{3,t-k+1} > .$$

Clearly if $t/2 \le k \le t$ then:

$$J_d =< \underline{w}^{d-t-1}\tilde{G}_{i,j}, \underline{w}^{d-t-1}G_l > \tag{9}$$

for $i = 1, 2, 3$, $j = 1, \dots t-k+1$, $l = 1, \dots, 2k-t$ and $\underline{w}^{d-t-1}G = \{w_1^{d-t-1}G, w_1^{t-d-2}w_2G, \dots, w_3^{d-t-1}G\}$.
If $0 \le k < t/2$ then:

$$J_d =< \underline{w}^{d-t-1}\tilde{G}_{i,j} > \tag{10}$$

for $i = 1, 2, 3$ and $j = 1, \dots, t-k+1$.
Denote

$$\begin{cases} z_1 := w_1^{d-t-1}, \\ z_2 := w_1^{t-d-2}w_2, \\ \vdots \\ z_u := w_3^{t-d-1} \end{cases}$$

where $u = \binom{d-t+1}{2}$; or $z_{\underline{\alpha}}$ for $\underline{w}^{\underline{\alpha}} = w_1^{\alpha_1}w_2^{\alpha_2}w_3^{\alpha_3}$, if $\underline{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$, $|\underline{\alpha}| = d-t-1$ and we assume that the $\underline{\alpha}$'s are ordered by the lexicographic order.

Let N be the number of generators of J_d, and let $K[\tilde{x}_{h;i,j}, x_{h,l}]$ be a ring of coordinates on $\mathbb{P}^{N-1}$ with $l = 1, \dots, 2k-t$ only if $t/2 \le k \le t$ (in the other case the variables $x_{h,l}$ do not exist at all) and $h = 1, \dots, u$; $i = 1, 2, 3$; $j = 1, \dots, t-k+1$ in any case. The morphism $\varphi : \mathbb{P}^2 \setminus Z \to \mathbb{P}^{N-1}$ such that

$$\varphi([w_1, w_2, w_3]) = [z_1\tilde{G}_{1,1}, \dots, z_u\tilde{G}_{3,t-k+1}, z_1G_1, \dots, z_uG_{2k-t}], \text{ if } t/2 \le k \le t,$$

or

$$\varphi([w_1, w_2, w_3]) = [z_1\tilde{G}_{1,1}, \dots, z_u\tilde{G}_{3,t-k+1}], \text{ if } 0 \le k < t/2,$$

gives a parameterization of $X_{Z,d}$ into $\mathbb{P}^{N-1}$. Observe that $X_{Z,d} = \overline{\varphi_{J_d}(\mathbb{P}^2 \setminus Z)}$ is naturally embedded into $\mathbb{P}^{\binom{d+2}{2}-s-1}$, because $\dim_K(J_d) = \binom{d+2}{2} - s$. In terms of the $\tilde{x}_{h;i,j}$'s and the $x_{h,l}$'s, since the parameterization of $X_{Z,d}$ is:

$$\begin{cases} \tilde{x}_{h;i,j} = z_h\tilde{G}_{i,j}, \\ x_{h,l} = z_hG_l, \end{cases} \tag{11}$$

the independent linear relations between the generators of J_d will give the subspace $\mathbb{P}(< Im(\varphi_{\tilde{J}_d}) >) = \mathbb{P}^{\binom{d+2}{2}-s-1}$ of $\mathbb{P}^{N-1}$. The number of such relations has to be $N - \binom{d+2}{2} + s$.

If $t/2 \le k \le t$, the number of generators of J_d given by (9) is $\binom{d-t+2}{2}(t-k+1) + \binom{d-t-1+2}{2}(2k-t)$; hence there must be $\binom{d-t}{2}k$ independent relations between those generators of J_d.

If $0 \le k < t/2$, the number of generators of J_d in (10) is $\binom{d-t+2}{2}(t-k+1)$, hence there must be $\binom{d-t+1}{2}(t-k) - k(d-t)$ independent relations between those generators of J_d.

There is a very intuitive way of finding exactly those numbers of relations between the generators of J_d and this is what we are going to describe (then we will prove that such relations are also independent).

If $\mathbf{t/2 \le k \le t}$, assume that $\underline{\beta} = (\beta_1, \beta_2, \beta_3)$ with $|\underline{\beta}| = d-t-2$. The determinant obtained by adding to the matrix $\mathcal{L}$ defined in (7) a row $\begin{pmatrix} \underline{w}^{\underline{\beta}}L_{i,1} & \cdots & \underline{w}^{\underline{\beta}}L_{i,2k-t} & \underline{w}^{\underline{\beta}}Q_{i,1} & \cdots & \underline{w}^{\underline{\beta}}Q_{i,t-k+1} \end{pmatrix}$ clearly vanish for all $i = 1, \dots, k$:

$$\det \begin{pmatrix} \underline{w}^{\underline{\beta}}L_{i,1} & \cdots & \underline{w}^{\underline{\beta}}L_{i,2k-t} & \underline{w}^{\underline{\beta}}Q_{i,1} & \cdots & \underline{w}^{\underline{\beta}}Q_{i,t-k+1} \\ & & & \mathcal{L} & & \end{pmatrix} = 0.$$

