Fully Prime and Fully Coprime Modules

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Abstract

In this work we study fully prime and fully coprime modules by defining product and coproduct of fully invariant submodules in a module and characterize them. Moreover we look over the relation between fully prime (fully coprime modules) and another definition of primeness (coprimeness) such as prime and endo-prime (coprime and endo-coprime) modules. The primeness of the endomorphism ring is also of interest.

Key Words : fully prime modules, fully coprime modules.

We recall a notion of product of fully invariant submodules of a module M studied by Raggi, Ríos, Rincón, Fernández-Alonso and Signoret [?]. Such a product is defined in Bican et.al. [?] for every pair of submodules $K, L \subset M$ (not necessary fully-invariant). That product is used to define "prime module" and the condition is more restrictive than the one we consider here.

We dualize this product by define coproduct of fully invariant submodules of a module M. Notice that such a coproduct is considered in Bican et.al. [?] for any pair of submodules $K, L \subset M$ (not necessary fully invariant) and then a definition of "coprime modules" is derived from this coproduct.

We give some notions which are important in our investigation. M is called *(fi-)retractable* if for any non-zero (fully invariant) submodule K of M and $S = \text{End}_R(M)$, $\text{Hom}_R(M, K) \neq 0$. Dually, M is called *(fi-)coretractable* if for any proper (fully invariant) submodule K of M, $\pi_K \diamond \text{Hom}_R(M/K, M) \neq 0$, where $\pi_K : M \to M/K$ the canonical projection.

1 Fully prime modules

A module M is called *fully prime* if for any non-zero fully invariant submodule K of M, M is K-cogenerated.

Some characterizations are given in Proposition 2.3 of [?] and we have similar characterizations here.

1.1 Fully prime modules. *The following are equivalent for an R-module M :* [Primbic1]

(a) *M is a fully prime module.*

(b) $\text{Rej}(M, K) = 0$ *for any non-zero fully-invariant submodule $K \subset M$.*

(c) $K *_M L \neq 0$ *for any non-zero fully-invariant submodules $K, L \subset M$.*

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(d) $\text{Rej}(-, M) = \text{Rej}(-, K)$ *for any non-zero fully-invariant submodule K of M, i.e., any M-cogenerated module is also K-cogenerated.*

Proof. (a) $\Longleftrightarrow$ (b) $\Longleftrightarrow$ (d) are obvious by the definition of cogenerating.

(b) $\Longrightarrow$ (c) For any nonzero fully-invariant submodules $K, L \subset M$ we assume that $K\text{Hom}_R(M, L) = 0$. Then $0 \neq K \subset \text{Rej}(M, L)$.

(c) $\Longrightarrow$ (b) Assume $\text{Rej}(M, K) = U \neq 0$ for some non-zero fully-invariant submodule $K \subset M$. Then $U\text{Hom}_R(M, K) = U *_M K = 0$. □

For any fully invariant submodules K, L of M, consider the product

$$K *_M L := K\text{Hom}_R(M, L).$$

Based on the $*_M$-product we define fully prime submodules. A fully invariant submodule N of M is *fully prime* in M if for any fully invariant submodules K, L of M, the relation $K *_M L \subseteq N$ implies $K \subseteq N$ or $L \subseteq N$. Thus the module M is fully prime if the zero submodule is fully prime in M.

Proposition 18 of [?] provides a relationship between a fully prime submodule N of M and the factor module M/N. As a special case, consider R as a left R-module and let I, J be ideals of R. Then $I *_R J = IJ$. Since every ideal of R is a fully invariant R-submodule, we get :

1.2 Proposition. *The following are equivalent for a two-sided ideal I :*

(a) *R/I is a prime ring.*

(b) *I is a fully prime submodule in R.*

(c) *I is a prime ideal.*

In general prime modules need not be fully prime. For the following relationship we adopt the proof of [?], Proposition 13.2.

