The rigidity of embedded constant mean curvature surfaces

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Abstract

We study the rigidity of complete, embedded constant mean curvature surfaces in $\mathbb{R}^3$. Among other things, we prove that intrinsic isometries of such a surface extends to isometries of $\mathbb{R}^3$ or its isometry group contains an index two subgroup of isometries that extend, when the surface has finite genus.

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1 Introduction.

In this paper we discuss some global results for certain complete, embedded surfaces M in $\mathbb{R}^3$ which have constant mean curvature[1]. If this mean curvature is zero, we call M a minimal surface and if it is nonzero, we call M a CMC surface. Our main theorems deal with the rigidity of complete, embedded constant mean curvature surfaces in $\mathbb{R}^3$ with finite genus.

Recall that an isometric immersion $f\colon \Sigma \to \mathbb{R}^3$ of a Riemannian surface Σ is *congruent* to another isometric immersion $h\colon \Sigma \to \mathbb{R}^3$, if there exists an isometry $I\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $f = I \circ h$. Also recall that an isometric immersion $f\colon \Sigma \to \mathbb{R}^3$ is *rigid*, if whenever $h\colon \Sigma \to \mathbb{R}^3$ is another isometric immersion, then f is congruent to h.

In general, if $f\colon M \to \mathbb{R}^3$ is an isometric immersion of a simply-connected surface with constant mean curvature H_M and $f(M)$ is not contained in a round sphere or a plane, then there exists a smooth one-parameter deformation of the immersion f through non-congruent isometric immersions with mean curvature H_M; this family contains all isometric immersions of M into $\mathbb{R}^3$ with constant mean curvature H_M. Thus, the rigidity

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[1]We require that M is equipped with a Riemannian metric and that the inclusion map $i\colon M \to \mathbb{R}^3$ preserves this metric.

of simply-connected, constant mean curvature immersed surfaces fails in a rather natural way. On the other hand, the main theorems presented in this paper affirm that some complete, nonsimply-connected, embedded constant mean curvature surfaces in $\mathbb{R}^3$ are rigid. More precisely, we prove the following theorems.

Theorem 1.1 (Finite Genus Rigidity Theorem) *Suppose $M \subset \mathbb{R}^3$ is a complete, embedded constant mean curvature surface of finite genus. If M has bounded Gaussian curvature and M is not a helicoid, then any isometric immersion of M into $\mathbb{R}^3$ with the same constant mean curvature is congruent to the inclusion map of M into $\mathbb{R}^3$.*

Theorem 1.2 (Finite Genus Isometry Extension Theorem) *Let $M \subset \mathbb{R}^3$ be a complete, embedded constant mean curvature surface of finite genus and let $\sigma \colon M \to M$ be an isometry. Then the following statements hold.*

1. *If the isometry $\sigma \colon M \to M$ fails to extend to an isometry of $\mathbb{R}^3$, then the isometry group of M contains a subgroup of index two, consisting of those isometries which do extend to $\mathbb{R}^3$. In particular, if σ fails to extend, then σ^2 extends.*

2. *If M has bounded curvature or if M is minimal, then σ extends to an isometry of $\mathbb{R}^3$.*

The first relevant result in the direction of revealing the rigidity of certain constant mean curvature surfaces is a theorem of Choi, Meeks, and White. In [2] they proved that any properly embedded minimal surface in $\mathbb{R}^3$ with more than one end admits a unique isometric minimal immersion into $\mathbb{R}^3$. One of the outstanding conjectures in this subject states that, except for the helicoid, the inclusion map of a complete, embedded constant mean curvature surface M into $\mathbb{R}^3$ is the unique such isometric immersion with the same constant mean curvature up to congruence. Since extrinsic isometries of the helicoid extend to ambient isometries, the validity of this conjecture implies the closely related conjecture that the intrinsic isometry group of any complete, embedded constant mean curvature surface in $\mathbb{R}^3$ is equal to its ambient symmetry group. These two rigidity conjectures were made by Meeks; see Conjecture 15.12 in [7] and the related earlier Conjecture 22 in [6] for properly embedded minimal surfaces. The theorems presented in this paper demonstrate the validity of these rigidity conjectures under some additional hypotheses.