Computing those determinants, for $i = 1, \dots, k$, one gets:

$$\sum_{r=1}^{2k-t} \underline{w}^{\underline{\beta}} L_{i,r} G_r + \sum_{p=1}^{t-k+1} \underline{w}^{\underline{\beta}} Q_{i,p} F_p = 0 \tag{12}$$

where the G_r's and the F_p's are defined as minors of (7).
Since $L_{i,r} \in S_1$, there exist some $\lambda_{i,r,l} \in K$, for $i = 1, \dots, k$, $r = 1, \dots, 2k-t$ and $l = 1, 2, 3$, such that

$$L_{i,r} = \sum_{l=1}^{3} \lambda_{i,r,l} w_l;$$

analogously, since $Q_{i,p} \in S_2$, there exist some $\gamma_{i,p,l,h} \in K$, for $i = 1, \dots, k$, $p = 1, \dots, t-k+1$ and $l, h = 1, 2, 3$, such that

$$Q_{i,p} = \sum_{l,h=1}^{3} \gamma_{i,p,l,h} w_l w_h.$$

Before rewriting the equations (12), observe that

$$Q_{i,p} F_p = \left(\sum_{l,h=1}^{3} \gamma_{i,p,l,h} w_l w_h \right) F_p = \sum_{l,h=1}^{3} \gamma_{i,p,l,h} w_l \tilde{G}_{h,p},$$

and set:

- $\mu_{i,\underline{\alpha},r} = \begin{cases} \lambda_{i,r,l}, & \text{if } \underline{\alpha} = \underline{\beta} + \underline{e}_l, \\ 0 & \text{otherwise,} \end{cases}$
  for $i = 1, \dots, k$; $|\underline{\alpha}| = t-d-1$ and $l = 1, 2, 3$ and where $\underline{e}_1 = (1,0,0)$, $\underline{e}_2 = (0,1,0)$ and $\underline{e}_3 = (0,0,1)$;
- $\tilde{\mu}_{i,\underline{\alpha},p,h} = \begin{cases} \gamma_{i,p,l,h}, & \text{if } \underline{\alpha} = \underline{\beta} + \underline{e}_l, \\ 0 & \text{otherwise,} \end{cases}$
  for $i = 1, \dots, k$; $p = 1, \dots, t-k+1$; $l, h = 1, 2, 3$ and $|\underline{\alpha}| = d-t-2$.

Therefore the equations (12), for $i = 1, \dots, k$, can be rewritten as follows:

$$\sum_{\substack{|\underline{\alpha}| = d-t-1 \\ 1 \le r \le 2k-t}} \mu_{i,\underline{\alpha},r} \underline{w}^{\underline{\alpha}} G_r + \sum_{\substack{|\underline{\alpha}| = d-t-1 \\ 1 \le p \le t-k+1 \\ h=1,2,3}} \tilde{\mu}_{i,\underline{\alpha},p,h} \underline{w}^{\underline{\alpha}} \tilde{G}_{h,p} = 0, \tag{13}$$

which, for $i = 1, \dots, k$, in terms of $x_{\underline{\alpha},r}$ and $\tilde{x}_{\underline{\alpha},h,p}$ defined in (11) becomes:

$$\sum_{\substack{|\underline{\alpha}| = d-t-1 \\ 1 \le r \le 2k-t}} \mu_{i,\underline{\alpha},r} x_{\underline{\alpha},r} + \sum_{\substack{|\underline{\alpha}| = d-t-1 \\ 1 \le p \le t-k+1 \\ h=1,2,3}} \tilde{\mu}_{i,\underline{\alpha},p,h} \tilde{x}_{\underline{\alpha},h,p} = 0. \tag{E_1}$$

There are exactly k of such relations for each $\underline{\beta}$ and the number of $\underline{\beta}$'s is $\binom{d-t}{2}$. Hence in (13) we have found precisely the number of relations between the generators of J_d that we were looking for; we need to prove that they are independent.

If $\mathbf{0 \le k < t/2}$, the way of finding the relations between the generators of J_d is completely analogous to the previous one. The only difference is that in this case they come from the vanishing of two different kinds of determinants:

$$\det \begin{pmatrix} \underline{w}^{\underline{\beta}} L_{i,1} & \cdots & \underline{w}^{\underline{\beta}} L_{i,t-k+1} \\ & \mathcal{L} & \end{pmatrix} = 0 \tag{14}$$

for $i = 1, \dots, t-2k$, $|\underline{\beta}| = d-t-1$ and $\mathcal{L}$ defined as in (8); and

$$\det \begin{pmatrix} \underline{w}^{\underline{\beta}'} Q_{j,1} & \cdots & \underline{w}^{\underline{\beta}'} Q_{j,t-k+1} \\ & \mathcal{L} & \end{pmatrix} = 0 \tag{15}$$

for $j = 1, \dots, k$, $|\underline{\beta}'| = d-t-2$ and $\mathcal{L}$ defined as in (8).

Proceeding as in the previous case one finds that the relations coming from (14) are of the form

$$\sum_{\substack{|\underline{\alpha}| = d-t-1 \\ 1 \le r \le t-k-1 \\ l,h = 1,2,3}} \tilde{\lambda}_{i,\underline{\alpha},r,l} z_{\underline{\alpha}} \tilde{G}_{h,r} = 0 \tag{E}$$

for some $\tilde{\lambda}_{i,\underline{\alpha},r,l} \in K$ and the number of them is $\binom{d-t+1}{2}(t-2k)$.

