Generalized Non-Homogeneous Morrey Spaces And Olsen Inequality

I. Sihwaningrum*), H. Gunawan, Y. Soeharyadi, W. S. Budhi

Analysis and Geometry Group
Faculty of Mathematics and Natural Sciences
Bandung Institute of Technology, Bandung 40132, Indonesia
hanidha@students.itb.ac.id, hgunawan, yudish, wono@math.itb.ac.id

Abstract

In this paper, we shall discuss some properties of generalized non-homogeneous Morrey spaces. In addition, we will also prove the Olsen inequality in the non-homogeneous setting. Our proof utilizes the result of (García-Cuerva and Martell, 2001) on the boundedness of the fractional integral operator on Lebesgue spaces of non-homogeneous type.

Keywords: *Olsen inequality, fractional integral operator, non-homogeneous Lebesgue spaces, generalized non-homogeneous Morrey spaces.*

1. Introduction

We shall study here the fractional integral operator I_α^n (for $0 < \alpha < n \le d$), on non-homogeneous spaces, which is defined by the formula

$$I_\alpha^n f(x) := \int_{\mathbf{R}^d} \frac{f(y)}{|x-y|^{n-\alpha}} d\mu(y) .$$

The formula reduced to the classical version of (Hardy and Littlewood, 1927; Hardy and Littlewood, 1932; and Sobolev, 1938) when $n = d$ and μ is the usual Lebesgue measure. By a non-homogeneous space we mean a metric space -- here we will consider only the Euclidean space $\mathbf{R}^d$ -- equipped with an *n*-dimensional measure (García-Cuerva and Martell, 2000). A positive Borel measure μ satisfies *n*-dimensional measure (for $0 < n \le d$) if there exists a constant $C > 0$ such that

$$\mu(B(a,r)) \le Cr^n$$

*) Permanent address: Department of Mathematics General Soedirman University, Purwokerto 53122.

for every open ball $B(a,r)$ centered at $a \in \mathbf{R}^d$ with radius $r > 0$ (García-Cuerva and Gatto, 2004). This condition -- also known as the *growth condition* of order *n* (Sawano, 2005) -- replaces the *doubling condition*, which is the key property for a metric space to be a homogeneous space. Notice that a positive Borel measure μ satisfies the *doubling condition* if there exists a constant $C > 0$ such that for every ball $B(a,r)$ we have

$$\mu(B(a,2r)) \le C\mu(B(a,r))$$

(Coifman and Gusmán, 1970/1971). The ball $B(a,2r)$ is concentric to $B(a,r)$ with radius *2r*. We may consult (Krantz, 1999) for examples of the spaces of homogeneous type and (Verderra, 2002) for that of non-homogeneous type.

Now, let $L^p(\mu) = L^p(\mathbf{R}^d, \mu)$, $1 \le p < \infty$, denote the non-homogeneous Lebesgue spaces. It is well known from (García-Cuerva and Martell, 2001) that I_α^n is a bounded operator from $L^p(\mu)$ to $L^q(\mu)$ for $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Further, the following Olsen inequality

$$\left\| WI_\alpha f : L^p(\mu) \right\| \le C \left\| W : L^{n/\alpha}(\mu) \right\| \left\| f : L^p(\mu) \right\|,$$

for $W \in L^{n/\alpha}(\mu)$, can be viewed as a consequence of the $L^p(\mu) - L^q(\mu)$ boundedness of I_α^n (Sihwaningrum, *et.al.*, 2008b). The inequality was first introduced -- in homogeneous setting -- by (Olsen, 1995) to study the solution of the Schrödinger equation with a small perturbed potential *W* on Morrey spaces. Later on, (Kurata *et al*., 2002; Gunawan and Eridani, 2008) extended the Olsen's result to the homogeneous generalized Morrey spaces. In this paper, we will extend further the Olsen's result to the generalized Morrey spaces of non-homogeneous type.