The proofs of Theorems 1.1 and 1.2 rely on the classification of isometric immersions of simply-connected constant mean curvature surfaces in $\mathbb{R}^3$. A key ingredient in the proof of these theorems is our Dynamics Theorem for CMC surfaces in $\mathbb{R}^3$ which is proven in [11]. Among other things, this Dynamics Theorem implies that, under certain hypotheses, a CMC surface in $\mathbb{R}^3$ contains a Delaunay surface[2] at infinity. The fact that Delaunay

[2]The Delaunay surfaces are CMC surfaces of revolution which were discovered and classified by Delaunay [3] in 1841. When these surfaces are embedded, they are called *unduloids* and when they are nonembedded, they are called *nodoids*.

surfaces are rigid is applied in the proofs of our main theorems.

Additionally, using techniques similar to those applied to prove Theorems 1.1 and 1.2, we also demonstrate the following related rigidity theorem. This theorem is well known in the special case that the surface also has finite topology (see for example, [4]).

Theorem 1.3 *Suppose that $M \subset \mathbb{R}^3$ is a complete, embedded CMC surface with bounded Gaussian curvature. If for some point $p \in M$, the genus of the intrinsic balls $B_M(p, R)$ grows sublinearly in R, then any isometric immersion of M into $\mathbb{R}^3$ with constant mean curvature H_M is congruent to the inclusion map of M into $\mathbb{R}^3$.*

This paper is organized as follows. In Section 2 we provide the statement and the necessary background to understand our new Dynamics Theorem. Next, we discuss the known isometric classification of simply-connected constant mean curvature surfaces in $\mathbb{R}^3$. Finally, in Section 4 we demonstrate Theorems 1.1, 1.2 and 1.3.

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2 Background on the Dynamics Theorem.

Before stating our Dynamics Theorem for CMC surfaces in $\mathbb{R}^3$, we discuss the definitions needed to understand it. This is a theorem that we proved in [11]. The statement and proof of this theorem were motivated by the related Dynamics Theorem for minimal surfaces in $\mathbb{R}^3$ in [8].

Definition 2.1 Suppose W is a complete flat three-manifold with boundary $\partial W = \Sigma$ together with an isometric immersion $f : W \to \mathbb{R}^3$ such that f restricted to the interior of W is injective. We call the image surface $f(\Sigma)$ a *strongly Alexandrov embedded CMC surface* if $f(\Sigma)$ is a CMC surface and W lies on the mean convex side of Σ.

We note that, by elementary separation properties, any properly embedded CMC surface in $\mathbb{R}^3$ is always strongly Alexandrov embedded. Furthermore, by item *1* of Theorem 2.3 below, any strongly Alexandrov embedded CMC surface in $\mathbb{R}^3$ with bounded Gaussian curvature is properly immersed in $\mathbb{R}^3$. We remind the reader that the Gauss equation implies that a surface M in $\mathbb{R}^3$ with constant mean curvature has bounded Gaussian curvature if and only if its principal curvatures are bounded in absolute value; thus, M having bounded second fundamental form is equivalent to M having bounded Gaussian curvature.

Definition 2.2 Suppose $M \subset \mathbb{R}^3$ is a strongly Alexandrov embedded CMC surface with bounded Gaussian curvature.

1. $\mathcal{T}(M)$ is the set of all connected, strongly Alexandrov embedded CMC surfaces $\Sigma \subset \mathbb{R}^3$, which are limits of compact domains Δ_n in the translated surfaces $M - p_n$ with $\lim_{n\to\infty} |p_n| = \infty$, $\vec{0} \in \Sigma$, and such that the convergence is of class C^2 on compact subsets of $\mathbb{R}^3$. Actually we consider the immersed surfaces in $\mathcal{T}(M)$ to be *pointed* in the sense that if such a surface is not embedded at the origin, then we consider the surface to represent two different surfaces in $\mathcal{T}(M)$ depending on a choice of one of the two preimages of the origin.