The relations coming from (15) are of the form

$$\sum_{\substack{|\underline{\alpha}| = d-t-1 \\ 1 \le r \le t-k+1 \\ l,h = 1,2,3}} \tilde{\mu}_{i,\underline{\alpha},r,l} z_{\underline{\alpha}} \tilde{G}_{h,r} = 0 \tag{EE}$$

for some $\tilde{\mu}_{i,\underline{\alpha},r,l} \in K$ and the number of them is $\binom{d-t}{2}k$.

The equations (E) and (EE) allow to observe that $X_{Z,d}$ is contained in the projective subspace of $\mathbb{P}^{N-1}$ defined by the following linear equations in the variables $\tilde{x}_{\underline{\alpha},h,r}$:

$$\begin{cases} \sum_{\substack{|\underline{\alpha}| = d-t-1 \\ 1 \le r \le t-k-1 \\ l,h = 1,2,3}} \tilde{\lambda}_{i,\underline{\alpha},r,l} \tilde{x}_{\underline{\alpha};h,r} = 0 \\ \sum_{\substack{|\underline{\alpha}| = d-t-1 \\ 1 \le r \le t-k+1 \\ l,h = 1,2,3}} \tilde{\mu}_{i,\underline{\alpha},r,l} \tilde{x}_{\underline{\alpha};h,r} = 0 \end{cases} \tag{E_2}$$

The number of relations (E_2) is $\binom{d-t+1}{2}(t-2k) + \binom{d-t}{2}k$, that is exactly the number of independent relations we expect in the case $0 \le k < t/2$.

Now we have to prove that the relations (E_1), respectively (E_2), are independent.

Notation: Let M be the matrix of order $\left(\binom{d-t}{2}k\right) \times \left(\binom{d-t+1}{2}(2t-k+3)\right)$ given by the $\mu_{i,\underline{\alpha},r}$ and the $\tilde{\mu}_{i,\underline{\alpha},p,h}$ appearing in all the equations (E_1). We have already observed that there exists an equation of type (E_1) for each multi-index over three variables $\underline{\beta}$ of weight $|\underline{\beta}| = d-t-2$, and for each $i = 1, \dots, k$. We construct the matrix M by blocks $M_{\underline{\beta},\underline{\alpha}}$ (the triple multi-index $\underline{\alpha}$ is such that $|\underline{\alpha}| = d-t-1$):

$$M = (M_{\underline{\beta},\underline{\alpha}})_{|\underline{\beta}|=d-t-2, |\underline{\alpha}|=d-t-1} \tag{16}$$

and the orders on the $\underline{\beta}$'s and the $\underline{\alpha}$'s are the respective decreasing lexicographic orders. For each fixed $\underline{\beta}$ and $\underline{\alpha}$, the block $M_{\underline{\beta},\underline{\alpha}}$ is the following matrix:

$$M_{\underline{\beta},\underline{\alpha}} = \begin{pmatrix} \mu_{1,\underline{\alpha},1} & \cdots & \mu_{1,\underline{\alpha},2k-t} & \tilde{\mu}_{1,\underline{\alpha},1,1} & \cdots & \tilde{\mu}_{1,\underline{\alpha},t-k+1,3} \\ \vdots & & \vdots & \vdots & & \vdots \\ \mu_{k,\underline{\alpha},1} & \cdots & \mu_{k,\underline{\alpha},2k-t} & \tilde{\mu}_{k,\underline{\alpha},1,1} & \cdots & \tilde{\mu}_{k,\underline{\alpha},t-k+1,3} \end{pmatrix}.$$

Analogously we construct the matrix N of order $\left(\binom{d-t+1}{2}(t-2k) + \binom{d-t}{2}k\right) \times \left(3\binom{t-d+1}{2}(t-k+1)\right)$:

$$N := \begin{pmatrix} N_{\underline{\beta},\underline{\alpha}} \\ N_{\underline{\beta}',\underline{\alpha}} \end{pmatrix}_{|\underline{\alpha}|=|\underline{\beta}|=d-t-1,\, |\underline{\beta}'|=d-t-2} \tag{17}$$

where

$$N_{\underline{\beta},\underline{\alpha}} := \begin{pmatrix} \tilde{\lambda}_{1,\underline{\alpha},1,1} & \cdots & \tilde{\lambda}_{1,\underline{\alpha},t-k-1,3} \\ \vdots & & \vdots \\ \tilde{\lambda}_{t-2k,\underline{\alpha},1,1} & \cdots & \tilde{\lambda}_{t-2k,\underline{\alpha},t-k-1,3} \end{pmatrix} \quad \text{and} \quad N_{\underline{\beta}',\underline{\alpha}} := \begin{pmatrix} \tilde{\mu}_{1,\underline{\alpha}1,1} & \cdots & \tilde{\mu}_{1,\underline{\alpha}t-k+1,3} \\ \vdots & & \vdots \\ \tilde{\mu}_{k,\underline{\alpha}1,1} & \cdots & \tilde{\mu}_{k,\underline{\alpha}t-k+1,3} \end{pmatrix}$$

where the $\tilde{\lambda}_{i,\underline{\alpha},r,l}$'s and the $\tilde{\mu}_{i,\underline{\alpha},r,l}$'s are those appearing in (E) and in (EE) respectively.

Proposition 4.1. *The matrices M and N defined in (16) and (17), respectively, are of maximal rank.*

Proof. Without loss of generality we may assume that $P = [0, 0, 1] \notin Z$ and that F_1 (i.e. the first minor of the matrix $\mathcal{L}$ defined either in (7) or in (8)) does not vanish at P.

