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Testing gravity at cosmological scales

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Abstract

Our understanding of the Universe is based on the ΛCDM model which, although the best cosmological model so far, relies on the presence of major unknown components – dark matter, dark energy, and an inflationary field – which in turn play a crucial role in the evolution of the Universe. These limitations of the model suggest that we may need to introduce modifications at cosmological scales. Indeed, a large variety of modified gravity theories have been proposed (see [1] for a review) and in order to better understand the behaviour of gravity in this regime, we must begin by constructing theoretical and observational tests of the ΛCDM model and the various alternative proposals.

This thesis is concerned with testing gravity on cosmological scales, by analysing the viability of alternative gravitational theories, and scrutinising their theoretical consistency. In order to do this, we take two approaches. On the one hand, we explore the viability of a specific modified gravity theory, namely massive bigravity. The evolution of a perfectly homogeneous and isotropic Universe has been previously studied in detail in this model, and has been found to fit observational data. Hence, in this thesis we analyse the evolution of linear cosmological perturbations, where we find a number of interesting instabilities. On the other hand, we take a broader view and develop a method for parametrising linear cosmological perturbations that stays agnostic about the underlying theory of gravity. We apply this method to three classes of models: scalar-tensor, vector-tensor and bimetric theories, and as a result, in this case, we identify the complete forms of the quadratic actions for linear perturbations, and the number of free parameters that need to be defined, to cosmologically characterise these broad classes of gravity theories.

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Declaration

This thesis is mainly based on original research, and contains all references to previous works when appropriate. Specifically, Chapters 3 and 6 are based on work done in collaboration with Pedro G. Ferreira, which were published in JCAP (2014) [2] and (2017) [3], respectively. In addition, Chapters 4 and 5 are based on one paper done in collaboration with Pedro G. Ferreira, Tessa Baker and Johannes Noller, which was published in JCAP (2016) [4].

Contents

1 Introduction **11**

1.1 Notations and conventions . . . 13

1.2 General relativity . . . 14

1.3 Modern cosmology . . . 18

1.3.1 Background . . . 18

1.3.2 Perturbations . . . 21

1.4 Modified gravity . . . 26

2 Massive bigravity **31**

2.1 Linear massive gravity . . . 32

2.2 Non-linear massive gravity . . . 36

3 Cosmological perturbations in massive bigravity **42**

3.1 Cosmological perturbations . . . 43

3.1.1 Background . . . 43

3.1.2 Scalar perturbations . . . 46

3.1.3 Vector perturbations . . . 49

3.1.4 Tensor perturbations . . . 51

3.2 Scalar perturbations . . . 53

3.2.1 Expanding branch . . . 53

3.2.2 Bouncing branch . . . 57

3.3 Vector perturbations . . . 61

3.3.1 Expanding branch . . . 62

3.3.2 Bouncing branch . . . 64

3.4 Tensor perturbations . . . 67

3.4.1 Expanding branch . . . 67

3.4.2 Bouncing branch . . . 70

3.5 Discussion . . . 73

4 Tools for testing gravity: Noether identities **78**

4.1 The method: Noether identities and constraints . . . 81

4.1.1 Introductory example . . . 82

4.1.2 Gravitational action . . . 83

4.2 Recovering general relativity . . . 90

4.3 Discussion . . . 97

5 A general theory of linear cosmological perturbations: scalar-tensor and vector-tensor theories **99**

5.1 Recovering linearised Beyond Horndeski theory and beyond . . . 100

5.1.1 Horndeski and beyond . . . 101

5.1.2 Fourth-order extensions . . . 105

5.2 Vector-tensor theories . . . 107

5.2.1 General case . . . 107

5.2.2 Einstein-Aether theory . . . 111

5.3 Discussion . . . 114

6 A general theory of linear cosmological perturbations: bimetric theories **116**

6.1 Parametrising bimetric theories . . . 117

6.2 A reduced case: excluding derivative interactions . . . 127

6.3 Discussion . . . 133

7 Conclusions **135**

A Introduction **158**

A.1 Gauge transformations . . . 158

B Cosmological perturbations in massive bigravity **161**

B.1 Scalar perturbation equations . . . 161

B.1.1 Auxiliary variables . . . 161

B.1.2 Complete equations of motion . . . 163

B.1.3 Exponential instabilities . . . 165

B.1.4 Density contrast . . . 166

B.2 Ghost-like instabilities . . . 167

B.2.1 Scalar fields . . . 168
B.2.2 Vector fields . . . 170

C Tools for testing gravity: Noether identities **172**
C.1 3+1 decomposition . . . 172
C.2 Scalar perturbations . . . 173
C.3 New coefficients . . . 174
C.4 Background equations . . . 175

D A general theory of linear cosmological perturbations: scalar-tensor and vector-tensor theories **178**
D.1 New coefficients for scalar-tensor action . . . 179
D.2 Fourth order action for scalar-tensor theories . . . 180
D.3 New coefficients for vector-tensor action . . . 180

E A general theory of linear cosmological perturbations: bimetric theories **182**
E.1 Scalar perturbations . . . 182
E.2 Dictionary of parameters . . . 183
E.3 Complete action without derivative interactions . . . 185
E.4 The effect of derivative interactions . . . 187

List of Figures

3.1 Evolution of E_1 and E_2 as a function of τ for a given sub-horizon scales at early times during the radiation-dominated era. We have set $\beta_1 m^2 = 10^{-2}$ and the other βs vanishing. 55

3.2 Evolution of E_1 and E_2 as a function of τ in the radiation-dominated era for a given scale sub-horizon k. At later times, both perturbation fields are growing exponentially fast, becoming several orders of magnitude larger than their early time value. 56

3.3 Evolution of scalar perturbations as a function of the conformal time during the de-Sitter phase at late times in the matter-dominated era. 57

3.4 Evolution of scalar perturbations as a function of τ, during early times in the radiation-dominated for a given sub-horizon scales. 59

3.5 Evolution of scalar perturbations as a function of τ, during early times in the matter-dominated era for a given sub-horizon scale. 60

3.6 Evolution of scalar perturbations during early times in the matter-dominated for a given sub-horizon scale. 60

3.7 Evolution of density constrast as a function of the conformal time τ in the matter-dominated era for a given sub-horizon scale. 61

3.8 Evolution of vector perturbations as a function of τ, during early times in the radiation-dominated era for a given sub-horizon scale. 63

3.9 Evolution of vector perturbations during late times in the de-Sitter phase for a sub-horizon scale. 64

3.10 Evolution of vector perturbations as a function of τ, during early times in the radiation-dominated era for a sub-horizon scale. 65

3.11 Evolution of vector perturbations as a function of τ, during early times in matter-dominated era for a given sub-horizon scale. 67

3.12 Evolution of tensor perturbations as a function of the conformal time during early times in the radiation-dominated era for a given sub-horizon scale. 68
3.13 Evolution of tensor perturbations as a function of the conformal time during the de-Sitter phase at late times in the matter-dominated era. 70
3.14 Evolution of tensor perturbations as a function of the conformal time, during early times in the radiation-dominated era for a given sub-horizon scale. 72
3.15 Evolution of tensor perturbations during early times in the matter-dominated era. 73

List of Tables

5.1 Summary parametrised actions: scalar-tensor and vector-tensor theories 114

6.1 Summary parametrised action: bimetric theories 134

7.1 Conclusions: summary parametrised actions 137

CHAPTER 1

Introduction

Einstein's general theory of relativity (GR) is the foundation of gravity. It is widely accepted to be the model that describes the behaviour of gravity on small and large scales, in events such as planetary motion, black hole mergers, galaxy formation and the evolution of our Universe. On Solar System scales, not only do its predictions show remarkable agreement with astrophysical data, but precise measurements of phenomena such as light deflection around the Sun, the perihelion shift of Mercury, and others, rule out many modifications to GR [5, 6]. In addition, GR has recently been shown to be consistent with the first direct detection of gravitational waves from the merger of two black holes [7]. Nonetheless, GR exhibits weaknesses both at very high and very low energy regimes. At high energies, unavoidable singularities arise during gravitational collapses and the so-called renormalisation problem limits our understanding of quantum gravity [8–10]. On cosmological scales, GR relies on the presence of exotic unknown matter components in order to fit observational data. These issues show the current limitations in our understanding of how gravity behaves and interacts with matter in extreme energy regimes. As a result, a number of modified gravity theories have been proposed [1, 11, 12], and it is imperative that we analyse their consistency and viability, and develop tools to construct theoretical and observational tests of GR.

Cosmology seems to be a particularly promising regime for testing gravity. We are currently reaching an era of "precision cosmology", in which measurements of the anisotropy of the Cosmic Microwave Background (CMB), of the large-scale structure in galaxy distribution, and of the overall expansion of the Universe will reach unprecedented precision. Even though the CMB plays an essential role in constraining gravity on cosmological scales, most of the focus is currently on galaxy surveys and weak lensing data. In this context, over the next three years we will see a dramatic improvement in cosmological observations from ongoing experiments such as the Dark Energy Survey (DES) [13], the extended Baryon Oscillation

Spectroscopic Survey (eBOSS) [14, 15], Mapping Nearby Galaxies at APO (MaNGA) [16], and the Hobby-Eberly Telescope Dark Energy Experiment (HETDEX) [17]. Furthermore, in the next decade a next generation of surveys will come online, including the Euclid satellite [18], the Dark Energy Spectroscopic Instrument (DESI) [19], the Large Synoptic Survey Telescope (LSST) [20], the Square Kilometre Array (SKA) [21], and the Wide-Field InfrarRed Survey Telescope (WFIRST) [22], which will provide an opportunity to perform ultimate tests of gravity on the largest scales in our Universe. The time is ripe then to pin down some of the fundamental properties of nature on large scales.

The standard cosmological model is the ΛCDM model, which is based on GR and is the most successful cosmological model proposed so far. However, the ΛCDM model relies on the presence of crucial unknown components for the evolution of the Universe, namely dark matter, dark energy, and an inflationary field. Dark matter is essential for galaxy formation, dark energy is responsible for the late-time accelerated expansion of the Universe, and the inflationary field is responsible for the early-time expansion that led to the highly homogeneous Universe that we see today on large scales. The exotic properties of these components raise a number of questions that we need to answer to understand the physics dominating large scales: what is the origin and nature of these components? Does the gravitational force behave differently on large scales? Are there new forces coming into play on large scales? In particular, the origin of the late-time accelerated expansion of the Universe has become one of the most challenging problems in theoretical physics. In the ΛCDM model, dark energy corresponds to 69% of all the energy content of the Universe, and is described by a single constant, known as the cosmological constant. In order to explain the observed acceleration of the Universe, the value of this constant must be incredibly small. From particle physics arguments we can predict the existence of vacuum energy, which provides a value for the cosmological constant, but current estimates imply that it is more than 50 orders of magnitude larger than the observed value. This constitutes the so-called *cosmological constant problem* and shows how the ΛCDM model is at odds with well-established and robust particle physics theory.

This thesis is concerned with testing gravity on cosmological scales, by analysing possible modifications to GR and the ΛCDM model. In order to do this, we consider two approaches. On the one hand, we look at a specific theoretically well-motivated gravity theory, and analyse its viability. Specifically, we analyse the cosmological predictions of a theory known as massive bigravity, which can explain the late-time accelerated expansion of the Universe in a natural way and agree with background experimental data, without introducing a cosmological constant. We focus on the evolution of linear cosmological perturbations, where we find the presence of instabilities on the evolution of gravitational waves that could seriously jeopardise the viability of massive bigravity. Furthermore, we find that even though the prediction for the rate of formation of large-scale structures, such as clusters of galaxies, might agree with observations, one of the physical degrees of freedom driving this evolution has negative kinetic energy, rendering the model likely to be unstable in other regimes.

On the other hand, we stay agnostic regarding the particular underlying gravity theory, and analyse the consistency and phenomenology of cosmological models that can help test gravity and falsify GR. In this context, we develop a method for constructing parametrised cosmological models, in such a way that specific values for the parameters give a description of specific gravity theories. This method allows us to describe a broad range of theories on cosmological scales in a unified and efficient manner. The method is sufficiently general that we can identify the number of cosmologically relevant free parameters characterising families of theories, which can in turn ultimately be constrained with future observational data, and hence be used to find phenomenologically viable cosmological models.

This thesis is organised as follows. In the rest of this chapter we review the theory of general relativity as well as the ΛCDM cosmological model. In addition, we discuss some generic characteristics of modified gravity theories. In Chapter 2 we introduce and describe massive bigravity, and in Chapter 3 we examine its predictions on the evolution of cosmological perturbations, and show that a number of instabilities appear. In Chapter 4 we take a broader view and explain a method for constructing parametrised cosmological models that is able to encompass a large class of gravity theories, and we recover GR as an illustrative example of the method. In Chapters 5 and 6 we apply the previously developed method and consider three families of gravity theories: scalar-tensor, vector-tensor and bimetric theories. We discuss some properties of these models, and determine the number of cosmologically relevant free parameters. Finally, in Chapter 7 we summarise the main results of this thesis and discuss their applicability and relevance for present and future cosmological analyses.

1.1 Notations and conventions

Throughout this thesis we will be using the metric signature $(-,+,+,+)$. Greek indices such as μ and ν run over four dimensions of spacetime from 0 to 3, whereas Latin indices such as i and j run over three spatial dimensions from 1 to 3. The Einstein summation convention is implied, as usual. We will denote the Minkowski spacetime metric by $\eta_{\mu\nu} = \mathrm{Diag}(-1,1,1,1)$, and generic metrics by $g_{\mu\nu}$. In some chapters we will make use of two metrics, in which case indices of tensors depending on one metric only will be raised and lowered with their corresponding metric, whereas raising and lowering procedures for quantities depending on more than one metric will be explicitly specified where required. Partial derivatives are denoted by ∂ and covariant derivatives by ∇. Commas and semicolons will be used to denote partial and covariant derivatives, respectively. For instance, for a scalar field ϕ we would write derivatives as $\partial_\mu\phi = \phi_{,\mu}$ and $\nabla_\mu\phi = \phi_{;\mu}$. Symmetrisation and anti-symmetrisation on a pair of indices will be denoted by parenthesis and squared brackets, respectively, as follows:

$$S_{(\mu\nu)} = \frac{1}{2}\left(S_{\mu\nu} + S_{\nu\mu}\right); \quad A_{[\mu\nu]} = \frac{1}{2}\left(A_{\mu\nu} - A_{\nu\mu}\right), \tag{1.1}$$

where S and A represent generic tensors. We will describe the time coordinate of spacetime with physical time t or conformal time τ. Differentiation with respect to t will be denoted by a dot, and with respect to τ by a prime. The Hubble rate will be denoted as H and $\mathcal{H}$ in physical and conformal time, respectively. Finally, we mention that in all chapters we will set the speed of light to unity $c = 1$, whereas in the following chapters we also set the Planck mass to unity $M_{\mathrm{P}}^2 = 1/(8\pi G) = 1$, where G is the gravitational Newton's constant.

1.2 General relativity

General relativity is a geometric theory of gravity, in which gravity and matter interact in such a way that it leads to a nontrivial structure of space and time. In fact, in the presence of matter, physical distances between bodies change, and time lapses at different rates. All information about the effects of matter distribution on space and time are encoded in a 4-dimensional symmetric metric tensor $g_{\mu\nu}$ that describes a Lorentizan manifold with a line interval ds between two events given by:

$$ds^2 = g_{\mu\nu}\left(\mathbf{x}\right) dx^\mu dx^\nu, \tag{1.2}$$

where x^μ are the coordinates of spacetime, such that $x^0 = t$ is the time coordinate and $x^{\{1,2,3\}}$ are three spatial coordinates. Here, $\mathbf{x}$ stands for the 4-dimensional coordinate vector. Broadly speaking, the gravitational force is determined by the curvature of spacetime and the role of the metric is to determine how matter moves in this curved spacetime, whilst in turn matter determines how the spacetime curves. Let us next illustrate explicitly how this close relation between matter and spacetime occurs.

In general relativity, the gravitational interactions are described by the Einstein-Hilbert action, which is given by:

$$S_{\mathrm{EH}} = \frac{1}{16\pi G} \int d^4x \, \sqrt{-g}\, (R - 2\Lambda) \, , \tag{1.3}$$

where R is the Ricci scalar (that describes the 4-dimensional curvature of spacetime), G is known as the gravitational Newton's constant, and g is the determinant of the metric. Since we will be interested in the cosmological description of gravity, here we have also added Λ, which is a priori an arbitrary constant known as the cosmological constant. In addition, in GR we couple matter fields to the metric by adding a matter action of the following form:

$$S_{\mathrm{M}} = \int d^4x \, \sqrt{-g}\mathcal{L}_{\mathrm{M}} \left(\varphi, g_{\mu\nu}\right) , \tag{1.4}$$

where $\mathcal{L}_{\mathrm{M}}$ is some matter Lagrangian which is a functional of the metric $g_{\mu\nu}$ and some matter fields that we symbolically represent here with a single scalar field φ. The equations of motion of GR with matter are then given by:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.5}$$

where $G_{\mu\nu}$ is known as the Einstein tensor given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \tag{1.6}$$

where $R_{\mu\nu}$ is the Ricci tensor, which is related to the Ricci scalar by $R = g^{\mu\nu} R_{\mu\nu}$. In addition, we have defined $T_{\mu\nu}$ as the stress-energy momentum tensor of matter given by:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta\left(\sqrt{-g}\mathcal{L}_{\mathrm{M}}\right)}{\delta g^{\mu\nu}}, \tag{1.7}$$

where δ denotes a functional variation. Eq. (1.5) are known as the Einstein field equations and form a closed set of equations when complemented with the matter equation from action (1.4):

$$\nabla^{\mu} T_{\mu\nu} = 0. \tag{1.8}$$

This equation is known as the covariant energy and momentum density conservation equation. We notice that since any known form of matter builds up a stress-energy tensor that is conserved, eq. (1.8) is physically desirable and, while in some alternative gravity theories (such as massive gravity) it has to be imposed externally, in GR it is a consistency requirement. We can illustrate this last point by taking the covariant derivative of eq. (1.5):

$$\nabla^\mu G_{\mu\nu} = 8\pi G \nabla^\mu T_{\mu\nu}, \tag{1.9}$$

where we have used that $\nabla^\mu g_{\mu\nu} = 0$. From the definition of the Einstein tensor we actually have that:

$$\nabla^\mu G_{\mu\nu} = 0. \tag{1.10}$$

This equation is known as the *Bianchi identity*, and is satisfied by construction. From eq. (1.9) we can then see that, provided the Bianchi identity, eq. (1.8) must be satisfied in GR.

As we have previously mentioned, the Einstein field equations define how matter curves spacetime while eq. (1.8) define how matter moves in spacetime. In order to make this explicit, let us consider the trace of eq. (1.5) in the case without a cosmological constant ($\Lambda = 0$):

$$R = -8\pi G T^\mu{}_\mu, \tag{1.11}$$

from where it is clear that matter sources the curvature of spacetime. On the other hand, if we consider $T_{\mu\nu}$ to be given by a test particle, from eq. (1.8) we find that this particle moves in a geodesic of the spacetime metric $g_{\mu\nu}$:

$$\ddot{y}^\mu + \Gamma^\mu_{\alpha\beta} \dot{y}^\alpha \dot{y}^\beta = 0, \tag{1.12}$$

where $y^\mu(\lambda)$ describes the 4-dimensional position of the particle parametrised with an affine parameter λ. Here we have used dots to denote derivatives with respect to λ and we have introduced the Christoffel symbols $\Gamma^\mu_{\alpha\beta}$ given by:

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\beta\alpha,\nu} \right). \tag{1.13}$$

From here it is also clear to see that matter moves according to how spacetime is curved, which in this case is encoded in the Christoffel symbols.

General relativity has key characteristics that will be of importance for this thesis, and we hence mention. First, the Einstein field equations contain up to second derivatives of the metric in $G_{\mu\nu}$, and since the metric is symmetric, in principle there are at most 10 propagating gravitational degrees of freedom (DoFs). However, it turns out that it propagates only two DoFs[1] corresponding to two polarisations of a massless spin-2 particle (henceforth called graviton), which propagates at the speed of light in vacuum. Second, we notice that the Einstein field equations might resemble the Maxwell theory of electromagnetism and the Klein-Gordon equation for scalar matter, nonetheless they are non-linear (there are higher powers of the metric in $G_{\mu\nu}$), and thus GR is a much more complex theory. Fortunately, these terms are dynamically relevant only in strong gravitational fields. Third, we notice that the Einstein field equations (and the Einstein-Hilbert action) are invariant under general coordinate transformations $x^\mu \to \tilde{x}^\mu(\mathbf{x})$, which induce a transformation in the metric tensor given by:

$$\tilde{g}_{\mu\nu}(\tilde{\mathbf{x}}) = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\frac{\partial x^\nu}{\partial \tilde{x}^\beta} g_{\alpha\beta}(\mathbf{x}), \tag{1.14}$$

where $\tilde{g}_{\mu\nu}$ is the metric in the new coordinates, and a similar transformation in the stress-energy tensor $T_{\mu\nu}$. Since the equations are invariant, all metrics related by a general coordinate transformation are physically equivalent, and any apparent differences should not be ascribed to physical effects. Accordingly, any physical observable should be invariant under general coordinate transformations. This invariance is often referred to as general covariance, and corresponds to a gauge symmetry of the theory.

The principle of general covariance and the requirement that in the weak field non-relativistic limit the Einstein field equations should reproduce Newton's theory of gravity, were the two main ingredients that lead Einstein to the discovery of the theory of general relativity. However, we emphasise that these two ingredients are not unique to GR. As previously mentioned, general covariance is a gauge symmetry, which only represents a redundancy in the description and thus it is not a physical property of a theory. Similarly, having Newton's gravity limit is not unique to GR. For instance, this is achieved in the Einstein-Fokker theory [23–25], by iteratively coupling a canonical massless scalar to its own energy momentum tensor. Such a theory provides a universally attractive force, and reduces to Newton's gravity in the non-relativistic limit. Furthermore, this theory could even be made invariant under general coordinate transformations through the Stueckelberg trick (which will be illustrated in the next chapter). The modern view of GR is that it is the theory of a non-linearly interacting massless spin-2 particle. Indeed, it can be shown that the only non-linear Lorentz invariant theory that propagates two

[1]GR propagates only two degrees of freedom instead of ten because it has a gauge symmetry: diffeomorphism invariance. The presence of this symmetry means that there is a redundancy in the description of gravity in GR, which appears in the form of constraint equations.

polarisations of a massless spin-2 particle is GR [26–29].

1.3 Modern cosmology

Cosmology describes the global structure and evolution of the Universe. The standard cosmological model is known as the ΛCDM model and is based on GR. A key feature of this model is that the Universe can be described using perturbation theory, where the background is a perfectly homogeneous and isotropic expanding Universe and perturbations describe deviations from this idealised setting. In this section, we summarise the main ingredients of the ΛCDM model; we describe and show the relevant equations determining the background evolution as well as its linear perturbations.

1.3.1 Background

The standard cosmological model is based on the cosmological principle, which states that there should be neither a preferred location nor a preferred direction in space. Indeed, this assumption embodies the observational fact that the Universe is nearly homogeneous and isotropic on scales larger than 100 Mpc, as seen in galaxy distribution surveys and the CMB [30, 31]. Therefore, as an approximation, an infinitesimal line element for the spacetime of the Universe can be written as:

$$d\bar{s}^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j, \tag{1.15}$$

where t is the physical time, $a(t)$ is known as the scale factor and determines how the Universe expands, δ_{ij} is the Kronecker delta, and the bar superscript signals the fact that this is only the background metric. Here, we have assumed a spatially flat metric, i.e. a Universe with an Euclidean spatial geometry. This assumption is supported by observations from CMB and large-scale structures [32]. The background metric given in eq. (1.15) is known as the Friedmann-Lemaitre-Robertson-Walker (FLRW) flat metric and is a maximally spatially symmetric metric. For the purpose of this thesis, it is relevant to mention that the FLRW metric is sometimes expressed in terms of the conformal time τ instead of the physical time t. In that case, the line element becomes:

$$d\bar{s}^2 = a(\tau)^2 \left[-d\tau^2 + \delta_{ij} dx^i dx^j\right], \tag{1.16}$$

where we have made a time coordinate transformation such that $a d\tau = dt$. Physical time corresponds to the time measured by observers comoving with the cosmic expansion, such as us and thus provides a natural coordinate to use for observations. However, conformal time is often used for mathematical convenience as in this case the metric is conformally related to Minkowski spacetime $\bar{g}_{\mu\nu} = a(\tau)^2\eta_{\mu\nu}$. In this chapter though we will be using physical time.

The contribution of matter to the Universe is well described by an ideal hydrodynamic approximation. For such ideal fluid, with the same symmetries as the metric, the stress-energy tensor is given by:

$$\bar{T}_{\mu\nu} = (P_0 + \rho_0)\,\bar{u}_\mu\bar{u}_\nu + P_0\bar{g}_{\mu\nu}, \tag{1.17}$$

where $P_0 = P_0(t)$ is the pressure of the fluid, $\rho_0 = \rho_0(t)$ its rest energy density, and $\bar{u}^\mu = (1,0,0,0)$ its isotropic 4-velocity. Here, the subscripts 0 stand for background quantities.

From the Einstein field equations of motion given in eq. (1.5) and the conservation equation (1.8) we find that in this background:

$$H^2 = \frac{1}{3}\left(8\pi G\rho_0 + \Lambda\right), \tag{1.18}$$

$$\dot{\rho}_0 = -3H\left(\rho_0 + P_0\right), \tag{1.19}$$

where dots denote derivatives with respect to the physical time t, and $H \equiv \dot{a}/a$ is the Hubble parameter. Eq. (1.18) is known as the Friedmann equation and is the main equation governing the expansion of the Universe. These two equations give a closed system to be solved, provided a matter equation of state for ρ_0 and P_0. In the standard cosmological model, the different components in the Universe have a barotropic equation of state:

$$P_0 = w\rho_0, \tag{1.20}$$

where w is constant. We can distinguish three types of components:

- Radiation (relativistic matter): $w = 1/3$,
- Dust (non-relativistic matter): $w = 0$,
- Dark energy (cosmological constant): $w = -1$.

From the equations of motion (1.18)-(1.19) we find that the energy density and scale factor evolve in the following way, depending on the type of matter:

- Radiation (relativistic matter): $\rho_0 \sim a^{-4}; \quad a \sim t^{1/2}$,
- Dust (non-relativistic matter): $\rho_0 \sim a^{-3}; \quad a \sim t^{2/3}$,
- Dark energy (cosmological constant): $\rho_0 = \Lambda/(8\pi G); \quad a \sim e^{H_0 t}$,

where H_0 is the Hubble parameter evaluated today. A realistic description of the Universe should include all the different matter components at the same time, such as photons, neutrinos, baryons, and cosmological constant. In that case, the energy density determining the expansion of the Universe is given by:

$$\rho_0 = \rho_\gamma + \rho_\nu + \rho_b + \rho_\Lambda + \ldots , \tag{1.21}$$

where the ellipsis stands for other possible matter components. In this case we see that, when $\Lambda > 0$, as the Universe expands in time, radiation dominates first, followed by the domination of non-relativistic matter and finally by the cosmological constant. These are the three main distinct stages of the evolution of the Universe in the ΛCDM model.

The name ΛCDM stands for the assumption on the presence of two particular components: cosmological constant (Λ) and Cold Dark Matter (CDM). The cosmological constant is introduced to drive an exponentially fast growing scale factor and thus explain current observations of the accelerated expansions of the Universe. As we have previously mentioned, this is a very exotic component as it has an equation of state $P_0 = -\rho_0$, and hence it has negative pressure. Cold Dark Matter is a non-relativistic matter component and its name refers to the fact that it does not interact with electromagnetic radiation (or at least very weakly) and behaves like dust. It is introduced to explain that – as seen in the motion of visible matter, gravitational lensing, large-scale structure formation and the CMB – gravity seems to be stronger than predicted by GR around large-scale structures such as galaxies and clusters if we only take into account the presence of baryonic non-relativistic matter. CDM would correspond to additional non-baryonic (non-visible) matter that enhances the gravitational attraction of large-scale structures.

It has been shown by data that neutrinos and photons at present make up about 0.001% of all the energy of the Universe, baryonic matter about 5%, whereas CDM makes up about 26% and dark energy 69% [32]. Therefore, we can see that CDM and the cosmological constant make up about 95%, but their origin and nature are still not understood. Whereas it is widely accepted that CDM is made up of massive particles that interact predominantly via gravity and there are particle candidates such as axions, sterile neutrinos or Weakly Interacting Massive Particles (WIMPs) [33–36], the origin of the cosmological constant is certainly less clear and has become one of the most challenging problems in theoretical physics, and one of the the main motivations to study modified gravity theories.

All the equations shown in this subsection give the basis of the ΛCDM model. This model can make accurate and testable hypotheses in key areas such as: expansion of the Universe, origin of the cosmic microwave background, and nucleosynthesis of light elements. The remarkable agreement with observational data gives considerable confidence in the model. However, the presence of structures such as galaxies and clusters of galaxies show that we do not live in a perfectly homogeneous and isotropic Universe. In order to obtain a more realistic description of the Universe and its constituents we introduce perturbations around this background. As a first approximation we can consider only linear perturbations, and predict the entire time evolution of linear inhomogeneities at large scales, which certainly goes beyond what we can currently observe. Higher-order perturbations are relevant to study the distribution of matter and radiation at smaller scales, and include the effect of screening mechanisms in modified gravity theories, but in this thesis we only focus on first-order perturbations.

1.3.2 Perturbations

In the standard cosmological model we suppose that small deviations from homogeneity and isotropy were generated during the early Universe due to quantum fluctuations. Since gravity is an attractive force, these small perturbations would grow in time, forming structures through the mechanism of gravitational collapse. In this section, we present and describe the main ingredients to analyse cosmological perturbations.

Let us consider linear perturbations of the spacetime metric and matter on a given background. In general, we can split the perturbed metric into two parts:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \quad \text{such that} \quad |\delta g_{\mu\nu}| \ll |\bar{g}_{\mu\nu}|, \tag{1.22}$$

where $\bar{g}_{\mu\nu}$ corresponds to the background metric and $\delta g_{\mu\nu}$ to the first order metric perturbation. Similarly, we can write the perturbed stress-energy momentum tensor as:

$$T^{\mu}{}_{\nu} = \bar{T}^{\mu}{}_{\nu} + \delta T^{\mu}{}_{\nu}, \quad \text{such that} \quad |\delta T^{\mu}{}_{\nu}| \ll |\bar{T}^{\mu}{}_{\nu}|, \tag{1.23}$$

where $\bar{T}^{\mu}{}_{\nu}$ gives the background value and $\delta T^{\mu}{}_{\nu}$ its perturbation. In general, the background and first order terms satisfy equations (1.5) and (1.8), and thus these equations provide all the necessary information to find the evolution of the background and perturbations (provided some initial conditions).

We notice that once the background is fixed the system is no longer invariant under general coordinates transformations. Instead, it is invariant only under linear infinitesimal coordinate transformations: $x^{\mu} \to x^{\mu} + \epsilon^{\mu}(x)$, where $|\epsilon^{\mu}| \ll |x^{\mu}|$ is an arbitrary perturbation. Under this transformation the background remains unchanged but the linear perturbations transform in the following way:

$$\delta g_{\mu\nu} \to \delta g_{\mu\nu} - \bar{g}_{\mu\beta}\partial_\nu \epsilon^\beta - \bar{g}_{\beta\nu}\partial_\mu \epsilon^\beta + \epsilon^\alpha \bar{g}_{\mu\beta}\bar{g}_{\nu\gamma}\left(\partial_\alpha \bar{g}^{\beta\gamma}\right), \tag{1.24}$$

and similarly for $\delta T_{\mu\nu}$. The derivation of this transformation of the metric can be found in Appendix A.1. The presence of this gauge freedom means that not all apparently perturbed metrics are physical perturbed spacetimes. For instance, a homogeneous and isotropic form of the metric can be transformed to an inhomogeneous form by performing particular coordinate transformations. Thus, in order to be certain whether we are considering a homogeneous and isotropic spacetime or a perturbed one, we will need to distinguish physical inhomogeneities (geometrical) and coordinate artifacts.

One approach to this problem is to work in a manifestly gauge-invariant framework. It consists in defining a new set of gauge-invariant perturbation fields and rewrite all the actions (or equations of motion) in terms of these new fields. Since, in general, not all the gauge-invariant perturbation fields are physical, one must also find the physical degrees of freedom by finding only relevant gauge-invariant fields which the action depends on. Another approach, which will be adopted for the most part in this thesis, consists on fixing all the gauge freedoms in a special way such that we eliminate all non-physical perturbation fields, and remain with physical degrees of freedom only [37].

In the following discussion, we focus on first order perturbations in the idealised homogeneous and isotropic background Universe previously presented. We start by noticing that all perturbations are a priori arbitrary, so even if the background is homogeneous and isotropic, in general the perturbed metric is not. Usually, metric perturbations are categorised into three different types: scalar, vector and tensor. This classification is known as the Scalar-Vector-Tensor (SVT) decomposition and refers to the way perturbation fields transform under spatial coordinate transformations [38]. It can be shown that, at linear order around a homogenous and isotropic background, the three type of perturbations decouple from each other, and hence evolve independently. This property has a particular advantage: we can simplify the original long mathematical problem by dividing it into three shorter sets of problems for scalar, vector and tensor perturbations.

As previously mentioned, since the metric tensor lives in a 4-dimensional spacetime and it is symmetric, it propagates at most ten DoFs. We shall see that these ten DoFs are divided into four scalar perturbations, two vector perturbations (that propagate two DoFs each), and one tensor perturbations (that propagates two DoFs). Furthermore, in the standard cosmological model, scalar perturbations couple directly to matter, and exhibit the growing modes needed to cause the formation of large-scale structure in the Universe. For this reason, throughout this thesis we will mainly be interested in analysing the evolution of scalar perturbations. In addition, vector perturbations decay in an expanding Universe and thus they do not affect any cosmological observable in a relevant way. Finally, tensor perturbations do not couple matter[2], and thus they travel freely most of the time and describe the evolution of primordial gravitational waves.

In what follows we present explicitly the separation into scalar, vector and tensor perturbations.

Scalar Perturbations

The most general way to construct metric perturbations around the background (1.15) with scalar perturbations is using four scalar fields Φ, Ψ, E, and B. In this case, the infinitesimal line element of the perturbed spacetime can be written as:

$$ds^2 = -\left(1+2\Phi\right)dt^2 + 2B_{,i}dx^i dt + a^2\left[(1-2\Psi)\delta_{ij} + 2E_{,ij}\right]dx^i dx^j. \tag{1.25}$$

We emphasise that whereas the scale factor depends only on time, the four scalar perturbations in general depend on space *and* time.

Similarly, we can write a decomposition of the perturbations for the stress-energy momentum tensor. For a general perfect fluid, the perturbed tensor is given by:

$$\begin{aligned} &T^0{}_0 = -(\rho_0 + \delta\rho); \quad T^i{}_0 = -(\rho_0 + P_0)v^{,i}; \\ &T^0{}_i = (\rho_0 + P_0)\left(B_{,i} + a^2 v_{,i}\right); \quad T^i{}_j = (P_0 + \delta P)\delta^i_j, \end{aligned} \tag{1.26}$$

where there are three scalar perturbations[3]: the perturbed energy density $\delta\rho$, the perturbed pressure δP, and the perturbed curl-free spatial velocity of the fluid $v^{,i}$. Here, spatial indices are lowered and raised with the Kronecker delta.

We notice that the scalar perturbations defined in this section in general are not gauge invariant. Indeed, we find that for an infinitesimal coordinate transformation the scalar fields

[2] They only couple to an anisotropic stress-energy tensor of matter, which is negligible at late times.

[3] We could in principle also add an anisotropic stress perturbation which would add a fourth scalar perturbation, but we ignore it here as it is irrelevant at late times.

transform as:

$$\Phi \rightarrow \Phi - \dot{\pi}; \quad \Psi \rightarrow \Psi + H\pi; \quad B \rightarrow B + \pi - a^2\dot{\epsilon}; \quad E \rightarrow E - \epsilon, \tag{1.27}$$

$$\delta\rho \rightarrow \delta\rho - \pi\dot{\rho}_0; \quad \delta P \rightarrow \delta P - \pi\dot{P}_0; \quad v \rightarrow v + \dot{\epsilon}, \tag{1.28}$$

where we have also used the SVT decomposition for the gauge parameter ϵ^μ in such a way that its scalar components are explicitly given by:

$$\epsilon^\mu = (\pi, \epsilon^{,i}). \tag{1.29}$$

Now that we have decomposed the metric and the stress-energy tensor we can write the equation of motion for perturbations. Replacing expressions (1.25)-(1.33) into eq. (1.5)-(1.8), and choosing the Newtonian gauge $E = B = 0$, we find a set of equations that when combined lead to:

$$\nabla^2\Psi = 4\pi G\rho_0\Delta_{\rm M}; \quad \Phi = \Psi, \tag{1.30}$$

where $\Delta_{\rm M} = \delta\rho/\rho_0 + \dot{\rho}_0/\rho_0 a^2 v$ is a gauge-invariant matter density perturbation field. Here, the first equation is known as the Poisson equation, and shows explicitly how the metric scalar perturbations are coupled to the matter density perturbation, which is responsible for the formation of the observed large-scale structures today.

Vector Perturbations

Metric vector perturbations can be constructed by using two vectors that live in the 3-dimensional space: S_i and F_i. These vectors satisfy the following condition:

$$S_i{}^{,i} = F_i{}^{,i} = 0, \tag{1.31}$$

which states that these two vectors have no scalar contributions, and then they carry only two independent DoFs each. Specifically, the infinitesimal line element of the perturbed spacetime can be written as follows:

$$ds^2 = -dt^2 - 2S_i dx^i dt + a^2\left[\delta_{ij} + F_{i,j} + F_{j,i}\right] dx^i dx^j. \tag{1.32}$$

For a perfect fluid, there will be only one vector perturbation: the divergence-free spatial velocity of the fluid v^{Ti}, such that $v_i^{Ti} = 0$. Specifically, the stress-energy tensor can be written as:

$$T^0{}_0 = -\rho_0; \quad T^i{}_0 = -(\rho_0 + P_0) v^{Ti};$$
$$T^0{}_i = (\rho_0 + P_0)\left(S_i + a^2 v_i^T\right); \quad T^i{}_j = P_0 \delta^i_j. \tag{1.33}$$

From the Einstein field equations we can find that the three vector perturbations, S_i, F_i and v^{Ti}, couple to each other but there is only one physical vector DoF propagating, namely the velocity of the fluid. The amplitude of this perturbation decays as the Universe expands, so it could only be observable today if its initial amplitude was so large that it completely spoiled the isotropy of the very early Universe. However, in an inflationary Universe there is no room for such large primordial vector perturbations and hence vector perturbations are not cosmologically relevant (see [39]).

For completeness, we mention that in general vector perturbations are not gauge invariant. Indeed, under infinitesimal coordinate transformations they transform as:

$$F_i \to F_i - \epsilon_i^T; \quad S_i \to S_i + a^2 \dot{\epsilon}_i^T,$$
$$v^{Ti} \to v^{Ti} + \dot{\epsilon}^{Ti}, \tag{1.34}$$

where we have decomposed the gauge parameter in its vector part as $\epsilon^\mu = (0, \epsilon^{Ti})$, such that $\epsilon^{Ti}{}_{,i} = 0$.