2. $\Delta \subset \mathcal{T}(M)$ is called $\mathcal{T}$*-invariant*, if $\Sigma \in \Delta$ implies $\mathcal{T}(\Sigma) \subset \Delta$.

3. A nonempty subset $\Delta \subset \mathcal{T}(M)$ is called a *minimal* $\mathcal{T}$-invariant set, if it is $\mathcal{T}$-invariant and contains no smaller $\mathcal{T}$-invariant subsets; it turns out that a nonempty $\mathcal{T}$-invariant set $\Delta \subset \mathcal{T}(M)$ is a minimal $\mathcal{T}$-invariant set if and only if whenever $\Sigma \in \Delta$, then $\mathcal{T}(\Sigma) = \Delta$.

4. If $\Sigma \in \mathcal{T}(M)$ and Σ lies in a minimal $\mathcal{T}$-invariant subset of $\mathcal{T}(M)$, then Σ is called a *minimal element* of $\mathcal{T}(M)$.

With these definitions in hand, we now state our Dynamics Theorem from [11]; in the statement of this theorem, $\mathbb{B}(R)$ denotes the open ball of radius R centered at the origin in $\mathbb{R}^3$.

Theorem 2.3 (Dynamics Theorem for CMC surfaces in $\mathbb{R}^3$) *Let M be a connected, noncompact, strongly Alexandrov embedded CMC surface with bounded Gaussian curvature. Then:*

1. *M is properly immersed in $\mathbb{R}^3$. More generally,* $\mathrm{Area}(M \cap \mathbb{B}(R)) \leq cR^3$*, for some constant $c > 0$.*

2. *$\mathcal{T}(M)$ is nonempty.*

3. *$\mathcal{T}(M)$ has a natural metric $d_{\mathcal{T}(M)}$ induced by the Hausdorff distance between compact subsets of $\mathbb{R}^3$. With respect to $d_{\mathcal{T}(M)}$, $\mathcal{T}(M)$ is a compact metric space.*

4. *Every nonempty $\mathcal{T}$-invariant subset of $\mathcal{T}(M)$ contains minimal elements of $\mathcal{T}(M)$. In particular, since $\mathcal{T}(M)$ is itself a nonempty $\mathcal{T}$-invariant set, $\mathcal{T}(M)$ always contains minimal elements.*

5. *A minimal $\mathcal{T}$-invariant set in $\mathcal{T}(M)$ is a compact connected subspace of $\mathcal{T}(M)$.*

6. *If M has finite genus, then every minimal element of $\mathcal{T}(M)$ is a Delaunay surface passing through the origin.*

3 Background on Calabi's and Lawson's Rigidity Theorems.

In this section we review the classical rigidity theorems of Calabi and Lawson for simply-connected constant mean curvature surfaces in $\mathbb{R}^3$. The rigidity theorem of Lawson is motivated by the earlier result of Calabi [1] who classified the set of isometric minimal immersions of a simply-connected Riemannian surface Σ into $\mathbb{R}^3$; we now describe Calabi's classification theorem.

Suppose $f\colon \Sigma \to \mathbb{R}^3$ is a isometric minimal immersion and Σ is simply-connected. In this case the coordinate functions f_1, f_2, f_3 are harmonic functions which are the real parts of corresponding holomorphic functions h_1, h_2, h_3 defined on Σ. For any $\theta \in [0, \pi)$, the map $f_\theta = \mathrm{Re}(e^{i\theta}(h_1, h_2, h_3))\colon \Sigma \to \mathbb{R}^3$ is an isometric minimal immersion of Σ into $\mathbb{R}^3$; the immersions f_θ are called *associate immersions* to f. Many classical examples of minimal surfaces arise from this associate family construction. For example, simply-connected regions on a catenoid are the images of regions in the helicoid under the associate map for $\theta = \frac{\pi}{2}$; in this case the corresponding coordinate functions on these domains are conjugate harmonic functions and consequently, the catenoid and the helicoid are called *conjugate* minimal surfaces. Calabi's classification or rigidity theorem states that if Σ is not flat, then for any isometric minimal immersion $F\colon \Sigma \to \mathbb{R}^3$, there exists a *unique* $\theta \in [0, \pi)$ such that F is congruent to f_θ, i.e. there exists an isometry $I\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that as mappings, $F = I \circ f_\theta$. This notion of rigidity does *not* mean that the image surface $f(\Sigma)$ cannot be congruent to the image of an associate surface $f_\theta(\Sigma)$, where $\theta \in (0, \pi)$. For example, let $f\colon \mathbb{C} \to \mathbb{R}^3$ be a parametrization of the classical Enneper surface and let $f_{\frac{\pi}{2}}$ be the conjugate mapping. Then the images of these immersions are congruent as subsets of $\mathbb{R}^3$ but these immersions are not congruent as mappings. In fact, the rotation $R_{\frac{\pi}{2}}$ counter clockwise by $\frac{\pi}{2}$ in the usual parameter coordinates $\mathbb{C}$ for Enneper's surface is an intrinsic isometry of this surface which does not extend to an isometry of $\mathbb{R}^3$ and $f \circ R_{\frac{\pi}{2}}$ is congruent to the immersion $f_{\frac{\pi}{2}}$.