1.3 Proposition. *For an R-module M with $(*fi)$, the following are equivalent :*

(a) *M is prime and fi-retractable.*

(b) *M is fully prime.*

Notice that for any ring R, $\text{End}_R(R) \simeq R$ and as a left R-module, R satisfies $(*fi)$ and is fi-retractable. If $M = R$, Proposition ?? yields

1.4 Corollary. *For the ring R the following assertions are equivalent :*

(a) *R is a prime ring.*

(b) *$_RR$ is a prime module.*

(c) *$_RR$ is a fully prime module.*

1.5 Proposition. *Let M be a module with $\text{Soc}(M) \neq 0$. If M is fully prime, then*

(i) *M is cogenerated by a simple module.*

(ii) *$\overline{R} := R/\text{Ann}_R(M)$ is a left primitive ring.*

Proof. (i) Let K be a simple submodule of M. Then $\text{Tr}(K, M)$ is a fully invariant submodule and hence M is $\text{Tr}(K, M)$-cogenerated. $\text{Tr}(K, M)$ is K-cogenerated, and hence M is K-cogenerated.

(ii) $\overline{R}$ is cogenerated by M and hence by the simple module K (from (i)). □

2 Fully coprime modules

A module M is called *fully coprime* if for any proper fully invariant submodule K of M, M is M/K-generated. An inner coproduct of fully invariant submodules of M can be defined in the following way. For any fully invariant submodules $K, L \subset M$, put

$$\begin{aligned} K :_M L &:= \bigcap\{(L)f^{-1} \mid f \in \text{End}_R(M), K \subseteq \text{Ker} f\} \\ &= \text{Ker } \pi_K \diamond \text{Hom}_R(M/K, M) \diamond \pi_L, \end{aligned}$$

where $\pi_K : M \to M/K$ and $\pi_L : M \to M/L$ denote the canonical projections. $K :_M L$ is also a fully invariant submodule. Notice that such a coproduct is considered in Bican et.al. [?] for any pair of submodules $K, L \subset M$ (not necessary fully invariant) and then a definition of "coprime modules" is derived from this coproduct.

We characterize fully coprime modules in the proposition below. This is similar to Proposition 4.3 of [?] but here we consider proper fully invariant submodules.

2.1 Fully coprime modules. *The following are equivalent for an R-module M :*

(a) *M is a fully coprime module.*

(b) *If $K :_M L = M$, then $K = M$ or $L = M$, for any fully invariant submodules K, L of M.*

(c) *$K :_M L \neq M$ for any proper fully invariant submodules K, L of M;*

(d) *$\text{Tr}(M/K, -) = \text{Tr}(M, -)$ for any proper fully invariant submodules K of M, i.e. any M-generated module is also M/K-generated.*

Proof. (a) $\Longleftrightarrow$ (d) and (b) $\Longleftrightarrow$ (c) are trivial.

(c) $\Longrightarrow$ (a) Let $K \subset M$ be a proper fully invariant submodule such that

$$N = \text{Tr}(M/K, M) = (M)\ \pi_K \diamond \text{Hom}_R(M/K, M) \neq M.$$

Then $0 = (M)\ \pi_K \diamond \text{Hom}_R(M/K, M) \diamond \pi_N$ and $K :_M N = M$.

(d) $\Longrightarrow$ (c) Let K, L be any proper fully invariant submodules of M and assume $(M)\ \pi_K \diamond \mathrm{Hom}_R(M/K, M) \diamond \pi_L = 0$. Then

$$M = \mathrm{Tr}(M, M) = \mathrm{Tr}(M/K, M) \subset L. \quad \Box$$

2.2 Fully coprime rings. *For the ring R the following are equivalent :*

(a) *${}_RR$ is coprime.*

(b) *${}_RR$ is fully coprime.*

(c) *R is a simple ring.*

2.3 Lemma. *Let M be fully coprime, $S = \mathrm{End}_R(M)$. Then M is indecomposable as (R, S)-bimodule.*

Proof. Assume $M = U \oplus V$ where U, V are (R, S)-subbimodules of M. Then $\mathrm{Hom}_R(U, V) = 0$. Since M is fully coprime, M is generated by $M/U \simeq V$. It means V also generates U, thus contradicts $\mathrm{Hom}_R(U, V) = 0$. $\Box$

2.4 Corollary. *Let M be a fully coprime module. If M is semilocal, then M is homogeneous semisimple.*

Proof. $\mathrm{Rad}(M)$ is a fully invariant submodule of M, hence M is generated by $M/\mathrm{Rad}(M)$ which is semisimple. Thus M is semisimple and now apply Lemma ??. $\Box$

A fully invariant submodule $N \subset M$ is called *fully coprime in M* if for any fully invariant submodules $K, L \subset M$, $N \subseteq K :_M L$ implies $N \subset K$ or $N \subset L$. By ??, M is fully coprime if and only if M is fully coprime in M. An immediate consequence of the definition is (compare with Proposition ??)