Tensor perturbations

Tensor perturbations can be constructed by using one 3-dimensional spatially symmetric tensor h_{ij}, which satisfies:

$$h_i{}^i = 0; \quad h_{ij}{}^{,i} = 0. \tag{1.35}$$

These two conditions ensure that this tensor does not have any scalar or vector contributions, and hence carries only two independent DoFs. In this case, the infinitesimal line element of spacetime can be written as:

$$ds^2 = -dt^2 + a^2 \left[\delta_{ij} + h_{ij}\right] dx^i dx^j. \tag{1.36}$$

A perfect fluid does not have any tensor perturbation (unless it has anisotropic stress), and thus h_{ij} does not couple with matter. From the Einstein field equations we find the following equation of motion for h_{ij}:

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{1}{a^2}\partial^k\partial_k h_{ij} = 0 \tag{1.37}$$

which has oscillatory solutions that describe the propagation of primordial gravitational waves. Finally, we mention that tensor perturbations do not transform under infinitesimal coordinate transformations, and hence they are gauge-independent fields.

To finish this section, we note that in order to completely fix the solutions for scalar, vector and tensor perturbations we need initial conditions. These are provided by the inflationary theory, which describes an early period of accelerated expansion of the Universe when the Universe was dominated by a scalar field: the inflationary field or inflaton. Quantum fluctuations in the metric and the scalar field created small inhomogeneities and sourced primordial matter perturbations and gravitational waves. This inflationary model predicts nearly scale-invariant initial conditions for perturbations, which agree with observations from the Planck Satellite on the CMB and large-scale matter distribution [40].

So far, perturbation theory in the ΛCDM model agrees with all cosmological data, specifically on the statistical distribution of matter and light in the Universe, with only mild tensions between Planck and other astrophysical observations [32]. However, as previously mentioned, there are fundamental unanswered questions regarding the origin of dark matter and dark energy. Furthermore, the inflationary theory has also been criticised for assuming the presence of an as of yet unobserved scalar field with very specific properties and the need of fine-tuned initial conditions [41, 42]. In addition, questions have arisen regarding the validity of usual inflation and the presence of observable trans-Planckian scales [43]. As previously mentioned, these issues, and specially the dark energy, have driven the production of many modified gravity theories in the last decades. In the next section we discuss the cosmological constant problem as well as the main characteristics that most alternative gravity theories have.

1.4 Modified gravity

The cosmological constant is one of the central topics of cosmology nowadays, and the main motivation to consider modified gravity theories. It is only visible at large scales, where the curvature is small enough (i.e. $R \leq 2\Lambda$), and thus it can only be probed cosmologically. As shown in the previous sections, the cosmological constant drives the present accelerated expansion of the Universe. Observations show that the cosmological constant represents about 69% of all the energy content of the Universe, and its measured value is $\Lambda \sim 10^{-123} M_{\rm P}^2$. According to standard lore, this measured constant is an effective cosmological constant that has a contribution from a geometrical constant (inserted ad hoc into the Einstein-Hilbert action) and the gravitating quantum vacuum energy of all particles in the standard model of particle physics. However, estimates of this vacuum energy with Quantum Field Theory give a value more than 50 orders of magnitude larger than the measured cosmological constant [44]. This results in a fine-tuning problem, in which the geometrical constant must cancel out to a high degree of accuracy the vacuum energy in order to lead to the measured value of the effective cosmological constant, even though, a priori, they are independent from each other. This issue constitutes the so-called cosmological constant problem and is one of the most important challenges in theoretical cosmology nowadays.

Furthermore, it has been observed that the fractional energy densities of the effective cosmological constant and non-relativistic matter are comparable at present. However, since the latter evolves in time as $\rho \sim a^{-3}$, the period of time in which these two are comparable is extremely short. This constitutes the so-called coincidence problem and has been explained so far by assuming that we are privileged observers, living in a special time. This assumption however, is contrary to the previously mentioned Cosmological Principle. Therefore, even though the cosmological constant provides a simple way of recreating the current accelerated expansion of the Universe, it poses major theoretical issues.

In the last decades there has been a number of proposals to avoid the cosmological constant and coincidence problems with modified gravity theories, in which dynamical mechanisms can create an accelerated expansion of the Universe. Usually, we can distinguish two types of gravity models: those where the matter stress-energy tensor is modified in such a way that the equation of state approaches $w \simeq -1$ at late times, and those where the left-hand side of the Einstein field equations is modified. The former type of models are commonly known as dark energy theories of gravity, and the latter ones as modified gravity theories. While there are models which unambiguously belong to one category or the other, in reality there is a continuum of models between the two types (see [45] for a review). For this reason, the division between these two classes is a matter of personal preference and in this thesis we will be generically referring as modified gravity theories to any modification to GR and the ΛCDM model.

Before discussing some general characteristics of modified gravity theories, it is interesting to mention that modifications of gravity have an old history. In fact, one of the events that led to the success of GR was that the accepted theory of gravity at the time – Newton's law of Universal attraction – did not describe gravity properly in certain regimes. In the 19th century, Newtonian gravity predicted only half of the observed advance angle of precession of Mercury's orbit. At the time, the mathematician Le Verrier hypothesised the presence of a new planet – Vulcan – perturbing Mercury's orbit and hence responsible for this discrepancy. Vulcan was never found, and the solution was actually found in the early 20th century when Einstein showed that GR could predict exactly the observed amount of advance of Mercury's perihelion, without any recourse to additional planets. Therefore, it was then found that gravity needed to be modified: while Newtonian gravity was very accurate in the regime of small velocities and weak fields, it did not describe appropriately gravitational forces at larger velocities and stronger fields.

In general, modified gravity theories can be constructed in different ways and in order to understand how they can modify GR it is important to mention Lovelock's theorem [46, 47]. This theorem states the following:

> "*In four dimensions, the only local diffeomorphism invariant action which leads to 2^{nd} order field equations of motion and which depends only on a metric is a linear combination of the Einstein-Hilbert action with a cosmological constant up to a total derivative.*"

This theorem shows that in order to modify GR we can change the dimensions of spacetime, consider non-local terms, change the number of degrees of freedom (or fields), give up diffeomorphism invariance, and/or add higher derivatives. In many cases these changes are degenerate, and for this reason many modified gravity theories perform more than one change. For instance, as we will see in the next section, massive gravity breaks diffeomorphism invariance and also propagates more degrees of freedom than GR.

Once we have considered a certain modification of gravity, we need to assess its viability. In general, any model must pass certain theoretical consistency checks and also satisfy observational constraints. The main theoretical requirements on any modified gravity theory are that it is free from unstable degrees of freedom, such as ghosts, gradient, or tachyon instabilities. In general, any of these instabilities can make cosmological perturbations grow too fast, and either invalidate the perturbative approach taken in the previous section or simply not fit data. Furthermore, ghosts are particularly dangerous because they are modes with negative kinetic energy, and hence make any vacuum state unstable as an arbitrary number of these modes can be produced. For this reason, special care must be taken when considering higher-derivative gravity theories, as Ostrogradsky's theorem [48] tells us that equations of motion with higher than second derivatives will lead to a Hamiltonian which is unbounded from below due to the presence of ghosts.

Regarding observational constraints, modified gravity theories need to first reproduce the success of GR on small scales, specifically from Solar System experiments. However, many theories propagate an extra scalar field that couples (directly or indirectly) to matter and thus, a priori, modifies gravity by producing a fifth force. On the one hand, this fifth force could be an advantage and modify GR on cosmological scales, potentially driving the late-time accelerated expansion of the Universe without a cosmological constant. On the other hand, this fifth force could also affect predictions at small scales, but models must reduce to GR in this regime as any modification is severely constrained by experimental data. In fact, typically the strength of the fifth force is required to be orders of magnitude weaker than gravity at small scales. For this reason, most modified gravity theories are equipped with a mechanism that suppresses the effect of the scalar field in the Solar System, called *screening mechanism*. This is done by making some property of the scalar field dependent on the background environment under consideration. Next, we summarise the three main types of screening mechanisms (see [49, 50] for a detailed review):

1. *Chameleon.*

   In the Chameleon mechanism the scalar field has a mass that depends on its environment in such a way that it becomes heavy in dense environments, such as in the Solar System. This makes the Compton wavelength of this mode small, and hence effectively an undetectable short-range force. On the contrary, around diffuse environments, the scalar field becomes light with a large Compton wavelength. In this case, the mode becomes detectable and potentially relevant for cosmological observations.

2. *Symmetron.*

   In the Symmetron mechanism the scalar field has a potential that is density-dependent, which effectively changes the coupling to matter. In dense environments the vacuum expectation value of the potential is near zero, and thus the field does not couple to matter. As the density drops, the potential undergoes a spontaneously symmetry-breaking phase and takes a non-zero expectation value, resulting in a coupling between the scalar field and matter.

3. *Vainshtein.*

   In the Vainshtein mechanism the scalar field has non-linear kinetic terms. These terms become large at small scales and effectively make negligible the coupling to matter (and hence the fifth force). At large scales these non-linear terms become negligible and the fifth force becomes relevant, inducing potential modifications at cosmological scales. As we will mention in the next section, the scale at which the fifth force starts getting suppressed is known as the Vainshtein radius.

We emphasise that, by construction, all these screening mechanisms operate in the regime of non-linear perturbations. Therefore, even though they play an important role in the analysis of viable modified gravity theories, we will not consider them further in this thesis as we will focus on cosmological scales where perturbations are linear.

Modified gravity theories also need to fit cosmological data, and whereas many models can reproduced exactly the same evolution for a homogeneous and isotropic Universe as ΛCDM, they usually differ in the evolution of cosmological perturbations. For instance, in the Dvali-Gabadadze-Porrati (DGP) gravity model, the rate of structure formation can differ from other models by a few percent for identical Hubble functions [51]. Therefore, cosmological perturbations provide a relevant tool for discriminating between gravity theories, and potentially support or refute some of them. For this reason, this thesis is mainly concerned with predictions on the evolution of perturbations of gravity theories.

Even though the CMB plays an essential role in constraining cosmological perturbations, most of the focus is currently going to galaxy surveys and weak lensing data, in which one of the key probes will be the growth-rate of large scale structures, defined as:

$$f = \frac{d \ln \Delta_{\rm M}}{d \ln a}. \tag{1.38}$$

As we showed in the previous sections, the evolution of $\Delta_{\rm M}$ depends on the gravitational potentials, and if there is any presence of fifth force, then the growth rate f will change. A number of experiments will come online within the next decade, including Euclid, DESI, SKA, LSST and WFIRST, which will allow us to make important improvements in constraints on gravity at large scales. Indeed, current attempts on constraining gravity models have shown that present data is not conclusive yet [52]. We note that it has been suggested that there might be some evidence for deviations from GR, but the systematic uncertainties in the experimental data are too large to consider such deviations conclusive [53]. From all the future combined data, preliminary studies [54–58] have shown that we could improve the precision on modified gravity constraints by order 10. Therefore, over time we expect to see a dramatic increase in the precision with which we can constrain gravity on cosmological scales, and the work presented in this thesis aims to analyse models and develop tools to test gravity at large scales in preparation for future data.

CHAPTER 2

Massive bigravity

The history of massive gravity dates back to 1939, when Fierz and Pauli developed the linear theory of a massive spin-2 field in Minkowski space [59]. It consists of a covariant quadratic action, known as the Fierz-Pauli action, describing a free massive graviton which propagates five degrees of freedom – namely, modes with helicity ± 2, ± 1 and 0. This action is the only possible instability-free quadratic action for a massive graviton [60]. In this chapter we show explicitly this action and some of its properties. We show that in the presence of matter, the Fierz-Pauli action exhibits the so-called van Dam, Veltman, Zakharov (vDVZ) discontinuity [61, 62], which arises when taking the massless limit as, contrary to expectations, the model does not reduce to the theory of a massless graviton (that is, GR). This happens because in this limit the helicity-0 mode still propagates and couples to the trace of the stress-energy tensor, hence modifying the behaviour of matter.

We then focus on non-linear completions of the Fierz-Pauli action. Non-linear massive gravity theories (that reduce to the Fierz-Pauli action at the linear level) were studied extensively following the non-linear proposal by Vainshtein in 1972 [63]. It was then argued that non-linearities could cure the vDVZ discontinuity as non-linear interactions would become comparable to the linear terms for small values of m. Such non-linear interactions would give rise to a screening of the helicity-0 mode at small scales, rendering the theory compatible with Solar System tests of gravity [63, 64]. Despite this resolution of the vDVZ discontinuity, Vainshtein's model was flawed as it contained an instability, a Boulware-Deser ghost [65], i.e. an extra scalar degree of freedom whose kinetic term had the wrong sign.

Finally, in this chapter we show the most successful current non-linear theory of massive gravity. In 2010 major progress was made when a particular family of ghost-free interaction potentials was constructed by de Rham, Gabadadze and Tolley in [66] and confirmed to be ghost-free by Hassan and Rosen in [67] (see also [68]). dRGT massive gravity [69–71], as it is known, contains the spacetime metric $g_{\mu\nu}$ as well as a fixed non-dynamical second metric $f_{\mu\nu}$. A bimetric ghost-free extension of the dRGT massive gravity was proposed by Hassan and Rosen in [72] (see also [73]), where the new metric $f_{\mu\nu}$ is also dynamical. For a more detailed review on massive gravity and its origins, see [74, 75].

2.1 Linear massive gravity

Let us start constructing the linear action for a massless graviton. Consider a symmetric Lorentz spin-2 field $h_{\mu\nu}$ in 4-dimensional Minkowski space. It has been previously shown that the most general local covariant quadratic kinetic Lagrangian that avoids the propagation of unstable degrees of freedom (modulo boundary terms and an overall normalisation factor) is given by:

$$\mathcal{L}_{\mathrm{GR,L}} = -\frac{1}{4} h^{\mu\nu} \hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu} h_{\alpha\beta}, \tag{2.1}$$

where $\hat{\mathcal{E}}$ is known as the Lichnerowicz operator and it is given by:

$$\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} = -\frac{1}{2}\left[\Box h_{\mu\nu} - 2\partial_{(\mu}\partial_{\alpha} h_{\nu)}{}^{\alpha} + \partial_\mu \partial_\nu h - \eta_{\mu\nu}\left(\Box h - \partial_\alpha \partial_\beta h^{\alpha\beta}\right)\right], \tag{2.2}$$

where we raise and lower indices with the Minkowski metric, also $h = h^\mu_\mu$ is the trace of the tensor field, and $\Box = \partial^\mu \partial_\mu$ is the d'Alembert operator. This action corresponds to the linearised Einstein-Hilbert action around Minkowski space when the metric is expanded as:

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_{\mathrm{P}}} h_{\mu\nu}; \quad h_{\mu\nu}/M_{\mathrm{P}} \ll \eta_{\mu\nu}, \tag{2.3}$$

and only quadratic terms in $h_{\mu\nu}$ are kept in the action. Therefore, we conclude that, as in GR, the field $h_{\mu\nu}$ describes a massless graviton and hence propagates only two polarisations and the action is invariant under linearised diffeomorphism transformations:

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu. \tag{2.4}$$

We note that diffeomorphism invariance plays a crucial role as it ensures that the action does not propagate an unstable mode, known as Boulware-Deser ghost [65]. In other words, if the action was not diffeomorphism invariant it would propagate a ghost, which would be an additional physical degree of freedom with a wrong sign in its kinetic term in the action. The presence of such a ghost leads to a Hamiltonian which is unbounded from below, which renders quantum vacua unstable as states with arbitrarily negative energy can be created.

Next, let us consider the possibility of giving a mass to the graviton. In order to do this we add non-derivative terms to the action in eq. (2.1). A priori, we could write two kind of interactions:

$$\mathcal{L}_{\mathrm{m}} = -\frac{1}{8} m^2 \left(h^{\mu\nu} h_{\mu\nu} - a h^2 \right) , \tag{2.5}$$

where m is a mass parameter and a is a free dimensionless constant. It is possible to show that the entire Lagrangian $\mathcal{L}_{\mathrm{GR,L}} + \mathcal{L}_{\mathrm{m}}$ propagates six degrees of freedom, one of which is a Boulware-Deser ghost. In order to avoid the presence of this unstable mode we can only choose $a = 1$, in which case eq. (2.5) simply reduces to:

$$\mathcal{L}_{\mathrm{FP,m}} = -\frac{1}{8} m^2 \left(h^{\mu\nu} h_{\mu\nu} - h^2 \right) , \tag{2.6}$$

which is known as the Fierz-Pauli mass term. The total Fierz-Pauli Lagrangian is the given by $\mathcal{L}_{\mathrm{FP}} = \mathcal{L}_{\mathrm{GR,L}} + \mathcal{L}_{\mathrm{FP,m}}$, whose equations of motion are given by:

$$\left(\Box - m^2 \right) h_{\mu\nu} = 0; \quad \partial^{\mu} h_{\mu\nu} = 0, ; \quad h = 0. \tag{2.7}$$

From here we can identify m as the mass of the graviton and we can see that the field $h_{\mu\nu}$ propagates now only five degrees of freedom (instead of six)[1], corresponding to the five polarisations of a massive graviton: the helicity-0, ± 1 and ± 2 modes.

We note that the presence of the mass terms now breaks the gauge symmetry of the Einstein-Hilbert action shown in eq. (2.4), while Lorentz invariance is still preserved. An important consequence of this is that the mass parameter m is technically natural, i.e. it is stable under quantum corrections because these corrections are proportional to the parameter m itself [74, 75]. This means that even if this parameter is set to a very small value (which might be needed if this theory is expected to describe a modified gravity theory explaining the cosmological constant problem), there is no fine-tuning problem. In general, we say that a small parameter is technically natural if there is a symmetry that appears as the small parameter is set to zero. In the case of massive gravity, this symmetry is linearised diffeomorphism invariance that protects a zero value of the mass from quantum corrections.

The construction of an action for a massive spin-2 field is interesting already from a particle

[1] Explicity, the field $h_{\mu\nu}$ in principle has ten independent DoFs because it is a 4D symmetric tensor. However, from eq. (2.7) we see that it is traceless and divergenceless, and hence it satisfies five constraints that reduce the total number of DoFs to five.

physics point of view, but in this thesis we will focus on considering such action as a candidate for a gravity theory. If we were to consider a massive graviton to be the mediator of the gravitational force we would need first to find how this field interacts with matter. Let us consider a linear coupling to matter:

$$\mathcal{L}_{\mathrm{M}} = \frac{1}{2M_{\mathrm{P}}} h_{\mu\nu} T^{\mu\nu}, \tag{2.8}$$

where $T^{\mu\nu}$ is the energy-momentum tensor of some external source. In this case, it was shown that this model exhibits a discontinuity – the vDVZ discontinuity [61, 62] – which refers to the fact that in the $m \to 0$ we do not recover the model of a massless graviton, that is GR. Indeed, calculations have shown that in this limit the model predicts a magnitude of the light bending angle around a point source that is 25% smaller than in GR [74]. At first sight, this discontinuity might seem odd as by taking that limit we make $\mathcal{L}_{\mathrm{FP,m}} = 0$ and the Fierz-Pauli actions reduces to the Einstein-Hilbert action of GR. However, this limit is not smooth as degrees of freedom are lost in the process. The correct way of taking the limit is by performing the so-called Stueckelberg trick. As we shall see in a moment, this trick makes explicit the origin of the discontinuity: the massless limit of massive gravity is not massless gravity (GR), but rather massless gravity plus extra degrees of freedom, which correspond to a massless vector and a massless scalar fields. While this massless vector propagates freely (without any coupling to the graviton or matter), the scalar does couple to the trace of the energy momentum tensor, hence changing the behaviour of matter and causing the vDVZ discontinuity.

The main idea behind the Stueckelberg trick is that gauge freedoms are not physical, but instead simply a redundancy in the description of a given system. Thus, we notice that Fierz-Pauli action previously given is not explicitly gauge invariant but this symmetry can be restored by introducing redundant fields. Let us introduce two redundant fields A_μ and ϕ, known as the Stueckelberg fields, by performing the following replacement in the Fierz-Pauli action[2]:

$$h_{\mu\nu} \to h_{\mu\nu} + \frac{2}{m}\partial_{(\mu}A_{\nu)} + \frac{2}{m^2}\partial_\mu\partial_\nu\phi. \tag{2.9}$$

[2]Notice that in this field replacement we have renormalised the fields A_μ and ϕ with powers of m in such a way that these fields are canonically normalised in the resulting action. This renornalisation is crucial as, even though the transformation becomes singular in the $m \to 0$ limit, it allows us to take a smooth limit in the action when $m \to 0$.

In this case, the total Fierz-Pauli Lagrangian $\mathcal{L}_{\mathrm{FP}}$, including matter, becomes:

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}h^{\mu\nu}\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} - \frac{1}{8}m^2\left(h^{\mu\nu}h_{\mu\nu} - h^2\right) - \frac{1}{2}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}h^{\mu\nu}\left(\Pi_{\mu\nu} - \Pi\eta_{\mu\nu}\right)\\ &- m\left(h^{\mu\nu} - h\eta^{\mu\nu}\right)\partial_\mu A_\nu + \frac{1}{2M_{\mathrm{P}}}h_{\mu\nu}T^{\mu\nu} - \frac{1}{m}\frac{2}{M_{\mathrm{P}}}A_\mu\partial_\nu T^{\mu\nu} + \frac{1}{m^2}\frac{1}{M_{\mathrm{P}}}\phi\partial_\mu\partial_\nu T^{\mu\nu},\end{aligned} \tag{2.10}$$

where we have defined $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$, $\Pi_{\mu\nu} = \partial_\mu\partial_\nu\phi$, and Π as the trace of $\Pi_{\mu\nu}$. This action is now invariant under the following gauge transformations:

$$\delta h_{\mu\nu} = \partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu; \quad \delta A_\mu = -\frac{m}{2}\epsilon_\mu + \partial_\mu\Lambda; \quad \delta\phi = -2m\Lambda, \tag{2.11}$$

where ϵ^μ and Λ are arbitrary gauge parameters. These transformations correspond to linearised diffeomorphisms with a $U(1)$ symmetry. We emphasise that eq. (2.10) has the exact same physical information as the previously shown Fierz-Pauli Lagrangian, and the difference lies in the presence of gauge symmetries.

As mentioned in the previous chapter in eq. (1.8), in GR all matter sources are covariantly conserved as that is a consistency condition. In massive gravity this is not the case, and matter could well be or not be conserved. However, since any known form of matter is indeed conserved, we impose conservation of matter as an external equation to be satisfied: $\partial_\nu T^{\mu\nu} = 0$. In this case, we take the $m \to 0$ limit of eq. (2.10) and find that the Lagrangian becomes:

$$\mathcal{L} = \mathcal{L}_{\mathrm{GR,L}}(h) - \frac{1}{2}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}h^{\mu\nu}\left(\Pi_{\mu\nu} - \Pi\eta_{\mu\nu}\right) + \frac{1}{2M_{\mathrm{P}}}h_{\mu\nu}T^{\mu\nu}, \tag{2.12}$$

where $\mathcal{L}_{\mathrm{GR,L}}(h)$ is the Lagrangian in eq. (2.1) for the field $h_{\mu\nu}$. From here we see that the model propagates one tensor, one vector and one scalar field, and thus, after performing the Stueckelberg trick, the massless limit is smooth because the number of degrees of freedom is conserved. In addition, we see that the vector field has become a free field, but the tensor and scalar fields are still mixed. In order to see explicitly how fields interact with each other and matter, let us perform the following field redefinition:

$$h_{\mu\nu} = h'_{\mu\nu} + \phi\eta_{\mu\nu}, \tag{2.13}$$

so the Lagrangian in eq. (2.12) becomes unmixed:

$$\mathcal{L} = \mathcal{L}_{\mathrm{GR,L}}(h') - \frac{1}{2}F^{\mu\nu}F_{\mu\nu} - \frac{3}{4}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2M_{\mathrm{P}}}h'_{\mu\nu}T^{\mu\nu} + \frac{1}{4M_{\mathrm{P}}}\phi T, \tag{2.14}$$

where T is the trace of the energy-momentum tensor. As we can see, matter is coupled to a massless graviton but also to the scalar mode ϕ that corresponds to the helicity-0 mode of the massive graviton. Due to the presence of this extra scalar-matter coupling this model does not reduce to GR in the massless limit. A resolution to the vDVZ discontinuity is provided by the Vainshtein screening mechanism [63] which involves including non-linear kinetic terms in the action. In the limit $m \to 0$, these non-linear terms screen the effects of the troublesome helicity-0 mode. This mechanism highlights the importance of constructing an appropriate non-linear theory of massive gravity, which is the main topic of the next section.

2.2 Non-linear massive gravity

The development of a non-linear theory for a massive graviton started with the proposal of Vainshtein in 1972 [63]. He suggested the following simple extension to the Fierz-Pauli Lagrangian:

$$\mathcal{L} = \frac{M_{\mathrm{P}}^2}{2}\sqrt{-g}R - \frac{1}{8}m^2\eta^{\alpha\mu}\eta^{\nu\beta}\left(h_{\mu\nu}h_{\alpha\beta} - h_{\mu\alpha}h_{\nu\beta}\right), \tag{2.15}$$

where the metric $g_{\mu\nu}$ is related to the tensor $h_{\mu\nu}$ through the relation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}/M_{\mathrm{P}}$. Vainshtein found spherically symmetric solutions around a point object of mass M, and showed that for $r \gg r_V$ the linear terms dominate and for $r \ll r_V$ the non-linear terms dominate, where r is the distance to the source and r_V is given by:

$$r_V = \left(\frac{GM}{m^4}\right)^{1/5}, \tag{2.16}$$

where G is the gravitational Newton's constant. This radius is known as the Vainshtein radius, and determines when non-linearities become relevant. In the limit $m \to 0$, we see that $r_V \to \infty$, and thus the non-linear terms always dominate, and hence there is no regime in which we can trust the Fierz-Pauli theory. This suggests that the vDVZ discontinuity is an artifact of the linear perturbation theory, but the solutions to the full non-linear theory should be smooth in that limit. In fact, the correct limit of GR should be in the regime where the non-linearities dominate, i.e. for $r \ll r_V$ at distances near the source, whereas modifications to GR should appear far away from the source where the description is given by the linear theory. This dynamical mechanism in which GR can be restored due to non-linearities is known as the Vainshtein mechanism, and it happens because the non-linear kinetic terms in eq. (2.15) suppress the coupling between the helicity-0 mode and external matter, making it effectively negligible.

In 1972 Boulware and Deser [65] studied this and other specific non-linear extensions to the Fierz-Pauli theory, and showed that they propagated six degrees of freedom – namely, five polarisations of the massive graviton and one extra Boulware-Deser ghost. This ghost would be infinitely heavy on flat backgrounds, and hence would not be excited in Minkowski space, and would not propagate in the Fierz-Pauli theory. However, the ghost would become light and hence propagate around nontrivial solutions [70], including cosmological backgrounds [76] and weak-field solutions around static matter [77–79], rendering these models unstable. The presence of this ghost became the main obstacle in the process of building non-linear massive gravity for four decades. Furthermore, other pathologies such as strong classical non-linear couplings and a very low cutoff energy for the theory (determining its region of trustability) also came into play.

In 2010 considerable progress was made with the first ghost-free proposal by de Rham, Gabadadze, and Tolley [66] of a non-linear massive gravity theory, known as dRGT massive gravity. This model was motivated by constructions of massive gravity with auxiliary extra dimensions [80], and its action is explicitly given by:

$$S = \frac{M_g^2}{2}\int d^4x\,\sqrt{-g}R(g) - m^2 M_g^2 \int d^4x\,\sqrt{-g}\sum_{n=0}^{4}\beta_n e_n\left(\sqrt{g^{-1}f}\right), \tag{2.17}$$

where $g_{\mu\nu}$ is a dynamical metric with a Ricci scalar $R(g)$, whereas $f_{\mu\nu}$ is a fixed reference metric. Here we have that β_n are free dimensionless coefficients, while M_g, and m are arbitrary mass scales (although it is customary to set $M_g = M_{\mathrm{P}}$). In addition, this action contains interactions between both metrics, which are expressed in terms of the functions $e_n\left(\sqrt{g^{-1}f}\right)$, which correspond to the elementary symmetric polynomials of the eigenvalues λ_n of the matrix $\sqrt{g^{-1}f}$, which satisfies $\sqrt{g^{-1}f}\sqrt{g^{-1}f} = g^{\mu\lambda}f_{\lambda\nu}$[3]. Explicitly, the functions $e_n(\mathbb{X})$ are given by:

$$\begin{aligned}
e_0 &= 1,\\
e_1 &= [\mathbb{X}],\\
e_2 &= \frac{1}{2}([\mathbb{X}]^2 - [\mathbb{X}^2]),\\
e_3 &= \frac{1}{6}([\mathbb{X}]^3 - 3[\mathbb{X}][\mathbb{X}^2] + 2[\mathbb{X}^3]),\\
e_4 &= \det(\mathbb{X}) = \frac{1}{24}([\mathbb{X}]^4 - 6[\mathbb{X}]^2[\mathbb{X}^2] + 3[\mathbb{X}^2]^2 + 8[\mathbb{X}][\mathbb{X}^3] - 6[\mathbb{X}^4]),
\end{aligned} \tag{2.18}$$

[3] Note that there is an ambiguity in $\sqrt{g^{-1}f}$, as different matrices may result in $g^{\mu\lambda}f_{\lambda\nu}$ when squared.

where $\mathbb{X}$ is a matrix and $[\mathbb{X}]$ stands for the trace of $\mathbb{X}$. From these functions we notice that the terms β_0 and β_4 in eq. (2.17) simply describe the presence of cosmological constant terms for each metric, whereas the parameters $\beta_{1,2,3}$ describe genuine interactions between both metrics.

dRGT massive gravity has been studied in detail and it has been proved to be fully ghost free by Hassan and Rosen in [67] (see also [68]). Therefore, it propagates only five degrees of freedom, corresponding to the five polarisations of a massive graviton. In addition, it also equipped with the Vainshtein mechanism, and thus reduces to GR in the massless limit. However, this model is not renormalisable, that is, it contains non-renormalisable operators suppressed by a scale given by:

$$\Lambda_3 = \left(M_g m^2\right)^{1/3}. \tag{2.19}$$

Scattering amplitudes are proportional to powers of (E/Λ_3), where E is the energy of the system. Thus, these amplitudes become order one and strongly coupled when $E \sim \Lambda_3$, and hence perturbative unitarity is broken at that scale. For this reason, massive gravity is considered to be an effective field theory with a cutoff scale given by Λ_3[4]. As an estimation, if $M_g = M_{\mathrm{P}}$ and $m \sim H_0 \sim 10^{-33}$eV (in order to make the mass relevant at cosmological scales), $\Lambda_3 \sim 10^{-13}$.

We notice that, similarly to the Fierz-Pauli action, massive gravity is not diffeomorphism invariant. This is due to the fact that the reference metric $f_{\mu\nu}$ is fixed and thus it does not transform under a change of coordinates, hence making the potential interactions in eq. (2.17) not invariant. However, there is a natural extension to this theory in which the reference metric $f_{\mu\nu}$ is indeed promoted to be a dynamical field. This extension is called massive bigravity and it is manifestly diffeomorphism invariant.

Massive bigravity is a bimetric theory of gravity proposed by Hassan et. al. in [72], and its action is given by:

$$S = \frac{M_g^2}{2}\int d^4x\,\sqrt{-g}R(g) + \frac{M_f^2}{2}\int d^4x\,\sqrt{-f}R(f) - m^2 M_g^2 \int d^4x\,\sqrt{-g}\sum_{n=0}^{4}\beta_n e_n\left(\sqrt{g^{-1}f}\right). \tag{2.20}$$

Here we have two dynamical metric fields: $g_{\mu\nu}$ and $f_{\mu\nu}$, with their associated Ricci scalars $R(g)$ and $R(f)$ and mass scales M_g and M_f, respectively. Massive bigravity has the same potential interactions as dRGT massive gravity, and thus is also free from the Boulware-Deser ghosts.

[4]Strictly speaking, Λ_3 is the strong coupling scale of dRGT massive gravity, but not necessarily its cutoff scale. The cutoff of a theory corresponds to the scale at which the given theory breaks down and new physics is required to describe nature. However, in the case of massive gravity we only know that perturbation theory breaks down at Λ_3, and whether new physics come into play is not yet known. In most cases both scales coincide (like in GR) but examples of theories in which the strong and cutoff scales differ from each other have been found [81].

However, it propagates more degrees of freedom: five polarisations of a massive graviton and two of a massless graviton. In order to illustrate this we calculate the quadratic action of massive bigravity around Minkowski space. We start by considering linear perturbations of both metrics:

$$
\begin{aligned}
g_{\mu\nu} &= \eta_{\mu\nu} + \frac{1}{M_g} h_{\mu\nu}; \quad \frac{1}{M_g} h_{\mu\nu} \ll \eta_{\mu\nu}, \\
f_{\mu\nu} &= \eta_{\mu\nu} + \frac{1}{M_f} l_{\mu\nu}; \quad \frac{1}{M_f} l_{\mu\nu} \ll \eta_{\mu\nu},
\end{aligned}
\tag{2.21}
$$

where $h_{\mu\nu}$ and $l_{\mu\nu}$ are perturbations to be kept only up to second order in eq. (2.20). For simplicity, we consider the so-called "minimal model", where $\beta_0 = 3$, $\beta_1 = -1$, $\beta_2 = \beta_3 = 0$ and $\beta_4 = 1$. In this case, the quadratic action for both perturbations is given by:

$$
\begin{aligned}
S = -\int d^4x \; \Bigg\{ & \frac{1}{4} h^{\mu\nu} \hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} + \frac{1}{4} l^{\mu\nu} \hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu} l_{\alpha\beta} \\
& + \frac{1}{8} m^2 M_{\text{eff}}^2 \left[\left(\frac{h_{\mu\nu}}{M_g} - \frac{l_{\mu\nu}}{M_f} \right) \left(\frac{h^{\mu\nu}}{M_g} - \frac{l^{\mu\nu}}{M_f} \right) - \left(\frac{h}{M_g} - \frac{l}{M_f} \right) \right] \Bigg\},
\end{aligned}
\tag{2.22}
$$

where we have defined an effective mass $M_{\text{eff}}^{-2} = M_g^{-2} + M_f^{-2}$, and h and l denote the trace of the perturbations $h_{\mu\nu}$ and $l_{\mu\nu}$, respectively. We notice that indices for both metrics are raised and lowered with the background Minkowski metric. From eq. (2.22) we see that we have two linearised Einstein-Hilbert kinetic terms for each metric, as well as two mixed Fierz-Pauli-like mass terms. The mixed terms can be diagonalised by performing the following field redefinition:

$$
\frac{u_{\mu\nu}}{M_{\text{eff}}} = \frac{h_{\mu\nu}}{M_g} + \frac{l_{\mu\nu}}{M_f}; \quad \frac{v_{\mu\nu}}{M_{\text{eff}}} = \frac{h_{\mu\nu}}{M_g} - \frac{l_{\mu\nu}}{M_f}. \tag{2.23}
$$

Action (2.22) in terms of the fields $u_{\mu\nu}$ and $v_{\mu\nu}$ becomes:

$$
S = -\int d^4x \; \left\{ \frac{1}{4} u^{\mu\nu} \hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu} u_{\alpha\beta} + \frac{1}{4} v^{\mu\nu} \hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu} v_{\alpha\beta} + \frac{1}{8} m^2 \left[v^{\mu\nu} v_{\mu\nu} - v^2 \right] \right\}, \tag{2.24}
$$

where v denotes the trace of the field $v_{\mu\nu}$. From here it is clear that this model propagates one massless graviton, described by the field $u_{\mu\nu}$, and one massive graviton of mass m, described by $v_{\mu\nu}$. We emphasise that the last term in this action has the Fierz-Pauli structure and thus it is free of ghosts.

As previously mentioned, massive bigravity is now manifestly invariant under coordinate transformations $x^\mu \to \tilde{x}^\mu(\mathbf{x})$

$$\tilde{g}_{\mu\nu}(\tilde{\mathbf{x}}) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(\mathbf{x}); \quad \tilde{f}_{\mu\nu}(\tilde{\mathbf{x}}) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} f_{\alpha\beta}(\mathbf{x}). \tag{2.25}$$

We emphasise that, due to the potential interactions, there is only one copy of diffeomorphism invariance, i.e. both metrics transform with the same coordinate transformation. We note that even though massive bigravity is diffeomorphism invariant, the mass parameter m is still technically natural as its quantum corrections are protected by a second copy of coordinate transformations (that is recovered when $m = 0$). In addition, we mention that massive bigravity is also considered to be an effective field theory, and has the same strong coupling scale Λ_3 as massive gravity.

Next, we discuss how to couple matter to gravity. Whereas in massive gravity it seems natural to couple matter to the dynamical metric $g_{\mu\nu}$ (although alternatives have been considered as well [82–105]), in massive bigravity it is not clear. As a first analysis, in the next chapter we consider the case in which matter is minimally coupled to $g_{\mu\nu}$ only:

$$S_{\mathrm{M}} = \int d^4x \sqrt{-g}\mathcal{L}_{\mathrm{M}}, \tag{2.26}$$

where $\mathcal{L}_{\mathrm{M}}$ is some matter Lagrangian. Therefore, $g_{\mu\nu}$ describes the evolution of spacetime. In this case, as shown in [71], the equations of motion for $g_{\mu\nu}$ and $f_{\mu\nu}$ are given by:

$$R(g)_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R(g) + \frac{m^2}{2}\sum_{n=0}^{3}(-1)^n\beta_n\left[g_{\mu\lambda}Y^\lambda_{(n)\nu}\left(\sqrt{g^{-1}f}\right) + g_{\nu\lambda}Y^\lambda_{(n)\mu}\left(\sqrt{g^{-1}f}\right)\right] = \frac{T_{\mu\nu}}{M_g^2}, \tag{2.27}$$

$$R(f)_{\mu\nu} - \frac{1}{2}f_{\mu\nu}R(f) + \frac{m^2}{2M_*^2}\sum_{n=0}^{3}(-1)^n\beta_{4-n}\left[f_{\mu\lambda}Y^\lambda_{(n)\nu}\left(\sqrt{f^{-1}g}\right) + f_{\nu\lambda}Y^\lambda_{(n)\mu}\left(\sqrt{f^{-1}g}\right)\right] = 0, \tag{2.28}$$

where we have defined $M_*^2 \equiv M_f^2/M_g^2$, and we have used the following relation for the interaction terms:

$$\sqrt{-g}\sum_{n=0}^{4}\beta_n e_n\left(\sqrt{g^{-1}f}\right) = \sqrt{-g}\sum_{n=0}^{4}\beta_n\frac{e_{4-n}\left(\sqrt{f^{-1}g}\right)}{\det\left(\sqrt{g^{-1}f}\right)} = \sqrt{-f}\sum_{n=0}^{4}\beta_{4-n}e_n\left(\sqrt{f^{-1}g}\right), \tag{2.29}$$

where the matrix $\sqrt{f^{-1}g}$ is the inverse of $\sqrt{g^{-1}f}$. Note that to satisfy this relation we need to have $\sqrt{-g}\det(\sqrt{g^{-1}f}) = \sqrt{-f}$. Otherwise, we would have a minus sign in the RHS of eq. (2.29), and hence a minus sign in the interaction terms of eq. (2.28). In addition, we have defined the matrices $Y^{\lambda}_{(n)\mu}(\mathbb{X})$ as:

$$\begin{aligned}
Y_{(0)} =&\mathbb{I}, \\
Y_{(1)} =&\mathbb{X} - \mathbb{I}[\mathbb{X}], \\
Y_{(2)} =&\mathbb{X}^2 - \mathbb{X}[\mathbb{X}] + \frac{1}{2}\mathbb{I}\left([\mathbb{X}]^2 - [\mathbb{X}^2]\right), \\
Y_{(3)} =&\mathbb{X}^3 - \mathbb{X}^2[\mathbb{X}] + \frac{1}{2}\mathbb{X}\left([\mathbb{X}]^2 - [\mathbb{X}^2]\right) - \frac{1}{6}\mathbb{I}\left([\mathbb{X}]^3 - 3[\mathbb{X}][\mathbb{X}^2] + 2[\mathbb{X}^3]\right),
\end{aligned} \tag{2.30}$$

where $\mathbb{I}$ is the identity matrix. In addition, we have to complement the gravitational equations (2.27)-(2.28) with a matter equation. In massive bigravity, due to the presence of diffeomorphism invariance, matter conservation is satisfied:

$$\nabla^{\mu}_{g} T_{\mu\nu} = 0, \tag{2.31}$$

where ∇^{μ}_{g} is the covariant derivative with respect to the metric $g_{\mu\nu}$. Contrary to GR, this equation does not follow from the conservation of the equations of motion. Indeed, if we take the covariant derivative of the equations of motion (2.27)-(2.28) we find that the derivatives of the interaction terms do not vanish, and in fact they lead to a set of consistency constraints which will be referred to as Bianchi constraints.