Lawson's Rigidity Theorem that we referred to in the previous paragraph appears in Theorem 8 in [5]. We will not need his theorem in its full generality and we state below the special case which we will apply in the next section.

Theorem 3.1 (Lawson's Rigidity Theorem for CMC surfaces in $\mathbb{R}^3$) *If $f\colon \Sigma \to \mathbb{R}^3$ is an isometric CMC immersion with mean curvature H and Σ is simply-connected, then there exists a differentiable 2π-periodic family of isometric immersions*

$$f_\theta\colon \Sigma \to \mathbb{R}^3$$

of constant mean curvature H called associate immersion to f. Moreover, up to congruences the maps f_θ, for $\theta \in [0, \pi]$, represent all isometric immersions of Σ into $\mathbb{R}^3$ with constant mean curvature H and these immersions are non-congruent to each other if $f(M)$ is not contained in a sphere.

4 Proofs of the main theorems.

Our rigidity theorems are motivated by several classical results on the rigidity of certain complete embedded minimal surfaces. The first relevant result in this direction is a theorem of Choi, Meeks, and White [2] who proved that any properly embedded minimal surface in $\mathbb{R}^3$ with more than one end admits a unique isometric minimal immersion into $\mathbb{R}^3$; their result proved a special case of the conjecture of Meeks [6] that the inclusion map of a properly embedded, nonsimply-connected minimal surfaces in $\mathbb{R}^3$ is the unique minimal immersion of the surface into $\mathbb{R}^3$ up to congruence.

We now prove the following important special case of Theorem 1.1 regarding properly embedded minimal surfaces.

Theorem 4.1 *If M is a connected, properly embedded minimal surface in $\mathbb{R}^3$ with finite genus which is not a helicoid, then the inclusion map of M into $\mathbb{R}^3$ represents the unique isometric minimal immersion of M into $\mathbb{R}^3$ up to congruence.*

Proof. If M has more than one end, then the result of Choi, Meeks, and White implies that M satisfies the conclusions of the theorem.

Suppose M has finite genus, one end and M is not a helicoid. The main theorem of Meeks and Rosenberg in [10] then implies that M is asymptotic to a helicoid H with a finite positive number of handles attached or M is a plane. Since Theorem 4.1 holds for planes, assume now that M is not a plane. Since M is asymptotic to a helicoid H, any plane P orthogonal to the axis of H intersects M in an analytic set with each component of $M \cap P$ having dimension one and such that outside of some ball in $\mathbb{R}^3$, $M \cap P$ consists of two proper arcs asymptotic to the line $H \cap P$. Since M is a helicoid with a finite positive number of handles attached, elementary Morse theory implies that for a certain choice of P, $M \cap P$ is a one-dimensional analytic set with a vertex contained in the intersection of two open analytic arcs in $M \cap P$. An elementary combinatorial argument implies that $P \cap M$ contains a simple closed oriented curve Γ bounding a compact disk whose interior is disjoint from M. It follows that the integral of the conormal to Γ has a nonzero dot product with the normal to P. The existence of Γ implies that for at least one of the coordinate functions x_i of M, the conjugate harmonic function of x_i is not well-defined (for example, if Γ lies in the (x_1, x_2)-plane, then the conjugate harmonic function of the x_3-coordinate function is not well-defined on M). From our discussion of the Calabi Rigidity Theorem in the previous section, it follows that the inclusion map of M into $\mathbb{R}^3$ is the unique isometric minimal immersion of M into $\mathbb{R}^3$ up to congruence. This completes the proof of the theorem. $\square$