2.5 Proposition. *If a module M is fully coprime, then M is coprime and fi-coretractable.*

In view of later use for comodules and coalgebras (wedge product), we consider another coproduct of two proper fully invariant submodules $K, L \subset M$. Put

$$\begin{aligned} K \wedge^M L &:= \mathrm{Ker}\ \pi_K \diamond \mathrm{Hom}_R(M/K, M) \diamond \pi_L \diamond \mathrm{Hom}_R(M/L, M) \\ &= \mathrm{Ker}\ (\mathrm{Ann}_S(K) \diamond \mathrm{Ann}_S(L)), \end{aligned}$$

a fully invariant submodule of M. Obviously, $K :_M L \subseteq K \wedge^M L$. If M is a self-cogenerator, then the equality holds.

2.6 Proposition. *Let M be a self-cogenerator and $S = \mathrm{End}_R(M)$. If S is prime, then M is fully coprime.*

Proof. Let K, L be proper fully invariant submodules of M and $M = K :_M L$. Then $(M)\ \pi_K \diamond \mathrm{Hom}_R(M/K, M) \diamond \pi_L \diamond \mathrm{Hom}_R(M/L, M) = 0$. Since S is prime, $\pi_K \diamond \mathrm{Hom}_R(M/K, M) = 0$ or $\pi_L \diamond \mathrm{Hom}_R(M/L, M) = 0$. Hence $K = M$ or $L = M$ since M is a self-cogenerator. □

Let I, J be ideals in $\mathrm{End}_R(M)$ and put Ker $I = K$, Ker $J = L$. Then

$$I \subseteq \mathrm{Hom}_R(M/K, M), \quad J \subseteq \mathrm{Hom}_R(M/L, M). \tag{1}$$

For the converse of Proposition ?? the equalities in (??) are of interest.

2.7 Proposition. *Let M be a self-cogenerator and $S = \mathrm{End}_R(M)$.*

(i) *If M is self-injective and fully coprime, then S is prime and M is coprime as a right S-module.*

(ii) *If M is coprime as a right S-module, then M is fully coprime.*

Proof. (i) Let M be a fully coprime module and I, J finitely generated right ideals in S with $IJ = 0$. Put $K = \mathrm{Ker}\ I$ and $L = \mathrm{Ker}\ J$. Then by Lemma ??, $\mathrm{Hom}(M/K, M) = I$ and $\mathrm{Hom}(M/L, M) = J$ and $K :_M L = M$. Then $M = K$ or $M = L$, thus $I = 0$ or $J = 0$. Hence the ring S is prime. Since $_RM$ is fi-coretractable, M is coprime as a right S-module.

(ii) We assume that M_S is coprime, hence S is prime. Then the assertion follows from Proposition ??. □

2.8 Corollary. *If M is a self-injective self-cogenerator and $S = \mathrm{End}_R(M)$, then the following assertions are equivalent :*

(a) *M is fully coprime.*

(b) *M is coprime as a right S-module.*

(c) *S is a prime ring.*

Proof. The equivalence holds by Proposition ?? and Proposition ??. □

2.9 Proposition. *Let M be a fully coprime module with* $\mathrm{Rad}(M) \neq M$. *Then :*

(i) *M is generated by a module that is cogenerated by a simple module.*

(ii) *For any projective module P in $\sigma[M]$,* $\mathrm{Rad}(P) = 0$.

(iii) $\overline{R} := R/\mathrm{Ann}_R(M)$ *is a left primitive ring.*

Proof. (i) By assumption there is a maximal submodule K in M. Consider the fully invariant submodule $\mathrm{Rej}(M, M/K) \subset K \neq M$. By assumption M is $M/\mathrm{Rej}(M, M/K)$-generated, where $M/\mathrm{Rej}(M, M/K)$ is cogenerated by the simple module M/K.

(ii) By (i), P is subgenerated by $M/\mathrm{Rej}(M, M/K)$ which is cogenerated by M/K. Hence P is subgenerated by a product Q of copies of (M/K), and $P \subset Q^{(\Lambda)}$, for some index Λ (see 18.4 of [?]). Thus P is M/K-cogenerated and $\mathrm{Rad}(P) = 0$.

(iii) It is a consequence of Proposition ??, since M fully coprime implies that M is coprime. □

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