To summarise, in this chapter we have briefly reviewed the history of massive bigravity. The linear theory for a massive graviton is given by the Fierz-Pauli action, which propagates five polarisations of a spin-2 particle with mass m. This linear model presented a discontinuity in the limit $m \to 0$, known as the vDVZ discontinuity, which could be solved by means of the Vainshtein mechanism. In this mechanism non-linear interactions play a crucial role as they are the ones dominating the massless limit, hence rendering the linear theory inappropriate to describe this limit. In general, these non-linear interactions introduce a Boulware-Deser ghost; a degree of freedom with negative kinetic energy that makes the model unstable. However, specific non-linear terms can be constructed to avoid this ghost, which are the ones given in the dRGT action. This action incorporates the presence of a fixed reference metric, but a natural extension can be made by promoting it to a dynamical field. The resulting model is massive bigravity, a theory that propagates one massive graviton and one massless graviton. In the next chapter we focus on analysing cosmological solutions in massive bigravity.

CHAPTER 3

Cosmological perturbations in massive bigravity

Massive bigravity, as proposed by Hassan and Rosen in [72], is an alternative to general relativity, and an extension of the dRGT massive gravity [66]. One of the main attractions of this model is that it can predict viable cosmological homogeneous and isotropic solutions with late time self-acceleration *without* including a cosmological constant. Furthermore, if one assumes the presence of a large vacuum energy in this model, it has been argued that an appropriate value for the graviton's mass may lead to screening of long wavelength modes, reconciling the value of the *measured* cosmological constant with quantum field theory [74]. As such, massive bigravity seems to be an appealing candidate for a theory of the Universe.

Massive bigravity has five more degrees of freedom than general relativity – due to an extra massive graviton propagating – which could be a source of concern. Only recently has GR been shown to be well-behaved, i.e. that the initial value problem is sufficiently well posed that the theory can be considered classically predictive [106]. With an extra five degrees of freedom, it is conceivable that massive bigravity will not be as obliging. A possible hint of there being any problem would be the presence of classical instabilities and a natural first step would be to study linear cosmological perturbations. A first analysis of such perturbations has been undertaken in [107–110], where unstable solutions on sub-horizon scales were found for some parameters of the theory in homogeneous and isotropic backgrounds[1]. A subsequent analysis in [111] identified a particular class of parameters that lead to stable solutions and, as such, might be used to construct a viable cosmology. In this chapter, we undertake an independent analysis of the evolution and stability of linear cosmological perturbations using the gauge fixing method proposed in [112]. We confirm previous results for scalar perturbations but also analyse vector and tensor perturbations finding a number of interesting instabilities.

[1] As of now, these type of backgrounds have been the only ones considered on cosmological studies of massive gravity.

Our results confirm the obvious: that it is a phenomenologically rich theory which needs to be studied in great detail if it is to be cosmologically considered on par with GR.

The outline of this chapter is as follows. In Section 3.1 we review the standard FLRW cosmological background in the presence of a perfect fluid, and we find the equations of motion for first order cosmological perturbations. Here, we use the formalism developed in [112] to fix the gauge, simplify the problem, and to identify the physical degrees of freedom. In Section 3.2 we study the evolution of the two physical scalar degrees of freedom, in Section 3.3 we study vector perturbations, and in Section 3.4 we study tensor perturbations. Finally, in Section 3.5 we summarise our findings and discuss the prospects of massive bigravity as a viable theory of gravity and cosmology. Throughout this chapter we will be using Planck units and conformal time.

3.1 Cosmological perturbations

In this section we first review previous results on solutions for a homogeneous and isotropic Universe in massive bigravity. We then consider general linear cosmological perturbations and use the standard SVT decomposition [38].

3.1.1 Background

For simplicity we will assume that both metrics share the same characteristics: homogeneous, isotropic and flat:

$$ds_f^2 = Y(\tau)^2[-X(\tau)^2 d\tau^2 + \delta_{ij} dx^i dx^j], \tag{3.1}$$

$$ds_g^2 = a(\tau)^2[-d\tau^2 + \delta_{ij} dx^i dx^j], \tag{3.2}$$

where τ is the conformal time, $a(\tau)$ is the scale factor of the spacetime metric that is coupled to matter, and $X(\tau)$ with $Y(\tau)$ describe the evolution of the metric second $f_{\mu\nu}$ [2].

In addition, we will assume that matter is only coupled to $g_{\mu\nu}$ and is described by a perfect fluid:

$$\bar{T}^{\mu}{}_{\nu} = (P_0 + \rho_0)\bar{u}^{\mu}\bar{u}_{\nu} + P_0 \delta^{\mu}{}_{\nu}, \tag{3.3}$$

where $P_0 = P_0(\tau)$ is the pressure of the fluid, $\rho_0 = \rho_0(\tau)$ its rest energy density and $\bar{u}^{\mu} =$

[2] Note that we do not have the freedom to set $X = 1$, as any time coordinate redefinition that absorbs the dependence on X in the metric $f_{\mu\nu}$, will add a new dependence on X into the metric $g_{\mu\nu}$.

$(1/a, 0, 0, 0)$ its isotropic 4-velocity.

If we replace eq. (3.1)-(3.3) into eq. (2.27)-(2.28), we find the following equations of motion:

$$\mathcal{H}^2 = \frac{a^2}{3}\left[\frac{\rho_0}{M_g^2} + m^2\left(\beta_0 + 3\beta_1 N + 3\beta_2 N^2 + \beta_3 N^3\right)\right], \tag{3.4}$$

$$\mathcal{H}' = \frac{a^2}{2}\left[-\frac{P_0}{M_g^2} - \frac{\mathcal{H}^2}{a^2} + m^2\left(\beta_0 + \beta_1 N\left[2+X\right] + \beta_2 N^2\left[1+2X\right] + \beta_3 N^3 X\right)\right], \tag{3.5}$$

$$h^2 = \frac{a^2}{3}\left(\frac{X^2}{N}\right)\nu^2\left(\beta_1 + 3\beta_2 N + 3\beta_3 N^2 + \beta_4 N^3\right), \tag{3.6}$$

$$h' = \frac{a^2}{2}\left[\frac{2}{a^2}h_x h - \frac{h^2}{a^2} + \left(\frac{X}{N}\right)\nu^2\left(\beta_1 + \beta_2 N[2+X] + \beta_3 N^2[1+2X] + \beta_4 N^3 X\right)\right] \tag{3.7}$$

where it is implicit that all variables depend only on τ and all primes represent conformal time derivatives. We have defined $\mathcal{H} = a'/a$, $h = Y'/Y$, $h_x = X'/X$, $\nu = m/M_*$, and $N = Y/a$. Note that the parameter M_* is redundant, as we can rescale the metric $f_{\mu\nu}$ to make M_* take any value we want and redefine βs such that the action remains invariant. For simplicity, from now on we will use $M_* = 1$.

It is important to note that in order to obtain the previous equations, we had to make a choice for the matrix $\sqrt{g^{-1}f}$. For simplicity, we have chosen the diagonal form: $\sqrt{g^{-1}f} = \mathrm{diag}\,(NX, N, N, N)$. As we will explain later, some solutions allow X to change sign, and therefore this matrix can change sign at some point. Then, in order to satisfy $\sqrt{-g}\,\det(\sqrt{g^{-1}f}) = \sqrt{-f}$, and therefore eq. (2.29), we need to make the unconventional (multivalued) choice of $\sqrt{-g}$ and $\sqrt{-f}$ *without* absolute values, allowing them to change signs. As explained in [113, 114] if $\sqrt{g^{-1}f}$ can change sign we can find continuous solutions through singularities in $f_{\mu\nu}$.

We also have the matter equation of motion:

$$\rho_0' = -3\mathcal{H}(\rho_0 + P_0), \tag{3.8}$$

which has the standard form, as matter has been minimally coupled to the metric $g_{\mu\nu}$. In addition, we have Bianchi constraints for both metrics, resulting from the Bianchi identity of each metric and the local conservation of the matter stress-energy tensor. However, due to the diffeomorphism invariance, they are both equivalent, so we have only one relevant Bianchi constraint, given in this case by:

$$(X\mathcal{H} - h)\left(\beta_1 + 2\beta_2 N + \beta_3 N^2\right) = 0. \tag{3.9}$$

We can easily identify two cases for the solutions:

$\underline{(\beta_1 + 2\beta_2 N + \beta_3 N^2) = 0}$**:** This case leads to a constant $N = \bar{N}$, such that

$$\beta_1 + 2\beta_2 \bar{N} + \beta_3 \bar{N}^2 = 0. \tag{3.10}$$

As a consequence, $\mathcal{H} = h$ and the Friedmann equation becomes:

$$\mathcal{H}^2 = \frac{a^2}{3}\left[\frac{\rho_0}{M_g^2} + \Lambda\right];\ \Lambda = m^2\left(\beta_0 + 3\beta_1\bar{N} + 3\beta_2\bar{N}^2 + \beta_3\bar{N}^3\right), \tag{3.11}$$

which corresponds to general relativity with a cosmological constant. This case is not particularly interesting at the background level as it does not bring new features. Furthermore, as pointed out in [115], when studying first order perturbations, the interaction terms between $g_{\mu\nu}$ and $f_{\mu\nu}$ vanish when imposing the constraint (3.10), and the model results in just two copies of general relativity.

$\underline{(X\mathcal{H} - h) = 0}$**:** This constraint can be replaced into eq. (3.6), and then compared to eq. (3.4), to find the following consistency equation:

$$\tilde{\rho} \equiv \frac{\rho_*}{m^2} = \frac{\beta_1}{N} + 3\beta_2 - \beta_0 + 3N(\beta_3 - \beta_1) + N^2(\beta_4 - 3\beta_2) - N^3\beta_3; \quad \rho_* = \rho_0/M_g^2, \tag{3.12}$$

which relates N and the density ρ_0.

For a standard equation of state $P_0 = w\rho_0$ (with w constant), according to eq. (3.12), at late times ($\tilde{\rho} \ll 1$), N will approach a constant value, and both metrics enter an accelerated de-Sitter phase. However, at early times ($\tilde{\rho} \gg 1$), two types of behaviours can be identified: one where $N \ll 1$ (and $\beta_1 \neq 0$) and another where $N \gg 1$. The branch characterised by $N \ll 1$ at early times will be called *expanding branch*, as in this case both metrics expand in time. While the branch characterised by $N \gg 1$ will be called *bouncing branch*, as in this case $g_{\mu\nu}$ expands but $f_{\mu\nu}$ bounces.

The expanding branch is usually identified as the physical one as, in this case, the contribution of the graviton mass to the Friedmann equation will always be small (for appropriate choices of parameters), as expected. However, in the bouncing branch, the contribution of the graviton mass may be comparable to the matter energy density ρ_0 at early times. Furthermore, in the bouncing branch, if $w > 0$ at early times, then $X < 0$ at early times, and tend to $X = 1$ at late times. This means that X crosses a zero point, where $f_{00} = 0$, and therefore $f^{-1}_{\mu\nu}$ diverges. At this point also $\det(\sqrt{g^{-1}f}) = 0$. As explained in [113, 114], this divergence stays hidden from the matter sector as $g_{\mu\nu}$ does not experience any divergence, and the corresponding vielbein fields are continuous through this point. We confirm this at the level of the background, where no divergence is present in the set of equations of motion eq. (3.4)-(3.7), nor in their solutions[3]. In addition, in the next sections we find non-divergent solutions for linear perturbations through this point. Therefore, our results suggest that this divergence might have a mathematical origin instead of a physical one[4]. Then, even though solutions in the bouncing branch are exotic, they will be analysed in this chapter at the level of perturbations. However, it is clear to us that further research is needed to understand completely the nature of this branch.

Throughout this chapter we will focus on the second branch of solutions satisfying $X\mathcal{H} = h$, as this one brings relevant modifications to general relativity. Background solutions and viable cosmologies in this branch have been studied in detail in [115–120]. Given these results, the next logical step is the study of cosmological perturbations in this background. We will use the standard SVT decomposition, and find the relevant equations of motion for these three types of perturbations.

3.1.2 Scalar perturbations

Let us consider linear scalar perturbations [107–111]. We use the following Ansatz for the perturbed metrics:

$$ds_f^2 = Y^2[-X^2(1 + 2\Phi_1)d\tau^2 + 2B_{1,i}X dx^i d\tau + [(1 - 2\Psi_1)\delta_{ij} + 2E_{1,ij}]dx^i dx^j], \tag{3.13}$$

$$ds_g^2 = a^2[-(1 + 2\Phi_2)d\tau^2 + 2B_{2,i}dx^i d\tau + [(1 - 2\Psi_2)\delta_{ij} + 2E_{2,ij}]dx^i dx^j], \tag{3.14}$$

where ds_f^2 and ds_g^2 are the line elements for the metrics $f_{\mu\nu}$ and $g_{\mu\nu}$ respectively. We read from here that we have four scalar perturbation fields for each metric: Φ_1, B_1, E_1, Ψ_1 for $f_{\mu\nu}$ and Φ_2, B_2, E_2, Ψ_2 for $g_{\mu\nu}$.

For matter, we have a perfect fluid and the scalar perturbations of the perturbed stress-

[3]One might worry about eq. (3.7), as the first term in the RHS contains h_x, which diverges when $X = 0$. However, the full relevant quantity in that equation is $h_x h$, which is finite. This can be seen from eq. (3.6), where we observe that $h \propto X$, cancelling the X in the denominator of h_x and rendering the relevant term finite.

[4]The Ricci scalar associated to the metric $f_{\mu\nu}$ diverges, while the one for $g_{\mu\nu}$ does not. Given that the latter one represents the spacetime metric, it will determine the relevant physical properties of spacetime. Furthermore, the Ricci scalar of $f_{\mu\nu}$ will always appear multiplied by the determinant of $f_{\mu\nu}$, rendering it finite.

energy with an equation of state $P = w\rho$ can be written as:

$$
\begin{aligned}
\delta T^0{}_0 =& -(\rho_0 + P_0)(3\Psi_2 - E_{2,ii} - \chi_{,ii}),\\
\delta T^i{}_0 =& -(\rho_0 + P_0)\chi'_{,i},\\
\delta T^0{}_i =& (\rho_0 + P_0)(B_{2,i} + \chi'_{,i}),\\
\delta T^i{}_j =& w(\rho_0 + P_0)(3\Psi_2 - E_{2,ll} - \chi_{,ll})\delta^i{}_j.
\end{aligned}
\tag{3.15}
$$

Note that we describe matter in a different way to the one presented in Chapter 1. Here, scalar perturbations are written in terms of only one field χ, in a non-conventional but useful way proposed in [38]. Consequently, we have nine scalar fields describing first order perturbations in this theory. As we will see later, from these nine fields there will be only two propagating physical degrees of freedom: one coming from matter perturbations and another one from the helicity-0 mode of the massive graviton. All the other seven scalar fields are simply auxiliary fields, i.e. they appear without time derivatives and therefore they are not physical dynamical fields. This unconventional description for perfect fluid perturbations is useful in order to apply the tools developed in [112] to eliminate ambiguities related to the gauge-symmetry present in the theory.

The action given in eq. (2.20) is invariant under diffeomorphisms, and the nine perturbation scalar fields in the model transform under this symmetry as:

$$
\begin{aligned}
&\tilde{\Phi}_2 = \Phi_2 - \mathcal{H}\pi - \pi'; \quad \tilde{\Psi}_2 = \Psi_2 + \mathcal{H}\pi; \quad \tilde{B}_2 = B_2 + \pi - \epsilon'; \quad \tilde{E}_2 = E_2 - \epsilon,\\
&\tilde{\Phi}_1 = \Phi_1 - [h + h_x]\,\pi - \pi'; \quad \tilde{\Psi}_1 = \Psi_1 + h\pi; \quad \tilde{B}_1 = B_1 - \frac{\epsilon'}{X} + \pi X; \quad \tilde{E}_1 = E_1 - \epsilon,\\
&\tilde{\chi} = \chi + \epsilon,
\end{aligned}
\tag{3.16}
$$

where the fields with tilde denote the fields in the new set of coordinates, and ϵ with π are the two scalar gauge parameters defining an infinitesimal coordinate transformation. As these fields are gauge-dependent, anything you calculate from them will depend on your gauge choice. This ambiguity is usually eliminated by defining a new set of independent gauge-invariant scalar fields. In this chapter we will approach this problem by fixing the gauge in a convenient way, as in [112]. First, we look at the Noether identities associated to the gauge symmetry:

$$
\begin{aligned}
&\mathcal{E}'_{\Phi_1} - \mathcal{E}_{\Phi_1}[h + h_x] + \mathcal{E}_{\Psi_1}h + \mathcal{E}_{B_1}X + \mathcal{E}'_{\Phi_2} + (\mathcal{E}_{\Psi_2} - \mathcal{E}_{\Phi_2})\,\mathcal{H} + \mathcal{E}_{B_2} = 0,\\
&\mathcal{E}_\chi - \mathcal{E}_{E_1} + \left(\frac{\mathcal{E}_{B_1}}{X}\right)' - \mathcal{E}_{E_2} + \mathcal{E}'_{B_2} = 0,
\end{aligned}
\tag{3.17}
$$

where we have denoted $\mathcal{E}_x$ as the equation of motion for the field x. From here we can recognise those fields with redundant equations of motions, and therefore the ones that are good candidates to be fixed with the gauge-freedom. The appropriate candidates are the following:

$$(\Psi_1, \Psi_2) + (E_1, E_2, \chi), \tag{3.18}$$

which means that we can use our two gauge parameters to fix one field of the first parenthesis and one of the second parenthesis. Specifically, we will choose the gauge such that $\Psi_1 = \chi = 0$. The advantages of fixing the gauge, and particularly in this way, is that: (1) we easily simplify the problem by reducing the number of fields by two, (2) we eliminate the redundant equations of motion, and all the remaining ones form the independent set of relevant equations, (3) all the remaining dynamical fields are still gauge-invariant, in the sense that the following gauge-invariant variables:

$$\zeta \equiv \Psi_2 - \frac{1}{3(\rho_0 + P_0)}\delta\rho = \frac{1}{3}(E_{2,ii} + \chi_{,ii}), \tag{3.19}$$

$$\zeta_1 \equiv \frac{1}{3}(E_{1,ii} + \chi_{,ii}), \tag{3.20}$$

become E_2 and E_1 in our gauge-choice, and as we will see later, these two fields are the only physical ones.

After fixing the gauge, let us consider the equation of motions for the seven remaining fields in Fourier space:

$$\begin{aligned}
&2\mathcal{H}\left(3\Psi_2' + k^2 E_2'\right) + \left((1+w)\rho_*(3\Psi_2 + k^2 E_2) + m^2 N Z(3\Psi_2 + k^2(E_2 - E_1))\right) a^2 \\
&+ 2\left(\Psi_2 k^2 + \mathcal{H}(3\Phi_2\mathcal{H} - k^2 B_2)\right) = 0, && (3.21)\\
&2(X+1)\Psi_2' + 2\mathcal{H}(X+1)\Phi_2 - m^2 Z N(X B_1 - B_2) + (1+w)\rho_*(1+X)B_2 = 0, && (3.22)\\
&2(k^2 E_2'' + 3\Psi_2'') + 2\mathcal{H}(3\Phi_2' + 6\Psi_2' + 2k^2 E_2') - 2k^2 B_2' + 3Z a^2 m^2 N(\Phi_1 + \Phi_2)X \\
&+ a^2\left(-3(1+w)\rho_*(2\Phi_2 + w(3\Psi_2 + k^2 E_2)) + 2Nm^2(-3\Phi_2 Z + (3\Psi_2 + k^2(E_2 - E_1))\tilde{Z})\right) \\
&+ 2(9\mathcal{H}^2 - k^2)\Phi_2 + 2k^2(\Psi_2 - 2\mathcal{H}B_2) = 0, && (3.23)\\
&E_2'' - B_2' + 2\mathcal{H}E_2' + (E_2 - E_1)a^2 m^2 N\tilde{Z} - \Phi_2 - 2\mathcal{H}B_2 + \Psi_2 = 0, && (3.24)
\end{aligned}$$

and also

$$2Nhk^2E_1' - a^2\nu^2 Z(k^2E_2 - k^2E_1 + 3\Psi_2)X^2 - 2Nhk^2B_1X + 6\Phi_1h^2N = 0, \tag{3.25}$$

$$2h\Phi_1N(X+1) + \nu^2Xa^2Z(XB_1 - B_2) = 0, \tag{3.26}$$

$$NXE_1'' - N(-2Xh + X')E_1' - X^2\left(B_1'N + N\Phi_1X + 2NB_1h + \nu^2a^2\tilde{Z}(E_2 - E_1)\right) = 0, \tag{3.27}$$

where we have defined $Z = \beta_1 + 2\beta_2N + \beta_3N^2$, and $\tilde{Z} = \beta_1 + \beta_2N(1+X) + \beta_3N^2X$. We have omitted the explicit dependence of variables, but it should be clear that the perturbation fields are now in Fourier space and depend on the conformal time τ and the wavenumber k.

From equations (3.21), (3.22), (3.25), and (3.26) we can see that B_1, B_2, Φ_1 and Φ_2 appear as auxiliary variables, as they do not have any time derivatives and therefore they can be easily worked out in terms of Ψ_2, E_1 and E_2 (see Appendix B.1.1). After replacing these four fields into the remaining three equations, we notice from eq. (3.23) that Ψ_2 becomes an auxiliary variable as all its time derivatives are cancelled. Therefore, we can now work out Ψ_2 in terms of E_1 and E_2. If we do this, we end up with two equations for the only two physical scalar degrees of freedom:

$$E_a'' + c_{ab}E_b' + d_{ab}E_b = 0, \tag{3.28}$$

where the indices a and b can take the values $(1,2)$, and the coefficients c_{ab} and d_{ab} depend on the background functions and the wavenumber k. More specifically, these coefficients depend only on k, $\mathcal{H}$, N and a, which are the four relevant quantities. The explicit expressions for these equations are given in the Appendix B.1.2.

3.1.3 Vector perturbations

Let us consider vector perturbations for both metrics:

$$ds_f^2 = Y^2[-X^2d\tau^2 - 2S_{1i}Xdx^id\tau + (\delta_{ij} + F_{1i,j} + F_{1j,i})dx^idx^j], \tag{3.29}$$

$$ds_g^2 = a^2[-d\tau^2 - 2S_{2i}dx^id\tau + (\delta_{ij} + F_{2i,j} + F_{2j,i})dx^idx^j]. \tag{3.30}$$

From here we can see that the vector perturbations are S_{1i} and F_{1i} for the metric $f_{\mu\nu}$, and S_{2i} and F_{2i} for $g_{\mu\nu}$. These vector fields satisfy:

$$S_i{}^{,i} = F_i{}^{,i} = 0, \tag{3.31}$$

which means that they are purely transverse vectors with no scalar contributions. Here we lower and raise three-space indices by using the Kronecker delta, δ_{ij}, and its inverse, δ^{ij}. The perturbed stress-energy tensor for a perfect fluid coupled to vector perturbations can be written as:

$$
\begin{aligned}
&\delta T^0{}_0 = 0, \\
&\delta T^i{}_0 = -(\rho_0 + P_0)\chi^{iT'}, \\
&\delta T^0{}_i = (\rho_0 + P_0)(\chi^{iT'} - S_{2i}), \\
&\delta T^i{}_j = 0,
\end{aligned}
\tag{3.32}
$$

where $v^{iT} \equiv \chi^{iT'}$ represents the vorticity of the fluid and satisfies $v^{iT}{}_{,i} = 0$. Here, we have used the same non-conventional decomposition as for scalar perturbations, as proposed in [38]. We see that have five vector perturbation fields: two for each metric and one for matter. In GR we only have one propagating vector degree of freedom but it is cosmologically irrelevant as it decays with the expansion of the Universe. However, in massive gravity we will have three degrees of freedom: one from matter and two polarisations from the massive graviton.

In analogy to scalar perturbations, we analyse the gauge symmetry present in the massive bigravity action to fix a gauge. In this case, vector fields transform as:

$$
\tilde{F}_{2i} = F_{2i} - \epsilon_i^T,\ \tilde{S}_{2i} = S_{2i} + \epsilon_i^{T'},\ \tilde{v}_i^T = v_i^T + \epsilon_i^{T'},\ \tilde{F}_{1i} = F_{1i} - \epsilon_i^T,\ \tilde{S}_{1i} = S_{1i} + \epsilon_i^{T'}, \tag{3.33}
$$

where ϵ^{iT} is an infinitesimal arbitrary gauge field, also satisfying $\epsilon^{iT}{}_{,i} = 0$. Consequently, the Noether identity associated to this gauge parameter is:

$$
\mathcal{E}_{F_{2i}} + \mathcal{E}_{F_{1i}} + \mathcal{E}'_{S_{2i}} + \mathcal{E}'_{S_{1i}} + \mathcal{E}'_{v_i^T} = 0, \tag{3.34}
$$

and we can use the gauge freedom to fix either F_{1i} or F_{2i}. With the gauge choice $\tilde{F}_{1i} = 0$, the relevant equations of motion are:

$$
\left((k^2 N + 2m^2 a^2 Z)X + k^2 N\right) S_{1i} - 2m^2 a^2 S_{2i} Z = 0, \tag{3.35}
$$

$$
\begin{aligned}
&- 2a^2 \rho_*(1+X)(1+w)v_i^T + k^2(1+X)F'_{2i} + 2S_{2i}\rho_*(1+X)(1+w)a^2 + S_{2i}k^2(1+X) \\
&+ 2Zm^2 N a^2 (S_{2i} - X S_{1i}) = 0,
\end{aligned}
\tag{3.36}
$$

$$
F''_{2i} + 2\mathcal{H}F'_{2i} + S'_{2i} + m^2 N a^2 \tilde{Z} F_{2i} + 2\mathcal{H}S_{2i} = 0, \tag{3.37}
$$

$$
v_i^{T'} - \mathcal{H}(3w-1)v_i^T - S'_{2i} + (3w-1)\mathcal{H}S_{2i} = 0. \tag{3.38}
$$

We see that S_{1i} and S_{2i} appear as auxiliary variables in equations (3.35) and (3.36). Therefore they can be worked out in terms of the remaining fields. When doing that we obtain only two relevant equations for F_{2i} and the vorticity field v_i^T.

The full equations for the vector field F_{2i} and the vorticity field v_i^T are the following:

$$
\begin{aligned}
&v_i^{T\prime} + \frac{1}{D_v}\Big[-2a^2\rho_* Z(Nk^2+2m^2a^2Z)(1+w)X' - \mathcal{H}\Big(-4k^2a^2\tilde{Z}\rho_* N(1+X)(1+w) \\
&+2a^2m^2\left(-4\rho_* X^2(1+w)a^2+(3w-1)(N^2+X)k^2\right)Z^2 \\
&+(1+X)k^2N\left((3w-1)k^2-2\rho_*(1+X)(1+w)a^2\right)Z\Big)\Big]v_i^T \\
&-\frac{k^2}{D_v}\Big[-X'Z(Nk^2+2m^2a^2Z)+\mathcal{H}\Big(2a^2XZ^2(2X-1+3w)m^2+(X+3w)(1+X)k^2NZ \\
&+2Nk^2\tilde{Z}(1+X)\Big)\Big]F'_{2i} - \frac{\tilde{Z}(k^2(1+X)N+2Xm^2a^2Z)}{2ZN}F_{2i}=0,
\end{aligned}
\tag{3.39}
$$

$$
\begin{aligned}
&F''_{2i} + \frac{1}{D_v}\Big[-X'k^2Z(k^2N+2a^2m^2Z)+\mathcal{H}\Big(4a^2m^2\left(k^2N^2+X(2\rho_*(1+w)a^2+k^2X)\right)Z^2 \\
&+(1+X)\left(4\rho_*(1+w)a^2+(1+X)k^2\right)k^2NZ+2Nk^4\tilde{Z}(1+X)\Big)\Big]F'_{2i} \\
&-\frac{2a^2(1+w)\rho_*}{D_v}\Big[-X'Z(Nk^2+2m^2a^2Z)+\mathcal{H}\Big(4a^2Xm^2Z^2(X-1)+Nk^2(X-1)(1+X)Z \\
&+2Nk^2\tilde{Z}(1+X)\Big)\Big]v_i^T + \frac{\tilde{Z}(2m^2a^2ZN^2+k^2(1+X)N+2Xm^2a^2Z)}{2NZ}F_{2i}=0,
\end{aligned}
\tag{3.40}
$$

where D_v is given by:

$$
D_v = Z[4\rho_* m^2ZX(1+w)a^4+2k^2(m^2N^2Z+\rho_*(1+X)(1+w)N+m^2ZX)a^2+k^4N(1+X)]. \tag{3.41}
$$

Since v_i^T and F_{2i} satisfy $v_i^Tk^i = F_{2i}k^i = 0$, and the equation for v_i^T is of first order, these set of equations actually propagate three degrees of freedom, as expected.

3.1.4 Tensor perturbations

Let us consider tensor perturbations for both metrics:

$$
ds_f^2 = Y^2[-X^2d\tau^2+(\delta_{ij}+h_{1ij})dx^idx^j], \tag{3.42}
$$

$$
ds_g^2 = a^2[-d\tau^2+(\delta_{ij}+h_{2ij})dx^idx^j], \tag{3.43}
$$

such that

$$
{h_{bi}}^i = 0, \quad {h_{bij}}^{,i} = 0\ ;\ b=(1,2). \tag{3.44}
$$

From here we can see that the tensor perturbations are h_{1ij} for the metric $f_{\mu\nu}$, and h_{2ij} for $g_{\mu\nu}$. Here, we use the metric δ_{ij} and its inverse δ^{ij} to lower and raise spatial indices. Since, in the perfect fluid model, there are no tensor matter perturbations, the perturbed stress-energy tensor to be considered here coupled to tensor perturbations h_{ij} is zero.

Because of the constraints given in eq. (3.44), each h_{bij} has two degrees of freedoms, or polarisations, which are usually indicated as $p = +, \times$. More precisely,

$$h_{bij}(\vec{x},\tau) = \int \frac{d^3k}{(2\pi)^{3/2}} h_{bij}(k,\tau) e^{i\vec{k}\cdot\vec{x}}, \quad h_{bij}(k,\tau) = h_{b+}(k,\tau)e^{+}_{ij}(k) + h_{b\times}(k,\tau)e^{\times}_{ij}(k), \tag{3.45}$$

where e^{+}_{ij} and $e^{\times}_{ij}$ are the polarisation tensors, which have the following properties:

$$\begin{aligned} e^{p}_{ij} &= e^{p}_{ji}, \quad k^i e^{p}_{ij} = 0, \quad e^{p}_{ii} = 0, \\ e^{p}_{ij}(k) &= e^{p*}_{ij}(-k), \quad e^{p*}_{ij}(k) e^{p'}_{ij}(k) = 2\delta_{pp'}. \end{aligned} \tag{3.46}$$

Notice also that h_{bij} are gauge invariant and therefore they represent physical degrees of freedom. For simplicity, and without loss of generality, we choose a specific direction $\vec{k} = k\hat{z}$ so tensor perturbations lie in the xy plane. As a result, tensor metric perturbations can be written as:

$$ds^2_f = Y^2 \left[-X^2 d\tau^2 + \left[(1+h_{1+})dx^2 + (1-h_{1+})dy^2 + dz^2 + 2h_{1\times} dxdy \right] \right], \tag{3.47}$$

$$ds^2_g = a^2 \left[-d\tau^2 + \left[(1+h_{2+})dx^2 + (1-h_{2+})dy^2 + dz^2 + 2h_{2\times} dxdy \right] \right], \tag{3.48}$$

where these tensor perturbations now depend only on τ and z. If we replace this Ansatz in the equations of motion (2.27) and (2.28) we find:

$$h''_{2p} + 2\mathcal{H}h'_{2p} + h_{2p}k^2 + m^2 a^2 N \tilde{Z}(h_{2p} - h_{1p}) = 0, \tag{3.49}$$

$$h''_{1p} - (h_x - 2h)h'_{1p} + X^2 k^2 h_{1p} + \frac{X m^2 a^2 \tilde{Z}}{N}(h_{1p} - h_{2p}) = 0. \tag{3.50}$$

Summarising, in this section we described the possible background cosmological solutions in the massive bigravity theory, and found the relevant equations for first order cosmological perturbations. Note that for scalar and vector perturbations all the coefficients in their equations of motion are continuous and finite in the expanding and bouncing branches. However, if we recall that $h_x = X'/X$, for the tensor perturbations we see in eq. (3.50) that the coefficient of h'_{1p} diverges when $X = 0$ in the bouncing branch. Nevertheless, this coefficient is not a problem given that $h'_{1p} = 0$ when $X = 0$, in such a way that the complete second term in eq. (3.50) stays finite, regardless of the initial conditions. We can see this analytically near the bounce time, τ_b, where $X(\tau_b) = 0$. For the large k limit, eq. (3.50) is approximated by:

$$h''_{1p} - \frac{h'_{1p}}{(\tau - \tau_b)} + x_0^2 k^2 (\tau - \tau_b)^2 h_{1p} = 0, \tag{3.51}$$

where we have used that $h = 0$ and $X = x_0 (\tau - \tau_b)$ near τ_b. The solution to this equation is $h_{1p} \propto e^{\pm ikx_0(\tau-\tau_b)^2/2}$, and its derivative is $h'_{1p} \propto (\tau - \tau_b)e^{\pm ikx_0(\tau-\tau_b)^2/2}$, which goes to zero as fast as X when $\tau \to \tau_b$. Similarly, for the small k limit, eq. (3.50) is approximated by:

$$h''_{1p} - \frac{h'_{1p}}{(\tau - \tau_b)} = 0, \tag{3.52}$$

where we have ignored the interaction term with h_{2p}, as this one is proportional to $(\tau - \tau_b)$, and is then negligible. The solution to this equation is $h_{1p} \propto (\tau - \tau_b)^2$, whose derivative also goes to zero as fast as X when $\tau \to \tau_b$.

3.2 Scalar perturbations

In order to study the evolution of the two physical scalar fields, we need to analyse the form of the coefficients given in eq. (3.28). Since it is not possible to find exact analytical solutions to these equations, we focus on a number of different relevant regimes and use suitable approximations in order to have a better understanding of the evolution of perturbations.

3.2.1 Expanding branch

As mentioned before, the expanding branch is characterised by $N \ll 1$ at early times and a de-Sitter phase at late times.

Early times

Let us assume that the early times are dominated by radiation. At this stage, we have $\tilde{\rho} \gg 1$, and $N \ll 1$, therefore, we can expand the solutions in powers of N. For example, at first order we have that eq. (3.12) becomes:

$$\tilde{\rho} = \frac{\beta_1}{N} + \mathcal{O}(N^0);\ \beta_1 > 0 \tag{3.53}$$

Note that at early times this equation is solely characterised in terms of β_1, and therefore all models in this branch will behave in the same way at early times, regardless of the specific values for the other βs. We can then find approximate equations of motion for super-horizon and sub-horizon scales when considering only the leading order terms in $1/N$ in eq. (3.28).

1. **Super-Horizon scales ($x = k\mathcal{H}^{-1} \ll 1$):** the evolution equations reduce to

$$
\begin{aligned}
&E_2'' + 2\mathcal{H}E_2' - \frac{x^2}{15}\mathcal{H}N^2E_1' + 3N^2\mathcal{H}^2(E_2 - E_1) = 0, \\
&E_1'' + 10\mathcal{H}E_1' - \frac{5}{3}\mathcal{H}x^2E_2' + 15\mathcal{H}^2(E_1 - E_2) = 0,
\end{aligned} \tag{3.54}
$$

and when considering only lowest orders in x^2 and N, the solutions are:

$$
\begin{aligned}
&E_2 = c_1 + \frac{c_2}{\tau}, \\
&E_1 = c_1 + \frac{15}{7}\frac{c_2}{\tau} + c_\pm\tau^{n_\pm} \quad n_\pm = \frac{1}{2}(-11 \pm \sqrt{21}) < 0,
\end{aligned} \tag{3.55}
$$

where c_1, c_2 and $c_\pm$ are some integration constants. As we can observe, in this regime both functions are decaying to the same constant c_1.

2. **Sub-Horizon scales ($x = k\mathcal{H}^{-1} \gg 1$):** the evolution equations reduce to

$$
E_2'' + \frac{12\mathcal{H}}{x^2}E_2' - \frac{27}{2}\frac{N^2\mathcal{H}}{x^4}E_1' + \frac{x^2\mathcal{H}^2}{3}E_2 - \frac{45}{2}\frac{N^2\mathcal{H}^2}{x^2}E_1 = 0, \tag{3.56}
$$

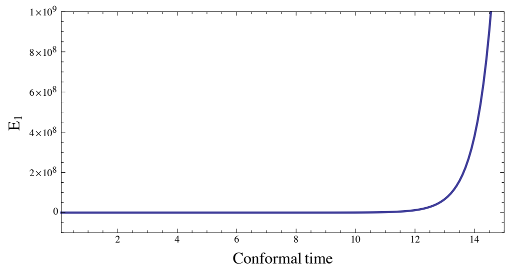
$$
E_1'' + 6\mathcal{H}(E_1' - E_2') - \frac{5}{3}x^2\mathcal{H}^2E_1 + 2x^2\mathcal{H}^2E_2 = 0, \tag{3.57}
$$

and when considering only the highest orders in x^2 the solutions are:

$$
\begin{aligned}
&E_2 \propto e^{\pm ik\tau/\sqrt{3}}, \\
&E_1 = \frac{1}{(k\tau)^3}c_\pm e^{\pm\frac{\sqrt{15}}{3}k\tau} + E_2,
\end{aligned} \tag{3.58}
$$

where $c_\pm$ are some integration constants. We can see that E_2 is oscillating, while E_1 has an exponential instability.

We confirm the general behaviour previously described with numerical plots given in Fig. 3.1, obtained solving the full equations of motion. In this figure we show the evolution of E_1 and E_2 as a function of conformal time (with arbitrary units) at early times during the radiation-dominated era for a given sub-horizon scale- we have set $m^2\beta_1 = 10^{-2}$, with the other βs vanishing, and arbitrary initial conditions of order 1 for both fields. For this plot and all the following numerical plots in this chapter we will set $M_g = 1$. As we expected, E_2 oscillates while E_1 grows exponentially fast, increasing its value in many orders of magnitude, and eventually breaking the validity of linear perturbations. Note that large scales will not be affected by the exponential growth as much as small scales, as the former ones enter the horizon later, and therefore, experience the exponential expansion for a shorter period. Note also that in eq. (3.57), the exponential solution for E_1 is due to the minus sign in the coefficient E_1, which when calculated for a general w, will be negative for $w > -1/2$. Therefore, during the matter-dominated era, there will also be an exponential growth in E_1.



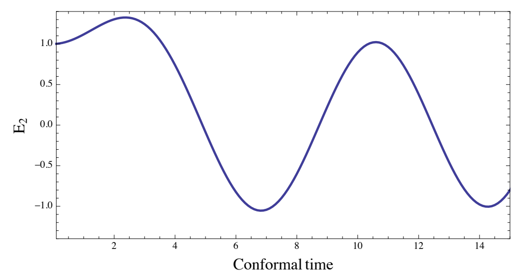


Figure 3.1: Evolution of E_1 and E_2 as a function of τ for a given sub-horizon scales at early times during the radiation-dominated era. We have set $\beta_1 m^2 = 10^{-2}$ and the other βs vanishing.