Recall that a recent result of Meeks, Perez, and Ros [8] proves that if M is a complete, connected, embedded minimal surface with finite genus and countably many ends, then it is properly embedded in $\mathbb{R}^3$. Moreover, the Structure Theorem of minimal laminations described in [10] implies that if M has bounded curvature, then it is likewise properly embedded in $\mathbb{R}^3$ (also see [9] or [12] for a proof of this result). Thus, Theorem 4.1 together with the aforementioned results implies the following.

Theorem 4.2 *Suppose that M is a complete, connected, embedded minimal surface in $\mathbb{R}^3$ with finite genus and M is not a helicoid. If M has countably many ends or has bounded curvature, then the inclusion map of M into $\mathbb{R}^3$ is proper and represents the unique isometric minimal immersion of M into $\mathbb{R}^3$ up to congruence.*

We will now apply the above theorems and the results described in Sections 2 and 3 to prove Theorems 1.1, 1.2 and 1.3.

Let $M \subset \mathbb{R}^3$ be a complete, embedded constant mean curvature surface of finite genus.

Proof of Theorem 1.1. Suppose M has bounded Gaussian curvature and it is not a helicoid. We want to show that the inclusion map $i\colon M \to \mathbb{R}^3$ is the unique isometric immersion of M into $\mathbb{R}^3$ with constant mean curvature H_M.

If M is a minimal surface, then Theorem 1.1 follows from Theorem 4.2. Suppose M is a CMC surface. Without loss of generality we will assume $H_M = 1$. By item *4* of Theorem 2.3, $\mathcal{T}(M)$ contains an embedded Delaunay surface Σ. More precisely, for $n \in \mathbb{N}$, there exist compact annular domains $\Delta_n \subset M$ and points $p_n \in \Delta_n$ such that the translated surfaces $\Delta_n - p_n$ converge C^2 to Σ on compact subsets of $\mathbb{R}^3$. For concreteness, suppose g_1 denotes the inclusion map of Σ into $\mathbb{R}^3$. First we show that the immersion g_1 is rigid.

Let $\pi\colon \widetilde{\Sigma} \to \Sigma$ denote the universal covering of Σ. Consider $\widetilde{\Sigma}$ with the induced Riemannian metric and let $f = g_1 \circ \pi\colon \widetilde{\Sigma} \to \mathbb{R}^3$ be the related isometric immersion. Let $f_\theta\colon \widetilde{\Sigma} \to \mathbb{R}^3$ be the associate immersion for angle $\theta \in [0, \pi]$, given in Theorem 3.1; note $f_0 = f$. Suppose g_2 is another isometric immersion of Σ into $\mathbb{R}^3$ which is not congruent to g_1. This implies that $g_2 \circ \pi$ is congruent to $f_{\overline{\theta}}$ for a certain $\overline{\theta} \in (0, \pi]$.

Let $\widetilde{\gamma} \subset \widetilde{\Sigma}$ be a lift[3] of the shortest geodesic circle $\gamma \subset \Sigma$. We will prove that for any $\theta \in (0, \pi]$ the endpoints of $f_\theta(\widetilde{\gamma})$ are distinct, which means that the associate immersions to g_1 do not exist. We will accomplish this by describing the geometry of $f_\theta(\widetilde{\gamma})$.

Note first that if A represents the second fundamental form of f and A_θ the one of f_θ, it follows from Theorem 8 in [5] (see also [13]) that these forms are related by the following equation. Recall that the mean curvature of Σ is equal to H_M which we are assuming to be one.

$$A_\theta = \cos(\theta)(A - H_M I) + \sin(\theta) J(A - H_M I) + H_M I, \tag{1}$$

where I is the identity matrix and J is the almost complex structure on M.