For the M case, one can observe that every time $\underline{\alpha} \neq \underline{\beta} + \underline{e}_l$, $l = 1, 2, 3$, the block $M_{\underline{\beta},\underline{\alpha}}$ is identically zero, and we denote $M_{\underline{\beta},\underline{\beta}+\underline{e}_l}$ with A_l for $l = 1, 2, 3$.

Consider $\tilde{M}$ the maximal square submatrix of M obtained by deleting the last columns of M (recall that we have ordered both the columns and the rows of M with the respective decreasing lexicographic orders).

All the blocks $M_{\underline{\beta},\underline{\alpha}}$ on the diagonal of $\tilde{M}$ are such that the position of $\underline{\beta}$ is the same position of $\underline{\alpha}$ in their respective decreasing lexicographic orders. Since $|\underline{\beta}| = |\underline{\alpha}| - 1$, then the blocks appearing on the diagonal of $\tilde{M}$ are $M_{\underline{\beta},\underline{\beta}+\underline{e}_1} = A_1$ for all $\underline{\beta}$'s.
If $\underline{\beta} = (\beta_1, \beta_2, \beta_3)$ and $\underline{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, the blocks $M_{\underline{\beta},\underline{\alpha}}$ under the diagonal are all such that $\beta_1 < \alpha_1 - 2$, hence they are all equal to zero.
This is clearly sufficient to prove that $\tilde{M}$ has maximal rank; then M has maximal rank too.

The N case is completely analogous. □

With this discussion we have proved the following:

Proposition 4.2. *The variety $X_{Z,d} \subset \mathbb{P}^{N-1} = \mathbb{P}((K[\tilde{x}_{h;i,j}, x_{h,l}]_1)^*)$ verifies either (E_1) if $t/2 \leq k \leq t$, or (E_2) if $0 \leq k < t/2$. Moreover the relations (E_1), respectively (E_2), are linearly independent.*

Remark: There exist other linear relations between the $\tilde{x}_{\underline{\alpha};i,j}$'s and the $x_{\underline{\alpha},l}$ coming from the fact that $w_i\tilde{G}_{h,j} = w_h\tilde{G}_{i,j}$ for $i, h = 1, 2, 3$ and all j's. If we denote $z_{\underline{\beta}+\underline{e}_i} = \underline{w}^{\underline{\beta}}w_i$ (with $|\underline{\beta}| = d - t - 2$), we have that $z_{\underline{\beta}+\underline{e}_i}\tilde{G}_{h,j} = z_{\underline{\beta}+\underline{e}_h}\tilde{G}_{i,j}$, that is equivalent to:

$$\tilde{x}_{\underline{\beta}+\underline{e}_i;h,j} = \tilde{x}_{\underline{\beta}+\underline{e}_h;i,j}.$$

The proposition just proved and the fact that the span $< Im(\varphi_{\tilde{J}_d}) >$ has the same dimension of the subspaces of $\mathbb{P}^N$ defined by either (E_1) or by (E_2), imply that those relations are linear combinations of either the (E_1), or the (E_2).

Now the study moves from the linear dependence among generators of J_d to the dependence in higher degrees.

4.2 Quadratic relations

Remark:

1. Let $X := (\tilde{x}_{h;i,j}x_{h,l})_{h;i,j,l}$ be the matrix whose entries are the variables of the coordinate ring $K[\tilde{x}_{h;i,j}, x_{h,l}]_1$ where the index $h = 1, \ldots, \binom{d-t+1}{2}$ indicates the rows of X, and the indicies (i, j, l) indicate the columns and are ordered via the lexicographic order, $i = 1, 2, 3$, $j = 1 \ldots, t - k + 1$, $l = 1, \ldots, 2k - t$ (when it occurs).
   The 2-minors of X are annihilated by points of $X_{Z,d}$. Denote this set of equations with (XM).

2. The z_i's satisfy the equations of the Veronese surface $Y_{2,d-t-1}$, i.e. the 2-minors of the following catalecticant matrix:

$$C := \begin{pmatrix} z_1 & z_2 & z_3 & \cdots & z_{u-2} \\ z_2 & z_4 & z_5 & \cdots & z_{u-1} \\ z_3 & z_5 & z_6 & \cdots & z_u \end{pmatrix} \tag{18}$$

with $u = \binom{d-t+1}{2}$.
Multiplying C either by $\tilde{G}_{i,j}$, or by G_l, for each $i = 1, 2, 3$; $j = 1, \dots, t-k+1$ and $l = 1, \dots, 2k-t$, one obtains either

$$\begin{pmatrix} \tilde{x}_{1;i,j} & \cdots & \tilde{x}_{u-2;i,j} \\ \tilde{x}_{2;i,j} & \cdots & \tilde{x}_{u-1;i,j} \\ \tilde{x}_{3;i,j} & \cdots & \tilde{x}_{u;i,j} \end{pmatrix}, \text{ or } \begin{pmatrix} x_{1,l} & \cdots & x_{u-2,l} \\ x_{2,l} & \cdots & x_{u-1,l} \\ x_{3,l} & \cdots & x_{u,l} \end{pmatrix}.$$

Therefore on $X_{Z,d} \subset \mathbb{P}^{N-1}$, the coordinates $\tilde{x}_{1;i,j}, \dots, \tilde{x}_{u;i,j}$, for all $i = 1, 2, 3$ and $j = 1, \dots, t-k+1$, or $x_{1,l}, \dots, x_{u,l}$, for all $l = 1, \dots, 2k-t$, annihilate the 2-minors of those catalecticant matrices, respectively. Denote the set of all these equations with (Cat).