Furthermore, we can see that in eq. (3.56), we ignored the terms with E_1 and E_1' to find the analytical solutions in eq. (3.58). However, as time goes on, E_1 will grow many orders of magnitude and it will not be possible to discard the coupling between the two fields; E_1 will feed back into the equation for E_2, making this latter field grow as well. Roughly, we expect that to happen when the terms for E_1 are larger than those of E_2 in eq. (3.56), i.e. when $x^7 e^{-\sqrt{15}x/3} \ll N^2$. Fig. 3.2 is a continuation of Fig. 3.1, as it shows the evolution at later times, where we can see the unstable behaviour in E_2.

We have studied the behaviour of scalar perturbations at early times during the radiation-dominated era, showing that generically, there is an exponential instability at early times in both scalar perturbations. During the matter era, the same instability appears. This exponential growth breaks the validity of first order perturbations and therefore we cannot trust the results. This instability could correspond to an actual physical problem of the model, or could be cured by higher order perturbations. A further analysis is needed to understand the nature of this instability and what it tells us, more generally, about the theory.

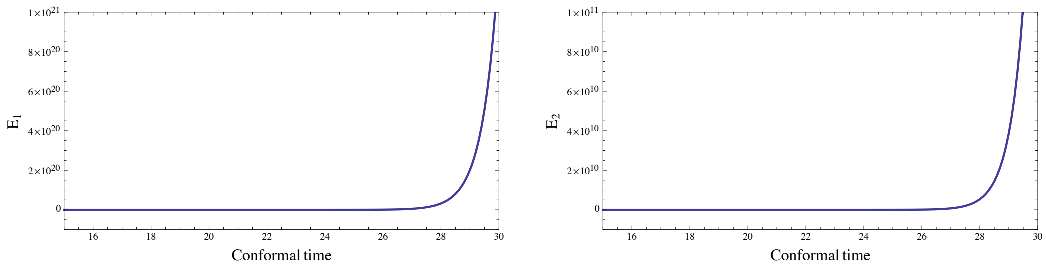


Figure 3.2: Evolution of E_1 and E_2 as a function of τ in the radiation-dominated era for a given scale sub-horizon k. At later times, both perturbation fields are growing exponentially fast, becoming several orders of magnitude larger than their early time value.

Late times

At late times, the background will approach a de-Sitter phase, where $N \to \bar{N}$, $X \to 1$, $\tilde{Z} \to Z = \bar{Z}$, and $\mathcal{H} \to aH_0$, with $\bar{N}$, $\bar{Z}$ and H_0 constants. Notice that the exact value of $\bar{N}$ depends on the parameters β, and also

$$\begin{aligned}
\bar{Z} &= \beta_1 + 2\beta_2\bar{N} + \beta_3\bar{N}^2, \\
H_0^2 &= \frac{1}{3}\frac{m^2}{\bar{N}}\left(\beta_1 + 3\beta_2\bar{N} + 3\beta_3\bar{N}^2 + \beta_4\bar{N}^3\right), \\
a &= \frac{1}{-H_0\tau},
\end{aligned} \tag{3.59}$$

where, in these coordinates, the infinite future is characterised by $\tau \to 0$.

We now study the evolution for super-horizon and sub-horizon scales in this de-Sitter phase, assuming $w = 0$.

1. Super-horizon scales: the evolution equations are now

$$\begin{aligned}
E_2'' + \left(\frac{2\bar{N}^2+1}{\bar{N}^2+1}\right)\mathcal{H}E_2' - \left(\frac{\bar{N}^2}{\bar{N}^2+1}\right)\mathcal{H}E_1' + q\bar{N}\mathcal{H}^2(E_2 - E_1) = 0, \\
E_1'' + \left(\frac{\bar{N}^2+2}{\bar{N}^2+1}\right)\mathcal{H}E_1' - \left(\frac{1}{\bar{N}^2+1}\right)\mathcal{H}E_2' + \left(\frac{q}{\bar{N}}\right)\mathcal{H}^2(E_1 - E_2) = 0,
\end{aligned} \tag{3.60}$$

where $q \equiv m^2\bar{Z}/H_0^2$. These equations are solved by:

$$E_1 = c_0 + c_1\tau^2 + c_\pm\tau^{n_\pm}, \tag{3.61}$$

$$E_2 = c_0 + c_1\tau^2 - \bar{N}^2c_\pm\tau^{n_\pm}, \tag{3.62}$$

where c_0, c_1 and $c_\pm$ are some integration constants, and $n_\pm$ is such that $Re(n_\pm) > 0$.

Therefore, both functions decay to the same constant.

2. Sub-horizon scales: the evolution equations are now

$$E_2'' + \mathcal{H}E_2' - \frac{9}{4}\frac{q[q(\bar{N}^2+1)-2\bar{N}]}{x^4}\mathcal{H}E_1' + \frac{1}{2}q\bar{N}\mathcal{H}^2(E_2 - E_1) = 0,$$
$$E_1'' + 6\mathcal{H}E_1' - 5\mathcal{H}E_2' + x^2\mathcal{H}^2(E_1 - E_2) = 0, \tag{3.63}$$

and when considering only the highest orders in x^2, the solutions are hypergeometric functions with power laws decaying to the same constant.

Figure 3.3 shows numerical results on the evolution of both scalar perturbations in the de-Sitter phase in the matter-dominated era for a given sub-horizon scale. As in previous plots, we considered $m^2\beta_1 = 10^{-2}$ and the other βs vanishing, and arbitrary initial conditions of the same order for both fields. Here both fields are oscillating and approaching the same constant value.

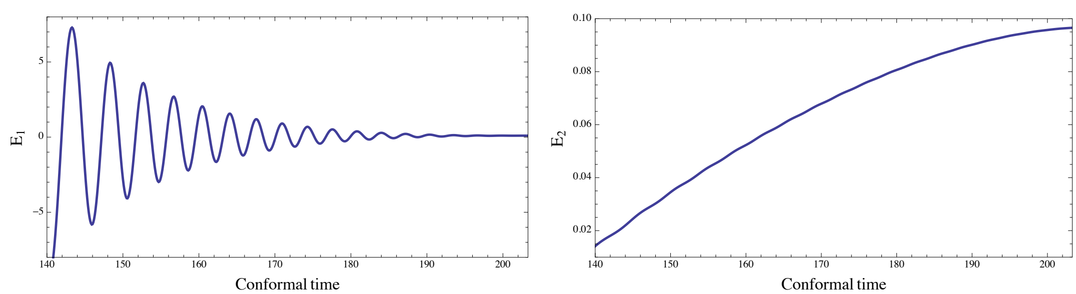


Figure 3.3: Evolution of scalar perturbations as a function of the conformal time during the de-Sitter phase at late times in the matter-dominated era.

3.2.2 Bouncing branch

In this subsection we will show some approximate analytical solutions for the two physical scalar fields in the bouncing branch. First of all, note that the differences between the background evolutions in the expanding and bouncing branches occur only at early times, as at late times in both cases the metrics will enter a de-Sitter phase. Consequently, in the bouncing branch the evolution of perturbations at late times is the same as in the expanding branch. For this reason, in this subsection we focus on early times only. It is relevant in this case to show the evolution during the radiation-dominated era and the matter-dominated era, as fields do not evolve in the same way in both stages.

In this branch we can have different background solutions depending on the parameter values. We will distinguish the following cases: (a) $\beta_3 \neq 0$; (b) $\beta_3 = 0$ and $(\beta_4 - 3\beta_2) \neq 0$; (c) $\beta_3 = 0$ and $(\beta_4 - 3\beta_2) = 0$; (d) $\beta_3 = \beta_2 = 0$. All the viable solutions with other combinations of null parameters are contained in these cases. As stated in [111], only case (d) is physically possible, as all the other cases have an exponential instability for sub-horizon scales at early times, similar to the one found in the expanding branch. For this reason, from now on we study perturbations for case (d) only. For more details about the other cases see Appendix B.1.3.

For case (d), notice that at early times $N \gg 1$ and then eq. (3.12) approximates to:

$$\tilde{\rho} = N^2 \beta_4, \tag{3.64}$$

and therefore we need to impose $\beta_4 > 0$. Conditions on the remaining parameters β_0 and β_1 are also present, as at late times the Friedmann equation (3.4) becomes:

$$\mathcal{H}^2 = \frac{a^2}{3} m^2 \left(\beta_0 + 3\beta_1 \bar{N} \right), \tag{3.65}$$

where $\bar{N}$ is the late time value of the function N. Consequently, we also need to impose $\beta_0 + 3\beta_1 \bar{N} > 0$. In general, we could satisfy this condition when both βs are positive or when one of them is negative (for some appropriate values). However, as we will see later, cases with $\beta_1 < 0$ bring instabilities in the solutions for scalars, vectors and tensor perturbations during the radiation-dominated era. Therefore, from now on we will assume $\beta_1 > 0$.

Early times radiation-dominated era

At early times $N \gg 1$, and therefore we consider only leading order terms in N in the equations of motion and we assume $w = 1/3$. We again study the evolution in super-horizon and sub-horizon scales, focusing on case (d), where $\beta_3 = \beta_2 = 0$.

Super-horizon scales: for super-horizon scales the equations become

$$E_2'' + 2\mathcal{H}E_2' + \frac{9}{2x^2}\frac{\mathcal{H}}{N}\frac{\beta_1}{\beta_4}E_1' - \frac{1}{3}x^2\mathcal{H}^2 E_2 - \frac{m^2\beta_1 a^2 N}{2}E_1 = 0, \tag{3.66}$$

$$E_1'' + 6\frac{\beta_1}{\beta_4}\frac{\mathcal{H}}{N}\left(E_1' - \frac{x^2}{6}E_2'\right) + \frac{x^2\mathcal{H}^2}{3}E_1 - 2m^2\beta_1 a^2 N\frac{x^2}{3}E_2 = 0 \tag{3.67}$$

and when keeping only the lowest orders of x^2 and the highest orders of N, the solutions are $E_2 = c_1 + c_2/\tau$ and $E_1 = c_3 + c_4 \mathrm{erf}(p\tau)$, where c_1, c_2, c_3 and c_4 are some integration constants and $p^2 = 3\beta_1/(\beta_4 N^2 \tau^2) = const$. Therefore, both functions decay to a constant in this regime.

Notice that if β_1 were negative, the solution for E_1 would be $E_1 = c_3 + c_4 \mathrm{erf}(i|p|\tau)$, which would grow exponentially fast, breaking the linear perturbation approximation.

Sub-horizon scales: the evolution equations are now

$$E_2'' + \frac{12}{x^2}\mathcal{H}E_2' + \frac{27}{x^4}\frac{\mathcal{H}}{N}\frac{\beta_1}{\beta_4}E_1' + \frac{1}{3}x^2\mathcal{H}^2E_2 - \frac{3m^2\beta_1 a^2 N}{x^2}E_1 = 0, \tag{3.68}$$

$$E_1'' + 6\frac{\beta_1}{\beta_4}\frac{\mathcal{H}}{N}\left(E_1' - E_2'\right) + \frac{1}{3}x^2\mathcal{H}^2E_1 - 4m^2\beta_1 a^2 N E_2 = 0, \tag{3.69}$$

and when keeping only the terms of order x^2, the solutions are $E_i \propto e^{\pm ik\tau/\sqrt{3}}$. Unlike in the expanding branch, in this case scalar perturbations are well behaved.

Fig. 3.4 shows numerical results for the evolution of scalar perturbations as a function of the conformal time (in arbitrary units), for a given sub-horizon scale during the radiation-dominated era at early times, confirming our previous analytical results. In this particular case we set $m^2\beta_1 = m^2\beta_4 = 10^{-2}$, and arbitrary initial conditions of order 1 for both fields.

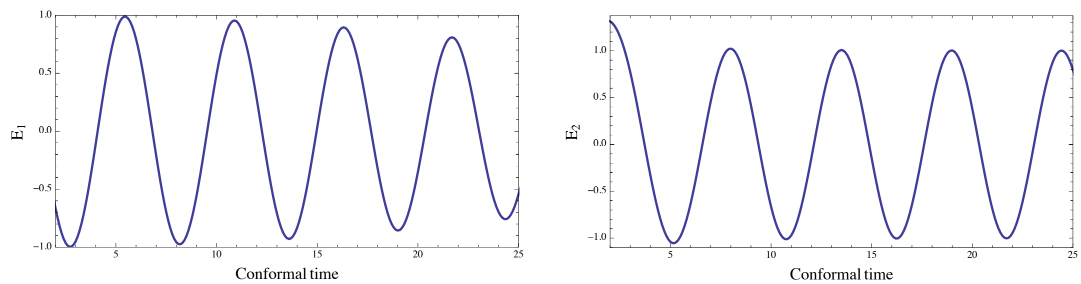


Figure 3.4: Evolution of scalar perturbations as a function of τ, during early times in the radiation-dominated for a given sub-horizon scales.

Early times matter-dominated era

As above, let us consider only leading terms in N but now assume $w = 0$.

Super-horizon scales: the evolution equations are

$$E_2'' + 2\mathcal{H}E_2' - 2\frac{\mathcal{H}}{N}\frac{\beta_1}{\beta_4}E_1' - \frac{1}{3}x^2\mathcal{H}^2E_2 - m^2\beta_1 a^2 N E_1 = 0, \tag{3.70}$$

$$E_1'' + \frac{5}{2}\mathcal{H}E_1' - \mathcal{H}\frac{x^2}{3}E_2' + \frac{5}{6}x^2\mathcal{H}^2E_1 - \frac{1}{3}x^2\mathcal{H}^2E_2 = 0, \tag{3.71}$$

and when keeping only terms with the lowest orders in x^2 (and highest powers in N) we get: $E_i = c_{1i} + c_{2i}/\tau^{n_i}$, where c_{1i} and c_{2i} are some integration constants, and $n_1 = 4$ and $n_2 = 3$.

Sub-horizon scales the evolution equations now reduce to

$$E_2'' + \mathcal{H}E_2' + \frac{27}{2x^4}\frac{\mathcal{H}}{N}\frac{\beta_1}{\beta_4}E_1' - \frac{3}{2}\mathcal{H}^2 E_2 - \frac{m^2\beta_1 a^2 N}{2}E_1 = 0, \tag{3.72}$$

$$E_1'' + \frac{3}{2}\mathcal{H}E_1' - \frac{1}{2}\mathcal{H}E_2' + \frac{1}{2}\mathcal{H}^2 x^2 E_1 - \mathcal{H}^2 E_2 = 0, \tag{3.73}$$

and when considering only the highest orders in x^2 (and highest powers in N) the solutions are $E_1 \propto e^{\pm ik\tau/\sqrt{2}}$ and $E_2 = c_1/\tau^3 + c_2\tau^2$, where c_1 and c_2 are some integration constants. Here we can see that E_2 grows as a power law in time, which will affect E_1 at later times, where this one will also start to grow as a power law.

Fig. 3.5 shows numerical solutions for both scalar fields for a given sub-horizon scale during early times in the matter-dominated era. In this case we set $m^2\beta_1 = m^2\beta_4 = 10^{-2}$, and arbitrary initial conditions of order one for both fields. As found in the analytical solutions, E_1 oscillates while E_2 grows as a power law.

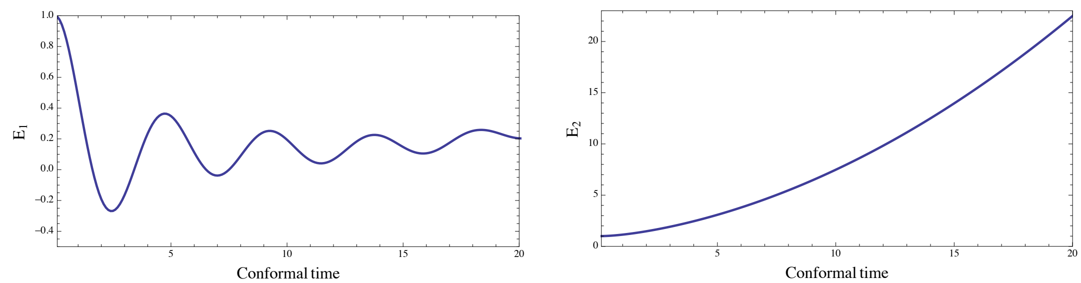


Figure 3.5: Evolution of scalar perturbations as a function of τ, during early times in the matter-dominated era for a given sub-horizon scale.

Analogous to the results for the expanding branch, in this case the quadratic growth in E_2 will affect E_1 at later times, making the latter field grow as a power law as well, as we observe in Fig. 3.6 (this figure is a continuation of Fig. 3.5).

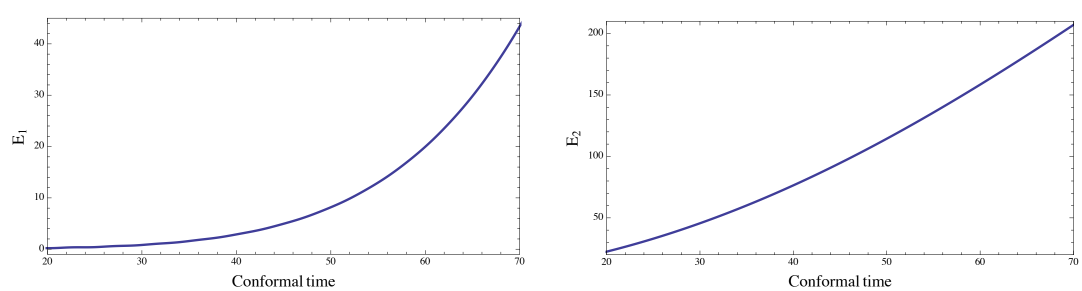


Figure 3.6: Evolution of scalar perturbations during early times in the matter-dominated for a given sub-horizon scale.

In addition, we can study the evolution of the gauge-invariant form for the density contrast $\delta_{GIk} = \delta\rho_{GI}/\rho_0$,

$$\delta_{GIk} = [\delta\rho + \rho_0'(B_2 - E_2')]/\rho_0, \tag{3.74}$$

where $\delta\rho$ is given by the $\delta T^0{}_0$ in eq. (3.15). After fixing the gauge, and eliminating the auxiliary variables, δ_{GIk} can be expressed entirely in terms of E_i and E_i' (see Appendix B.1.4). In Fig. 3.7 we see numerical results for the evolution of $d\ln\delta_{GIk}/d\ln a$ as a function of the conformal time (in arbitrary units) for a given sub-horizon scale during the matter-dominated era. In this case we have also set $m^2\beta_1 = m^2\beta_4 = 10^{-4}$. We observe that at early times δ_{GIk} grows nearly proportional to the scale factor a, and then it starts decaying faster as we enter into the de-Sitter phase, analogously to GR. A more detailed study on the comparison of this model with observations was done in [111].

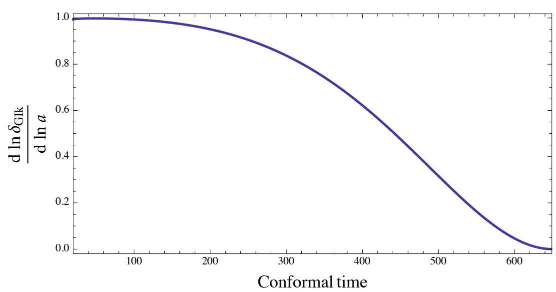


Figure 3.7: Evolution of density constrast as a function of the conformal time τ in the matter-dominated era for a given sub-horizon scale.

It is important to remark that even though classical scalar fields do not evidence exponential instabilities in this branch, they do not satisfy the Higuchi bound (see Appendix B.2 for details), and therefore one scalar field propagates as a ghost, i.e. with a negative kinetic term. Consequently, instabilities might appear when studying higher order perturbations, and negative norm states are expected to appear in a quantum theory of massive gravity (see [121]).

3.3 Vector perturbations

Analogously to the previous section, we now study the evolution of vector perturbations in different regimes, by making relevant approximations to the full equations of motion given by eq. (3.39)-(3.40).

3.3.1 Expanding branch

Recall that the expanding branch is characterised by $N \ll 1$ at early times and a de-Sitter phase at late times.

Early times radiation-dominated era

Considering $w = 1/3$ and leading order terms in $1/N$, the equations for vector perturbations become:

$$F''_{2i} + \frac{2(4x^4 + 33x^2 + 40)\mathcal{H}}{(8 + x^2)(x^2 + 5)} F'_{2i} - \frac{16(3x^2 + 20)\mathcal{H}}{(8 + x^2)(x^2 + 5)} v_i^T + 3(x^2 + 5)\mathcal{H}^2 F_{2i} = 0, \tag{3.75}$$

$$v_i^{T\prime} + \frac{8(8x^2 + 50)\mathcal{H}}{(8 + x^2)(x^2 + 5)} v_i^T - \frac{2(4x^2 + 25)\mathcal{H}x^2}{(8 + x^2)(x^2 + 5)} F'_{2i} - 3(x^2 + 5)\mathcal{H}^2 F_{2i} = 0, \tag{3.76}$$

where $x = k/\mathcal{H}$.

Super-horizon scales: the evolution equations reduce to

$$F''_{2i} + 2\mathcal{H}F'_{2i} - 8\mathcal{H}v_i^T + 15\mathcal{H}^2 F_{2i} = 0, \tag{3.77}$$

$$v_i^{T\prime} + 10\mathcal{H}v_i^T - \frac{5}{4}x^2\mathcal{H}F'_{2i} - 15\mathcal{H}^2 F_{2i} = 0, \tag{3.78}$$

and, ignoring terms of order x^2, the solutions are $F_{2i} = c_1/\tau + c_\pm \tau^{n_\pm}$ and $v_i^T = c_2/\tau^2 + b_\pm \tau^{n_\pm}$, where $n\pm < 0$, and where c_1, c_2, $c_\pm$ and $b_\pm$ are some integration constants related to each other. Therefore, both vector perturbations decay to zero in this regime.

Sub-horizon scales: the evolution equations reduce to

$$F''_{2i} + 8\mathcal{H}F'_{2i} - \frac{48}{x^2}\mathcal{H}v_i^T + 3x^2\mathcal{H}^2 F_{2i} = 0, \tag{3.79}$$

$$v_i^{T\prime} + \frac{64}{x^2}\mathcal{H}v_i^T - 8\mathcal{H}F'_{2i} - 3x^2\mathcal{H}^2 F_{2i} = 0, \tag{3.80}$$

and when ignoring terms of order x^{-2}, the solutions are

$$F_{2i} \propto e^{\pm ik\sqrt{3}\tau}/\tau^4, \tag{3.81}$$

$$v_i^T = c_1 - c_\pm e^{\pm ik\sqrt{3}\tau}/\tau^4, \tag{3.82}$$

where c_1 and $c_\pm$ are come integration constants related to those of F_{2i}. Therefore, in this regime both functions decay as a^4.

Figure 3.8 shows numerical results for the evolution of vector perturbations as a function of τ, during early times in the radiation-dominated era for a given sub-horizon scale; we have set $m^2\beta_1 = 10^{-2}$ while all other βs are vanishing, and we have chosen arbitrary initial conditions of the same order for both fields. We can clearly see that both fields decay in the same way, but while F_{2i} is oscillating around 0, v_i^T oscillates around a constant value. We find similar behaviour the matter-dominated era.

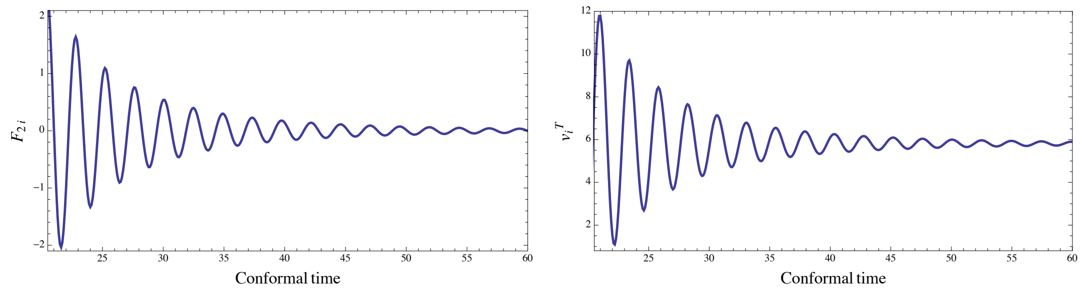


Figure 3.8: Evolution of vector perturbations as a function of τ, during early times in the radiation-dominated era for a given sub-horizon scale.

Late times

We now assume $w = 0$ and a de-Sitter spacetime where N takes the constant value $\bar{N}$, and $a \propto 1/\tau$, with $\tau \to 0$ being the infinite future.

Super-horizon scales: the evolution equations are

$$F_{2i}'' + 2\mathcal{H}F_{2i}' + a^2 x_2 F_{2i} = 0,\ x_2 = m^2\bar{Z}(\bar{N}^2+1)/\bar{N}, \tag{3.83}$$

$$v_i^{T'} + \mathcal{H}v_i^T - \frac{1}{\bar{N}^2+1}\mathcal{H}F_{2i}' - a^2 x_1 F_{2i} = 0,\ x_1 = m^2\bar{Z}/\bar{N}, \tag{3.84}$$

and are solved by $F_{2i} \propto \tau^{n_\pm}$ and $v_i^T = c_1\tau + c_\pm\tau^{n\pm-1}$; $Re(n_\pm) > 1$, while c_1 and $c_\pm$ are integration constants related to those of F_{2i}.

Sub-horizon scales: the evolution equations reduce to

$$F_{2i}'' + 4\mathcal{H}F_{2i}' + x^2\mathcal{H}^2 F_{2i} = 0, \tag{3.85}$$

$$v_i^{T'} + \mathcal{H}v_i^T - 3\mathcal{H}F_{2i}' - x^2\mathcal{H}^2 F_{2i} = 0, \tag{3.86}$$

and are solved by $F_{2i} \propto \tau^2 e^{\pm ik\tau}$ and $v_i^T = c_1\tau + c_\pm\tau^2 e^{\pm ik\tau}$, where c_1 and $c_\pm$ are integration constants related to those of F_{2i}. In this case, both perturbations are decaying.

Figure 3.9 shows numerical solutions for the evolution of vector perturbations as a function of τ, during late times for a given sub-horizon scale. Again, we have set $m^2\beta_1 = 10^{-2}$ and all other βs vanishing, and arbitrary initial conditions of the same order for both fields. Both fields oscillate and decay in the same way, but while F_{2i} is oscillating around 0, v_i^T oscillates around a decaying function.

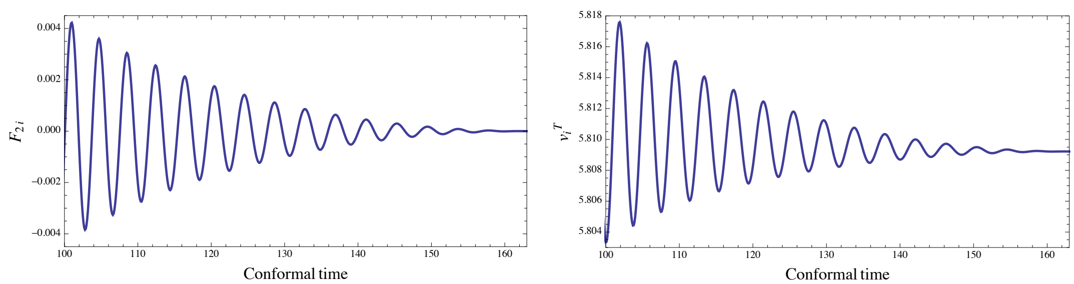


Figure 3.9: Evolution of vector perturbations during late times in the de-Sitter phase for a sub-horizon scale.

3.3.2 Bouncing branch

As we have mentioned before, the bouncing branch is characterised by $N \gg 1$ at early times and a de-Sitter phase at late times. Next, we study the evolution of vector perturbations at early times in the same way we previously did for scalar perturbations.

Early times radiation-dominated era

We will start by assuming $w = 1/3$. When considering only leading terms in N, the equations of motion become:

$$F_{2i}'' + \frac{20\mathcal{H}}{(x^2+10)}F_{2i}' + \frac{16\mathcal{H}}{x^2+10}v_i^T + \frac{3\beta_1}{2\beta_4}\frac{\mathcal{H}^2}{N}(2+x^2)F_{2i} = 0, \tag{3.87}$$

$$v_i^{T\prime} + \frac{8}{3}\frac{\mathcal{H}}{N}\frac{(9\beta_1^2 - 4\beta_0\beta_4)}{\beta_1\beta_4(x^2+10)}v_i^T - \frac{x^2}{3}\frac{\mathcal{H}}{N}\frac{(9\beta_1^2 - 4\beta_0\beta_4)}{\beta_1\beta_4(x^2+10)}F_{2i}' - \frac{3\beta_1}{2\beta_4}\frac{x^2\mathcal{H}^2}{N}F_{2i} = 0. \tag{3.88}$$

We now study these equations for sub-horizon and super-horizon scales.

Super-horizon scales: the evolution equations reduce to

$$F_{2i}'' + 2\mathcal{H}F_{2i}' + \frac{8}{5}\mathcal{H}v_i^T + \frac{3\beta_1}{\beta_4}\frac{\mathcal{H}^2}{N}F_{2i} = 0, \tag{3.89}$$

$$v_i^{T\prime} + \frac{4}{15}\frac{\mathcal{H}}{N}\frac{(9\beta_1^2 - 4\beta_0\beta_4)}{\beta_1\beta_4}v_i^T - \frac{x^2}{30}\frac{\mathcal{H}}{N}\frac{(9\beta_1^2 - 4\beta_0\beta_4)}{\beta_1\beta_4}F_{2i}' - \frac{3\beta_1}{2\beta_4}\frac{x^2\mathcal{H}^2}{N}F_{2i} = 0. \tag{3.90}$$

Ignoring terms of order x^2 and lowest order terms of N, the solutions are $F_{2i} = c_1 + c_2/\tau$ and $v_i^T \propto e^{-p^2\tau^2}$, where c_1 and c_2 are some integration constants, and $p^2 = 2(9\beta_1^2 - 4\beta_0\beta_4)/(15\beta_1\beta_4 N\tau^2) = const$. Notice that here we have assumed that $(9\beta_1^2 - 4\beta_0\beta_4)/(\beta_1\beta_4) > 0$, since otherwise v^{iT} would grow exponentially fast, creating an instability in the solutions.

Sub-horizon scales: the evolution equations reduce to

$$F_{2i}'' + \frac{20}{x^2}\mathcal{H}F_{2i}' + \frac{16\mathcal{H}}{x^2}v_i^T + \frac{3\beta_1}{2\beta_4}\frac{\mathcal{H}^2x^2}{N}F_{2i} = 0, \tag{3.91}$$

$$v_i^{T\prime} + \frac{8}{3}\frac{\mathcal{H}}{N}\frac{(9\beta_1^2 - 4\beta_0\beta_4)}{\beta_1\beta_4 x^2}v_i^T - \frac{1}{3}\frac{\mathcal{H}}{N}\frac{(9\beta_1^2 - 4\beta_0\beta_4)}{\beta_1\beta_4}F_{2i}' - \frac{3\beta_1}{2\beta_4}\frac{x^2\mathcal{H}^2}{N}F_{2i} = 0. \tag{3.92}$$

Considering only highest order terms in x^2, the solutions are $F_{2i} \propto e^{\pm iK\tau^2/2}/\sqrt{\tau}$ and $v_i^T \propto e^{\pm iK\tau^2/2}\sqrt{\tau}$, where $K^2 = \frac{3\beta_1}{2\beta_4}\frac{k^2}{N\tau^2}$. We then see that, contrary to GR, F_{2i} decays but the vorticity field v_i^T grows. This modification happens as the dominant term in eq. (3.92) corresponds to the interaction term with F_{2i} instead of the term with v_i^T.

Notice that if β_1 were negative, solutions for F_{2i} and v^{iT} would be combinations of Bessel I and K functions, which would grow exponentially fast, creating an instability in the solutions.

Figure 3.10 shows numerical results for the evolution of vector perturbations as a function of τ, during early times for a given sub-horizon scale in the radiation-dominated era. In this case we have set $m^2\beta_1 = m^2\beta_4 = 10^{-2}$, and arbitrary initial conditions of the same order for both fields. As expected due to the analytical solutions, F_{2i} decays in time while v_i^T grows.

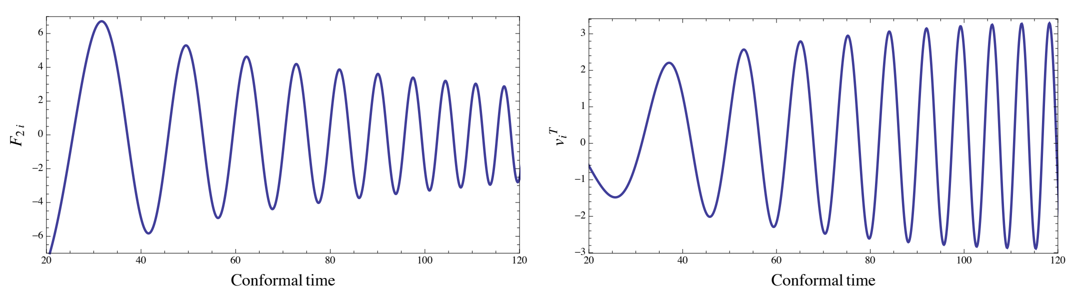


Figure 3.10: Evolution of vector perturbations as a function of τ, during early times in the radiation-dominated era for a sub-horizon scale.

Early times matter-dominated era

Let us now assume that $w = 0$, and consider only leading order terms in N in the equations of motion to find

$$F_{2i}'' + \mathcal{H}\frac{(5x^2+24)}{2(x^2+6)}F_{2i}' - \frac{3\mathcal{H}}{x^2+6}v_i^T + \frac{1}{4}x^2\mathcal{H}^2F_{2i} = 0, \tag{3.93}$$

$$v_i^{T\prime} + \mathcal{H}\frac{(x^2+15)}{(x^2+6)}v_i^T - \frac{3}{2}\mathcal{H}\frac{x^2}{(x^2+6)}F_{2i}' - \frac{1}{4}x^2\mathcal{H}^2F_{2i} = 0. \tag{3.94}$$

Super-horizon scales: the evolution equations become

$$F_{2i}'' + 2\mathcal{H}F_{2i}' - \frac{1}{2}\mathcal{H}v_i^T + \frac{1}{4}x^2\mathcal{H}^2F_{2i} = 0, \tag{3.95}$$

$$v_i^{T\prime} + \frac{15}{6}\mathcal{H}v_i^T - \frac{1}{4}\mathcal{H}x^2F_{2i}' - \frac{1}{4}x^2\mathcal{H}^2F_{2i} = 0, \tag{3.96}$$

and, when ignoring terms of order x^2, the solutions are $F_{2i} = c_1/\tau^4 + c_2/\tau^3 + c_3$ and $v_i^T \propto 1/\tau^5$, where c_1, c_2 and c_3 are some integration constants. Therefore, both functions decay in time.

Sub-horizon scales the evolution equations now reduce to

$$F_{2i}'' + \frac{5}{2}\mathcal{H}F_{2i}' - \frac{3\mathcal{H}}{x^2}v_i^T + \frac{1}{4}x^2\mathcal{H}^2F_{2i} = 0, \tag{3.97}$$

$$v_i^{T\prime} + \mathcal{H}v_i^T - \frac{3}{2}\mathcal{H}F_{2i}' - \frac{1}{4}x^2\mathcal{H}^2F_{2i} = 0, \tag{3.98}$$

and, when ignoring terms of order x^{-2}, the solutions are $F_{2i} \propto e^{\pm ik\tau/2}/\tau^{3/2}$ and $v_i^T = c_1/\tau^2 + c_\pm e^{\pm ik\tau/2}/\tau^{3/2}$, where c_1 and $c_\pm$ are integration constants.

Figure 3.11 shows numerical results for the evolution of vector perturbations as a function of τ, during early times for a given sub-horizon scale in the matter-dominated era. In this case we have set $m^2\beta_1 = m^2\beta_4 = 10^{-2}$, and arbitrary initial conditions of the same order for both fields. With these plots we confirm our analytical results.

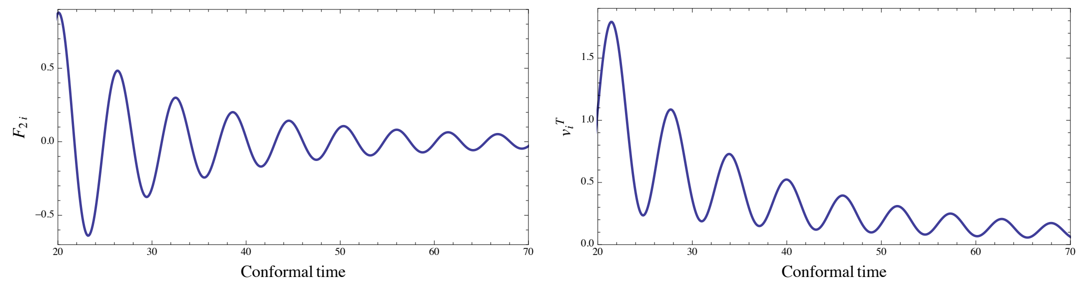


Figure 3.11: Evolution of vector perturbations as a function of τ, during early times in matter-dominated era for a given sub-horizon scale.

3.4 Tensor perturbations

In this section we find approximate analytical solutions for the tensor modes in the relevant regimes for both branches. As mentioned previously, in the bouncing branch, we restrict our study of the tensor modes for the case $\beta_3 = \beta_2 = 0$.

3.4.1 Expanding branch

As before, we study the solutions of tensor perturbations at early and late times.

Early times

Let us consider only leading order terms in $1/N$, as $N \ll 1$ at early times in this branch. In this approximation eq. (3.49)-(3.50) become:

$$h''_{2p} + 2\mathcal{H}h'_{2p} + x^2\mathcal{H}^2 h_{2p} + m^2 a^2 N\beta_1(h_{2p} - h_{1p}) = 0, \tag{3.99}$$

$$h''_{1p} + 2(4+3w)\mathcal{H}h'_{1p} + (4+3w)^2 x^2\mathcal{H}^2 h_{1p} + 3(4+3w)\mathcal{H}^2(h_{1p} - h_{2p}) = 0. \tag{3.100}$$

Super-horizon scales: the equations simplify to the form

$$h''_{2p} + 2\mathcal{H}h'_{2p} = 0, \tag{3.101}$$

$$h''_{1p} + 10\mathcal{H}h'_{1p} + 15\mathcal{H}^2(h_{1p} - h_{2p}) = 0, \tag{3.102}$$

and are solved by $h_{2p} = c_1 + c_2/\tau$ and $h_{1p} = c_3 + c_4/\tau + c_\pm\tau^{n_\pm}$, with $n_\pm = -(9\pm\sqrt{21})/2 < 0$, where c_1, c_2, c_3, c_4 and $c_\pm$ are integrations constants, related to each other. Therefore, both solutions decay to a constant.

Sub-horizon scales: the evolution equations become

$$h_{2p}'' + 2\mathcal{H}h_{2p}' + x^2\mathcal{H}^2 h_{2p} + \mathcal{O}(N^{3/2})(h_{2p} - h_{1p}) = 0, \tag{3.103}$$

$$h_{1p}'' + 10\mathcal{H}h_{1p}' + 25x^2\mathcal{H}^2 h_{1p} = 0, \tag{3.104}$$

with solutions $h_{2p} \propto e^{\pm ik\tau}/\tau$ and $h_{1p} \propto e^{\pm i5k\tau}/\tau^5$.