[3]The curve $\widetilde{\gamma}$ is a compact embedded arc in $\widetilde{\Sigma}$ which is the image of a lift of map $\gamma\colon [0,1] \to \Sigma$.

A computation shows that for the geodesic $\widetilde{\gamma}$ and the immersion f_θ, the curvature k_θ and torsion τ_θ of $f_\theta(\widetilde{\gamma})(t)$ are given by

$$k_\theta = \langle A_\theta(\widetilde{\gamma}'(t)), \widetilde{\gamma}'(t)\rangle \text{ and } \tau_\theta = -\langle A_\theta(\widetilde{\gamma}'(t)), J(\widetilde{\gamma}'(t))\rangle. \tag{2}$$

Furthermore, since $\widetilde{\gamma}$ is a lift of the shortest geodesic circle in Σ, there exist $s \leq 0 < 2 \leq r$, $r + s = 2$ such that in the $\widetilde{\gamma}'$, $J\widetilde{\gamma}'$ basis, the second fundamental form along $\widetilde{\gamma}$ is expressed as the matrix

$$A = \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}.$$

Consequently, equations 1 and 2 imply that

$$k_\theta = \cos(\theta)(r-1) + 1 \text{ and } \tau_\theta = \sin(\theta)(1-r). \tag{3}$$

In particular, k_θ and τ_θ are constants. If $\theta \neq \pi$, then $f_\theta(\widetilde{\gamma})$ is contained in a helix, while if $\theta = \pi$ it is contained in circle of radius $R = |2-r| = |s| < |r|$. Since the length of $f_\pi(\widetilde{\gamma})$ is $\frac{2\pi}{r}$, it follows that in either case the endpoints of $f_\theta(\widetilde{\gamma})$ are distinct.

Since the compact annuli $\Delta_n - p_n$ converge C^2 to the embedded Delaunay surface Σ as $n \to \infty$, we conclude that the associate immersions for $\theta \in (0, \pi]$ are not well-defined on $\Delta_n - p_n$ for n large. By Theorem 3.1, Theorem 1.1 now follows. □

Remark 4.3 *In the above proof, we showed that any Delaunay surface is rigid. However, the same computations prove that if f represents the inclusion map of a nodoid into $\mathbb{R}^3$, then the associate immersions f_θ, $\theta \in (0, \pi)$ are never well-defined and the associate immersions f_π are well-defined for an infinite countable set of nodoids.*

Proof of Theorem 1.2. If M has bounded curvature, then Theorem 1.1 implies Theorem 1.2. So, assume now that M does not have bounded Gaussian curvature and let $i\colon M \to \mathbb{R}^3$ be the inclusion map. We will show that if an intrinsic isometry of $\sigma\colon M \to M$ fails to extend to an ambient isometry of $\mathbb{R}^3$, then M is a CMC surface, the associate surface i_π is well-defined and $i \circ \sigma$ is congruent to i_π. It then follows that when M has an intrinsic isometry which does not extend, then the composition of any two isometries which do not extend, extends to an isometry of $\mathbb{R}^3$ and so, there is an homomorphism from the group $\mathrm{Isom}(M)$ of intrinsic isometries of M onto the group $\mathbb{Z}_2$. The theorem then follows from this discussion. Assume that $\sigma\colon M \to M$ is an isometry that does not extend to $\mathbb{R}^3$.

The local picture theorem on the scale of curvature (see Section 7 in [8]) states that there exists a sequence of points $p_n \in M$ and positive numbers ε_n, λ_n such that:

1. $\lim_{n\to\infty} \varepsilon_n \to 0$, $\lim_{n\to\infty} \lambda_n = \infty$ and $\lim_{n\to\infty} \lambda_n \varepsilon_n = \infty$.

2. The component M_n of $M \cap B(p_n, \varepsilon_n))$ that contains p_n is compact with $\partial M_n \subset \partial B(p_n, \varepsilon_n)$.