3. For all $h = 1, \dots, \binom{d-t+1}{2}$, on $X_{Z,d}$ we have that $\tilde{G}_{i,j} = \tilde{x}_{h,i,j}/z_h$ and $G_l = x_{h,l}/z_h$ therefore on $X_{Z,d} \times Y_{2,d-t-1}$ the following system of equations is satisfied for all h's:

$$\begin{cases} \tilde{x}_{h;i,j} z_1 = \tilde{x}_{1;i,j} z_h \\ \vdots \\ \tilde{x}_{h;i,j} z_u = \tilde{x}_{u;i,j} z_h \\ x_{h,l} z_1 = x_{1,l} z_h \\ \vdots \\ x_{h,l} z_u = x_{u,l} z_h \end{cases}. \tag{S_h}$$

Proposition 4.3. *Let $Q : [\tilde{x}_{h;i,j} x_{h,l}]$, $h = 1, \dots, \binom{d-t+1}{2}$, $i = 1, 2, 3$, $j = 1 \dots, t-k+1$ and $l = 1, \dots, 2k-t$, such that the equations (XM) are zero if evaluated in Q. Then there exists a point $P : [z_1, \dots, z_u] \in \mathbb{P}^{u-1}$ such that P and Q satisfy the equations (S_h) for all h's.*

Proof. Since $Q : [\tilde{x}_{1,;1,1}, \dots, x_{\binom{d-t+1}{2}, 2k-t}]$ annihilates all the equations (XM), the rank of X at Q is 1, i.e., if we assume that the first row of X is not zero, there exist $a_h \in K$, $h = 1, \dots, u$, such that the coordinates of Q verify the following conditions:

$$\tilde{x}_{h;i,j} = a_h \tilde{x}_{1;i,j} \quad \text{and} \quad x_{h,l} = a_h x_{1,l}$$

for $h = 1, \dots, \binom{d-t+1}{2}$, $i = 1, 2, 3$, $j = 1 \dots, t-k+1$ and $l = 1, \dots, 2k-t$.
We are looking for a point $P : [z_1, \dots, z_u]$ such that if the coordinates of Q are as above, then P and Q verify the systems (S_h). If Q verifies (S_h), then the coordinates of P are such that:

$$\begin{pmatrix} 0 & \cdots & \cdots & 0 \\ -a_2 & a_1 & \cdots & 0 \\ \vdots & & \ddots & \\ -a_u & 0 & \cdots & a_1 \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_u \end{pmatrix} = \underline{0},$$

that is to say $a_h z_1 = z_h$ for $h = 2, \dots, u$.
The solution of such a system is the point P we are looking for, i.e. $P : [a_1, \dots, a_u]$. □

4.3 The ideal of projections of Veronese surfaces from points

Theorem 4.4. *Let $X_{Z,d}$ be the projection of the Veronese d-uple embedding of $\mathbb{P}^2$ from $Z = \{P_1, \dots, P_s\}$ generic points, $s \leq \binom{d}{2}$. Then the equations (XM) and (Cat) together with either (E_1) if $t/2 \leq k \leq t$, or (E_2) if $0 \leq k < t/2$, describe set theoretically $X_{Z,d}$.*

Proof. Obviously $X_{Z,d}$ is contained in the support of the variety defined by the equations in statement of the theorem.

In order to prove the other inclusion we need to prove that if a point Q verifies all the equations required in the statement, then $Q \in X_{Z,d}$.

If $Q : [\tilde{x}_{h;i,j}, x_{h,l}]$ annihilates the equations (XM), then, by Proposition 4.3, there exists a point $P : [z_1, \dots, z_u]$ such that P and Q verify the systems (S_h). Solving those systems in the variables $\tilde{x}_{h;i,j}, x_{h,l}$ allows to write the point Q depending on the $z_1, \dots, z_u$. We do not write the computations for sake of simplicity, but what it turns out is that there exist $\tilde{c}_{i,j}, c_l \in K$, with $i = 1, 2, 3$, $j = 1, \dots, t-k+1$ and $l = 1, \dots, 2k-t$ (only if $t/2 \leq k \leq t$) such that the coordinates $\tilde{x}_{h;i,j}, x_{h,l}$ of Q are $\tilde{x}_{h;i,j} = \tilde{c}_{i,j} z_h$ and $x_{h,l} = c_l z_h$:

$$Q : [\tilde{x}_{h;i,j}, x_{h,l}] = [\tilde{c}_{i,j} z_h, c_l z_h].$$

Since such a Q, by hypothesis, verifies the equations (Cat), then there exists an unique point $R : [w_1, w_2, w_3] \in \mathbb{P}^2$ such that $z_1 = w_1^{d-t-1}, z_2 = w_1^{d-t-2} w_2, \dots, w_3^{d-t-1}$, therefore

$$Q : [\tilde{c}_{i,j} \underline{w}^{\underline{\alpha}}, c_l \underline{w}^{\underline{\alpha}}]$$

with $|\underline{\alpha}| = d - t - 1$.

Assume that $R \notin Z$, that corresponds to assuming that Q lies in the open set given by the image of $\varphi_{\tilde{J}_d}$ minus the exceptional divisors of $Bl_Z(\mathbb{P}^2)$.