Unlike scalar perturbations, tensor perturbations in the expanding branch are not unstable- they oscillate and decay. We find the same behaviour in the matter-dominated era. Fig. 3.12 shows numerical results for the evolution of both tensor perturbations as a function of τ (in arbitrary units), at early times during the radiation-dominated era for a given sub-horizon scale. In this particular case we set $m^2\beta_1 = 10^{-2}$, and all other βs vanishing, and arbitrary initial conditions of the same order for both fields. As expected due to the analytical solutions, we observe that h_{1p} decays faster than h_{2p}.

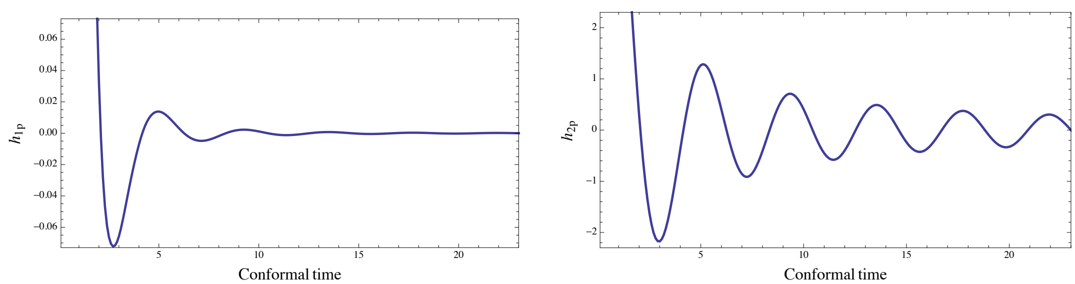


Figure 3.12: Evolution of tensor perturbations as a function of the conformal time during early times in the radiation-dominated era for a given sub-horizon scale.

Late times

Now, let us study the behaviour in the de-Sitter phase, in the matter-dominated era. In this phase N takes the constant value $\bar{N}$, and $a \propto 1/\tau$, with $\tau \to 0$ being the infinite future. The equations of motion become:

$$h_{2p}'' + 2\mathcal{H}h_{2p}' + k^2 h_{2p} + x_2\mathcal{H}^2(h_{2p} - h_{1p}) = 0, \tag{3.105}$$

$$h_{1p}'' + 2\mathcal{H}h_{1p}' + k^2 h_{1p} + x_1\mathcal{H}^2(h_{1p} - h_{2p}) = 0, \tag{3.106}$$

where $x_2 = m^2\bar{N}\tilde{\bar{Z}}/H_0^2$ and $x_1 = m^2\tilde{\bar{Z}}/(H_0^2\bar{N}) = x_2/\bar{N}^2$.

Super-horizon scales: the evolution equations simplify to

$$h''_{2p} + 2\mathcal{H}h'_{2p} + x_2\mathcal{H}^2(h_{2p} - h_{1p}) = 0, \tag{3.107}$$

$$h''_{1p} + 2\mathcal{H}h'_{1p} + x_1\mathcal{H}^2(h_{1p} - h_{2p}) = 0, \tag{3.108}$$

and are solved by $h_{1p} = c_1 + c_2\tau^3 + c_\pm\tau^{n_\pm}$ and $h_{2p} = c_1 + c_2\tau^3 - \frac{x_2}{x_1}c_\pm\tau^{n_\pm}$, where c_1, c_2 and $c\pm$ are integration constants and $n_\pm = \frac{1}{2}(3 \pm \sqrt{9 - 4x_1 - 4x_2})$. Since $Re(n_\pm) > 0$, both solutions decay in time to a constant.

Sub-horizon scales: the evolution equations now become

$$h''_{2p} + 2\mathcal{H}h'_{2p} + k^2 h_{2p} = 0, \tag{3.109}$$

$$h''_{1p} + 2\mathcal{H}h'_{1p} + k^2 h_{1p} = 0, \tag{3.110}$$

and are solved by $h_{bp} \propto e^{\pm ik\tau}\tau$, which are decaying, as in this regime $\tau \to 0$ in the infinite future.

Note that since h_{1p} decays considerably faster than h_{2p} during early times, h_{2p} could start in the de-Sitter phase being some orders of magnitude larger that h_{1p} (which will happen if the initial conditions at early times for both fields were of the same order of magnitude). In this case, there is an intermediate phase in the full solutions of eq. (3.105)-(3.106), when the $k^2 h_{1p} \sim x_1\mathcal{H}^2 h_{2p}$. In this phase h_{2p} could affect the evolution of h_{1p}, as h_{1p} will start growing,"reaching" the magnitude of h_{2p}, until $k^2 \ll x_1\mathcal{H}^2$, when the scale is super-horizon, and both fields will approach the same constant.

Fig. 3.13 shows numerical solutions for tensor perturbations as a function of τ (in arbitrary units) at late times for a given sub-horizon scale. In this particular case we set $m^2\beta_1 = 10^{-2}$ and all the other βs vanishing, and arbitrary initial conditions of the same order for both fields. In this case we observe that since h_{1p} starts in the de-Sitter phase being at least two orders of magnitude smaller than h_{2p}, the previously described intermediate phase occurs, where h_{1p} grows while h_{2p} decays as expected for a sub-horizon scale. Generically, for different initial conditions, we would see a phase where h_{1p} first decays and then it grows.

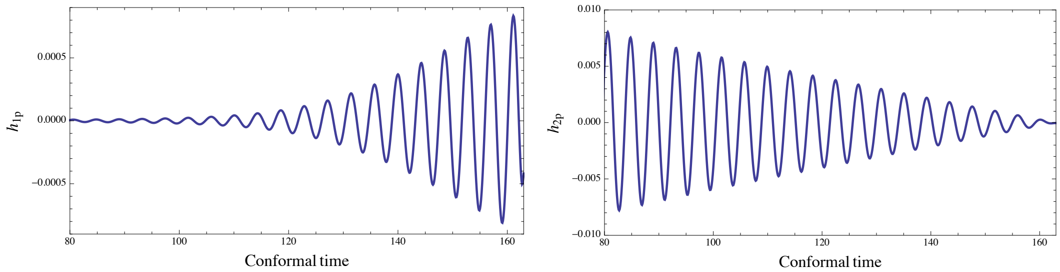


Figure 3.13: Evolution of tensor perturbations as a function of the conformal time during the de-Sitter phase at late times in the matter-dominated era.

3.4.2 Bouncing branch

As before, we only focus on early times as the evolution at late times will be the same as in the expanding branch. We study the radiation-dominated era and matter-dominated era. At early times, we consider only the leading order terms in N in all the coefficients in eq. (3.49)-(3.50), as $N \gg 1$ at early times in this branch:

$$h''_{2p} + 2\mathcal{H}h'_{2p} + x^2\mathcal{H}^2 h_{1p} + m^2a^2N\beta_1(h_{2p} - h_{1p}) = 0, \tag{3.111}$$

$$h''_{1p} - (1+3w)\mathcal{H}h'_{1p} + \left(\frac{1+3w}{2}\right)^2 x^2\mathcal{H}^2 h_{1p} - \frac{(1+3w)}{2}\frac{m^2a^2\beta_1}{N}(h_{1p} - h_{2p}) = 0. \tag{3.112}$$

Early times radiation-dominated era

Let us consider $w = 1/3$ in eq. (3.111)-(3.112), and find their solutions for super-horizon and sub-horizon scales.

Super-horizon scales: the evolution equations are now

$$h''_{2p} + 2\mathcal{H}h'_{2p} + m^2a^2N\beta_1(h_{2p} - h_{1p}) = 0, \tag{3.113}$$

$$h''_{1p} - 2\mathcal{H}h'_{1p} + \mathcal{O}(N^{-2})(h_{1p} - h_{2p}) = 0, \tag{3.114}$$

and are solved by

$$h_{2p} = c_\pm \frac{e^{\pm iK\tau}}{\tau} + c_3 + c_4\left[\tau^3 - 12\frac{\tau}{K^2} + \frac{24}{(K^4\tau)}\right],$$

$$h_{1p} = c_3 + c_4\tau^3, \tag{3.115}$$

where $c_\pm$, c_3 and c_4 are integration constants, and $K^2 = m^2a^2N\beta_1 = const$. Therefore,

h_{1p} and h_{2p} grow as a power of τ.

Notice that if β_1 were negative, the solution for h_{2p} would include $e^{\pm|K|\tau}$ instead of oscillating functions, which would correspond to an exponential instability.

Sub-horizon scales: the evolution equations are now

$$h''_{2p} + 2\mathcal{H}h'_{2p} + x^2\mathcal{H}^2 h_{2p} + m^2a^2N\beta_1(h_{2p} - h_{1p}) = 0, \tag{3.116}$$

$$h''_{1p} - 2\mathcal{H}h'_{1p} + x^2\mathcal{H}^2 h_{1p} - \frac{m^2a^2\beta_1}{N}(h_{1p} - h_{2p}) = 0, \tag{3.117}$$

and when considering highest orders in N only, the solutions are

$$\begin{aligned} h_{1p} &\propto (1 \mp ik\tau)e^{\pm ik\tau}, \\ h_{2p} &= \left(\frac{c_{1\pm}}{\tau} + c_{2\pm} + c_{3\pm}\tau\right)e^{\pm ik\tau} + \frac{c_{4\pm}}{\tau}e^{\pm i\omega\tau}, \end{aligned} \tag{3.118}$$

where $\omega^2 = k^2 + m^2\beta_1 a^2 N$, and where the coefficients $c_{1\pm}$, $c_{2\pm}$, $c_{3\pm}$ and $c_{4\pm}$ are integration constants, related to those of h_{1p}. Note that ω = constant as during the radiation-dominated era at early times $\tilde{\rho} \approx \beta_4 N^2 \propto a^{-4}$, and therefore a^2N is constant. Unlike GR, here we observe that h_{1p} grows linearly with time, while h_{2p} also includes a growing modes as a consequence of the interactions with h_{1p}. The growing mode in h_{1p} is a consequence of the fact that the metric $f_{\mu\nu}$ is bouncing, and therefore at early times the term with h'_{1p} in eq. (3.117) has a negative sign.

Notice that if β_1 were negative, ω^2 would be negative for some values of k, and for those cases there would be an exponential instability in the solution for h_{2p}.

Fig. 3.14 shows numerical solutions for the evolution of both tensor perturbations as a function of τ, at early times during the radiation-dominated era for a given sub-horizon scale. In this case we set $m^2\beta_1 = m^2\beta_4 = 10^{-2}$, and arbitrary initial conditions of order one for both fields. As expected due to the analytical solutions, we see a growth in both fields in this stage.

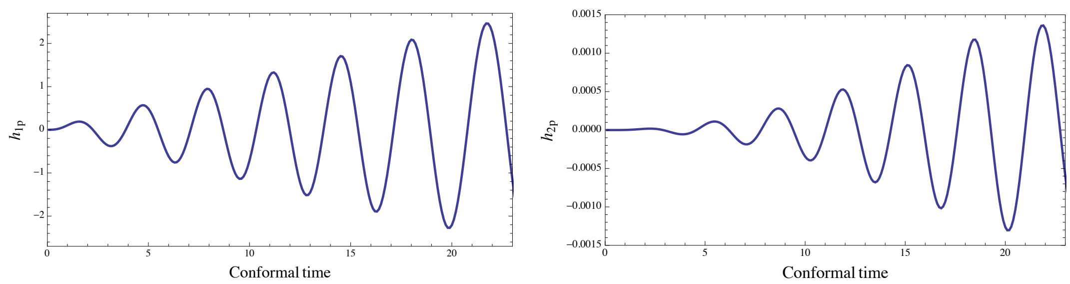


Figure 3.14: Evolution of tensor perturbations as a function of the conformal time, during early times in the radiation-dominated era for a given sub-horizon scale.

Early times matter-dominated era

Now, let us consider $w = 0$ in eq. (3.111)-(3.112), and find their solutions for super-horizon and sub-horizon scales. Note that during the matter-dominated era at early times $\tilde{\rho} \approx \beta_4 N^2 \propto a^{-3}$, and then $a^2 N \propto N^{-1/3}$ and $a^2/N \propto N^{-7/3}$. Therefore, mixing terms can be ignored in the equations of motion as $N \gg 1$ at early times in this branch.

Super-horizon scales: the evolution equations are now

$$h''_{2p} + 2\mathcal{H} h'_{2p} = 0, \tag{3.119}$$

$$h''_{1p} - \mathcal{H} h'_{1p} = 0, \tag{3.120}$$

and are solved by $h_{2p} = c_1 + c_2/\tau^3$; $h_{1p} = c_3 + c_4 \tau^3$, where c_1, c_2, c_3 and c_4 are integration constants. We find then that h_{1p} grows as a power of τ and h_{2p} decays, in a similar way to the radiation-dominated era solutions.

sub-horizon scales: the evolution equations simplify to

$$h''_{2p} + 2\mathcal{H} h'_{2p} + x^2 \mathcal{H}^2 h_{2p} + \mathcal{O}(N^{-1/3})(h_{2p} - h_{1p}) = 0, \tag{3.121}$$

$$h''_{1p} - \mathcal{H} h'_{1p} + \frac{x^2 \mathcal{H}^2}{4} h_{1p} + \mathcal{O}(N^{-7/3})(h_{1p} - h_{2p}) = 0, \tag{3.122}$$

and are solved by $h_{1p} \propto (1 \mp ik\tau/2) e^{\pm ik\tau/2}$ and $h_{2p} \propto \frac{(1 \mp ik\tau)}{\tau^3} e^{\pm ik\tau}$.

Fig. (3.15) shows numerical solutions for the evolution of tensor perturbations as a function of τ (in arbitrary units), at early times in the matter-dominated era for a given sub-horizon scale. Again, in this case we set $m^2\beta_1 = m^2\beta_4 = 10^{-2}$, and arbitrary initial conditions of the same order for both fields. Unlike during the radiation-dominated era, in this case h_{1p} grows linearly with time, but h_{2p} decays.

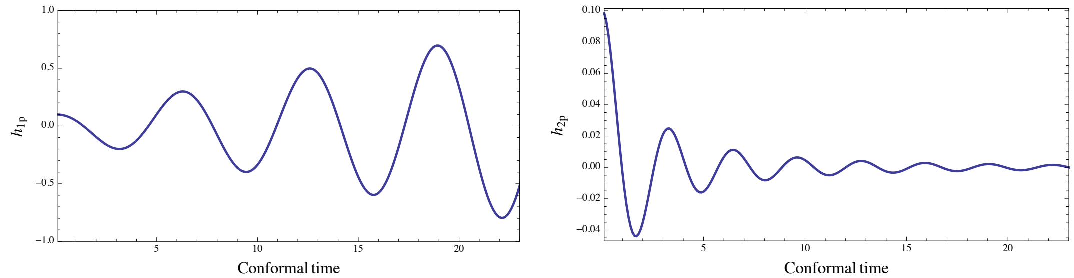


Figure 3.15: Evolution of tensor perturbations during early times in the matter-dominated era.

We have found that h_{1p} grows as a power law at early times for super-horizon and sub-horizon scales. At late times, this could mean that h_{1p} could start in de-Sitter phase being some orders of magnitude higher than h_{2p}. This would produce the same effect described previously for late times solutions in the expanding branch, but in this case h_{2p} would grow at late times due to h_{1p}.

3.5 Discussion

In this chapter we have undertaken a comprehensive analysis on the evolution of cosmological linear perturbations in massive bigravity and have found approximate analytical solutions in a wide range of regimes. We have confirmed the main results of previous works on linear perturbations, but also extended their analysis to vector and tensor modes. In doing so we have found that massive bigravity has a number of instabilities which manifest themselves as growing solutions. In particular, we have found that most choices of parameters generate exponential instabilities in the scalar, vector or tensor modes. A subset of model space does not have exponential instabilities: when $\beta_3 = \beta_2 = 0$ with β_1 and β_4 being positive, which corresponds to a particular case of the bouncing branch. However, even for this subset of models we have found growing power-law solutions in vector and tensor modes, contrary to GR, in addition to a violation of the Higuchi bound, which would likely bring instabilities when studying the model beyond the classical linear regime. For vector and tensor perturbations, this growth is a consequence of a bounce in $f_{\mu\nu}$ along with effects from the interaction terms between both metrics. Analogously to scenarios with exponential instabilities, these growing modes could be a source of concern as the validity of perturbation theory could break down at some early time. However, this latter case is not as bad because, as we will show later, we can prevent modes from growing too large by considering particular initial conditions. This resulting fine-tuning is much less restrictive than that required for the exponential solutions.

As previously mentioned, such growing modes may be a hint that all is not well and that the initial values problem may not be well-posed. If indeed this is the case, it would not be surprising as extra degrees of freedom may lead to such behaviour. For example there have been efforts in trying to determine whether scalar-tensor theories have a well-posed initial value problem, while a study of Einstein-Aether theories has shown that caustics will generically arise there [122]. We believe a detailed analysis of the initial value problem in massive bigravity is essential to place it on a firm footing.

An alternative view could be to take the solutions we have found and speculate on their cosmological consequences. To do this accurately, one would have to explore the correct set of initial conditions which would arise in such a theory due to (for example) inflation. One would then have to incorporate our equations into a complete and realistic model of the Universe that incorporates the various components, the correct thermal history and the Boltzmann equation for the relativistic degrees of freedom. Nevertheless, for now we can attempt to estimate the effect of the new solutions we have found by focusing on a few observables.

In what follows we will focus solely on tensor modes; we found a growing mode for vectors but we do not address its effect for now. Recall from the previous section that for super-horizon scales during the radiation-dominated era, h_{2p} grows as τ^3 due to the interaction terms with h_{1p}. Therefore, from the end of the inflationary era until the recombination era, h_{2p} might deviate substantially from its value in GR. As a result we might expect a larger effect from gravitational waves in the Cosmic Microwave Background (CMB). An estimate of how much h_{2p} could grow in this stage (on super-horizon scales) gives us:

$$h_{2\text{rec}} \approx h_{2i} + \frac{(K\tau_{\text{eq}})^2}{6}\left[h_{1i} - h_{2i} + \frac{\tau_r^3}{15}\tau_i h'_{1i}\right] + \frac{(K\tau_{\text{eq}})^2}{9}\left[h_{1i} - h_{2i} + \frac{\tau_r^3}{6}\tau_i h'_{1i}\right]\left(1 - \left(\frac{\tau_{\text{eq}}}{\tau_{\text{rec}}}\right)^3\right), \tag{3.123}$$

where $h_{2\text{rec}}$ is the value of the tensor perturbation h_{2p} at recombination given an initial value of h_{2i} at some initial time τ_i. The subindex eq corresponds to a value at the matter-equality time, and we have defined $\tau_r = \tau_{\text{eq}}/\tau_i = a_{\text{eq}}/a_i$. Here, we have also used that $K^2 = m^2 a^2 N \beta_1 \propto m$, and $K\tau_{\text{eq}} \ll 1$ (which would happen for a sufficiently small m), and calculated the first order corrections in K^2.

Note that in GR the value at the recombination era would be h_{2i} for a super-horizon scale, given that $h'_{2i} = 0$ and, therefore, the second and third terms in eq. (3.123) correspond to the modifications introduced by massive gravity to this tensor perturbation, which are proportional to m. Even though $K\tau_{eq} \ll 1$, the modification is not necessarily small as it depends also on the initial conditions for h_{1p}.

If we choose τ_i to be the end of the inflationary era (for example where $a_i \sim 10^{-28}$), we have that $\tau_r^3\tau_i \sim 10^{107}$. Therefore, we would need h_{1p} to be effectively zero at the end of the inflationary era, and h_{1p} would then be constant for super-horizon scales. Otherwise, h_{1p}, and as a consequence h_{2p}, could grow large and break the validity of perturbation theory. Assuming $h'_{1i} = 0$ and some preferred values found in [111] when constraining scalar perturbations with observational data, the largest modification introduced by massive gravity in h_{2p} at the epoch of the recombination, according to eq. (3.123), would be:

$$\Delta h_{2\text{rec}} = h_{2\text{rec}} - h_{2i} = 10^{-6}\left(h_{1i} - h_{2i}\right). \tag{3.124}$$

Further research at early times is needed in order to give exact numbers as we would need to know the initial condition for both tensor perturbations.

In a similar way, we can study the evolution for sub-horizon perturbations. For a scale that crosses the horizon during the radiation-dominated era, there will be a modification in the evolution of h_{2p}, with respect to GR, coming from the interaction with h_{1p}, as we can see in eq. (3.118). From the horizon crossing time τ_c until the recombination era τ_{rec}, the modification to h_{2p} is given by:

$$\Delta h_{2\text{rec}} = \left(\frac{\tau_{\text{eq}}}{\tau_{\text{rec}}}\right)^2 \left(\frac{K^2}{k^2}\right)\left[c_1 h_{2c} + x_{\text{eq}}^2\left(c_2 h_{1c} + c_3\frac{h'_{1c}}{k}\right)\right], \tag{3.125}$$

where $x_{\text{eq}} = k\tau_{\text{eq}}$, and the subindex c indicates that the quantity is evaluated at the horizon-crossing time. Here, again, we have considered only first order corrections in K^2, and the coefficients c_1, c_2, and c_3 are functions of $\sin(x_{\text{rec}})$ and $\cos(x_{\text{rec}})$, so they all roughly have the same order of magnitude.

Note that, since in eq. (3.118) h_{2p} has a linear growing mode, one could have expected to have larger modifications for larger k, as larger k enter the horizon before and consequently spend more time growing. However, as we observe in eq. (3.125), for larger k the modification is smaller. This happens because the coefficients $c_{1\pm}$, $c_{2\pm}$ and $c_{3\pm}$ in eq. (3.118) are related to those of h_{1p}. In particular, $c_{3\pm} \sim kh_{1c}$, $c_{2\pm} \sim k^2 h_{1c}/K^2$ and $c_{1\pm} \sim k^3 h_{1c}/K^4$. Therefore, for sufficiently small m, the dominant term will be $c_{1\pm}$ and therefore the growing mode will be suppressed compared to the decaying mode, which is what actually happens for observable scales with the preferred values found in [111].

In addition, note in eq. (3.125) that, since $x_{\text{eq}} \gg 1$, the contribution from h_{1c} to $\Delta h_{2\text{rec}}$ is much larger that the contribution from h_{2c}. A numerical estimate at a scale of order 1Mpc gives us

$$\Delta h_{2\mathrm{rec}} \sim 10^{-24} h_{2c} + 10^{-5} h_{1c} + 10^{45} h'_{1c}, \tag{3.126}$$

where, again, we see that some kind of mechanism is needed to get $h'_{1c} = 0$ at early times, in order to avoid large modifications to GR. In addition, since the value of $h_{2\mathrm{rec}}$ in GR is estimated to be $h^{(GR)}_{2\mathrm{rec}} \sim 10^{-10} h_{2c}$, the initial condition $h_{1c} \sim h_{2c}$ will not lead to a small modification to GR. In fact, it will lead to a correction 10^5 times larger than the GR value, contrary to what we found on super-horizon scales according to eq. (3.124).

It is clear that, without an appropriate set of initial conditions for cosmological perturbations, we are unable to make definitive statements about the observational viability of these models. They do, however, give us an indication as to what we might expect and it seems that there might be problems with both branches of massive bigravity. The full equations presented in this chapter are what is required to modify existing software packages for precise calculations of the growth of large scale structure and the evolution of the cosmic microwave background. With such machinery in hand it should be possible to explore what initial conditions are observationally viable and can be used to place stringent constraints on any theory of the early Universe in massive bigravity.

After we published the work presented in this chapter [2], the results on tensor instabilities were confirmed in [123, 124]. Furthermore, two papers were published on the analysis of initial conditions for tensor perturbations [125, 126] where it was found that inflation naturally generates a set of initial conditions that lead to a viable amplitude of primordial gravitational waves. However, the presence of the ghost in the helicity-0 mode of the massive graviton makes this model likely to exhibit instabilities in the scalar sector beyond linear classical perturbations anyway.

Finally, it is important to remark that there are simple modifications to the model analysed in this chapter that could be explored. One simple and interesting modification can arise if asymmetries in the background metrics are introduced. Indeed, such models have been studied in [127, 128] and have been found to be free of instabilities. More general consistent theories of metrics/spin-2 fields beyond massive (bi-)gravity with a single matter coupling have also been explored recently. New kinetic interactions were investigated in [129–135], generalisations of the potential interactions of massive bigravity to N multiple metrics in [86, 136–143], and new couplings to matter in [82–105]. These matter couplings allow matter to couple to both metrics and we therefore refer to them as ‘double matter couplings’. In general, such couplings re-introduce the Boulware-Deser ghost at an unacceptably low scale, however the specific couplings of [93, 94] stand out in that they are consistent ghost-free double matter couplings. In the context of this double coupling, some homogeneous and isotropic cosmological solutions were studied in [95], where viable (background) evolutions were found. However, at the level of linear perturbations, tachyonic, gradient, and ghost instabilities were found for these solutions [144, 145] for tensor, vector and scalar perturbations, respectively.

CHAPTER 4

Tools for testing gravity: Noether identities

The possibility that we might be able to constrain general relativity on cosmological scales is one of the science drivers behind future surveys [18]. In preparation, there have been a number of proposals on how to characterise deviations from GR (or to be more specific, deviations from the ΛCDM model) in as general a fashion as possible. To some extent, the idea has been to find an approach on cosmological scales analogous to that used in the weak field, non-relativistic regime, where the Parametrised Post-Newtonian (PPN) approach captures the behaviour of a wide range of theories on the scale of the Solar System or compact binaries [6]. Ultimately, one would like to have a similarly systematic method for describing a general swathe of the landscape of gravitational theories on cosmological scales (see [1] for a review on this topic). This description must be written in terms of a finite (and preferably small) number of "parameters" – really independent functions of time – which are easy to map onto specific theories.

The quest for a complete and efficient parametrisation is ongoing, and it is useful to briefly summarise the main approaches that have been considered, their strengths and weaknesses. The approach most widely used until now involves phenomenological corrections to the linear perturbation equations [146–150]. The Newton-Poisson equation is modified to include a time- and scale-dependent Newton's constant, and a "gravitational slip" allows the two metric scalar potentials to differ from each other. This two-parameter approach is remarkably effective, easily implemented in Einstein-Boltzmann solvers and, with a judicious choice of functional forms, can be shown to closely mimic specific extensions of GR. While it can be shown that this parametrisation is the limit of *any* viable theory in the quasi-static regime [151] (the cosmological equivalent of the Newtonian regime), there is no systematic way of relating it to any fundamental theory on large scales. In other words, constraints on the two parameters in this approach do not unambiguously lead to information about any putative underlying theory that might be responsible for deviations from GR. The same strengths and weaknesses can be found in attempts to parametrise deviations from GR in terms of a perturbed cosmological fluid [152]. In this case, the parameters are the equation of state, the sound speed and terms that control adiabaticity and shear. While these parameters have a clear meaning in terms of the physics of relativistic fluids, they tell us little about what the fundamental modifications to the GR field equations or to the Einstein-Hilbert action are.

There are a number of attempts at the construction of a more fundamental parametrisation. Two routes have been considered: a generalisation of the field equations, in what we have called the "Parametrised Post-Friedmann" (PPF) approach [153,154], or a generalisation of the gravitational action, of which the two main variants are the "Effective Action" (EA) approach [155, 156] and the "Effective Field Theory" (EFT) approach [157–161]. In the PPF approach one parametrises the most general gauge-invariant field equations, which include up to second-order derivatives of the two scalar metric potentials. When only one scalar DoF propagates, the PPF approach covers a very general class of theories; in [154] it was shown that scalar-tensor, Einstein-Aether and bigravity theories are all encompassed by this parametrisation. Unfortunately, as a result of its generality, there are a large number of free parameters that need to be included. Furthermore, these depend on time *and scale* due to the lack of knowledge of the field content of the underlying theory from which the scalar DoF comes. This makes the PPF approach potentially impractical for constraining GR on large scales.

Restricting oneself to theories that can be derived from a local fundamental action, as one does in the EFT and EA approaches, simplifies any potential parametrisation. The tools of EFT have been successfully applied to characterise scalar field perturbations during inflation, allowing a systematic study of non-Gaussianity arising from higher-order operators on a quasi-de Sitter background [162,163]. These ideas have been imported to late-time cosmology where, even though it is not strictly an EFT approach (one is looking at coupled but, effectively, free fields with no higher-order operators) it is useful in organising all possible terms in the action. The approach is constructed using Arnowitt-Deser-Misner (ADM) variables and the unitary gauge to build a general spatially-invariant quadratic action for cosmological perturbations in a scalar-tensor theory; one then performs a Stueckelberg transformation to make the scalar DoF explicit and recover time diffeomorphism invariance. This is an elegant approach which has already been implemented in a couple of Einstein-Boltzmann solvers [164–166], but is restricted to scalar-tensor theories (and particular forms of Horava-Lifschitz theory). The EA approach takes a covariant point of view *ab-initio*, constructing an effective action with all possible covariant combinations of the metric perturbations. It is more general than the EFT approach, is systematic and has also been implemented in existing Einstein-Boltzmann solvers [52].

In this chapter we would like to follow the spirit of the PPF approach and construct a systematic and general parametrisation procedure, but at the level of the action, instead of the equations of motion; it will be, in some sense, an *integrable* version of the PPF approach. With this procedure we will construct local, general, diffeomorphism-invariant quadratic actions for linear perturbations, around homogeneous and isotropic backgrounds, encompassing all possible gravitational theories with a given field content and derivative order. We will argue that the form of the quadratic action, crucially, depends on the gauge transformation properties of any extra fields that may arise in a modified gravity theory. An important feature of this approach is that the free parameters characterising the quadratic action, and thus the evolution of cosmological perturbations, are defined in terms of functional derivatives of an underlying, unknown, fundamental Lagrangian. One can then identify where, in the general space of parameters, a particular theory resides. As a consequence, mimicking the success of PPN, it should be straightforward to translate constraints on the general set of parameters into constraints on a particular theory (for example, Jordan-Brans-Dicke theory, Einstein-Aether gravity, bigravity, etc.).

We will use some of the tools proposed in the EFT approach and its variants; working in terms of the 3+1 decomposition and ADM variables, connecting free coefficients with properties of fundamental theories, and assuming linear diffeomorphism invariance. However, we will not restrict ourselves to scalar-tensor theories; we will not gauge fix and, therefore, will not Stueckelberg. While in the first steps of the procedure the action that we start with will seem more complex than those proposed in the EFT approach, we show that imposing the action to be diffeomorphism-invariant rapidly simplifies it to a manageable form that is equivalent to, but more general than, other formalisms. A tremendous strength of our approach is that it is completely systematic and easily generalised to any background, degrees of freedom and gauge symmetries.

The outline of this chapter is as follows. In Section 4.1 we explain our method, give a simple introductory example and then show the application to cosmological perturbations of a general local diffeomorphism-invariant gravitational theory, around a homogeneous and isotropic background. In Section 4.2 we apply the method to the simplest theory - a theory with a single metric. Here we will see how linearised GR can arise from a more general construction than one would have *a priori* thought, and in the next chapters we will apply this method to describe other families of gravity theories. Finally, in Section 4.3 we review and discuss our findings.

Finally, we mention that along with the paper published from this work we also released two pieces of code: firstly the *xIST* package, an extension of the *xAct* tensor algebra system [167], which implements a framework to investigate general scalar-tensor theories at the level of linear perturbations. Secondly, based on *xIST*, a Mathematica notebook we dub *COPPER* (COsmological Parametrized PERturbations), which reproduces the calculations carried out in the chapter of this thesis in detail, and can be straightforwardly adapted to investigate more complicated setups. The full code and documentation can be found and downloaded at `https://github.com/noller/xIST`.

4.1 The method: Noether identities and constraints

In this section we explain the method for obtaining general local quadratic actions for linear cosmological perturbations of gravitational theories with a given field content and gauge symmetries. The objective of this method is to find the maximum set of free functions parametrising the quadratic action, and thus the cosmological predictions, of different gravitational theories. One can then automatically translate observational constraints on the free functions into constraints on these theories. In this method we will be assuming a known form for the matter sector which couples to gravity.

Before explaining the method in detail, we first summarise the three main steps. Then we illustrate the method with a simple (non-cosmological) example of an action with a 4-vector field, invariant under $U(1)$ gauge transformations. We then proceed to analyse gravitational theories composed of at least one 2-rank tensor field, or metric, and invariant under linearised diffeomorphisms.

The main three steps of the method are the following:

1. Choose the fields present in the theory and the gauge symmetries to be satisfied, e.g. invariance under linear coordinate transformations.

2. Write down an action with all possible quadratic interactions between the fields, leading to a given maximum number of derivatives of the fields in the equations of motion.

3. Find the Noether identities associated to the required gauge symmetries, and impose the resulting constraints on the quadratic action.

Before explaining in detail, and generality, these three steps, we start with a simple example to illustrate the procedure. In particular, Step 3 above should be made clearer by this.

4.1.1 Introductory example

Step 1: Consider a covariant theory for a 4-vector A^μ on Minkowski space, invariant under the following gauge transformation:

$$A^\alpha \to A^\alpha + \partial^\alpha \varepsilon, \tag{4.1}$$

where ε is an arbitrary function of space and time.

Step 2: The most general quadratic action, leading up to second derivatives of the field in its equation of motion, can be written as:

$$S_A = \int d^4x \, \left[c_1 \partial_\alpha A^\beta \partial^\alpha A_\beta + c_3 \partial_\alpha A^\beta \partial_\beta A^\alpha + m^2 A^\alpha A_\alpha \right], \tag{4.2}$$

where c_1, c_3 and m are free constant parameters. Here we have included all possible covariant quadratic contractions of the field with an unknown coefficient in front. The structure of this action is that of the Proca-Einstein-Aether theory [168], where we have discarded a term proportional to $(\partial_\alpha A^\alpha)^2$ (known as the "c_2" term in curved space) as it is equivalent to the c_3 term through an integration by parts.

Step 3: If the action S_A is gauge-invariant under the transformation in eq. (4.1), then a variation of the action $\delta_\varepsilon S_A$, due to an infinitesimal gauge transformation of the field, must vanish. Specifically, if we make an infinitesimal variation $\delta A^\mu = \partial^\mu \varepsilon$, at linear order in ε we obtain:

$$\begin{aligned}
\delta_\varepsilon S_A &= \int d^4x \left[c_1 (\partial_\alpha \partial^\beta \varepsilon \partial^\alpha A_\beta + \partial_\alpha A^\beta \partial^\alpha \partial_\beta \varepsilon) + c_3 (\partial_\alpha \partial^\beta \varepsilon \partial_\beta A^\alpha + \partial_\alpha A^\beta \partial_\beta \partial^\alpha \varepsilon) \right. \\
&\quad \left. + m^2 (\partial^\alpha \varepsilon A_\alpha + A^\alpha \partial_\alpha \varepsilon) \right] \\
&= 2 \int d^4x \left[(c_1 + c_3) \partial^2 \partial^\beta A_\beta - m^2 \partial^\alpha A_\alpha \right] \varepsilon,
\end{aligned} \tag{4.3}$$

where the last line comes from an integration by parts. From eq. (4.3) we obtain a condition that must be satisfied if the action is gauge-invariant. This condition corresponds to the *Noether identity* associated to the gauge transformation in eq. (4.1), and is given by:

$$(c_1 + c_3) \partial^2 \partial^\beta A_\beta - m^2 \partial^\alpha A_\alpha = 0, \tag{4.4}$$

where we have used the fact that ε is an arbitrary parameter, and therefore the entire bracket must vanish in order to satisfy $\delta_\varepsilon S_A = 0$. In addition, since the action must be gauge-invariant off-shell, i.e. for any field configuration A^μ, this identity must be satisfied off-shell as well. Thus, the terms with different derivatives acting on A_α must vanish separately, leading to two independent constraints for the parameters:

$$\begin{aligned} c_1 + c_3 &= 0, \\ m^2 &= 0. \end{aligned} \tag{4.5}$$

From now on, the individual constraints following from the Noether identities will be called *Noether constraints*. In this example, these constraints reduce the action in eq. (4.2) to that of classical electromagnetism (for an appropriate choice of normalization), which is then the most general quadratic action invariant under eq. (4.1) for a vector field with second derivatives in its equation of motion. We have systematically constructed this action by using the Noether identities to find a set of constraints on the coefficients of the original general quadratic action in eq. (4.2).

4.1.2 Gravitational action

We will now use this method to construct the most general, linearly diffeomorphism-invariant and local quadratic action for linear perturbations of gravitational theories on a cosmological background. As already seen in the previous example, the result depends strongly on the field content and the number of their derivatives. In other words, we will be parametrising gravitational theories with the same fields, derivative order and gauge symmetries. In general, theories that deviate from general relativity have extra degrees of freedom, either explicitly or emerging from higher-derivative operators, extra dimensions, etc. If our method is to encompass these theories, we need to account for extra degrees of freedom.

In order to be concrete, we will sometimes refer to scalar-tensor theories to explain our procedure, but we emphasise that the method is easily generalisable to other gravitational theories. In fact, in the next chapter we will apply the procedure to vector-tensor theories. We follow the same three steps as above.

Step 1: Consider a gravitational theory composed of one rank-2 tensor field (or metric) and possibly some additional fields. We focus on linear perturbations around a homogeneous and isotropic cosmological background.

The tensor degrees of freedom arise from the following perturbed metric:

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \delta g_{\alpha\beta}, \tag{4.6}$$

where $\bar{g}_{\alpha\beta}$ describes the background metric, assumed to be a spatially-flat FLRW metric with a line element given by:

$$d\bar{s}^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j, \tag{4.7}$$

where $a(t)$ is the scale factor and t is the physical time, which will be used throughout this chapter. $\delta g_{\alpha\beta}$ describes small first-order perturbations around the background. For all the additional fields, we assume the same linearly perturbed form, with a background solution satisfying the same symmetries as $\bar{g}_{\mu\nu}$ (isotropy and homogeneity, in this case). In the case of scalar-tensor theories, with an extra scalar field χ, we have

$$\chi = \chi_0 + \delta\chi, \tag{4.8}$$

where $\chi_0(t)$ is the background solution of the scalar field χ, and $\delta\chi$ its first-order perturbation.

We will be looking for actions which are quadratic in these perturbations and invariant under linear general coordinate transformations of the form $x^\mu \to x^\mu + \epsilon^\mu$, where ϵ^μ is a first-order arbitrary perturbation to the coordinates x^μ. Under linear coordinate transformations the background stays the same, while the linear perturbations of the metric $\delta g_{\alpha\beta}$ transform as in eq. (1.24). For the scalar perturbation $\delta\chi$, the corresponding transformation is

$$\delta\chi \to \delta\chi - \dot{\chi}_0 \pi, \tag{4.9}$$

where π is an arbitrary gauge parameter.

In addition, we couple the gravitational action to matter fields. In this chapter, for simplicity, we consider the matter sector to be comprised of a scalar field φ minimally coupled to the metric, with the same gauge transformation rule as the scalar field χ in eq. (4.9). However, all the results found in this chapter will also hold for a general perfect fluid. The formalism can also be extended to non-minimally coupled matter (for an attempt at doing this in the context of PPF see [169] and in the context of EFT see [170]).

Step 2: In this step we construct the most general local quadratic action for all the gravitational perturbation fields $\delta g_{\mu\nu}$, and any other extra field present. This quadratic action will lead to equations of motion which are linear in the perturbation fields.

We start by assuming the existence of an underlying non-perturbative, fundamental, gravitational action $S_{\rm G}$, that leads to the quadratic action we are interested in. In this chapter we work in the 3+1 ADM formalism – see Appendix C.1 for notation. We do this for three main reasons: i) in a cosmological setting there is a straightforward 3+1 split; ii) for ease of comparison with EFT approaches in which time diffeomorphism invariance is broken; iii) it is straightforward to construct terms with different numbers of maximum derivatives for time and space. We emphasise, though that the procedure presented here could also be used without the ADM formalism (in a "fully covariant" approach), although we would be forced to consider the same number of time and space derivatives. For a similar (but not identical) approach with explicit 4-dimensional covariance see [156].