3. The second fundamental forms of the surfaces $\widetilde{M_n} = \lambda_n M_n \subset \lambda_n B(p_n, \varepsilon_n) \subset \mathbb{R}^3$ are uniformly bounded and are equal to one at the related points $\widetilde{p}_n$.

4. The translated surfaces $\widetilde{M} - \widetilde{p}_n$ converge with multiplicity one to a connected, properly embedded minimal surface $M_\infty \subset \mathbb{R}^3$ with bounded curvature and genus zero.

First suppose that M_∞ is not a helicoid. In this case, up to congruence, it has a unique isometric minimal immersions into $\mathbb{R}^3$ by Theorem 4.1. It follows that if M is minimal, then it has a unique isometric immersion into $\mathbb{R}^3$. Thus, to complete this case assume that M is not minimal. Suppose M is a CMC surface, and $i \circ \sigma$ is congruent to i_θ for some $\theta \in (0, \pi]$. Then a rescaling argument and equation (1) imply that $(M_\infty)_\theta$ is well-defined which contradicts Theorem 4.1 unless $\theta = \pi$. Hence, when M_∞ is not a helicoid and M is CMC, then $\theta = \pi$.

It remains to prove our claim in the case when M_∞ is a helicoid and to do this we will use the embeddedness property of M. In this case $i \circ \sigma \colon M \to \mathbb{R}^3$ must be congruent to an associate immersion $i_\theta \colon M \to \mathbb{R}^3$, where $\theta \in (0, \pi]$ and $\theta \neq \pi$ if M is minimal. It follows that the associate minimal surfaces $(\widetilde{M} - \widetilde{p}_n)_\theta$ can be chosen to approximate a large region of the related associate surface $(M_\infty)_\theta$ to the helicoid M_∞. If $\theta \neq \pi$, then $(M_\infty)_\theta$ intersects itself transversely which means that $(\widetilde{M} - \widetilde{p}_n)_\theta$ is not embedded. This is a contradiction which proves that intrinsic isometries of M must extend to ambient isometries of $\mathbb{R}^3$ when $\theta \neq \pi$. As observed earlier, this completes the proof of the theorem. $\square$

Proof of Theorem 1.3. Suppose $M \subset \mathbb{R}^3$ is a complete, embedded surface with constant mean curvature H_M, bounded Gaussian curvature and the genus of $B_M(p, R)$ is growing sublinearly in R. We want to show that any isometric immersion of M into $\mathbb{R}^3$ with constant mean curvature H_M is congruent to the inclusion map of M into $\mathbb{R}^3$. First note that M has bounded curvature, which implies M is properly embedded in $\mathbb{R}^3$ (see [9] for the proof of this result or the proof of item *1* in Theorem 2.3 which is given in [11]). The hypothesis that the genus of $B_M(p, R)$ grows sublinearly in R implies that there exist points $p_n \in M$ which diverge in $\mathbb{R}^3$ and a related divergent sequence of positive numbers R_n, such that there is a uniform bound on the genus of the balls $B_M(p_n, R_n)$. The proof of item *2* of Theorem 2.3 implies that there exist compact domains $\Delta_n \subset M - p_n$ which converge C^2 on compact subsets of $\mathbb{R}^3$ to a strongly Alexandrov embedded CMC surface M_∞ of finite genus. With some extra care taken in the choice of the domains Δ_n, item *4* of Theorem 2.3 implies such a M_∞ can be produced which is a Delaunay surface. By the proof of Theorem 1.1, we observe that the fact that M_∞ is a Delaunay surface implies that the inclusion map of M into $\mathbb{R}^3$ is the unique isometric immersion of M into $\mathbb{R}^3$ with the constant mean curvature as M. This completes the proof of Theorem 1.3. $\square$

Remark 4.4 *The conclusions of the Finite Genus Rigidity Theorem (Theorem 1.1) should hold without the hypothesis that M have bounded Gaussian curvature. This improvement would be based on techniques we are developing in [11] to prove curvature estimates for certain embedded CMC surfaces in $\mathbb{R}^3$ and the conjectured properness of complete, embedded finite genus minimal surfaces in $\mathbb{R}^3$.*

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