Now, if $t/2 \leq k \leq t$, the point Q verifies also the equations (E_1), while if $0 \leq k < t/2$ the point Q verifies the equations (E_2). Therefore if $t/2 \leq k \leq t$, then $\tilde{c}_{i,j} = b\tilde{G}_{i,j}$ and $c_l = bG_l$ for $i = 1, 2, 3$, $j = 1, \dots, t-k+1$ and $l = 1, \dots, 2k-t$; if $0 \leq k < t/2$, then $\tilde{c}_{i,j} = b\tilde{G}_{i,j}$ for $i = 1, 2, 3$ and $j = 1, \dots, t-k+1$, for some $b \in K$. This proves that $Q \in X_{Z,d}$. □

Now we want to construct a weak generic hypermatrix of indeterminates $\mathcal{A}$ in the variables $\tilde{x}_{h;i,j}, x_{h,l}$ in such a way that the vanishing of its 2-minors coincide with the equations (XM) and (Cat). Then $I_2(\mathcal{A})$ will be a prime ideal because of Proposition 3.10. so it will only remain to show that the generators of $I_2(\mathcal{A})$, together with the equations either (E_1) or (E_2), are generators for the defining ideal of $X_{Z,d}$.

Let $C = (c_{i_1,i_2}) \in M_{3,d-t-3}(K)$ be the Catalecticant matrix defined in (18). Let the $\tilde{x}_{h;i,j}$ and the $x_{h,l}$ be defined as in (11). For all $i_1 = 1, 2, 3$, $i_2 = 1, \dots, d-t-3$ and $i_3 = 1, \dots, r$ where $r = 2t-k+3$ if $t/2 \leq k \leq t$ and $r = 3(t-k+1)$ if $0 \leq k < t$, construct the hypermatrix

$$\mathcal{A} = (a_{i_1,i_2,i_3}) \tag{19}$$

in the following way:

$a_{i_1,i_2,i_3} = \tilde{x}_{h,i,j}$ if $c_{i_1,i_2} = z_h$ for $h = 1, \dots, \binom{d-t+1}{2}$, and $i_3 = 1, \dots, 3(t-k+1)$ is the position of the index (i,j) after having ordered the $\tilde{G}_{i,j}$ with the lexicographic order,

$a_{i_1,i_2,i_3} = x_{h,i_3-3(t-k+1)}$ if $c_{i_1,i_2} = z_h$ for $h = 1, \dots, \binom{d-t+1}{2}$ and $i_3 - 3(t-k+1) = 1, \dots, 2k-t$ if $t/2 \leq k \leq t$.

Proposition 4.5. *The hypermatrix $\mathcal{A}$ defined in (19) is a weak generic hypermatrix of indeterminates.*

Proof. We need to verify that all the properties of weak generic hypermatrices hold for such an $\mathcal{A}$.

1. The fact that $\mathcal{A} = (\tilde{x}_{h;i,j}, x_{h,l})$ is a hypermatrix of indeterminates is obvious.
2. The variable $\tilde{x}_{1,1,1}$ appears only in position $a_{1,1,1}$.
3. The ideals of 2-minors of the sections obtained fixing the third index of $\mathcal{A}$ are prime ideals because those sections are Catalecticant matrices and their 2-minors are the equations of a Veronese embedding of $\mathbb{P}^2$. The sections obtained fixing either the index i_1 or the index i_2 are generic matrices of indeterminates, hence their 2-minors generate prime ideals.

□

Corollary 4.6. *Let $\mathcal{A}$ be defined as in (19). The ideal $I_2(\mathcal{A})$ is a prime ideal.*

Proof. This corollary is a consequence of Proposition 4.5 and of Proposition 3.10 . □

Now, we need to prove that the vanishing of the 2-minors of the hypermatrix $\mathcal{A}$ defined in (19) coincide with the equations (XM) and (Cat).

Theorem 4.7. *Let $X_{Z,d}$ be as in Theorem 4.4, then the ideal $I(X_{Z,d}) \subset K[\tilde{x}_{h;i,j}, x_{h,l}]$, with $h = 1, \dots, \binom{d-t+1}{2}$, $i = 1, 2, 3$, $j = 1 \dots, t-k+1$ and $l = 1, \dots, 2k-t$ is generated by all the 2-minors of the hypermatrix $\mathcal{A}$ defined in (19) and the linear formss appearing either in (E_1) if $t/2 \le k \le t$ or in (E_2) if $0 \le k < t/2$.*

Proof. In Corollary 4.6 we have shown that $I_2(\mathcal{A})$ is a prime ideal; in Theorem 4.4 we have proved that the equations (XM), (Cat) and either the equations (E_1) if $t/2 \le k \le t$ or the equations (E_2) if $0 \le k < t/2$ define $X_{Z,d}$ set-theoretically. Then we need to prove that the vanishing of the 2-minors of $\mathcal{A}$ coincide with the equations (XM) and (Cat) and that either $(I_2(\mathcal{A}), (E_1))$ for $t/2 \le k \le t$, or $(I_2(\mathcal{A}), (E_2))$ is actually equal to $I(X_{Z,d})$ for $0 \le k \le t/2$.

Denote with I the ideal defined by $I_2(\mathcal{A})$ and the polynomials appearing either in (E_1) in one case or in (E_2) in the other case. Denote also $\mathcal{V}$ the variety defined by I.