In the ADM formalism we have that the metric $g_{\mu\nu}$ can be decomposed into a lapse function N, shift functions N^i and a 3-dimensional spatial metric h_{ij} in the following way:

$$g_{00} = -N^2 + h_{ij}N^iN^j, \quad g_{0i} = h_{ij}N^j, \quad g_{ij} = h_{ij}. \tag{4.10}$$

The underlying fundamental action will be a local functional of N, N^i, h_{ij} and the extra fields, as well as their multiple time and spatial derivatives:

$$S_{\rm G} \;=\; \int d^4x\; N\sqrt{|h|}\; L_{\rm G}\left[N, N^i, h_{ij}, K^i{}_j, R^i{}_j, \chi, \cdots\right], \tag{4.11}$$

where $L_{\rm G}$ is a Lagrangian functional, $|h|$ is the determinant of h_{ij}, and the ellipses encompass higher derivatives of the metric and any extra field(s). Given that we are seeking a linearly diffeomorphism-invariant action, we have replaced time derivatives and secondary spatial derivatives of the 3-dimensional metric in $L_{\rm G}$ by the extrinsic curvature tensor $K^i{}_j$ and the intrinsic 3-dimensional curvature $R^i{}_j$, respectively. In general, we will consider $S_{\rm G}$ to be a functional of a set of building blocks $\vec{\Theta} = (N, N^i, h_{ij}, K^i{}_j, R^i{}_j, \chi, \cdots)$. It is important to note that, since the building blocks include all time and spatial derivatives of the fields, we have to make an extra assumption on $S_{\rm G}$, otherwise we could have infinitely many of these terms. We will choose a maximum number of (combined space and time) derivatives allowed for the fields in the equations of motion (and thus in the action) and truncate at that order.

Given that we are interested in linear perturbations of the gravitational fields, we need the quadratic expansion of $S_{\rm G}$ in δN, δN^i, δh_{ij}, $\delta\chi$ and the rest of the extra fields. To do so, we take the functional Taylor expansion of $L_{\rm G}$ around the background fields in terms of the perturbed set of building blocks $\delta\vec{\Theta} = (\delta N, \delta N^i, \delta h_{ij}, \delta K^i{}_j, \delta R^i{}_j, \delta\chi, \cdots)$, so that:

$$L_{\rm G} \;\; \simeq \;\; \bar{L} + L_{\Theta_A}\delta\Theta_A + \frac{1}{2}L_{\Theta_A\Theta_B}\,\delta\Theta_A\delta\Theta_B \tag{4.12}$$

where $\bar{L}$ is the zeroth order Lagrangian ($L_{\rm G}$ evaluated at the background), and the subindices A and B label the different building blocks, and repeated such indices are summed over. The terms L_{Θ_A} and $L_{\Theta_A\Theta_B}$ are what we call *coefficients*, and are given by functional derivatives of $L_{\rm G}$ evaluated at the background; therefore they generally depend on time. Explicitly, $L_{\Theta_A} \equiv \partial L_{\rm G}/\partial\Theta_A$ and $L_{\Theta_A\Theta_B} \equiv \partial^2 L_{\rm G}/\partial\Theta_A\partial\Theta_B$. Notice that even though the fields ($g_{\mu\nu}$, χ, etc) have only linear perturbations, the perturbed building blocks could have higher-order perturbations as result. Thus, we clarify that $\delta\vec{\Theta}$ contains both first and second-order perturbative pieces.

We can now find the Taylor expansion of the gravitational action, which is given by:

$$\begin{aligned} S_{\rm G} \simeq \int d^4x \; \Big[& a^3\bar{L} + \delta_1\left(N\sqrt{|h|}\right)\bar{L} + a^3 L_{\Theta_A}\delta\Theta_A + \delta_2\left(N\sqrt{|h|}\right)\bar{L} \\ & + \delta_1\left(N\sqrt{|h|}\right)L_{\Theta_A}\delta\Theta_A + \frac{1}{2}a^3 L_{\Theta_A\Theta_B}\delta\Theta_A\delta\Theta_B \Big], \end{aligned} \tag{4.13}$$

where δ_n stands for an n$^{\rm th}$ order perturbation. In addition, we include a matter action:

$$S_{\rm M} \;\; = \;\; \int d^4x \; N\sqrt{|h|}\; L_M\left[N, N^i, h_{ij}, \varphi, \cdots\right], \tag{4.14}$$

where L_M is once again a Lagrangian functional. Here, we have generically represented "matter fields" with φ, but they can be fields of any spin, perfect or imperfect fluids, etc. This action is assumed to be known, and therefore its Taylor expansion can be carried out straightforwardly. The linear terms of the Taylor expansion of the total action $S_{\rm G} + S_{\rm M}$ will be zero, and will lead to the background equations of motion (see Appendix C.4), while the quadratic terms will give the total quadratic action $S^{(2)}_{\rm G} + S^{(2)}_{\rm M}$ determining the evolution of the cosmological perturbations. Explicitly, the second-order gravitational action will be given by:

$$S^{(2)}_{\rm G} = \int d^4x \; \left[a^3 L_{\Theta_A}\delta_2\Theta_A + \delta_2\left(N\sqrt{|h|}\right)\bar{L} + \delta_1\left(N\sqrt{|h|}\right)L_{\Theta_A}\delta_1\Theta_A + \frac{1}{2}a^3 L_{\Theta_A\Theta_B}\delta_1\Theta_A\delta_1\Theta_B \right], \tag{4.15}$$

where we have used eq. (4.13) and the fact that a given perturbed building block can be separated into a first and second-order perturbation as: $\delta\Theta_A = \delta_1\Theta_A + \delta_2\Theta_A$. As we will see in the next section, only some building blocks Θ_A will have a second-order perturbation, (for example, the lapse N or the 3-curvature, R). As we have already mentioned, the coefficients

L_{Θ_A} and $L_{\Theta_A\Theta_B}$ can be derived from the fundamental non-perturbative action. However, we will assume that such an action is not known and thus these coefficients will be left as free functions to be fixed by the Noether constraints, in a way analogous to the coefficients c_1, c_3 and m^2 in the example presented in Section 4.1.1.

Step 3: In this step we impose that the total quadratic action (from gravity and matter) is invariant under linear coordinate transformations. We do so by finding the relevant Noether identities, and solving the resulting Noether constraints.

To find the Noether identities, we write down all the perturbed building blocks $\delta\Theta_A$ in terms of the perturbation fields $\delta g_{\mu\nu}$, $\delta\chi$, etc. and vary the quadratic action with regards to them. Specifically, in this chapter, we vary the quadratic action in terms of the scalar perturbation fields, according to the standard SVT decomposition of fields [171]. We focus only on these types of perturbations, as they are the seeds of large-scale structure in the density field, and therefore cosmologically relevant. We can ignore the vector and tensor perturbations as they decouple from the scalar perturbations on a homogeneous and isotropic background. Thus, we write $\delta g_{\mu\nu}$ as:

$$\delta g_{00} = -2\Phi,\ \delta g_{0i} = \partial_i B,\ \delta g_{ij} = a^2\left[-2\Psi\delta_{ij} + 2\partial_i\partial_j E\right], \tag{4.16}$$

where we have four scalar perturbation fields Φ, B, Ψ and E, which transform as in eq. (1.27) under linear coordinate transformations. With this decomposition in hand, we can rewrite all the perturbed building blocks depending on the metric in terms of these four scalar perturbations (see Appendix E.1 for a full list), and obtain a quadratic action $S^{(2)}_{\rm G} + S^{(2)}_{\rm M}$ in terms of Φ, B, Ψ, E, $\delta\varphi$, and the rest of the perturbed extra fields.

We now take an infinitesimal variation of the total quadratic action with regards to each one of the scalar perturbation fields. For scalar-tensor theories, where the matter sector is comprised by a scalar field φ, the variation of the quadratic action can be written as:

$$\delta S^{(2)}_{\rm G} + \delta S^{(2)}_{\rm M} = \int d^4x\ \left[\mathcal{E}_\Phi\delta\Phi + \mathcal{E}_B\delta B + \mathcal{E}_\Psi\delta\Psi + \mathcal{E}_E\delta E + \mathcal{E}_\chi\delta\chi + \mathcal{E}_\varphi\delta\varphi\right] \tag{4.17}$$

where $\mathcal{E}_X$ is the equation of motion for the perturbation field X. To find the Noether identities, we replace the variations of the fields by the corresponding gauge transformations in eq. (1.27) and (4.9), and integrate by parts to end up with:

$$\begin{aligned} \delta_g S^{(2)}_{\rm G} + \delta_g S^{(2)}_{\rm M} &= \int d^4x\ \left[\mathcal{E}_B + H\mathcal{E}_\Psi + \dot{\mathcal{E}}_\Phi - \mathcal{E}_\chi\dot{\chi}_0 - \mathcal{E}_\varphi\dot{\varphi}_0\right]\pi & (4.18)\\ &+ \int d^4x\ \left[-\mathcal{E}_E + \frac{d}{dt}\left(a^2\mathcal{E}_B\right)\right]\epsilon, & (4.19)\end{aligned}$$

where the expression δ_g stands for a variation of the action due to a gauge transformation. We have used the fact that the matter perturbation field $\delta\varphi$ transforms in an analogous way to $\delta\chi$. Given that the total quadratic action is invariant under these gauge transformations, and given that both π and ϵ are arbitrary and independent, each set of brackets must be zero; this gives us the two *Noether identities* associated to the two scalar gauge parameters of the model π and ϵ. Furthermore, each combination of coefficients, inside each of the brackets, multiplying the perturbation fields and their derivatives such as Φ, $\dot{\Phi}$, $\partial^2\Phi$, Ψ, etc, must be individually zero for the Noether identities to be satisfied off-shell, giving a set of *Noether constraints*. These constraints will be, in general, linear ordinary differential equations of the coefficients L_{Θ_A} and $L_{\Theta_A\Theta_B}$. However, for all the cases presented in this chapter and the following, these Noether constraints can be solved algebraically. Solving all of these constraints and replacing the solutions in the quadratic action allows us to determine the number of independent free coefficients and the number of degrees of freedom of the theory. The resulting action will be the most general linearly diffeomorphism-invariant local quadratic action, given the field content. It is important to remark that we only impose gauge invariance under the scalar gauge parameters π and ϵ, and the resulting action will not necessarily be gauge-invariant under the vector gauge parameter ϵ^{Ti} if the extra gravitational DoFs propagate vector perturbations.

We emphasise that the procedure described above is easily generalisable to different backgrounds, to include extra gravitational fields, and different gauge symmetries. To illustrate this, in the next sections we apply the procedure to gravitational actions including a metric and one extra scalar field or vector field. We will also briefly discuss a case in which we impose an extra gauge symmetry, in addition to linear diffeomorphism invariance, in the quadratic action.

As a comparison, we mention that in the EFT approach, the quadratic action for scalar-tensor theories is constructed by working in the unitary gauge, which simplifies calculations, as the dependence on the scalar field vanishes and thus the action only depends on the metric. In this situation one constructs a spatially gauge-invariant quadratic action, and in the end gauge transforms (or "Stueckelberg") to make the extra scalar field explicit, and recover the time gauge invariance [158–160]. However, a generalisation of this procedure is not straightforward. For example, in general, in a vector-tensor theory we would have to fix the spatial and time gauge invariance in order to eliminate the entire dependence on the vector field, and have a quadratic action depending only on the metric. It is not clear that the construction of such metric action is simple as now there would be no gauge symmetry satisfied, relating the different coefficients of the action. In addition, in general, in bimetric theories there is no way of using the gauge freedom to eliminate the entire dependence on the second metric field.

Returning to our procedure, it is possible to easily generalise the matter content to encompass fluids such as baryons, dark matter, etc. In such cases it is convenient to work at the level of the equations of motion instead of the quadratic action. We can do this by finding the first-order equations of motion $\mathcal{E}^{\mu\nu}$:

$$\mathcal{E}^{\mu\nu} \equiv \frac{\delta S_{\mathrm{G}}^{(2)}}{\delta g_{\mu\nu}} = -\frac{\delta_1(\sqrt{-g})}{2}\bar{T}^{\mu\nu} - \frac{a^3}{2}\delta_1 T^{\mu\nu}, \tag{4.20}$$

where we have expanded the energy-momentum tensor of the matter content up to first order, $T^{\mu\nu} = \bar{T}^{\mu\nu} + \delta_1 T^{\mu\nu}$. Note that for finding $\mathcal{E}^{\mu\nu}$ we make a variation of the quadratic gravitational action $S_{\mathrm{G}}^{(2)}$ only. We then have that the equations of motion for each one of the scalar metric perturbation fields become:

$$\begin{aligned}
\mathcal{E}_\Phi &= \left(\delta_1\sqrt{-g}\right)\bar{T}^{00} + a^3\delta_1 T^{00}, \\
\mathcal{E}_B &= \partial_i\left(\delta_1\sqrt{-g}\right)\bar{T}^{0i} + a^3\partial_i\left(\delta_1 T^{0i}\right), \\
\mathcal{E}_\Psi &= \left[\left(\delta_1\sqrt{-g}\right)\bar{T}^{ij} + a^3\delta_1 T^{ij}\right]\bar{h}_{ij}, \\
\mathcal{E}_E &= -a^2\left[\partial_i\partial_j\left(\delta_1\sqrt{-g}\right)\bar{T}^{ij} + a^3\partial_i\partial_j\left(\delta_1 T^{ij}\right)\right].
\end{aligned} \tag{4.21}$$

Naturally, we need to supplement the system with the equations of motion of the matter fields that constitute $T^{\mu\nu}$.

As we have mentioned before, this procedure is useful for translating cosmological constraints into constraints on fundamental gravitational actions, and for easily finding where a given gravity theory lies in the space of free parameters. However, we point out that even accurate observational constraints on the set of parameters do not lead uniquely to one fundamental theory. As we will see in the next sections, there is a considerable degeneracy of the parameters L_{Θ_A} and $L_{\Theta_A\Theta_B}$ that lead to the same observable combinations. The reason for this degeneracy is that we are only constraining the linear evolution of perturbations, but a corresponding higher-order theory could take different forms.

In the following section we apply the procedure presented above to the simplest (and well-established) case of general relativity, as it will allow us to illustrate the method in a familiar setting. In the following chapters we will apply our method to a wider range of theories.

4.2 Recovering general relativity

In this section we parametrise linearly diffeomorphism-invariant gravitational theories containing only one metric field, coupled minimally to a scalar field that constitutes our matter sector. However, the results presented in this section also hold for a perfect fluid matter sector. As in the previous section, we analyse linear perturbations of the fields around a homogeneous and isotropic background.

We follow Step 2 for constructing the most general quadratic gravitational action for a metric. We will allow, at most, second-order derivatives in the equations of motion for the perturbation fields. We start by writing down all the possible perturbed building blocks $\delta\Theta_A$ on which the Taylor-expanded Lagrangian $L_{\rm G}$ might depend. In the ADM formalism, we have $\delta\vec{\Theta} = (\delta N, \delta\dot{N}, \delta\partial_i N, \delta\partial_i\dot{N}, \delta\partial_i\partial_j N, \delta N^i, \delta\dot{N}^i, \delta\partial_j N^i,\ \delta\partial_j\dot{N}^i, \delta\partial_i\partial_j N^k, \delta h_{ij}, \delta K^i{}_j, \delta R^i{}_j)$, where the latter two terms replace $\dot{h}_{ij}$ and $\partial_k\partial_l h_{ij}$. As expected, here we have included all possible metric perturbations up to two derivatives[1]. Note that partial derivatives of the perturbation fields are taken with regards to the background metric, and thus we raise and lower the indices of the perturbed building blocks with $\bar{h}_{ij}$. Also, δ commutes with partial spatial derivatives and so, for instance, $\delta(\partial_i N) = \partial_i(\delta N)$. We emphasise that, contrary to GR, we are *a priori* assuming that δN and δN^i could in principle be dynamical fields (with time derivatives); we will let the Noether identities dictate whether they really are or not. As we will see later, the Noether constraints will indeed make δN and δN^i be non-dynamical fields.

We now proceed to Taylor expand $L_{\rm G}$ up to second order in the perturbed building blocks, as in eq. (4.12). A few comments are in order that will help us understand the notation in the calculations that follow. In the subscripts of the coefficients L_{Θ_A} and $L_{\Theta_A\Theta_B}$ (hereafter referred to as L_*), we use 'S' (for "Shift") as a proxy for N^k, and ∂^n to signal the number of spatial derivatives acting on the ADM metric variables. We recall that all coefficients L_* are evaluated at the level of the background and thus can only depend on $\bar{N} = 1$ and $\bar{h}_{ij}$. Therefore, statistical isotropy allows us to discard coefficients with an odd number of indices (it is not possible to construct such an object out of $\bar{h}_{ij}$ and $\bar{N}$ that respects the isotropy) and imposes symmetries on coefficients with an even number of indices. We then use the following notation for the coefficients L_*:

[1]For simplicity, we have not considered here terms with first spatial derivatives of δh_{ij}. However, they can be systematically added, and the quadratic actions given in eq. (4.30)-(4.32) will not change except for the explicit relations between the coefficients T_* and L_*.

$$
\begin{aligned}
&L_{A^i{}_j} = L_A\delta^j{}_i, \quad L_{A_iB_j} = L_{AB}\bar{h}^{ij}, \quad L_{AB^i{}_j} = L_{AB}\delta^j{}_i, \\
&L_{A^i{}_jB^r{}_s} = L_{AB+}\delta^j{}_i\delta^s{}_r + L_{AB\times}\left(\delta^j{}_r\delta^s{}_i + \bar{h}^{js}\bar{h}_{ir}\right), \text{ where } A_{ij} = A_{ji} \text{ (and/or } B_{ij} = B_{ji}) \\
&L_{A^i{}_jB^r{}_s} = L_{AB+}\delta^j{}_i\delta^s{}_r + L_{AB\times 1}\delta^j{}_r\delta^s{}_i + L_{AB\times 2}\bar{h}^{js}\bar{h}_{ir}, \\
&L_{B^lA^i{}_{jk}} = L_{BA\times 2}\bar{h}_{li}\bar{h}^{jk} + L_{BA\times 1}\left(\delta^j{}_l\delta^k{}_i + \delta^k{}_l\delta^j{}_i\right), \text{ where } A^i{}_{jk} = A^i{}_{kj},
\end{aligned} \tag{4.22}
$$

where A, $A^i{}_j$, etc. represent any term of the building blocks with the corresponding index structure. Two exceptional cases that do not follow the previous definitions are these:

$$
\begin{aligned}
L_{\partial_i\partial_jN\partial_sN^r} &= \frac{1}{3}L_{\partial^2N\partial S}\left(\bar{h}^{ji}\delta^s{}_r + \delta^j{}_r\bar{h}^{si} + \bar{h}^{js}\delta^i{}_r\right), \\
L_{\partial_lN\partial_j\partial_kN^i} &= \frac{1}{3}L_{\partial^2S\partial N}\left(\delta^l{}_i\bar{h}^{jk} + \bar{h}^{jl}\delta^k{}_i + \bar{h}^{kl}\delta^j{}_i\right).
\end{aligned} \tag{4.23}
$$

With all these definitions in hand we can Taylor expand $L_{\rm G}$ up to second order. Recall that the perturbed building blocks can contain first- and second-order perturbations of the metric. However, we find that only N, $\sqrt{|h|}$, R and K have second-order terms; thus from now on we use δ for first-order perturbations and δ_2 for second-order perturbations, unless explicitly stated otherwise.

We then can find the Taylor expansion of the gravitational action $S_{\rm G}$, as in eq. (4.13). We require the first and second-order perturbations of the metric density:

$$
\begin{aligned}
\delta_1\left(N\sqrt{|h|}\right) &= \delta\sqrt{|h|} + a^3\delta N, \\
\delta_2\left(N\sqrt{|h|}\right) &= \delta_2\sqrt{|h|} + a^3\delta_2 N + \delta\sqrt{|h|}\delta N,
\end{aligned} \tag{4.24}
$$

where we have used $\sqrt{|\bar{h}|} = a^3$ and $\bar{N} = 1$. To simplify notation we introduce $\delta h^i{}_j \equiv \bar{h}^{ik}\delta h_{kj}$ and $\delta h \equiv \bar{h}^{ij}\delta h_{ij}$, and thus δ(trace of h_{ij}) $\neq$ (trace of δh_{ij}), which will be used later.

Finally, the action for our matter sector scalar field φ is given by:

$$
S_{\rm M} = -\int d^4x\sqrt{-g}\left(\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + V(\varphi)\right), \tag{4.25}
$$

where $V(\varphi)$ is some potential. This action can be straightforwardly written in terms of the ADM variables, and Taylor expanded up to second order in the linear perturbations of the metric and scalar field.

As mentioned in the previous section, from the linear expansion of the total action (gravity and matter) we find the background equations. In this case, from the metric perturbations, we get:

$$\begin{aligned}\bar{L} + L_N - 3HL_{\dot{N}} - \dot{L}_{\dot{N}} - 3HL_K &= \rho_0, \\ \bar{L} - 3HL_K - \dot{L}_K + 2L_h &= -P_0, \end{aligned} \tag{4.26}$$

where ρ_0 and P_0 are the energy density and pressure of the fluid, respectively. Explicitly,

$$\rho_0 = \frac{1}{2}\dot{\varphi}_0^2 + V_0, \quad P_0 = \frac{1}{2}\dot{\varphi}_0^2 - V_0, \tag{4.27}$$

where the subscript 0 indicates the background value. Equations (4.26) are a generalisation of the background equations shown in [161], whose explicit derivation can be found in Appendix C.4. Note that we will also have an additional background equation from the linear terms in the matter sector field:

$$\ddot{\varphi}_0 + 3H\dot{\varphi}_0 + V_0' = 0, \tag{4.28}$$

where V_0' is the derivative of the potential with regards to the scalar field, evaluated at the background.

On the other hand, from the quadratic terms of the total action, we obtain the action that governs the evolution of the cosmological perturbations. In this case, the full quadratic gravitational action in eq. (4.15) can be written as:

$$S_{\mathrm{G}}^{(2)} = \int d^4x \sum_{i=0}^{2} \mathcal{L}_T^i. \tag{4.29}$$

The subscript T here stands for "tensor", as in the present case we only have a tensor field. The $\mathcal{L}_T^i$ are quadratic Lagrangians leading to i derivatives of the perturbation fields in the equations of motion. Explicitly, we have:

$$\begin{aligned}\mathcal{L}_T^0 = \frac{a^3}{2}\left[T_{hh+}(\delta h)^2 + 2T_{hh\times}\delta h^i_j \delta h^j_i\right] + \bar{T}\delta_2\sqrt{|h|} + a^3\Bigg[\frac{1}{2}T_{SS}\bar{h}_{ij}\delta N^i \delta N^j \\ +\frac{1}{2}T_{NN}(\delta N)^2 + T_{Nh}\delta N \delta h + T_N\left(\delta_2 N + \delta N \frac{\delta\sqrt{|h|}}{a^3}\right)\Bigg],\end{aligned} \tag{4.30}$$

$$\begin{aligned}\mathcal{L}_T^1 = a^3\big[T_{\partial Sh+}\partial_i \delta N^i \delta h + 2L_{\partial Sh\times}\delta h_{ij}\partial^i \delta N^j + 2L_{hK\times}\delta h^j_i \delta K^i_j + L_{hK+}\delta K \delta h \\ +T_{NK}\delta N \delta K + T_{N\partial S}\delta N \partial_i \delta N^i\big],\end{aligned} \tag{4.31}$$

$$
\begin{aligned}
\mathcal{L}_T^2 = a^3 \Big[& 2L_{hR\times}\delta h_i^j \delta R_j^i + T_{hR+}\delta R\delta h + L_R \delta_2 R + T_{\partial\dot{S}h+}\partial_i \delta \dot{N}^i \delta h + 2L_{\partial\dot{S}h\times}\delta h_{ij}\partial^i \delta \dot{N}^j \\
& + \frac{1}{2}L_{\dot{S}\dot{S}}\bar{h}_{ij}\delta\dot{N}^j\delta\dot{N}^i + \frac{1}{2}T_{\partial S\partial S+}(\partial_i \delta N^i)(\partial_j \delta N^j) + \frac{1}{2}T_{\partial S\partial S\times}\bar{h}_{lj}(\partial_i \delta N^l)(\partial^i \delta N^j) \\
& + \frac{1}{2}L_{KK+}(\delta K)^2 + L_{KK\times}\delta K_j^i \delta K_i^j + \frac{1}{2}L_{\dot{N}\dot{N}}(\delta\dot{N})^2 + \frac{1}{2}T_{\partial N\partial N}\partial^i \delta N \partial_i \delta N \\
& + T_{h\partial^2 N+}\delta h \partial^2 \delta N + 2L_{h\partial^2 N\times}\delta h_{ij}\partial^i\partial^j \delta N + T_{NR}\delta N \delta R + L_{\dot{N}K}\delta K \delta \dot{N} \\
& + T_{N\partial\dot{S}}(\partial_j \delta \dot{N}^j)\delta N + L_{\partial SK+}\delta K \partial_i \delta N^i + 2L_{\partial SK\times}\delta K_j^i \partial_i \delta N^j \Big] ,
\end{aligned} \tag{4.32}
$$

where, for simplicity, we have integrated by parts, grouped coefficients together and relabeled them as T_* (a dictionary that translates between L_* and T_* can be found in Appendix C.3). To understand the derivative structure above, we remind the reader that δK_j^i contains one spatial derivative, and δR contains two – see the definitions in Appendix C.1. For writing these actions we have also made use of the relations $\delta\sqrt{|h|} = \frac{1}{2}\sqrt{\bar{h}}\bar{h}^{ij}\delta h_{ij} = \frac{1}{2}a^3\delta h$ and $\delta_2\sqrt{|h|} = \frac{1}{8}a^3(\delta h)^2 - \frac{1}{4}a^3\delta h^i{}_j \delta h^j{}_i$. In addition, we have rewritten the term $L_K \delta K$ that comes from the expansion of L_{G} (where now δK includes first- and second-order perturbations). Following [161] we have made an integration by parts so that:

$$
L_K \delta K \to -3HL_K - \dot{L}_K + \dot{L}_K \delta N + \dot{L}_K \delta_2 N - \dot{L}_K (\delta N)^2. \tag{4.33}
$$

From the matter action we find the following quadratic action:

$$
\begin{aligned}
S_{\mathrm{M}}^{(2)} = - \int d^4x \Bigg\{ & -P_0 \delta_2 \sqrt{|h|} + a^3 \rho_0 \left(\delta_2 N + \delta N \frac{\delta_1 \sqrt{|h|}}{a^3} \right) - \frac{a^3}{2}(P_0 + \rho_0)(\delta N)^2 \\
& + a^3 \left[\frac{1}{2} V_0'' \delta\varphi^2 + (V_0'\delta\varphi + \dot{\varphi}_0 \delta\dot{\varphi})\delta N + \dot{\varphi}_0 \partial_i \delta\varphi \delta N^i - \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}\bar{h}^{ij}\partial_j \delta\varphi \partial_i \delta\varphi \right] \\
& + \delta\sqrt{|h|}\,(V_0'\delta\varphi - \delta\dot{\varphi}\dot{\varphi}_0) \Bigg\} .
\end{aligned} \tag{4.34}
$$

Note that $S_{\mathrm{M}}^{(2)}$ leads to quadratic terms in the perturbations of the metric, as well as linear and quadratic terms on the perturbations of the matter field. We have isolated $\delta_2\sqrt{|h|}$ and $\left(\delta_2 N + \delta N \frac{\delta\sqrt{|h|}}{a^3}\right)$ in eq. (4.30), as their corresponding coefficients ($\bar{T}$ and T_N) will exactly cancel the corresponding terms in the matter action in eq. (4.34) due to the background equations (4.26), which can be re-expressed as:

$$
T_N = \rho_0, \quad \bar{T} = -P_0, \tag{4.35}
$$

where we have used the dictionary in Appendix C.3.

We can now apply Step 3 of the procedure described previously. First we focus only on scalar perturbations. We write down the total quadratic action in terms of the four metric scalars Φ, B, Ψ and E, and the matter perturbation field $\delta\varphi$. Note that even though we allow up to two derivatives of the metric perturbations δN, δN^i and δh_{ij}, this means that we will have higher-order derivatives of the four scalar metric perturbations. We proceed to find the Noether identities that arise for *both* spatial and temporal linear gauge invariance, given in eq. (1.24) for the metric perturbations and analogous to eq. (4.9) for $\delta\varphi$. From each of these two *sets* of constraints (the Noether identities) we then extract the individual Noether constraints multiplying each individual perturbation and its derivatives. We solve the Noether constraints to find the following non-redundant conditions on the parameters of the quadratic action:

$$L_{\dot{S}\dot{S}} = T_{N\partial\dot{S}} = L_{K\dot{N}} = L_{\dot{N}\dot{N}} = T_{SS} = 0, \tag{4.36}$$

$$T_{NN} = T_{\partial N\partial N} = T_{NK} = 0, \tag{4.37}$$

$$L_{KK+} = -2L_{KK\times}, \tag{4.38}$$

$$2L_{\partial\dot{S}h\times} - L_{K\partial S\times} = 2T_{\partial\dot{S}h+} - L_{K\partial S+} = 0, \tag{4.39}$$

$$T_{N\partial S} = 3HL_{K\partial S+} + 2HL_{K\partial S\times}, \tag{4.40}$$

$$2(L_{h\partial S\times} - L_{hK\times}) = \dot{L}_{K\partial S\times} + 3HL_{K\partial S\times}, \tag{4.41}$$

$$2(T_{h\partial S+} - L_{hK+}) = \dot{L}_{K\partial S+} + 3HL_{K\partial S+}, \tag{4.42}$$

$$T_{\partial S\partial S\times} + T_{\partial S\partial S+} = 2L_{K\partial S+} + 4L_{K\partial S\times}, \tag{4.43}$$

$$T_{Nh} = 3HL_{hK+} + 2HL_{hK\times}, \tag{4.44}$$

$$2(T_{hh+} + T_{hh\times}) = \dot{L}_{hK+} + \dot{L}_{hK\times} + 3HL_{hK+} + 3HL_{hK\times}, \tag{4.45}$$

$$4T_{hR+} = L_R + L_{KK\times} + \dot{L}_{KK\times}/H, \tag{4.46}$$

$$2L_{hh\times} = \dot{L}_{hK\times} + 3HL_{hK\times}, \tag{4.47}$$

$$H(L_R - 4L_{hR\times}) = \dot{L}_{KK\times} + HL_{KK\times}, \tag{4.48}$$

$$T_{h\partial^2N+} = -2L_{h\partial^2N\times}, \tag{4.49}$$

$$T_{NR} = L_{KK\times} - 2L_{h\partial^2N\times}, \tag{4.50}$$

$$4\dot{H}L_{KK\times} = -(\rho_0 + P_0), \tag{4.51}$$

where we have used the background equations to simplify some of these constraints. We have written these equations in a form that will look the same for a minimally coupled scalar field and a general perfect fluid. Notice that all these constraints can be solved algebraically, by simply working out one coefficient without time derivatives in terms of the rest.

Via the constraints above, the number of free coefficients in our original action is greatly reduced. A straight substitution of the Noether constraints into the quadratic action reduces the original 32 free, time-dependent, functions (30 coefficient functions L_* and T_* in $S_{\rm G}^{(2)}$, along with the two background functions φ_0 and a) down to 8; after some integrations by parts we can collapse the number of the remaining free functions down further to only one: $L_{KK\times}$. In addition, we find that all terms involving time derivatives of δN and δN^i vanish, so they play the role of functional Lagrange multipliers – one of the key characteristics of general relativity.

It is apparent that the time dependence of the coefficients is intimately tied to that of the background, through H, ρ_0 and P_0. We can then take the final step of replacing our reduced set of coefficients into eq. (6.8), to get the following total quadratic action:

$$\begin{aligned} S_{\rm G}^{(2)} + S_{\rm M}^{(2)} &= \int d^4x\, a^3 \Big[-\dot\varphi_0 \delta\dot\varphi \left(\Phi + 3\Psi\right) - V'\delta\varphi \left(\Phi - 3\Psi\right) - \frac{1}{2} V'' \left(\delta\varphi\right)^2 - \frac{1}{2}\delta\varphi\partial^2\delta\varphi \\ &+ \frac{1}{2}\left(\delta\dot\varphi\right)^2 - \dot\varphi_0 a^2 \partial^2 E \delta\dot\varphi - \dot\varphi_0 \delta\varphi \partial^2 B + V'\delta\varphi a^2 \partial^2 E + M^2 \left(1 + \frac{d\ln M^2}{d\ln a}\right) \Psi\partial^2\Psi \\ &- 3M^2\dot\Psi^2 - 6HM^2\dot\Psi\Phi - 2M^2\Psi\partial^2\Phi - \left(\dot H + 3H^2\right) M^2\Phi^2 \\ &- 2M^2 a^2 \partial^2 \dot E \left(\dot\Psi + H\Phi\right) + 2M^2\dot\Psi\partial^2 B + 2HM^2\Phi\partial^2 B \Big], \end{aligned} \tag{4.52}$$

where we have redefined $M^2 \equiv 2L_{KK\times}$, so that one of the Noether constraints becomes,

$$M^2 = -\frac{\rho_0 + P_0}{2\dot H}. \tag{4.53}$$

It is instructive to further transform this quadratic action, by making the replacement $\delta\varphi \to \delta\varphi\dot\varphi_0$, using the background equation for the scalar field (eq. (4.28)), and making a few integrations by parts. We then find:

$$\begin{aligned} S_{\rm G}^{(2)} + S_{\rm M}^{(2)} &= \int d^4x\, a^3 M^2 \Big[-6\dot H\delta\varphi\dot\Psi - \dot H\left(\delta\dot\varphi\right)^2 + 2\dot H\Phi\delta\dot\varphi - 2\dot H a^2\partial^2\dot E\delta\varphi \\ &- 3\dot H^2\left(\delta\varphi\right)^2 - 6H\dot H\delta\varphi\Phi + 2\dot H\delta\varphi\partial^2 B + \dot H\delta\varphi\partial^2\delta\varphi + \left(1 + \frac{d\ln M^2}{d\ln a}\right)\Psi\partial^2\Psi \\ &- 3\dot\Psi^2 - 6H\Phi\dot\Psi - 2\Psi\partial^2\Phi - \left(\dot H + 3H^2\right)\Phi^2 \\ &- 2a^2\partial^2\dot E\left(\dot\Psi + H\Phi\right) + 2\dot\Psi\partial^2 B + 2H\Phi\partial^2 B \Big]. \end{aligned} \tag{4.54}$$

In other words, the final action in terms of the metric perturbations depends only on one free function of time, M^2; the scale factor does not count as a free function, as it is related to M through eq. (4.53) and (4.28). If the background equations were simply the Friedman equations then, from eq. (4.53) we would find $M^2 = M_{\rm P}^2$, and eq. (4.54) would become the quadratic action for general relativity. In general, however, M^2 is a completely free function of time. This illustrates a crucial feature of any approach based on finding general linearised theories at the perturbative level. For a single tensor, at the level of the *full* diffeomorphism-invariant theory, we know that there should be no overall free function of time left – GR is unique in this sense. Said another way, M^2 being a free function of time is an artefact of just taking into consideration the linearised action for perturbations. The consistency of a full theory requires background, linearised perturbative and higher-order perturbative contributions all to be consistent, i.e. to avoid the propagation of unstable degrees of freedom such as ghosts. And so, crucially, while all well-behaved theories will map onto the free functions in our linearised perturbation theory parametrisation, not all possible functional forms for these seemingly free functions are associated with healthy theories. This happens for the very simple reason that there is more to a full theory than the action it gives rise to for linear perturbations, and that there are additional constraints not captured by any formalism based on linearised perturbations. These extra constraints will reduce the free functions we recover further. A detailed analysis on the construction of possible fundamental consistent theories leading to the quadratic actions presented here is beyond the scope of this work, but it is certainly relevant and requires further work.

We have shown that it is possible to systematically recover the linearised action for the most general linearly diffeomorphism-invariant theory of gravity built from a metric, by starting from a completely general action and systematically applying gauge transformations to obtain the Noether constraints. We have found that M^2 is the only parameter that enters the final action and hence the equations of motion. Looking forward, this means that any attempt to constrain this action (with cosmological observations) boils down to constraining M^2. But, as we have seen, there are a number of degeneracies that remain between the original coefficients L_*. So, we can already see that it is impossible to individually constrain all the coefficients that we used to build the action in equations (4.30)-(4.32). In effect, we will never be able to completely pin down the landscape of theories to solely GR using only cosmological linear perturbation theory alone. At best we will be able to constrain these actions to a degenerate family of theories that includes GR.

Finally, we remark that since Action (4.54) leads to, at most, second-order differential equations in time, it is free of Ostrogradski instabilities associated to higher time-derivative terms [1,172]. Furthermore, this action propagates only one physical scalar DoF, which actually comes from the matter sector. It can be seen that Φ and B are auxiliary variables, i.e. without dynamics, and can be expressed in terms of the rest of the fields by using their own equations of motion. Therefore, they do not represent independent physical DoFs. In addition, the action has a gauge symmetry with two arbitrary parameters inducing two redundant fields in the action. Thus, from the original 5 scalars in eq. (4.54), only one field is physical.

4.3 Discussion

In this chapter we have constructed a method for parametrising the most general, local, quadratic actions for linear cosmological perturbations. This is a crucial step towards identifying how many free functions fully characterise the landscape of gravitational theories in the linear cosmological regime. Our systematic method for finding such actions, given a field content and (set of) gauge symmetries, consists of the following three main steps:

1. Assume a given number and type(s) of fields present in the theory (gravity and matter). Given an ansatz for the cosmological background, consider linear perturbations around that background for each field. Finally, choose what gauge symmetries to impose on the quadratic action determining the evolution of these perturbations.

2. Construct the most general local quadratic gravitational action, given the content field set in Step 1. Start with an unperturbed fundamental gravitational action $S_{\rm G}$, a functional of a set of building blocks $\vec{\Theta}$ containing all the fields and their derivatives (up to some truncating maximum order). Find the perturbed set of building blocks $\delta\vec{\Theta}$, given the linear perturbations of the fields, and Taylor expand $S_{\rm G}$ up to second order in $\delta\vec{\Theta}$. Finally, add some known matter action $S_{\rm M}$ and Taylor expand in the same way. The first-order total action $S_{\rm G}^{(1)} + S_{\rm M}^{(1)}$ leads to the background equations of motion, while the second-order total action $S_{\rm G}^{(2)} + S_{\rm M}^{(2)}$ determines the evolution of the linear cosmological perturbations. The form of $S_{\rm G}^{(2)}$ should be that of an action including all possible covariant quadratic interactions between the linear perturbation fields. Each term in this action has an *a priori* free coefficient in front, which is a functional derivative of the fundamental action $S_{\rm G}$ evaluated at the background.

3. Find the most general linearly gauge-invariant quadratic action for perturbations. Consider $S_{\rm G}^{(2)} + S_{\rm M}^{(2)}$ from Step 2, and find the Noether identities associated to the desired gauge symmetry. Each gauge parameter will lead to a Noether identity, which in turn will lead to a number of Noether constraints which are, in general, linear ordinary differential equations of the free coefficients in $S_{\rm G}^{(2)}$. After solving the system of Noether

constraints and replacing the results in $S_{\mathrm{G}}^{(2)}$, we obtain the most general quadratic gravitational action for linear cosmological perturbations for that particular field content and set of symmetries. From this result it is straightforward to identify the number of free parameters describing the linear cosmological evolution of the Universe, and the number of physical DoFs propagating.