2. Main Results

2.1 Generalized non-homogeneous Morrey spaces

For $1 \le p < \infty$ and $\phi : (0,\infty) \to (0,\infty)$, let us define the generalized non-homogeneous Morrey spaces $\mathrm{M}^{p,\phi}(\mu) = \mathrm{M}^{p,\phi}(\mathbf{R}^d, \mu)$ to be the set of all functions $f \in L^p_{loc}(\mu)$ for which $\left\| f : \mathrm{M}^{p,\phi}(\mu) \right\| < \infty$. Here,

$$\left\| f : M^{p,\varphi}(\mu) \right\| := \sup_{r>0} \frac{1}{\phi(r)} \left(\frac{1}{r^n} \int_{B(x,r)} |f(y)|^p \, d\mu \right)^{\frac{1}{p}}, \qquad 1 \le p < \infty .$$

Our definition is in line with the definition of Hardy-L ittlewood maximal operator M^n given by the formula

$$M^n f(x) := \sup_{r>0} \frac{1}{r^n} \int_{B(a,r))} |f(y)| \, d\mu(y) .$$

The reader may also refer (Gunawan, *et.al.,* 2007) and (Sawano, 2008) for other types of generalized non-homogeneous Morrey spaces – which are defined in accordance with the *k*-dilated Hardy-Littlewood maximal operator M_k :

$$M_k f(x) := \sup_{Q \ni x} \frac{1}{\mu(kQ)} \int_Q f(y) \, d\mu(y)$$

Note that along with our definition, if $\phi(r) = r^{-n/p}$, we obtain $M^{p,\phi}(\mu) = L^p(\mu)$. Meanwhile, for $1 < p < q < \infty$, we have $M^{q,\phi}(\mu) \subseteq M^{p,\phi}(\mu) \subseteq M^{1,\phi}(\mu)$ (see (Sihwaningrum, *et. al.,* 2008a) for the proof). Furthermore, the generalized non-homogeneous Morrey spaces obey the following property.

Fact 2.1 *If* $1 \le p < \infty$ *and* $\phi(r) \le C\psi(r)$ *(for* $r > 0$*), then* $M^{p,\psi}(\mu) \subseteq M^{p,\phi}(\mu)$ *and* $\left\| f : M^{p,\psi}(\mu) \right\| \le C \left\| f : M^{p,\phi}(\mu) \right\|$.

Proof. Notice first that $\frac{1}{\psi(r)} \le C \frac{1}{\phi(r)}$ for all $r > 0$. Then, for $f \in M^{p,\phi}(\mu)$, we get

$$\sup_{r>0} \frac{1}{\psi(r)} \left(\frac{1}{r^n} \int_{B(x,r)} |f(y)|^p \, d\mu \right)^{\frac{1}{p}} \le C \sup_{r>0} \frac{1}{\phi(r)} \left(\frac{1}{r^n} \int_{B(x,r)} |f(y)|^p \, d\mu \right)^{\frac{1}{p}} < \infty .$$

The above inequality implies that $\left\| f : M^{p,\psi}(\mu) \right\| \le C \left\| f : M^{p,\phi}(\mu) \right\|$, and so does $M^{p,\psi}(\mu) \subseteq M^{p,\phi}(\mu)$. □

As a consequence, $M^{p,\psi}(\mu) = M^{p,\phi}(\mu)$ and $\left\| f : M^{p,\psi}(\mu) \right\| \sim \left\| f : M^{p,\phi}(\mu) \right\|$ for $\phi \sim \psi$.

2.2 Olsen Inequality

As an extention of the Olsen's result, we will present here an Olsen inequality on generalized non-homogeneous Morrey spaces. The inequality simply says that a multiplication of operators W and I_α^n is bounded on $M^{p,\phi}(\mu)$. To proof the inequality, we use the the boundedness of I_α^n from $L^p(\mu)$ to $L^q(\mu)$.