The inclusion $\mathcal{V} \subseteq X_{Z,d}$ is obvious because, by construction of $\mathcal{A}$, the ideal $I_2(\mathcal{A})$ contains the equations (XM) and (Cat), therefore I contains the ideal defined by (XM), (Cat) and either (E_1) or (E_2).

For the other inclusion it is sufficient to verify that each 2-minor of $\mathcal{A}$ appears either in (XM) or in (Cat). This is equivalent to prove that if $Q \in X_{Z,d}$ then $Q \in \mathcal{V}$, i.e. if $Q \in X_{Z,d}$ then Q annihilates all the polynomials appearing in I.

An element of $I_2(\mathcal{A})$ with $\mathcal{A} = (a_{i_1,i_2,i_3})$ is, by definition of a 2-minor of a hypermatrix, one of the following:

1. $a_{i_1,i_2,i_3}a_{j_1,j_2,j_3} - a_{j_1,i_2,i_3}a_{i_1,j_2,j_3}$,
2. $a_{i_1,i_2,i_3}a_{j_1,j_2,j_3} - a_{i_1,j_2,i_3}a_{j_1,i_2,j_3}$,
3. $a_{i_1,i_2,i_3}a_{j_1,j_2,j_3} - a_{i_1,i_2,j_3}a_{j_1,j_2,i_3}$.

We write for brevity z_{i_1,i_2} instead of z_h if (i_1, i_2) is the position occupied by z_h in the catalecticant matrix C defined in (18). We also rename the $\tilde{G}_{i,j}$'s and the G_l's with $\overline{G}_l := \tilde{G}_{i,j}$ if $l = 1, \dots, 3(t-k+1)$ is the position of (i, j) ordered with the lexicographic order, and $\overline{G}_l := G_{l-3(t-k+1)}$ if $l - 3(t-k+1) = 1, \dots, 2k-t$.
With this notation we evaluate those polynomials on $Q \in X_{Z,d}$.

1. $a_{i_1,i_2,i_3}a_{j_1,j_2,j_3} - a_{j_1,i_2,i_3}a_{i_1,j_2,j_3} = \overline{G}_{i_3}\overline{G}_{j_3}(z_{i_1,i_2}z_{j_1,j_2} - z_{j_1,i_2}z_{i_1,j_2})$ that vanishes on $X_{Z,d}$ because, by definition, $z_1 = w_1^{d-t-1}$, $z_2 = w_1^{d-t-2}w_2$, ..., $z_u = w_3^{d-t-1}$, hence the $z_{i,j}$'s vanish on the equations of the Veronese surface $Y_{2,d-t-1}$. The polynomial inside the parenthesis above is a minor of the catalecticant matrix defining such a surface, so the minor of $\mathcal{A}$ that we are studying vanishes on $X_{Z,d}$.
2. The above holds also for the case $a_{i_1,i_2,i_3}a_{j_1,j_2,j_3} - a_{i_1,j_2,i_3}a_{j_1,i_2,j_3}$.
3. $a_{i_1,i_2,i_3}a_{j_1,j_2,j_3} - a_{i_1,i_2,j_3}a_{j_1,j_2,i_3} = z_{i_1,i_1}\overline{G}_{i_3}z_{j_1,j_2}\overline{G}_{j_3} - z_{i_1,i_2}\overline{G}_{j_3}z_{j_1,j_2}\overline{G}_{i_3} = 0$, evidently.

This proves that the vanishing of the 2-minors of $\mathcal{A}$ coincides with the equations (XM) and (Cat).

For the remaining part of the proof, we work as in ([**TH**]), proof of Theorem 2.6.

Consider, with the previous notation, the sequence of surjective ring homomorphisms:

$$\begin{array}{ccccc} K[x_{i,j}] & \xrightarrow{\phi} & K[\underline{w}^{\underline{\alpha}}t_j] & \xrightarrow{\psi} & K[\underline{w}^{\underline{\alpha}}\overline{G}_j] \\ x_{i,j} & \mapsto & \underline{w}^{\underline{\alpha}}t_j & \mapsto & \underline{w}^{\underline{\alpha}}\overline{G}_j \end{array}$$

where the exponent $\underline{\alpha}$ appearing in $\phi(x_{i,j})$ is the triple-index that is in position i after having ordered the $\underline{w}$'s with the lexicographic order.

The ideal $I_2(\mathcal{A})$ is prime, so $I_2(\mathcal{A}) \subseteq \ker(\phi)$.

Let $J \subset K[\underline{w}^{\underline{\alpha}}t_j]$ be the ideal generated by the images via ϕ of the equations appearing either in (E_1) or in (E_2). The generators of J are zero when $t_j = \overline{G}_j$, then $K[\underline{w}^{\underline{\alpha}}t_j]/J \simeq K[\underline{w}^{\underline{\alpha}}\overline{G}_j]$. Hence $J = \ker(\psi)$.
Since it is almost obvious that a set of generators for $\ker(\psi \circ \phi)$ can be chosen as the generators of $\ker(\phi)$ together with the preimages via ϕ of the generators of $\ker(\psi)$, then $I = \ker(\psi \circ \phi)$. This is equivalent to the fact that $I(X_{Z,d}) = I$. $\square$

5 Projection of Veronese varieties

Here we want to generalize the results of the previous section to projections of Veronese varieties from a particular kind of irreducible and smooth varieties $V \subset \mathbb{P}^n$ of codimension 2.