In this procedure, the free parameters characterising the quadratic action for perturbations are related to properties of fundamental gravitational theories. This makes the procedure useful for translating cosmological constraints into constraints on fundamental actions, as well as for straightforwardly finding where a given gravity theory lies in the space of these free parameters. In addition, since our method is very systematic, all the calculations presented in this chapter are easily generalisable to different backgrounds, to include extra gravitational fields, and different gauge symmetries.

We have applied the procedure to a purely metric theory, leading to second-order derivatives in the equations of motion. In this case we found one free coefficient M, a function of time, describing the cosmological background and linear evolution of the universe. We also found that these quadratic gravitational actions do not propagate any scalar DoF. When $M = M_{\mathrm{P}}$ we recover GR. We do not uniquely obtain the quadratic action for GR in this case, as GR is fully diffeomorphism-invariant, but we only required linear diffeomorphism invariance. In other words, there is more to a full theory than its quadratic action, and there are additional constraints not captured by any formalism based on linearised perturbations. Therefore, the fundamental theories described by the parameter M could break the full diffeomorphism invariance, or maybe propagate extra DoFs that are only present at higher perturbative order. This also means that, in general, not all the possible values of the free parameters will be associated to healthy fundamental theories. This first case then highlights the fact that even with an accurate measurement of the free parameters, we will never be able to completely pin down the landscape of fundamental theories to only one by using linear cosmological perturbation theory alone.

CHAPTER 5

A general theory of linear cosmological perturbations: scalar-tensor and vector-tensor theories

The simplest, non-trivial example of a theory which includes an extra degree of freedom and differs from general relativity is a scalar-tensor theory. The original, most elementary, formulation is Jordan-Brans-Dicke gravity, a theory in which the Planck mass is promoted to a dynamical scalar field [173–175]. Jordan-Brans-Dicke gravity has been one of the workhorses of modern cosmology and has been deployed in understanding both the early Universe (specifically inflation) and the late-time accelerated expansion of the Universe [1]. Over the past few years, renewed interest in scalar-tensor theories has emerged, on the one hand from the rediscovery of the Horndeski action [176] – the most general, non-degenerate, scalar-tensor action with second-order equations of motion – and on the other hand from various extensions of the class of covariant Galileons [177].

Most attempts at constructing a general parametrisation of linearised gravity have focused on scalar-tensor theories. A nuanced understanding of how scalar-tensor theories emerge has been developed, most notably in [178], where an economical parametrisation of such theories was proposed in terms of four free functions. These functions (the 'α' functions) can be easily related to specific physical properties of the fundamental action. Subsequent work has extended this parametrisation to five free functions [179–181]. In this chapter we use the method we previously developed to recover the parametrisation found in [178] and also find parametrised actions that include higher-derivative corrections.

In order to extend the landscape of parametrised modified gravity theories, we also apply the method to vector-tensor gravity theories. Vector-tensor theories have been studied in detail in attempts to understand spontaneous Lorentz violation [168, 182], to generate massive gravitons [183] and as models of dark matter and dark energy [184, 185]. In particular, we construct the quadratic action for perturbations that leads to general second-order equations of motion, and then specialise to the case in which the vector field is time-like (à la Einstein-Aether gravity). As a result, we identify the complete forms of the quadratic actions for perturbations, and the number of free parameters that need to be defined, to cosmologically characterise these two broad classes of theories.

This chapter is structured as follows. In Section 5.1 we apply the method to scalar-tensor theories and show that we recover the results of [161] and [178]. In particular, our approach includes the "Beyond Horndeski" parameter found in [161], and extra parameters allowing fourth spatial derivatives of the fields in their equations of motion. In Section 5.2 we apply our method to vector-tensor theories, with at most two derivatives of the fields. Here there are two propagating scalar DoFs, neither of which transforms as a scalar perturbation of a scalar-tensor theory. We show how to construct the most general quadratic action for perturbations with this field content and, as importantly, how to implement constraints so that we end up (as advertised) with only one propagating scalar DoF. Finally, in Section 5.3 we review our findings and discuss how to generalise the calculations presented in this chapter.

5.1 Recovering linearised Beyond Horndeski theory and beyond

In this section we will parametrise linearly diffeomorphism-invariant gravitational theories containing one metric and one scalar field, coupled minimally to a matter scalar field (although the results presented here also hold for a general matter perfect fluid). As in the previous chapter, we will analyse linear perturbations of the fields around a homogeneous and isotropic background. We will show that with our procedure we can reproduce previous work. In particular, we will show how the free functions describing such theories will emerge from the Noether constraints applied to a quadratic action with up to three time and space derivatives. Furthermore, we will then show that, if we include higher-order derivatives, a further set of functions must be included to completely cover the possible space of theories. To avoid any Ostrogradski instability, we allow at most two *time* derivatives of the fields, but higher *spatial* derivatives are permitted (this situation can arise in some Lorentz-violating theories, but also in some special Lorentz-invariant cases such as Beyond Horndeski theories). It would of course be possible to go beyond this and find theories that are higher-order in temporal derivatives as well, yet evade Ostrogradski ghosts via the presence of degeneracies [186, 187] or, equivalently, hidden constraints [188]. [1]

5.1.1 Horndeski and beyond

We now include an extra degree of freedom, a scalar field χ, whose perturbation transforms under linear coordinate transformations as in eq. (4.9). We proceed with Step 2 for constructing the most general quadratic action. Allowing at most three derivatives of the perturbation fields (two temporal but three spatial), we write down all possible perturbed building blocks $\delta\vec{\Theta} = (\cdots, \delta\chi, \delta\dot{\chi}, \partial_i\delta\chi, \partial_i\delta\dot{\chi}, \partial_i\partial_j\delta\chi, \partial_i\partial_j\delta\dot{\chi}, \partial_i\partial_j\delta_k\delta\chi, \delta\partial_i\partial_j\dot{N}, \delta\partial_i\partial_j\partial_k N, \delta\partial_i\partial_j\dot{N}^k, \delta\partial_i\partial_j\partial_k N^l)$ where the initial ellipses indicates all the building blocks used in Section 4.2 [2]. We will also introduce more definitions for the coefficients L_*, in addition to those given in eq. (4.22) and (4.23):

$$
\begin{aligned}
L_{A_iB_{jkl}} &= \frac{1}{3}L_{AB}\left(\bar{h}^{ij}\bar{h}^{kl} + \bar{h}^{ik}\bar{h}^{jl} + \bar{h}^{il}\bar{h}^{jl}\right), \text{ where } B_{ijk} \text{ is fully symmetric,} \\
L_{A^i{}_{jkl}} &= \frac{1}{3}L_{A}\left(\delta^j{}_i\bar{h}^{kl} + \delta^k{}_i\bar{h}^{jl} + \delta^l{}_i\bar{h}^{jl}\right), \text{ where } A^i{}_{jkl} \text{ is symmetric in 3 indices,} \\
L_{BA^i{}_{jkl}} &= \frac{1}{3}L_{BA}\left(\delta^j{}_i\bar{h}^{kl} + \delta^k{}_i\bar{h}^{jl} + \delta^l{}_i\bar{h}^{jl}\right), \text{ where } A^i{}_{jkl} \text{ is symmetric in 3 indices,}
\end{aligned} \tag{5.1}
$$

where A_i, B_{jkl}, etc. correspond to any possible building block with the corresponding index structure. An exceptional case is

$$
\begin{aligned}
L_{h_{ij}\partial_k\partial_l\partial_m N^n}\delta h_{ij}\partial_k\partial_l\partial_m\delta N^n = {}& L_{h\partial^3 S+}\delta h\partial^2\partial_i\delta N^i + 2L_{h\partial^3 S\times}\delta h_{ij}\partial^2\partial^i\delta N^j \\
&+ 2L_{h\partial^3 S\odot}\delta h_{ij}\partial^i\partial^j\partial_l\delta N^l.
\end{aligned} \tag{5.2}
$$

As in the previous section, we Taylor expand the gravitational and matter action up to second order in the perturbation fields. From the linear total action we derive the background equations. If we do so, we will obtain eq. (4.26) and eq. (4.28) for the metric evolution and matter field, which we now supplement with:

$$
L_\chi - 3HL_{\dot{\chi}} - \dot{L}_{\dot{\chi}} = 0, \tag{5.3}
$$

[1] In this context also note that we are interested in effective theories, which should be ghost-free within their regimes of validity. Any given model may "predict" ghost-like instabilities outside the regime of validity of that theory, i.e. instabilities coming with a mass/energy scale above the theory's cutoff. However, such instabilities are not physical and there is no reason to discard a theory.

[2] As in the previous section, we do not consider a term with three spatial derivatives of δh_{ij}. Such terms could be added but the form of the quadratic Lagrangians $\mathcal{L}^i_T$ and $\mathcal{L}^i_\chi$ would not change except in terms of the explicit relations between the coefficients T_* and L_*.

which corresponds to the background equation for the scalar field χ_0. These four background equations should not be all independent, as there are only three undetermined background functions: a, χ_0 and φ_0. This redundancy imposes a relation between the coefficients T_* and L_*, which is not relevant for this work, but would be important for the task of constructing non-perturbative fundamental actions allowing homogeneous and isotropic backgrounds.

We now extend the gravitational action considered in Section 4.2 such that

$$S_{\mathrm{G}}^{(2)} = \int d^4x \sum_{i=0}^{3} \left(\mathcal{L}_T^i + \mathcal{L}_\chi^i\right), \tag{5.4}$$

where $\mathcal{L}_\chi^i$ are quadratic Lagrangians involving $\delta\chi$, leading to i derivatives of the perturbation fields in the equations of motion. Up to second-order derivatives, we have the tensor Lagrangians given in the previous section, and we add the following Lagrangians involving the perturbation of the scalar field, $\delta\chi$:

$$\begin{aligned}
\mathcal{L}_\chi^0 &= \frac{a^3}{2}\left[T_{\chi\chi}(\delta\chi)^2 + 2T_{\chi h}\delta\chi\delta h + T_{\chi N}\delta N\delta\chi\right], && (5.5)\\
\mathcal{L}_\chi^1 &= a^3\left[T_{\dot\chi h}\delta\dot\chi\delta h + L_{\chi K}\delta\chi\delta K + T_{\chi\partial S}\delta\chi\partial_i\delta N^i + T_{\dot\chi N}\delta\dot\chi\delta N\right], && (5.6)\\
\mathcal{L}_\chi^2 &= a^3\Bigg[L_{\chi R}\delta\chi\delta R + T_{\partial^2\chi h+}\delta h\partial^2\delta\chi + 2L_{\partial^2\chi h\times}\delta h_{ij}\partial^i\partial^j\delta\chi + \frac{1}{2}L_{\dot\chi\dot\chi}(\delta\dot\chi)^2 + L_{K\dot\chi}\delta K\delta\dot\chi \\
&+ \frac{1}{2}T_{\partial\chi\partial\chi}\partial_i\delta\chi\partial^i\delta\chi + L_{\dot\chi\dot N}\delta\dot N\delta\dot\chi + T_{\dot\chi\partial S}\partial_i\delta N^i\delta\dot\chi + T_{\partial\chi\partial N}\partial_i\delta N\partial^i\delta\chi\Bigg], && (5.7)
\end{aligned}$$

For third-order derivatives we include the following tensor and scalar Lagrangians:

$$\begin{aligned}
\mathcal{L}_T^3 &= a^3\big[2L_{h\partial^3S\times}\delta h_{ij}\partial^2\partial^i\delta N^j + T_{h\partial^3S+}\delta h\partial^2\partial_j\delta N^j \\
&+ 2L_{h\partial^3S\odot}\delta h_{ij}\partial^i\partial^j\partial_l\delta N^l + L_{h\partial^2\dot N+}\delta h\partial^2\delta\dot N + 2L_{h\partial^2\dot N\times}\delta h_{ij}\partial^i\partial^j\delta\dot N + L_{\partial SR+}\delta R\partial_j\delta N^j \\
&+ 2L_{\partial SR\times}\delta R^i_j\partial_i\delta N^j + L_{KR+}\delta K\delta R + 2L_{KR\times}\delta K^i_j\delta R^j_i + L_{\dot NR}\delta R\delta\dot N + T_{K\partial^2N+}\delta K\partial^2\delta N \\
&+ 2L_{K\partial^2N\times}\delta K^i_j\partial^j\partial_i\delta N + L_{\partial\dot SK+}\delta K\partial_i\delta\dot N^i + 2L_{\partial\dot SK\times}\delta K^i_j\partial_i\delta\dot N^j + T_{\dot N\partial\dot S}\delta\dot N\partial_j\delta\dot N^j \\
&+ T_{\partial^2N\partial S}\partial_i\delta N^i\partial^2\delta N\big], && (5.8)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}_\chi^3 &= a^3\Big[L_{R\dot\chi}\delta R\delta\dot\chi + T_{\partial^2\dot\chi h+}\delta h\partial^2\delta\dot\chi + 2L_{\partial^2\dot\chi h\times}\delta h_{ij}\partial^i\partial^j\delta\dot\chi + T_{\partial^2\dot\chi N}\delta N\partial^2\delta\dot\chi + T_{\dot\chi\partial\dot S}\partial_i\delta\dot N^i\delta\dot\chi \\
&+ L_{\partial^2\chi K+}\delta K\partial^2\delta\chi + 2L_{\partial^2\chi K\times}\delta K^i_j\partial^j\partial_i\delta\chi + T_{\partial^2\chi\partial S}\partial_i\delta N^i\partial^2\delta\chi\big]. && (5.9)
\end{aligned}$$

Note that, as in Section 4.2, we have integrated by parts and redefined some of the coefficients to simplify notation; the dictionary to translate between the T_* and L_* is in Appendix D.1. Finally, for the matter scalar field φ we add the quadratic action shown in eq. (4.34).

We now follow Step 3 where, in addition to (4.36), (4.38)-(4.47) we get a new set of Noether constraints. We find that the end result depends on five free coefficients of the time: M^2, α_B, α_K, α_T, α_H. In terms of the coefficients L_* and T_*, arising solely in $\mathcal{L}_T^i$, these are:

$$M^2 = 2L_{KK\times}, \tag{5.10}$$

$$\alpha_B = \frac{1}{2}\frac{T_{NK}}{M^2 H}, \tag{5.11}$$

$$\alpha_K = \frac{T_{NN}+T_{SS}}{H^2M^2}, \tag{5.12}$$

$$\alpha_T = \frac{2}{M^2}\left(L_R + HL_{KR\times} + \dot{L}_{KR\times} + 4L_{Rh+}\right) - 1, \tag{5.13}$$

$$\alpha_H = \frac{2}{M^2}\left[-\dot{L}_{K\partial^2 N\times} + H\left(L_{KR\times} - L_{K\partial^2 N\times}\right) + T_{NR} + 2L_{h\partial^2 N\times}\right] - 1, \tag{5.14}$$

which are completely equivalent (and more general) than the expressions found in [161]. Note that the α_i can be neatly understood via the physical effects they parametrise [178]. Explicitly, the final quadratic gravitational action is then:

$$\begin{aligned}
S_{\rm G}^{(2)} &= \int d^4x\, a^3 M^2 \left\{\frac{1}{2}H^2\left(\alpha_K - 12\alpha_B - 6\right)\Phi^2 - 6H\left(1+\alpha_B\right)\Phi\dot{\Psi} + 2\left(1+\alpha_H\right)\Psi\partial^2\Phi\right.\\
&- 3\dot{\Psi}^2 - \left(1+\alpha_T\right)\Psi\partial^2\Psi + 2a^2H\left(1+\alpha_B\right)\Phi\partial^2\dot{E} - 2H\left(1+\alpha_B\right)\Phi\partial^2 B + 2a^2\Psi\partial^2\dot{E}\\
&- 2\dot{\Psi}\partial^2 B - 3\left(\frac{\rho_0+P_0}{M^2} + 2\dot{H}\right)\dot{\Psi}\delta\chi + 2\alpha_H\dot{\Psi}\partial^2\delta\chi + 6H\alpha_B\dot{\Psi}\delta\dot{\chi} + H^2\left(6\alpha_B - \alpha_K\right)\Phi\delta\dot{\chi}\\
&- 2H\left[\alpha_T - \alpha_H - \frac{d\ln M^2}{d\ln a}\left(\alpha_H+1\right) - \frac{d\alpha_H}{d\ln a}\right]\Psi\partial^2\delta\chi - 2H(\alpha_B-\alpha_H)\Phi\partial^2\delta\chi\\
&- 3H\left[\frac{(\rho_0+P_0)}{M^2} + 2\dot{H}\left(1+\alpha_B\right)\right]\Phi\delta\chi - \left[\frac{(\rho_0+P_0)}{M^2} + 2\dot{H}\right]\delta\chi\left(\partial^2 B - a^2\partial^2\dot{E}\right)\\
&+ 2H\alpha_B\delta\dot{\chi}\left(\partial^2 B - a^2\partial^2\dot{E}\right) - \left[3\left(\dot{H}^2 + H\ddot{H} + 3H^2\dot{H} + H^2\dot{H}\frac{d\ln M^2}{d\ln a}\right)\alpha_B + 3H\dot{H}\dot{\alpha}_B\right.\\
&+ \left.\frac{3}{2}\dot{H}\frac{(\rho_0+P_0)}{M^2} + 3\dot{H}^2\right]\delta\chi^2 - \left[\left(\dot{H} + H^2 + H^2\frac{d\ln M^2}{d\ln a}\right)\left(\alpha_B - \alpha_H\right) + H\left(\dot{\alpha}_B - \dot{\alpha}_H\right)\right.\\
&+ \left.H^2\alpha_T + \dot{H} - H^2\frac{d\ln M^2}{d\ln a} + \frac{1}{2}\frac{(\rho_0+P_0)}{M^2}\right]\delta\chi\partial^2\delta\chi + \frac{1}{2}H^2\alpha_K\delta\dot{\chi}^2\\
&- \left. P_0\left(\frac{3}{2}\Psi^2 - a^2\Psi\partial^2 E - \frac{a^4}{2}\partial^2 E\partial^2 E\right) - \rho_0\left(\frac{1}{2}\Phi^2 + \frac{1}{2}B\partial^2 B + 3\Phi\Psi - a^2\Phi\partial^2 E\right)\right\},
\end{aligned} \tag{5.15}$$

where we have redefined $\delta\chi \to \delta\chi\dot{\chi}_0$. Note that all the terms in the last line are those arising from $\delta_2\sqrt{|h|}$ and $(\delta_2 N + \delta\sqrt{|h|}\delta N)$, and they will all cancel with an equivalent counterpart from the matter action $S_{\rm M}^{(2)}$. Given that the background depends on two free functions a and χ_0 (φ_0 is not free as it will be related to a by means of eq. (4.28)), we have shown that this cosmological model is completely characterised by seven free functions of time, parametrising the evolution of the background and linear perturbations. Note that, in this case we do not have any extra relation such as eq. (4.53) relating the background functions to M. We emphasise that, even though we did our calculations with a matter scalar field, our expression for $S_{\rm G}^{(2)}$ is valid when the matter sector is a general perfect fluid instead of a scalar field. The equations of motion for this gravitational model coupled to a general perfect fluid can be derived from equations (4.21).

The action we have just determined includes up to third-order derivatives of the perturbation fields. The coefficients M^2, α_K, α_B and α_T multiply terms that have, at most, two derivatives, and therefore encompass fundamental theories such as Horndeski theory. But we also found a "Beyond Horndeski" coefficient, α_H, which multiplies a term of the form $\dot{\Psi}\partial^2\delta\chi$ that has three derivatives; therefore our results encompass the extensions from Beyond Horndeski theory.

We can recover the results of the previous section by setting $\chi = 0$. The free coefficients then take the following values:

$$\alpha_K = \alpha_B = \alpha_H = 0, \quad \alpha_T = \frac{d\ln M^2}{d\ln a}, \tag{5.16}$$

which corresponds to GR if M is constant ($\alpha_T = 0$). On the other hand, if we want $\delta\chi$ to describe the perturbations of a quintessence scalar, we set the coefficients to be:

$$\alpha_B = \alpha_H = 0, \quad \alpha_T = \frac{d\ln M^2}{d\ln a}, \quad \alpha_K = \frac{\dot{\chi}_0^2}{H^2 M^2}. \tag{5.17}$$

Note that by constraining the form of the terms for $\delta\chi$ in this way, we are also constraining the quadratic tensor terms – they are all related. In this case, the tensor action reduces to that of the generalised GR action shown in the previous section. If we restrict ourselves to GR, we find $\alpha_T = 0$ as in [178][3].

We finally comment on the fact that action (5.15) propagates only one physical scalar DoF. It can be seen that B and Φ are auxiliary variables, while the other three fields have time derivatives; and due to the redundancies induced by the two scalar gauge freedoms, the action contains only one physical, propagating, scalar DoF.

[3]Note that there is a typo in Table 1 of [178] - a factor of 3 is missing in the definition of α_K for quintessence.

5.1.2 Fourth-order extensions

It is interesting to go further to see what the structure of higher-order derivative terms might take and what new free coefficients must be included. Both [158] and [159] include a term of the form $(g^{\mu\nu}+n^\mu n^\nu)\,\partial_\mu g^{00}\partial_\nu g^{00}$ in the unitary gauge which, when Stueckelberg-ed, leads to a *fourth-order* derivative term of the scalar field in the quadratic action of the form $\alpha_P\partial^i\delta\dot{\chi}\partial_i\delta\dot{\chi}$ where α_P can be expressed as:

$$\alpha_P=\frac{\dot{\chi}_0^2 T_{\partial\dot{\chi}\partial\dot{\chi}}}{M^2H^4a^2}, \tag{5.18}$$

and $T_{\partial\dot{\chi}\partial\dot{\chi}}$ is the coefficient in the quadratic action multiplying a term of the form $\partial^i\delta\dot{\chi}\partial_i\delta\dot{\chi}$. More recently, in [186–188], the authors explored the possibility of enlarging the family of viable scalar-tensor theories by allowing fourth-order derivatives of the scalar field in the equations of motion, but avoiding Ostrogradski instabilities through additional (hidden) constraints.

We now go beyond "Beyond Horndeski", to see what kinds of terms arise by systematically including *all* possible fourth-order derivative terms in the quadratic action (i.e. including Lagrangians $\mathcal{L}_T^4+\mathcal{L}_\chi^4$, with up to four spatial derivatives but only two time derivatives). We find that the final action now depends on the five coefficients previously found as well as six *new* coefficients, one of which is the α_P found in [159]. The new coefficients are defined in the following way:

$$\alpha_{Q1}=\frac{H^2}{2M^2}\left(4L_{RR+}+3L_{RR\times}\right), \tag{5.19}$$

$$\alpha_{Q2}=\frac{2}{M^2}\left(L_{KK+}+2L_{KK\times}\right), \tag{5.20}$$

$$\alpha_{Q3}=\frac{H}{M^2}\left(L_{KR+}+L_{KR\times}\right), \tag{5.21}$$

$$\alpha_{Q4}=\frac{H}{M^2}\left(T_{\partial^2N\partial^2\chi}-\frac{2}{3}L_{K\partial^2\dot{\chi}\times}\right)\dot{\chi}_0, \tag{5.22}$$

$$\alpha_{Q5}=\frac{H}{M^2}T_{\partial^2N\partial^2\chi}\dot{\chi}_0, \tag{5.23}$$

$$\alpha_P=\frac{\dot{\chi}_0^2}{M^2H^4a^2}T_{\partial\dot{\chi}\partial\dot{\chi}}, \tag{5.24}$$

(Note that, as for equations (5.10)-(5.14), we could rewrite all these new coefficients in terms of L_* and T_* solely from the tensor part of the action but the expressions would be more cumbersome). These terms contribute with the following fourth-order derivative interaction terms to the final quadratic action (as well as contributing to lower-order derivative terms):

$$\alpha_{Q1} \to \{\partial^2\Psi\partial^2\delta\chi, \partial^2\delta\chi\partial^2\delta\chi, \partial^2\Psi\partial^2\Psi\}, \tag{5.25}$$

$$\alpha_{Q2} \to \{\partial^2\delta\chi\partial^2\delta\chi\}, \tag{5.26}$$

$$\alpha_{Q3} \to \{\partial^2\delta\chi\partial^2\delta\chi, \partial^2\delta\chi\partial^2\Psi\}, \tag{5.27}$$

$$\alpha_{Q4} \to \{\partial^i\delta\dot{\chi}\partial_i\dot{\Psi}\}, \tag{5.28}$$

$$\alpha_{Q5} \to \{\partial^2\delta\dot{\chi}\partial^2\dot{E}, \partial^2\delta\chi\partial^2\delta\chi, \partial^2\delta\chi\partial^2\Phi, \partial^2\delta\dot{\chi}\partial^2 B\}, \tag{5.29}$$

$$\alpha_P \to \{\partial^i\delta\dot{\chi}\partial_i\delta\dot{\chi}\}. \tag{5.30}$$

Notice that all these terms have four derivatives of the perturbation fields δh_{ij}, δN, δN^i and $\delta\chi$, but when using the SVT decomposition they have higher derivatives of the scalar perturbations. For completeness we list $\mathcal{L}_T^4$ and $\mathcal{L}_\chi^4$ in Appendix D.2.

The final quadratic action is lengthy, but can be found explicitly in the *xIST* notebook *COPPER*. Although this final action becomes more complex, it has the same structure that we see in the action of equation (5.15): B and Φ are auxiliary variables, while the other three fields have time derivatives; and after using the gauge freedom, the action contains only one physical, propagating, scalar DoF. The quadratic actions found here with four derivatives should encompass some specific cases of the scalar-tensor theories considered in [186–188].

It is important to remark that some scalar-tensor actions could have a different structure and allow the quadratic term $\dot{\Phi}^2$. As shown in [161], such actions could be obtained by performing a conformal transformation of the metric with a dependence on derivative terms of the scalar field χ to the action in eq. (5.15). Even though Φ would not be an auxiliary field anymore, these actions would propagate the same number of DoFs as the actions found in this section, due to the presence of additional (hidden) constraints. We do not find the term $\dot{\Phi}^2$ in our results because the presence of such term requires the presence of other quadratic terms (in order to have a gauge-invariant action) of the form $\ddot{\chi}^2$ that lead to four time derivatives in the equations of motion, which we ignored. Furthermore, in [188] it was shown explicitly that after conformal transformations with kinetic dependence on the scalar field, the action of Horndeski is mapped into a specific action that leads to fourth derivatives in the equations of motion. Thus, we emphasise that the absence of these terms in our results does not represent a restriction on the formalism but on the extra assumptions made for the specific cases we worked out instead. In fact, if we had allowed four time derivatives of the fields, we would have found the term $\dot{\Phi}^2$ in the final quadratic action.

Finally, the final action depends on a small set of parameters, (the "α" parameters and M) which define subspaces of the full set of coefficients L_* we used to build our complete action. This means that with measurements of linear cosmological perturbations, at best, we can restrict ourselves to a degenerate subspace that includes (but is not solely restricted to) GR coupled to a scalar field.

5.2 Vector-tensor theories

In the previous section we have focused on scalar-tensor modified gravity theories; in this section we show how the method can easily be extended to vector-tensor gravity theories.

5.2.1 General case

We aim to parametrise linearly diffeomorphism-invariant quadratic actions containing one metric and one vector field A^μ. As in the previous sections, we add a scalar field, minimally coupled to the metric, to represent the matter sector, and consider linear perturbations of all the fields around a homogeneous and isotropic background. For the vector field we will have:

$$A^\mu = \left(A, \vec{0}\right) + \alpha^\mu, \tag{5.31}$$

where $A(t)$ is the background solution of the vector field, and α^μ its first-order perturbation. Since we will be focusing on scalar perturbations, we use the SVT decomposition of the vector field to write:

$$\alpha^\mu = (\alpha^0, \alpha^i); \quad \alpha^i = \alpha^{Ti} + \bar{h}^{ij}\partial_j \alpha, \tag{5.32}$$

where we have two scalar perturbations α^0 and α, and one vector perturbation α^{Ti}, such that $\partial_i \alpha^{Ti} = 0$. Therefore there will only be two relevant perturbations (the two scalar modes) from the vector field in our calculations. As explained in Appendix A.1, these scalar perturbations transform in the following way under linear coordinate transformations:

$$\begin{aligned} \delta\alpha^0 &= \dot{\pi} A - \dot{A}\pi, \\ \delta\alpha &= a^2 A \dot{\epsilon}, \end{aligned} \tag{5.33}$$

while the scalar metric perturbations transform as in eq. (1.27) and the matter scalar field $\delta\varphi$ as the field χ in eq. (4.9).

We now follow Step 2 to construct the most general gravitational quadratic action. We will allow, at most, two derivatives of the perturbation fields in the equations of motion. All the possible perturbed building blocks in this case will be $\delta\vec{\Theta} = (\ldots, \alpha^0, \partial_i\alpha^0, \dot{\alpha}^0, \partial_i\partial_j\alpha^0, \partial_i\dot{\alpha}^0, \alpha_i, \partial_j\alpha_i, \dot{\alpha}_i, \partial_j\partial_k\alpha_i, \partial_j\dot{\alpha}_i)$, where the initial ellipses indicate all the building blocks used in Section 4.2. For simplicity we have defined $\alpha_i = \bar{h}_{ij}\alpha^i$ which, in terms of scalar perturbations, becomes $\alpha_i = \partial_i\alpha$.

Next we proceed to Taylor expand the gravitational Lagrangian $L_{\rm G}$ up to second order in the perturbation fields. We use the same definitions introduced in eq. (4.22) for the coefficients L_*. In addition, we use α as a proxy for α_i in the subscripts of the coefficients L_*. We also Taylor expand the matter action.

We recall that we obtain the background equations of motion from the linear Taylor expansion of the total action (gravity and matter). In this case, we find eq. (4.26) from varying the metric field, eq. (4.28) from the matter scalar field, and the following expression from varying the vector field:

$$L_{\alpha^0} - \dot{L}_{\dot{\alpha}^0} - 3HL_{\dot{\alpha}^0} = 0. \tag{5.34}$$

Similar to the case of scalar-tensor theories, we expect one of these four background equations to be redundant as there are only three undetermined background functions a, A and φ_0. Again, this redundancy leads to a relation between the parameters L_* and T_*, which is not relevant for the analysis of this chapter, but would be important in constructing non-perturbative, fundamental actions allowing homogeneous and isotropic backgrounds.

We now proceed to express the general quadratic gravitational action as:

$$S_{\rm G}^{(2)} = \int d^4x \, \sum_{i=0}^{2} \left(\mathcal{L}_T^i + \mathcal{L}_{\alpha^0}^i + \mathcal{L}_{\alpha}^i + \mathcal{L}_{\alpha^0\alpha}^i \right), \tag{5.35}$$

where $\mathcal{L}_{\alpha^0}^i$ and $\mathcal{L}_{\alpha}^i$ are the quadratic Lagrangians involving α^0 and α_i respectively, along with the metric perturbations, leading to i derivatives of the perturbation fields in the equations of motion. We also include the Lagrangian $\mathcal{L}_{\alpha^0\alpha}^i$ involving interactions between α^0 and α_i.

The Lagrangians $\mathcal{L}_T^i$ are given in Section 4.2, while $\mathcal{L}_{\alpha^0}^i$ are the same as $\mathcal{L}_{\chi}^i$, for $i = (0, 1, 2)$, given in Section 5.1, but with $\chi \to \alpha^0$. For $\mathcal{L}_{\alpha}^i$, we have that:

$$\mathcal{L}_{\alpha}^0 = a^3 \left[T_{\alpha S}\alpha_i\delta N^i + \frac{1}{2}T_{\alpha\alpha}\alpha_i\alpha^i \right], \tag{5.36}$$

$$\mathcal{L}^1_\alpha = a^3\left[T_{\partial\alpha h+}\delta h\partial^i\alpha_i + 2L_{\partial\alpha h\times}\delta h_{ij}\partial^i\alpha^j + T_{\dot\alpha S}\delta N^i\dot\alpha_i + T_{\alpha\partial N}\partial^i\delta N\alpha_i\right], \tag{5.37}$$

$$\begin{aligned}\mathcal{L}^2_\alpha = a^3\Big[&T_{\partial\dot\alpha h+}\delta h\partial^i\dot\alpha_i + 2L_{\partial\dot\alpha h\times}\delta h_{ij}\bar h^{jk}\partial^i\dot\alpha_k + T_{\partial\dot\alpha N}\delta N\partial^i\dot\alpha_i + \frac{1}{2}L_{\dot\alpha\dot\alpha}\dot\alpha_i\dot\alpha_j\bar h^{ij}\\ &+ L_{\dot\alpha\dot S}\delta\dot N^i\dot\alpha_i + L_{\partial\alpha K+}\delta K\partial^i\alpha_i + 2L_{\partial\alpha K\times}\delta K^i_j\partial^j\alpha_i + T_{\partial\alpha\partial S}\partial_i\delta N^i\partial^j\alpha_j\\ &+T_{\partial\alpha\partial\alpha+}\partial^i\alpha_i\partial^j\alpha_j + L_{\partial\alpha\partial\alpha\times}\partial^i\alpha^j\partial_i\alpha_j\Big]. \end{aligned}\tag{5.38}$$

We finally add the following interaction terms to the total gravitational action:

$$\mathcal{L}^1_{\alpha^0\alpha} = a^3 T_{\alpha\partial\alpha^0}\alpha^i\partial_i\alpha^0, \tag{5.39}$$

$$\mathcal{L}^2_{\alpha^0\alpha} = a^3 T_{\alpha\partial\dot\alpha^0}\alpha^i\partial_i\dot\alpha^0, \tag{5.40}$$

and $\mathcal{L}^0_{\alpha^0\alpha} = 0$. As in the previous sections, we have integrated by parts and grouped coefficients to simplify notation. In Appendix D.3 we give the dictionary for the coefficients T_* in terms of the L_* for the Lagrangians $\mathcal{L}^i_\alpha$ and $\mathcal{L}^i_{\alpha^0\alpha}$. Since we will also be coupling a matter scalar field φ, we must include the quadratic matter action given in eq. (4.34) in the total quadratic action.

Moving on to Step 3, we write the total quadratic action $S^{(2)}_{\mathrm{G}} + S^{(2)}_{\mathrm{M}}$ in terms of the scalar perturbation fields (Φ, B, Ψ, E, α^0, α and $\delta\varphi$), find the corresponding Noether identities and solve the associated Noether constraints. After solving the Noether constraints we find that the total quadratic action depends on the following 10 free coefficients:

$$M^2 = 2L_{KK\times}, \tag{5.41}$$

$$\alpha_{D1} = \frac{L_{\dot N\dot N}}{M^2} = \frac{A^2}{M^2}L_{\dot\alpha^0\dot\alpha^0}, \tag{5.42}$$

$$\alpha_{D2} = -2\frac{L_{K\dot N}}{M^2}, \tag{5.43}$$

$$\alpha_{D3} = -\alpha_{D2} + 2\frac{T_{N\partial\dot S}}{M^2}, \tag{5.44}$$

$$\alpha_T = \frac{2}{M^2}\left(L_R + 4L_{Rh\times}\right) - 1, \tag{5.45}$$

$$\alpha_H = \frac{2}{M^2}\left(T_{NR} + 2L_{h\partial^2N\times}\right) - 1, \tag{5.46}$$

$$\alpha_{V0} = \frac{1}{2HM^2}\left(T_{NK} - 3HL_{KK+}\right) - \frac{3}{2}, \tag{5.47}$$

$$\alpha_{V1} = \frac{1}{M^2}\left(T_{\partial S\partial S\times} + T_{\partial S\partial S+} - 4L_{K\partial S+} - 4T_{\partial\dot S h+} - L_{KK\times}\right) + 1, \tag{5.48}$$

$$\alpha_{V2} = \frac{T_{SS}}{2H^2M^2}, \tag{5.49}$$

$$\alpha_{V3} = \frac{1}{M^2} L_{\dot{S}\dot{S}} = \frac{A^2}{M^2} T_{\dot{\alpha}\dot{\alpha}}. \tag{5.50}$$

Note that we also have three unknown background functions a, A and φ_0, but one relation between a and φ_0 given by eq. (4.28). Thus, the linear cosmological evolution of the most general linearly diffeomorphism-invariant vector-tensor theory is parametrised by a total of twelve free functions of time. Note that in solving the Noether constraints we assumed $\dot{A} \neq 0$, and therefore for cases with constant A the free functions might change.

Since the final quadratic gravitational action is somewhat unwieldy, we do not show it explicitly here. Instead, we highlight some interesting aspects of its form. The first three coefficients α_{Di} appear in the action multiplying time derivatives of Φ. From eq. (5.42) we can see that the presence of the dynamical terms for Φ (δN) are tightly related to the presence of those for α^0, as $L_{\dot{N}\dot{N}}$ (the coefficient of $\dot{\Phi}^2$ as seen in eq. (4.32)) is proportional to $L_{\dot{\alpha}^0\dot{\alpha}^0}$ (the coefficient of $(\dot{\alpha}^0)^2$ as seen in eq. (5.7)). The same happens for B and α, as can be seen in eq. (5.50). All the terms we have mentioned are not present in scalar-tensor theories, as B and Φ are auxiliary variables in such cases. Furthermore, even if we eliminate all the terms leading to time derivatives of B and Φ, i.e. set $\alpha_{D1} = \alpha_{D2} = \alpha_{D3} = \alpha_{V3} = 0$, the remaining gravitational action still has different quadratic metric interaction terms, compared to the ones in eq. (5.15), namely $(\partial B)^2$, $(\partial\Phi)^2$, and $(\partial^2 \dot{E})^2$. Note also that two of the ten coefficients in the final action are the same as those present for a scalar-tensor theory: α_T and α_H. However, they do not enter the action in exactly the same way; for instance, both α_T and α_H multiply a term of the form $(\partial\Phi)^2$.

A detailed analysis of the physical propagating DoFs and the stability of this class of theories is beyond the scope of this thesis. However, we comment on the fact that if all the free coefficients are nonzero, we might naively think that this gravitational action propagates four physical scalar DoFs, as E, Ψ, Φ, B, α and α^0 are dynamical fields (and as there are two scalar gauge parameters). This would suggest the propagation of unstable modes, given that a well behaved vector-tensor (Lorentz-invariant) theory is expected to propagate at most two scalar DoFs: the helicity-0 modes from the massive spin-1 and spin-2 particles. For this reason it is instructive to make the following redefinition of the vector perturbation fields:

$$\tilde{\alpha} = \alpha + AB, \quad \tilde{\alpha}^0 = \alpha^0 + A\Phi. \tag{5.51}$$

With this redefinition the fields B and Φ become auxiliary variables in the action (i.e. do not have any time derivatives), while $\tilde{\alpha}$ and $\tilde{\alpha}^0$ are dynamical. In this way, it is clear that the action will propagate at most two scalar DoFs. Furthermore, well-known linearly diffeomorphism-invariant vector-tensor theories propagate only one healthy scalar DoF. In this context, we notice that extra conditions on the coefficients might reduce the number of physical DoFs to one. For instance, if we set $\alpha_{V3} = 0$, $\tilde{\alpha}$ becomes an auxiliary variable, or, alternatively, if we set $\alpha_{D1} = \alpha_{D2} = 0$ then $\tilde{\alpha}^0$ becomes an auxiliary variable. Such cases should encompass the generalisations of the Proca action studied in [189–191]. As we will see in the next subsection, there are alternative ways of constructing vector-tensor theories propagating only one physical scalar DoF, by incorporating extra constraints.

It is interesting to see what happens if we also impose a $U(1)$ gauge symmetry on the vector field. In this case the quadratic action is invariant under

$$\alpha^\mu \to \alpha^\mu + \partial^\mu \varepsilon, \tag{5.52}$$

where ε is an arbitrary infinitesimal parameter, independent of the other two scalar gauge parameters in the linear coordinate transformation. After solving the Noether constraints associated to the U(1) gauge symmetry we find that

$$\alpha_{D1} = \alpha_{D2} = \alpha_H = \alpha_{V0} = \alpha_{V2} = 0, \quad \alpha_{V1} = 1, \quad \alpha_T = \frac{d\ln M^2}{d\ln a}, \quad \alpha_{D3} = -4\alpha_{V3}, \tag{5.53}$$

along with eq. (4.53) and the final quadratic action depends on two free coefficients M and α_{V3}. In addition, we have only one free function describing the background A, as a is related to M through eq. (4.53). In general, this action does not propagate any physical scalar DoF, because it has three dynamical fields E, Ψ and $\tilde{\alpha}$ and three gauge parameters inducing redundancies rendering these fields unphysical. The Einstein-Maxwell theory is one example of this case.