Theorem 2.2. *Suppose that ϕ satisfies the doubling condition for function, that is there exists a constant C such that $\frac{1}{2} \le \frac{t}{s} \le 2 \Rightarrow \frac{1}{C} \le \frac{\phi(t)}{\phi(s)} \le C$. Suppose further that ϕ satisfies $\int_r^\infty t^{\alpha-1}\phi(t)dt \le Cr^\alpha \phi(r)$. Then, the inequality*

$$\|WI_\alpha f : M^{p,\phi}(\mu)\| \le C \left\|W : L^{n/\alpha}(\mu)\right\| \left\|f : M^{p,\phi}(\mu)\right\|$$

holds provided that $W \in L^{n/\alpha}$.

Proof. Let $B := B(a,r)$ dan $\hat{B} := B(a,2r)$ where $a \in \mathbf{R}^d$. Then, we decompose the function $f \in M^{p,\phi}(\mu)$ as $f = f_1 + f_2 = f_{\chi_{\hat{B}}} + f_{\chi_{\hat{B}^C}}$. Recall that I_α^n is a bounded operator from $L^p(\mu)$ to $L^q(\mu)$, so that we have

$$\left(\frac{1}{r^n}\int_B \left|I_\alpha^n f_1(x)\right|^q d\mu(x)\right)^{1/q} \le C\psi(r)\left\|f : M^{p,\phi}(\mu)\right\|.$$

Now, by using the Hölder inequality and $L^p(\mu)$-$L^q(\mu)$ boundedness of I_α^n, we obtain

$$\left(\int_B \left|WI_\alpha^n f_1(y)\right|^p d\mu(y)\right)^{1/p} \le Cr^{n/p}\phi(r)\left\|W : L^{n/\alpha}(\mu)\right\|\left\|f : M^{p,\phi}(\mu)\right\| . \qquad (2.1)$$

As the measure μ satisfies the growth condition, then for every $x \in B$ we could find the estimate

$$\left|I_\alpha^n f_2(x)\right| \le \int_{|x-y|\ge r} \frac{|f(y)|}{|x-y|^{n-\alpha}} d\mu(y) \le C\left\|f : M^{p,\phi}(\mu)\right\|\sum_{k=0}^{\infty}\left(2^k r\right)^\alpha \phi\left(2^k r\right).$$

We see that the right hand side of the inequality contains the summation from $k=0$ to $k=\infty$. So, we utilize the doubling condition of $\phi(t)$ and t^{α} to get

$$\left(2^k r\right)^{\alpha} \phi\left(2^k r\right) \le C \int_{2^{k-1}}^{2^k} t^{\alpha-1}\phi(t)dt$$

for $k = 0,1,2,\ldots$. As a result

$$\left|I_{\alpha}^{n} f_2(x)\right| \le C\left\| f : M^{p,\phi}(\mu)\right\| \sum_{k=0}^{\infty} \int_{2^{k-1}}^{2^k} t^{\alpha-1}\phi(t)dt \le Cr^{\alpha}\phi(r)\left\| f : M^{p,\phi}(\mu)\right\|.$$

Furthermore, the Hölder inequality allows us to obtain

$$\left(\frac{1}{r^n}\int_B \left|WI_{\alpha}^{n} f_1(y)\right|^p d\mu(y)\right)^{1/p} \le Cr^{n/p}\phi(r)\left\|W : L^{n/\alpha}(\mu)\right\| \left\| f : M^{p,\phi}(\mu)\right\| . \qquad (2.2)$$

Now, we apply the Minkowski inequality to the estimate (2.1) and (2.2) to get

$$\begin{aligned}
&\frac{1}{\phi(r)}\left(\frac{1}{r^n}\int_B \left|WI_{\alpha}^{n} f(y)\right|^p d\mu(y)\right)^{1/p} \\
&\le \frac{1}{\phi(r)}\left(\frac{1}{r^n}\int_B \left|WI_{\alpha}^{n} f_1(y)\right|^p d\mu(y)\right)^{1/p} + \frac{1}{\phi(r)}\left(\frac{1}{r^n}\int_B \left|WI_{\alpha}^{n} f_2(y)\right|^p d\mu(y)\right)^{1/p} \\
&\le C\left\|W : L^{n/\alpha}(\mu)\right\| \left\| f : M^{p,\phi}(\mu)\right\|
\end{aligned}$$

By taking the supremum over all $r>0$, we complete our proof. □

3. Concluding Remarks

We may also proof Theorem 2.2 by using the $M^{p,\phi}(\mu)$ - $M^{q,\psi}(\mu)$ boundedness of I_{α}^{n}, where $1 < p < q < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $r^{\alpha}\phi(r) \le C\psi(r)$. The alternative proof is simpler than that of presented here (see (Sihwaningrum, *et. al.,* 2008b) for detail) .