Since we want to generalize the case of s generic points in $\mathbb{P}^2$, we choose V of degree $s = \binom{t+1}{2} + k \leq \binom{d}{2}$ for some non negative integers t, k, d such that $0 < t < d-1$ and $0 \leq k \leq t$.

Moreover we want to define the ideal $I(V) \subset K[x_0, \ldots, x_n]$ of V as we defined $J \subset K[x_0, x_1, x_2]$ in Section 4.1 (with the obvious difference that the elements of $I(V)$ belong to $K[x_0, \ldots, x_n]$ instead to $K[x_0, x_1, x_2]$). To be precise: let $L_{i,j} \in K[x_0, \ldots, x_n]_1$ be generic linear forms, and let $Q_{h,l} \in K[x_0, \ldots, x_n]_2$ be generic quadratic forms for $i, h = 1, \ldots, k$, $j = 1, \ldots, 2k-t$ and $l = 1, \ldots, t-k+1$ if $t/2 \leq k \leq t$; and for $i = 1, \ldots, t-2k$, $j, l = 1, \ldots, t-k+1$ and $h = 1, \ldots, k$ if $0 \leq k < t/2$. Define the matrix $\mathcal{L}$ either as in (7) or as in (8). The forms F_j and G_l are the maximal minors of $\mathcal{L}$ as previously. For each index j there exist $n+1$ forms $\tilde{G}_{i,j} = w_i F_j$ with $i = 0, \ldots, n$, because now $\underline{w} = (w_0, \ldots, w_n)$. Then the degree d part of $I(V)$ is defined as J_d in (9) if $t/2 \leq k \leq t$ and as J_d in (10) if $0 \leq k < t/2$.
This will be the scheme:

$$(V, I(V)) \subset (\mathbb{P}^n, K[x_0, \ldots, x_n]). \tag{20}$$

Remark: Let $W \subset \mathbb{P}^n$ be a variety of codimension 2 in $\mathbb{P}^n$. Let Y_W be the blow up of $\mathbb{P}^n$ along W. Let E be the exceptional divisor of the blow up and H the strict transform of a generic hyperplane. In [**Co**] (Theorem 1) it is proved that if W is smooth, irreducible and scheme-theoretically generated in degree at most $\lambda \in \mathbb{Z}^+$, then $|dH - E|$ is very ample on the blow up Y_W for all $d \geq \lambda + 1$.

Remark: If $\deg(V) = s = \binom{t+1}{2} + k \leq \binom{d}{2}$, $0 < t < d-1$ and $0 \leq k \leq t$, then $I(V)$ is generated in degrees t and $t+1$.

A consequence of those remarks is the following:

Proposition 5.1. *Let $V \subset \mathbb{P}^n$ be defined as in (20), and let $d > t+1$. If E is the exceptional divisor of the blow up Y_V of $\mathbb{P}^n$ along V and H is the strict transform of a generic hyperplane of $\mathbb{P}^n$, then $|dH - E|$ is very ample.*

Let $X_{V,d} \subset \mathbb{P}(H^0(\mathcal{O}_{Y_V}(dH - E)))$ be the image of the morphism associated to $|dH - E|$.

The arguments and the proofs used to study the ideal $I(X_{Z,d})$ in the previous section can all be generalized to $I(X_{V,d})$ if $d > t+1$, $\deg(V) = \binom{t+1}{2} + k \leq \binom{d}{2}$.
Now let S' be the coordinate ring on $\mathbb{P}(H^0(\mathcal{O}_{Y_V}(dH - E)))$, constructed as $K[\tilde{x}_{i,j}, x_{h,l}]$ in the previous section: $S' = K[\tilde{x}_{i,j}, x_{h,l}]$ with $i = 0, \ldots, n$; $j = 1, \ldots, t-k+1$; $h = 1, \ldots, \binom{n+d-t-1}{2}$ and $l = 1, \ldots, 2k-t$ only if $t/2 \leq k \leq t$ (in the other case the variables $x_{h,l}$ do not exist).

Let (E') and (E'') be the equations in S' corresponding to (E_1) and (E_2), respectively.

Let C' be the catalecticant matrix used to define the Veronese variety $Y_{n,d-t-1}$.

The hypermatrix $\mathcal{A}'$ that we are going to use in this case is the obvious generalization of the hypermatrix $\mathcal{A}$ defined in (19); clearly one has to substitute C with C'.

Now the proof of the fact that $I_2(\mathcal{A}') \subset S'$ is a prime ideal is analogous to that one of Corollary 4.6, and pass through the fact that $\mathcal{A}'$ is a weak generic hypermatrix, hence we get the following:

Theorem 5.2. *Let $(V, I(V)) \subset (\mathbb{P}^n, K[x_0, \ldots, x_n])$ be defined as in (20), let Y_V be the blow up of $\mathbb{P}^n$ along V and let $X_{V,d}$ be the image of Y_V via $|dH - E|$, where $d > t+1$, $\deg(V) = \binom{t+1}{2} + k \leq \binom{d}{2}$, H is a generic hyperplane section of $\mathbb{P}^n$ and E is the exceptional divisor of the blow up. The ideal $I(X_{V,d}) \subset S'$ is generated by all the 2-minors of the hypermatrix $\mathcal{A}'$ and the polynomials appearing either in (E') if $t/2 \leq k \leq t$ or in (E'') if $0 \leq k < t/2$, where S', $\mathcal{A}'$, (E') and (E'') are defined as above.*

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