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4. References

Coifman, R.R. and M. de Gusmán, 1970/1971, Singular Integrals and Multiplier on Homogeneous Spaces, *Rev. Un. Mat. Argentina,* **25**, 137–143.

García-Cuerva, J. and A.E. Gatto, 2004, Boundedness Properties of Fractional Integral Operators Associated to Non-Doubling Measures, *Studia Math.,* **162,** no. 3, 245–261.

García-Cuerva, J. and J.M. Martell, 2000, Weighted Inequalities and Vector-Valued Calderón-Zygmund Operators on Non-homogeneous Spaces, *Public. Math. ,* **44**, no. 2, 613–640.

García-Cuerva, J. and J.M. Martell, 2001, Two-weight Norm Inequalities for Maximal Operators and Fractional Integrals on Non-homogeneous Spaces, *Indiana Univ. Math. J.,* **50**, no. 3, 1241–1280.

Gunawan, H. and Eridani, 2008, Fractional Integrals and Generalized Olsen Inequalities, to appear in *Kyungpook Math. J.*

Gunawan, H. and I. Sihwaningrum, 2007, Fractional Integral Operator and Their Boundedness on Various Spaces, *Jurnal Matematika dan Sains,* **12:4**, 118–125.

Krantz, S.G., 1999, *A Panorama of Harmonic Analysis*, The Carus Mathematical Monographs, no. 27, The Mathematical Association of America, USA.

Kurata, K., S. Nishigaki and S. Sugano, 2002, Boundedness of Integral Operators on Generalized Morrey Spaces and Its Application to Schrödinger Operatos, *Proc. Amer. Math. Soc*., **129**, 1125–1134.

Nazarov, F., S. Treil and A. Volberg, 1998, Weak Type Estimates and Cotlar Inequalities for Calderón-Zygmund Operators on Nonhomogeneous Space, *Internat. Math. Res. Notices*, **9**, 463–487.

Olsen, P.A., 1995, Fractional Integration, Morrey Spaces and a Schrödinger Equation, *Comm. Partial Differential Equations,* **20**, 2005–2055.

Sawano, Y., 2005, Sharp Estimates of the Modified Hardy Littlewood Maximal Operator on the Non-homogeneous Space via Covering Lemmas, *Hokkaido Math. J.,* **34**, 435–458.

Sawano, Y., 2008, Generalized Morrey Spaces for Non-doubling Measures, to appear in *NoDEA Nonlinear Differential Equation Appl.*

Sihwaningrum, I., H. Gunawan dan W. S. Budhi, 2008a, Operator Integral Fraksional dan Ketaksamaan Olsen di Ruang Morrey Tak Homogen yang Diperumum, *Prosiding Seminar Nasional Mahasiswa S3 Matematika se-Indonesia,* 31 Mei 2008.

Sihwaningrum, I., H. P. Suryawan, and H. Gunawan., 2008b, Fractional Integral Operators and Olsen Inequalities on Non-homogeneous Spaces, to appear in *Aust. J. Math. Anal. Appl.*

Sobolev, S.L., 1938, On a Theorem in Fuctional Analysis (Russian), *Mat. Sob.*, **46**, 471–497. [English tanslation in *Amer. Math. Soc. Transl. ser. 2*, **34** (1963), 39–68].

Verdera, J., 2002, The Fall of the Doubling Condition in Calderón-Zygmund Theory, *Pub. Mat.,* 275–292.