5.2.2 Einstein-Aether theory

As mentioned above, there are different ways of constructing a gravitational action with a vector and tensor field that propagates only one scalar DoF. Here we show one special case in which we introduce an additional constraint. Specifically, we will add the Einstein-Aether constraint:

$$S_c = \int d^4x\, \sqrt{-g}\lambda \left(A^\mu A_\mu + 1\right), \tag{5.54}$$

to the gravitational action S_{G}, where λ is a Lagrange multiplier, and gives the constraint $A^\mu A_\mu = -1$. In particular, λ is an extra scalar field whose perturbation transforms under linear coordinate transformations in the same way as $\delta\chi$ in eq. (4.9). The presence of the new field λ imposes an extra background equation of motion: $A = 1$, while all the other background equations are the same as those in the general vector-tensor case presented previously, where now all the coefficients L_* are functional derivatives of the total gravitational action (which now includes S_c).

The Einstein-Aether constraint will contribute the following second-order terms to the total action:

$$S_c^{(2)} = \int d^4x\, a^3 \left\{-2\delta\lambda\left(\alpha^0 + \delta N\right)\right\}, \tag{5.55}$$

where we have expanded the Lagrange multiplier as $\lambda = \lambda_0 + \delta\lambda$. Notice that the second-order Taylor expansion of eq. (5.54) will also lead to quadratic terms in the metric perturbations, but we do not show them in eq. (5.55) as they are taken into account in $\mathcal{L}_T^i$ (i.e. any quadratic metric term from S_c contributes to the action via changing the explicit from of the coefficients L_* and T_* in the metric Lagrangians). In addition, from eq. (5.55) we note that the equation of motion for $\delta\lambda$ gives the Lagrange constraint $\delta N + \alpha^0 = 0$. As expected, this constraint corresponds to the linear expansion of the full constraint $A^\mu A_\mu = -1$.

As in the general vector-tensor case, we follow Step 3 to express the total quadratic action $S_G^{(2)} + S_{\rm M}^{(2)} + S_c^{(2)}$ in terms of the scalar perturbations, and impose that it is invariant under linear infinitesimal gauge transformations. After solving the Noether constraints we find an action depending on ten free coefficients (of which three are different to those present in the general vector-tensor case). However, after solving the Lagrange constraint $\alpha^0 = -\delta N$ the dependence on some coefficients vanishes, while the rest combine in such a way that the final quadratic gravitational action depends on four coefficients only. The final action is:

$$\begin{aligned} S_{\rm G}^{(2)} = \int d^4x\, a^3 \Bigg\{ & M^2 \Bigg[\alpha_{V3}\frac{1}{2}\partial_i\dot{\hat{\alpha}}\partial^i\dot{\hat{\alpha}} - \alpha_{V3}\hat{\Phi}\partial^2\dot{\hat{\alpha}} + \frac{1}{2}\alpha_{V3}\partial_i\hat{\Phi}\partial^i\hat{\Phi} - \frac{2}{3}\partial^2\hat{B}\partial^2\hat{\alpha} \\ & + \frac{1}{3}\partial^2\hat{B}\partial^2\hat{B} + \frac{1}{3}\partial^2\hat{\alpha}\partial^2\hat{\alpha} + H^2\alpha_{V5}\partial_i\hat{\alpha}\partial^i\hat{\alpha} + H\alpha_{V4}\partial_i\hat{\alpha}\partial^i\hat{\Phi}\Bigg] + \frac{(\rho_0 + P_0)}{\dot{H}H}\Bigg[\frac{3}{2}H^3\hat{\Phi}^2 \\ & + H^2\hat{\Phi}\partial^2\hat{B} + \frac{1}{6}H\partial^2\hat{\alpha}\partial^2\hat{\alpha} - \frac{1}{3}H\partial^2\hat{\alpha}\partial^2\hat{B} + \frac{1}{6}H\partial^2\hat{B}\partial^2\hat{B} + \dot{H}\partial_i\Psi\partial^i\hat{B} - 3\dot{H}H\Psi\hat{\Phi} \\ & + \frac{\dot{H}}{2H}\partial_i\Psi\partial^i\Psi - \frac{3\dot{H}^2}{2H}\Psi^2\Bigg] - P_0\left(\frac{3}{2}\Psi^2 - a^2\Psi\partial^2E - \frac{a^4}{2}\partial^2E\partial^2E\right) \\ & - \rho_0\left(\frac{1}{2}\Phi^2 + \frac{1}{2}B\partial^2B + 3\Phi\Psi - a^2\Phi\partial^2E\right)\Bigg\}, \end{aligned} \tag{5.56}$$

where, to simplify our expression, we have defined:

$$\hat{\alpha} = \tilde{\alpha} - \frac{\Psi}{H}, \tag{5.57}$$

$$\hat{B} = B - \frac{\Psi}{H} - a^2\dot{E}, \tag{5.58}$$

$$\hat{\Phi} = \Phi + \frac{\dot{\Psi}}{H} - \Psi\frac{\dot{H}}{H^2}, \tag{5.59}$$

and where $\tilde{\alpha} = \alpha + B$. Note that in eq. (5.56) the last two sets of parentheses will cancel with their corresponding counterparts from the matter action given in eq. (4.34). Also, note that $S_c^{(2)} = 0$, as we have solved the Lagrange constraint. In the final quadratic action given by eq. (5.56) M^2 and α_{V3} are given by eq. (5.41) and (5.50) respectively, while the other two free coefficients are given by:

$$\begin{aligned}
\alpha_{V4} &= \frac{1}{M^2 H}\left[-2\dot{L}_{K\partial S\times} + 3\left(L_{KK+} - 2L_{K\partial S\times}\right)H - 4L_{hK\times} + 4L_{h\partial S\times}\right] + \frac{d\ln M^2}{d\ln a} \\
&\quad - \alpha_T + 3, \\
\alpha_{V5} &= \frac{3}{2M^2 H}\left(\dot{L}_{KK+} + HL_{KK+}\right) - \frac{1}{2}\left(\alpha_{V4} - 1\right)\frac{d\ln M^2}{d\ln a} - \frac{\dot{\alpha}_{V4}}{2H} - \frac{\alpha_{V4}\dot{H}}{2H^2} + \alpha_T \\
&\quad - \frac{\alpha_{V4}}{2} + \frac{3}{2},
\end{aligned} \tag{5.60}$$

where α_T is given by eq. (5.45). This final quadratic action encompass all vector-tensor theories that include the Einstein-Aether constraint in eq. (5.54).

From eq. (5.56) we can see that when solving the Lagrange constraint, Φ becomes an auxiliary variable. Thus the final action has two auxiliary fields, B and Φ, and three dynamical fields E, Ψ and $\tilde{\alpha}$, with no dependence on $\delta\lambda$ and α^0. This action is still gauge invariant under linear infinitesimal coordinate transformations; the Lagrange constraint does not fix any preferred gauge because $\Phi + \alpha^0$ is a gauge-invariant quantity. Therefore, this action propagates only one physical scalar DoF, as the two scalar gauge parameters render two dynamical fields unphysical.

The final action in eq. (5.56) depends explicitly on four coefficients, while the background has only one free function a. Therefore this cosmological model is parametrised by five free functions in total. Notice that we expect λ_0 to appear in the background equations of motion, but we do not count it as an extra free function, since it can be eliminated by appropriately combining the background equations. Thus λ_0 is not directly observable. In addition, A and φ_0 do not count as free parameters either because A is fixed to be $A = 1$, and φ_0 will be related to a by the matter background eq. (4.28).

5.3 Discussion

In this chapter we have constructed quadratic actions for cosmological perturbations for scalar-tensor and vector-tensor theories, by applying the procedure of Chapter 4. We summarise our findings in Table 5.1.

Fields	Der.	Free Functions	ST DoFs	Theories
$g_{\mu\nu}$, χ	2	M, $\alpha_{\{K,T,B\}}+2$	1	Horndeski
$g_{\mu\nu}$, χ	3	M, $\alpha_{\{K,T,B,H\}}+2$	1	Beyond Horndeski
$g_{\mu\nu}$, χ	4	M, $\alpha_{\{K,T,B,H,P\}}$, $\alpha_{Q_{\{1,2,3,4,5\}}}+2$	1	4^{th} Scalar-Tensor
$g_{\mu\nu}$, A^μ	2	M, $\alpha_{\{T,H\}}$, $\alpha_{D_{\{1,2,3\}}}$, $\alpha_{V_{\{0,1,2,3\}}}+2$	2	2^{nd} Vector-Tensor
$g_{\mu\nu}$, A^μ, λ	2	M, $\alpha_{V_{\{3,4,5\}}}+1$	1	Einstein-Aether

Table 5.1: In this table we summarize the results found throughout this chapter for cases in which invariance under linear coordinate transformations was assumed. The first column indicates the field content of the gravitational action. In all cases we also added a matter scalar field φ whose presence is omitted in this table. The second column indicates the maximum number of derivatives of the perturbation fields allowed in the equations of motion. Note that in the cases where this number is higher than 2, we assumed a maximum of two time derivatives, but allowed higher spatial derivatives. The third column shows the free coefficients parametrizing the quadratic action, while the +1 or +2 counts the number of extra free background functions. The fourth column shows the maximum number of scalar DoFs propagated by the quadratic gravitational action. In all cases the complete quadratic theory would propagate one more matter scalar DoF. The fifth column shows theories that are encompassed by the corresponding parametrisation. The three grey rows show new parametrisations of fourth-order derivative scalar-tensor theories and second-order derivative vector-tensor theories, including Einstein-Aether.

We applied our procedure to scalar-tensor gravity theories, leading to second, third and fourth-order derivatives in the equations of motion. The first two cases are well known, and the quadratic actions found encompass the theories of Horndeski and Beyond Horndeski. We also analysed the fourth-derivative case and identify a total of 13 free functions of time, describing the background (2) and linear (11) cosmological evolution of the Universe, of which 6 are new compared to the third-derivative case. In all these cases the quadratic gravitational action propagates only one scalar DoF. The procedure could also be applied systematically to allow higher-order derivatives, and we would most likely generate more free parameters encompassing even more theories.

Finally, we applied the procedure to vector-tensor theories, leading to second-order derivatives in the equations of motion. We found a total of 12 free functions of time describing the background (2) and linear (10) cosmological evolution of the Universe. In general, these quadratic gravitational actions propagate two scalar DoFs, although there could be only one when some specific parameters are zero. As an alternative case of a vector-tensor theory propagating only one scalar DoF, we applied the procedure to theories of gravity with an Einstein-Aether constraint. We found a total of only 5 free parameters of time describing the background (1) and linear perturbations (4).

In all the cases presented in this chapter we minimally coupled the metric field to a matter scalar field; however, the same results hold for a general perfect fluid. In addition, we only analysed scalar perturbations, but the same free parameters will also describe vector and tensor linear perturbations, around homogeneous and isotropic backgrounds. The specific form of the quadratic action of vector and tensor perturbations is left for future work.

We remark that the field content, and more specifically how all fields transform under a given gauge symmetry, is crucial in determining the final form of the quadratic action. For instance, we could apply the same procedure to a gravitational theory with a tensor coupled to a generalised scalar field χ, whose linear perturbation $\delta\chi$ transforms under linear coordinate transformations as:

$$\delta\chi \to \delta\chi + G_0\pi + G_1\dot{\pi} + G_2\epsilon + G_3\dot{\epsilon}, \tag{5.61}$$

where G_* are unknown functions of the background and π, ϵ are arbitrary functions defined in Appendix A.1. After applying the three steps above, focusing on scalar perturbations, and allowing up to two derivatives of the fields, we could get very different results to those of scalar-tensor theories. If $G_i \neq 0$ for $i = (0, 1, 2, 3)$ the final quadratic gravitational action reduces to that of generalised GR found in Section 4.2, and thus no scalar DoFs are propagated by the resulting action. On the contrary, if $G_2 = G_3 = 0$, and $G_0 = \dot{G}_1$, i.e. when $\delta\chi$ is the linear time-like scalar component of a perturbed vector field A^μ, the final gravitational quadratic action propagates a maximum of two scalar DoFs. $\delta\chi$, Φ, E and Ψ are dynamical fields in the quadratic action, but there are two scalar gauge parameters inducing two non-physical fields. As in scalar-tensor theories, in this case we introduced only one extra field χ to the gravitational theory, but the resulting number of propagating DoFs is different. We see then the importance of our Step 1 in determining the space of gravitational theories under consideration, by defining the gauge transformation properties of the extra degrees of freedom.

CHAPTER 6

A GENERAL THEORY OF LINEAR COSMOLOGICAL PERTURBATIONS: BIMETRIC THEORIES

In order to extend our previous results, and construct a parametrised model that spans as large a swathe of the landscape of gravitational theories as possible, in this chapter we apply the method to diffeomorphism-invariant bimetric theories. We show that, around homogeneous and isotropic backgrounds, the most general quadratic action is determined by 29 free parameters, and propagates at most four scalar degrees of freedom. However, if we do not allow derivative interactions between both metrics, the number of free parameters simplifies greatly, reducing to three. Furthermore, if we focus on actions that propagate only one DoF, the action has only two free parameters. Lately, bimetric theories have been studied extensively in a cosmological context [2,11,37,84–86,88,89,91,93,95,97,100,104,107–111,115–120,123–127,144,192–220], and in this chapter we show that we can recover the specific cases of massive bigravity [66,70,72] and Eddington-inspired Born Infeld (EiBI) theory [11, 221], as bimetric theories encompassed by the parametrisation found. Due to the no-go theorem for ghost-free Lorentz-invariant bimetric theories with derivative interactions in [132], we devote most of the work in this chapter to study non-derivative interactions, although we briefly discuss derivative interactions in the appendix.

This chapter is structured as follows. In Section 6.1 we explain in detail how to implement the method developed in Chapter 4 to bigravity theories, focusing on scalar perturbations. In Section 6.2 we present the results of the method in the specific case when the two metrics do not have any derivative interaction. We analyse the number of physical scalar DoFs propagating and the number of free parameters determining the general structure of this action. We also compare with massive bigravity and corroborate results found in previous works. Finally, in Section 6.3 we summarise and discuss the findings of this chapter.

6.1 Parametrising bimetric theories

In this section we first summarise the method developed in Chapter 4 and then show how to apply it to bimetric theories. As previously explained, the objective of the method is to obtain a general, local, quadratic action for linear cosmological perturbations for a class of gravitational theories, with a given field content and gauge symmetries. This action will be expressed in terms of parameters –functions of the background– in such a way that specific forms for the parameters lead to the action of a specific gravity theory. For simplicity, in what follows, we will be assuming a known form for the matter sector which couples to gravity, although it is straightforward to relax this assumption. We perform the following three steps to find the aforementioned parametrised action:

1. **Set up:** Assume a given number and type of fields present in the theory (gravity and matter). Define an ansatz for the cosmological background, and consider linear perturbations for each field around that background. Finally, choose a set of gauge symmetries that will leave the quadratic action invariant.

2. **General action:** Construct the most general local quadratic gravitational action, given the field content established in Step 1. Consider a set of perturbed building blocks $\delta\vec{\Theta}$, containing all the derivatives of the gravitational perturbation fields, up to a given maximum order. Use the building blocks to write down all possible quadratic terms, and construct the most general quadratic gravitational action $S_G^{(2)}$ by adding each one of these terms multiplied by an, a priori, free parameter (i.e. unknown function of the background). The resulting gravitational action will contain all possible interactions between the perturbation fields. Finally, form the known matter content, calculate the quadratic matter action $S_m^{(2)}$ to get the total quadratic action $S_G^{(2)} + S_m^{(2)}$ determining the evolution of linear cosmological perturbations.

3. **Gauge invariance:** Find the most general linearly gauge-invariant quadratic action for perturbations. Consider the total action $S_G^{(2)} + S_m^{(2)}$ from Step 2, and impose invariance under the desired gauge symmetries set in Step 1. In order to do this, find the Noether identities associated to the gauge symmetries: there will be one for each gauge parameter. Then, from each Noether identity, find the Noether constraints associated, which will, in general, be linear ordinary differential equations of the free parameters in $S_G^{(2)}$. Finally, solve the system of Noether constraints and replace the results in $S_G^{(2)}$. The resulting

gravitational action will satisfy the Noether identities and, as consequence, be gauge-invariant under the desired symmetries.

After performing these three steps, we will obtain the most general gravitational action for a class of theories, determined by a given field content and set of gauge symmetries, up to a maximum number of derivatives. From this result it is straightforward to identify the number of free parameters describing the linear cosmological evolution of the Universe for these theories, and the number of physical, propagating, DoFs. In addition, from this action, we are able to constrain the free parameters with observational data and automatically translate these constraints into constraints on the gravitational theories.

Before applying the steps to bimetric theories, we clarify that, for simplicity, we have slightly modified Step 2 compared to the one presented in the previous chapter. We have seen that in Step 2 the free parameters can be expressed as functional derivatives of the unknown underlying (non-linear) gravity theory. This is an interesting feature of the method as the parameters are related to characteristics of the fundamental theory, which is useful when building viable non-linear gravitational theories out of observational constraints on the parameters. However, in this chapter we have skipped that part of Step 2, and thus we will not give such relations between the free parameters and the underlying gravity theory.

We now implement the previous method for bigravity theories. We show each one of the three steps in detail.

Step 1: Consider a gravitational theory composed of two rank-2 tensor fields (or metrics) $g_{A\mu\nu}$ (with $A = \{1, 2\}$), coupled to some additional matter fields. We focus on linear perturbations around a homogeneous and isotropic cosmological background. The tensor degrees of freedom arise from the following metrics:

$$g_{A\mu\nu} = \bar{g}_{A\mu\nu} + \delta g_{A\mu\nu}, \tag{6.1}$$

where $\bar{g}_{A\mu\nu}$ describes the background of both metrics $A = 1$ and $A = 2$, assumed to be spatially-flat FLRW metrics with line elements given by:

$$ds_1^2 = -dt^2 + a(t)^2 d\vec{x}^2, \quad ds_2^2 = -\bar{N}(t)^2 dt^2 + b(t)^2 d\vec{x}^2. \tag{6.2}$$

Here a and b are the scale factors of the metrics, and we have also introduced a non-trivial lapse term $\bar{N}$ for metric 2, as in general both metrics cannot be brought into the standard form with trivial shifts at the same time. Since we do not know what the underlying gravity theory is, the background functions a, b and $\bar{N}$ are considered to be free functions of time. In addition, $\delta g_{A\mu\nu}$ are small first-order perturbations around the given background, which generally depend on space and time.

For simplicity and concreteness, we will assume that metric 1 is minimally coupled to a scalar field φ, which will represent our matter content. This means that metric 1 will be the physical metric describing the space-time, and therefore a will be the scale factor of the expansion of the Universe. The matter field can be expanded as follows:

$$\varphi = \bar{\varphi}(t) + \delta\varphi, \tag{6.3}$$

where $\bar{\varphi}$ is the background value of the field, which has the same symmetries as the metrics, and thus depends on time only, whereas $\delta\varphi$ is the first-order perturbation of the field and, in general, depends on space and time. We remark that even though we use a minimally-coupled scalar field as matter content, we expect the same results for a general perfect fluid. Further generalisations could be made straightforwardly, such as coupling both metrics to matter through the composite metric proposed in [93, 94], but such cases are left for future work.

We will be looking for actions which are quadratic in these perturbations and invariant under linear general coordinate transformations of the form $x^\mu \to x^\mu + \epsilon^\mu$, where ϵ^μ is an arbitrary first-order perturbation of the coordinates x^μ. Under these transformations the background stays the same, while the linear perturbations of the metrics $\delta g_{A\mu\nu}$ transform as in eq. (1.24):

$$\delta g_{A\mu\nu} \to \delta g_{A\mu\nu} - \bar{g}_{A\mu\beta}\partial_\nu \epsilon^\beta - \bar{g}_{A\beta\nu}\partial_\mu \epsilon^\beta + \epsilon^\alpha \bar{g}_{A\mu\beta}\bar{g}_{A\nu\gamma}\left(\partial_\alpha \bar{g}^{A\beta\gamma}\right), \tag{6.4}$$

whereas the scalar perturbation $\delta\varphi$ transforms as

$$\delta\varphi \to \delta\varphi - \dot{\bar{\varphi}}\pi. \tag{6.5}$$

Notice that, since we expect both metrics to be coupled, there will be only one copy of the diffeomorphism invariance, and thus both metrics transform with the same gauge parameter.

Step 2: In this step we construct the most general local quadratic action for all the gravitational perturbation fields $\delta g_{A\mu\nu}$. This quadratic action will lead to linear equations of motion, assumed to have second-order derivatives at most.

We use the Arnowitt-Deser-Misner (ADM) formalism, in which we separate space and time, and both metrics are decomposed into lapse functions N_A, shift functions N_A^i and 3-dimensional spatial metrics h_{Aij} in the following way:

$$g_{A00} = -N_A^2 + h_{Aij}N_A^i N_A^j,\; g_{A0i} = h_{Aij}N_A^j,\; g_{Aij} = h_{Aij}. \tag{6.6}$$

Next, we write the set of perturbed building blocks $\delta\vec{\Theta}$, which includes all the perturbations the gravitational action can depend on. Specifically, this set will include all possible gravitational perturbations up to second-order derivatives for both metrics[1]:

$$\begin{aligned}\delta\vec{\Theta} =&\Big(\delta N_A,\, \delta\dot{N}_A,\, \delta\partial_i N_A,\, \delta\partial_i\dot{N}_A,\, \delta\partial_i\partial_j N_A,\, \delta N_A^i,\, \delta\dot{N}_A^i, \delta\partial_j N_A^i,\, \delta\partial_j\dot{N}_A^i,\, \delta\partial_i\partial_j N_A^k,\\ &\delta h_{Aij},\, \delta\partial_i h_{Ajk},\, \delta\partial_i\partial_j h_{Akl},\, \delta K^i_{A\;j}\Big),\end{aligned} \tag{6.7}$$

where the previous list includes all the terms for both subindices $A = \{1, 2\}$. Here, we have replaced the time derivative terms $\delta\dot{h}_{Aij}$ of the spatial metrics by the extrinsic curvature tensors $\delta K^i_{A\;j}$ – this can always be done as there is a one-to-one relation between these two quantities. It is important to note that in the previous chapter we also replaced second-order spatial derivatives of the spatial metrics by the 3-dimensional intrinsic curvature tensors $\delta R^i_{A\;j}$, given that we were seeking linearly diffeomorphism-invariant actions, and thus spatial derivatives were expected to appear in that form. However, in this chapter we do not use $\delta R^i_{A\;j}$ because, in the case of bimetric theories, the interactions between the two metrics may cause the action to depend on a different combination of spatial derivatives, while still maintaining its diffeomorphism invariance. This is a new feature of the bimetric theories as in scalar-tensor and vector-tensor theories the spatial derivatives always appeared in the combination of $\delta R^i_{A\;j}$.

Note that in eq. (6.7) partial derivatives of the perturbation fields are taken with respect to the background metric of the corresponding perturbation field, and thus we raise and lower the indices of the perturbed building blocks with $\bar{h}_{Aij}$. Also, δ commutes with partial spatial derivatives and so, for instance, $\delta(\partial_i N_A) = \partial_i(\delta N_A)$. Finally, notice that even though the fields $g_{A\mu\nu}$ and φ have only linear perturbations, the perturbed building blocks could have higher-order perturbations as result. Thus, we clarify that $\delta\vec{\Theta}$ contains both first and second-order perturbative terms; throughout this chapter, however, δ refers only to first-order perturbations, unless stated otherwise.

[1]We have ignored second-order time derivatives as they will be related – via integration by parts – to terms with first-order time derivatives.

We now proceed to construct the most general quadratic bimetric action with second-order derivative equations of motion. In order to do that we write down all possible quadratic terms formed by the perturbed building blocks, and place an arbitrary parameter in front of each term. The resulting gravitational action can be written as follows:

$$S_G^{(2)} = \int d^4x \sum_{n=0}^{2} \mathcal{L}^n_{T_1} + \mathcal{L}^n_{T_2} + \mathcal{L}^n_{T_1T_2}. \tag{6.8}$$

Here, for a given n, $\mathcal{L}^n_{T_1}$ is the quadratic Lagrangian for the self-interactions of tensor 1, leading to n-order derivatives in the equations of motion. $\mathcal{L}^n_{T_2}$ is defined analogously for tensor 2, and finally $\mathcal{L}^n_{T_1T_2}$ is defined analogously for the interactions between both tensors. We have the following Lagrangians for the self-interactions:

$$\begin{aligned}\mathcal{L}^0_{T_A} = \sqrt{-\bar{g}_A}\Bigg[&\frac{1}{2}L_{ASS}\bar{h}_{Aij}\delta N^i_A\delta N^j_A + \frac{1}{2}L_{ANN}\left(\delta N_A\right)^2 + L_{AN}\left(\delta_2 N_A + \delta N_A\frac{\delta\sqrt{h_A}}{\sqrt{-\bar{g}_A}}\right)\\ &+L_{ANh}\delta N_A\delta h_A + \frac{1}{2}L_{Ahh+}\left(\delta h_A\right)^2 + L_{Ahh\times}\delta h^i_{Aj}\delta h^j_{Ai}\Bigg] + \bar{N}_A\bar{L}_A\delta_2\sqrt{h_A},\end{aligned} \tag{6.9}$$

$$\begin{aligned}\mathcal{L}^1_{T_A} = \sqrt{-\bar{g}_A}\,\big[&L_{A\partial Sh+}\partial_i\delta N^i_A\delta h_A + 2L_{A\partial Sh\times}\delta h_{Aij}\partial^i\delta N^j_A + 2L_{AhK\times}\delta h^j_{Ai}\delta K^i_{Aj}\\ &+L_{AhK+}\delta K_A\delta h_A + L_{ANK}\delta N_A\delta K_A + L_{AN\partial S}\delta N_A\partial_i\delta N^i_A\big],\end{aligned} \tag{6.10}$$

$$\begin{aligned}\mathcal{L}^2_{T_A} = \sqrt{-\bar{g}_A}\,\big[&L_{Ah\partial^2h\times}\delta h_A\partial^i\partial^j\delta h_{Aij} + L_{Ah\partial^2h+}\delta h_A\partial^2\delta h_A + L_{Ah\partial^2h\odot}\delta h_{Akl}\partial^k\partial^j\delta h_{Aij}\bar{h}^{li}_A\\ &+ 2L_{A\partial\dot{S}h\times}\delta h_{Aij}\partial^i\delta\dot{N}^j_A + L_{A\partial\dot{S}h+}\partial_i\delta\dot{N}^i_A\delta h_A + \frac{1}{2}L_{A\partial S\partial S+}\left(\partial_i\delta N^i_A\right)\left(\partial_j\delta N^j_A\right)\\ &+ \frac{1}{2}L_{A\dot{S}\dot{S}}\bar{h}_{ij}\delta\dot{N}^j_A\delta\dot{N}^i_A + \frac{1}{2}L_{A\partial S\partial S\times}\bar{h}_{lj}\left(\partial_i\delta N^l_A\right)\left(\partial^i\delta N^j_A\right) + \frac{1}{2}L_{AKK+}\left(\delta K_A\right)^2\\ &+ L_{AKK\times}\delta K^i_{Aj}\delta K^j_{Ai} + \frac{1}{2}L_{A\dot{N}\dot{N}}\left(\delta\dot{N}_A\right)^2 + \frac{1}{2}L_{A\partial N\partial N}\partial^i\delta N_A\partial_i\delta N_A\\ &+ L_{Ah\partial^2N+}\delta h_A\partial^2\delta N_A + 2L_{Ah\partial^2N\times}\delta h_{Aij}\partial^i\partial^j\delta N_A + L_{AK\dot{N}}\delta K_A\delta\dot{N}_A\\ &+L_{AN\partial\dot{S}}\left(\partial_j\delta\dot{N}^j_A\right)\delta N_A + L_{A\partial SK+}\delta K_A\partial_i\delta N^i_A + 2L_{A\partial SK\times}\delta K^i_{Aj}\partial_i\delta N^j_A\big].\end{aligned} \tag{6.11}$$

Here, the L parameters are free functions of time with a subscript indicating the type of self-interaction they determine. Also, $\sqrt{-\bar{g}_A}$ correspond to the square root of the determinant of the 4-dimensional background metrics $\bar{g}_{A\mu\nu}$. We remind the reader that spatial indices are raised and lowered with the 3-dimensional background metrics, so in the previous actions we have simplified notation by introducing the perturbation fields δh_A, which are given by

$\delta h_A = \bar{h}^{ij}\delta h_{Aij}$. Also, notice that we are using δ_2 to describe quadratic perturbations [2]. In eq. (6.9) we have introduced the perturbations $\delta\sqrt{h_A}$ and $\delta_2\sqrt{h_A}$, which are the linear and quadratic perturbations of the square root of the determinant of the 3-dimensional metrics.

We also show explicitly the interaction terms between both metrics:

$$\begin{aligned}\mathcal{L}^0_{T_1T_2} &= P_{N_2h_1}\delta N_2\delta h_1 + P_{N_2N_1}\delta N_1\delta N_2 + P_{S_2S_1}\delta N_2^j\delta N_1^i\bar{h}_{1ij} + P_{h_1h_2+}\delta h_1\delta h_2 \\ &\quad + P_{h_1h_2\times}\bar{h}_1^{ik}\bar{h}_1^{jl}\delta h_{1ij}\delta h_{2kl} + P_{h_2N_1}\delta h_2\delta N_1,\end{aligned} \tag{6.12}$$

$$\begin{aligned}\mathcal{L}^1_{T_1T_2} &= P_{\dot{N}_2h_1}\delta\dot{N}_2\delta h_1 + P_{N_2\partial S_1}\delta N_2\partial_i\delta N_1^i + P_{\dot{N}_2N_1}\delta\dot{N}_2\delta N_1 + P_{S_2\partial h_1+}\partial_i\delta h_1\delta N_2^i \\ &\quad + 2P_{S_2\partial h_1\times}\partial^i\delta h_{1ij}\delta N_2^j + P_{\dot{S}_2S_1}\delta N_1^i\dot{N}_2^j\bar{h}_{1ij} + P_{N_1\partial S_1}\delta N_1\partial_i\delta N_2^i + P_{h_2K_1+}\delta K_1\delta h_2 \\ &\quad + P_{h_2K_1\times}\bar{h}_1^{ik}\delta K^j_{i1}\delta h_{2jk} + P_{K_2N_1}\delta K_2\delta N_1 + P_{h_2\partial S_1+}\delta h_2\partial_i\delta N_1^i \\ &\quad + P_{h_2\partial S_1\times}\delta h_{2ij}\partial^i\delta N_1^j,\end{aligned} \tag{6.13}$$

$$\begin{aligned}\mathcal{L}^2_{T_1T_2} &= P_{h_1\partial^2N_2+}\delta h_1\partial^2\delta N_2 + P_{h_1\partial^2N_2\times}\delta h_{1ij}\partial^i\partial^j\delta N_2 + P_{K_1\dot{N}_2}\delta K_1\delta\dot{N}_2 \\ &\quad + P_{\dot{N}_1\dot{N}_2}\delta\dot{N}_1\delta\dot{N}_2 + P_{\dot{N}_2\partial S_1}\partial_i\delta N_1^i\delta\dot{N}_2 + P_{N_2\partial^2N_1}\delta N_2\partial^2\delta N_1 + P_{\partial S_2\dot{N}_1}\delta\dot{N}_1\partial_i\delta N_2^i \\ &\quad + P_{\dot{S}_1\dot{S}_2}\bar{h}_{1ij}\delta\dot{N}_1^i\delta\dot{N}_2^j + P_{\partial S_2K_1+}\delta K_1\partial_i\delta N_2^i + 2P_{\partial S_2K_1\times}\delta K^i_{j1}\partial_i\delta N_2^j \\ &\quad + P_{\partial S_1\partial S_2+}\partial_i\delta N_1^i\partial_j\delta N_2^j + P_{\partial S_1\partial S_2\times}\bar{h}_{1kj}\partial^i\delta N_1^k\partial_i\delta N_2^j + P_{K_2\dot{N}_1}\delta K_2\delta\dot{N}_1 \\ &\quad + P_{K_1K_2+}\delta K_1\delta K_2 + P_{K_1K_2\times}\delta K^i_{j1}\delta K^j_{i2} + P_{h_2\partial^2N_1+}\delta h_2\partial^2\delta N_1 \\ &\quad + P_{h_2\partial^2N_1\times}\delta h_{2ij}\partial^i\partial^j\delta N_1 + P_{K_2\partial S_1+}\delta K_2\partial_i\delta N_1^i + P_{K_2\partial S_1\times}\delta K^i_{j2}\partial_i\delta N_1^j \\ &\quad + P_{h_2\partial^2h_1+}\delta h_2\partial^2\delta h_1 + P_{h_2\partial^2h_1\times1}\delta h_2\partial^i\partial^j\delta h_{1ij} + P_{h_2\partial^2h_1\times2}\delta h_{2ij}\partial^i\partial^j\delta h_1 \\ &\quad + P_{h_2\partial^2h_1\odot}\delta h_{2il}\partial^i\partial^j\delta h_{1jk}\bar{h}^{lk},\end{aligned} \tag{6.14}$$

where the P parameters are free functions of time with a subscript indicating the type of interaction they determine. We clarify that $\partial^2 = \partial^i\partial_i$ where, in general, the derivatives acting on a given field have a index that is raised or lowered with the background metric of that given field. For instance, the term $\delta N_2\partial^2\delta N_1$ can be equivalently expressed as:

$$\delta N_2\partial^2\delta N_1 = \delta N_2\left(\partial_i\partial_j\bar{h}_1^{ij}\delta N_1\right) = \frac{\delta^{ij}}{a^2}\delta N_2\partial_i\partial_j\delta N_1. \tag{6.15}$$

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[2] We have ignored the term $\delta_2 K^i{}_j$ as it is related to other terms in the previous action.

Finally, we note that in all these Lagrangians (self-interactions and interactions between both metrics) we have integrated by parts, and written only the independent terms, so we express the action in terms of a minimal set of parameters.

Now that we have written the most general quadratic action for two metrics, leading to second-order derivative equations of motion, we calculate the total quadratic action by adding the matter contribution. As previously mentioned, we will assume that metric 1 is minimally coupled to a scalar field φ, and thus the matter action has the following form:

$$S_m = -\int d^4x \, \sqrt{-g_1}\left(\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + V(\varphi)\right), \tag{6.16}$$

where $V(\varphi)$ is some potential, and the indices of the partial derivatives are raised and lowered using $g_{1\mu\nu}$. This action can be straightforwardly written in terms of the ADM variables, and Taylor expanded up to second order in the linear perturbations of the metric and scalar field to get eq. (4.34) for metric 1. We note that $S_m^{(2)}$ does not only contain linear and quadratic terms on the perturbations of the matter field, but also quadratic terms in the perturbations of metric 1. In fact, the first two terms of the quadratic matter action in eq. (4.34) will cancel out exactly the similar two terms for metric 1 in eq. (6.9). This is because the coefficients L_{AN} and $\bar{L}_A$ are not arbitrary, and they are defined in such a way that:

$$L_{1N} = \rho_0, \; \bar{L}_1 = -P_0, \; L_{2N} = 0, \; \bar{L}_2 = 0. \tag{6.17}$$

These relations are background equations and come from the linear expansion of the underlying gravity theory. The derivation of these relations was done in the Appendix C.4 for metric 1 only, but the relations for metric 2 follow from those results straightforwardly.

Finally, we can see that the total quadratic action $S^{(2)} = S_m^{(2)} + S_G^{(2)}$ contains 99 free parameters Ls and Ps[3], and it is the most general quadratic action that can be written for linear perturbations of two metrics around a spatially-flat FLRW background, with a minimal coupling to a scalar field, and leading to second-order derivative equations of motion. In addition, we have four background functions a, b, $\bar{N}$ and $\bar{\varphi}$ determining the background evolution, and thus affecting the linear perturbations, from which only three are independent due to the equation for background matter in eq. (4.28). We remark that, contrary to most known bigravity theories, we have assumed that the perturbations δN_A and δN_A^i are dynamical fields (have time-derivatives) and we will let the Noether identities dictate the consistent kinetic structure

[3]Here we are not counting the coefficients L_{AN} and $\bar{L}$, as their values are given in eq. (6.17).

of these terms in a gauge-invariant action.

Step 3: We now proceed to impose diffeomorphism invariance on the total quadratic action $S^{(2)}$ obtained in Step 2. First, we follow the standard SVT decomposition [171] of perturbations. Since scalar, vector and tensor perturbations evolve independently at the linear level in a homogeneous and isotropic background, we can study each one separately. In this chapter, we only analyse scalar perturbations, as they are the seeds of large-scale structure in the energy density field, and therefore cosmologically relevant. In this case, the line element of both metrics (including the background and linear perturbations) will be expressed as:

$$ds_1^2 = -\left(1+\Phi_1\right)dt^2 + 2\partial_i B_1 dt dx^i + a^2\left[\left(1-2\Psi_1\right)\delta_{ij} + 2\partial_i\partial_j E_1\right]dx^i dx^j, \tag{6.18}$$

$$ds_2^2 = -\bar{N}^2\left(1+\Phi_2\right)dt^2 + 2\bar{N}\partial_i B_2 dt dx^i + b^2\left[\left(1-2\Psi_2\right)\delta_{ij} + 2\partial_i\partial_j E_2\right]dx^i dx^j, \tag{6.19}$$

where the fields Φ_A, B_A, Ψ_A and E_A for $A = \{1, 2\}$ are the first-order scalar perturbations for both metrics $g_{A\mu\nu}$. From eq. (6.18)-(6.19) we can find the perturbed ADM variables at linear order:

$$\delta N_1 = \Phi_1,\ \delta N_1^i = \bar{h}_1^{ij}\partial_j B_1,\ \delta h_{1ij} = a^2\left[-2\Psi_1\delta_{ij} + 2\partial_i\partial_j E_1\right], \tag{6.20}$$

$$\delta N_2 = \bar{N}\Phi_2,\ \delta N_2^i = \bar{N}\bar{h}_2^{ij}\partial_j B_2,\ \delta h_{2ij} = b^2\left[-2\Psi_2\delta_{ij} + 2\partial_i\partial_j E_2\right], \tag{6.21}$$

as well as the rest of the perturbed building blocks. See Appendix E.1 for a list of relevant quantities, that appear in the gravitational quadratic action $S_G^{(2)}$, in terms of the scalar perturbations. In these equations, $\bar{h}_A^{ij}$ are the inverse tensors of the 3-dimensional spatial background metrics.

We can now express the total quadratic action in terms of the 8 scalar perturbations, and the matter perturbation field $\delta\varphi$. Note that even though we only allow up to two derivatives of the metric perturbations δN, δN^i and δh_{ij}, this means that we will have higher-order derivatives of the scalar perturbations. Next, we impose that the action is invariant under the linear diffeomorphism transformations given in eq. (6.4)-(6.5). In terms of the scalar perturbations, the transformation of the metric perturbations is the following:

$$\tilde{\Phi}_1 = \Phi_1 - \dot{\pi}, \quad \tilde{B}_1 = B_1 + \pi - a^2\dot{\epsilon}, \quad \tilde{\Psi}_1 = \Psi_1 + H\pi, \quad \tilde{E}_1 = E_1 - \epsilon, \tag{6.22}$$

$$\tilde{\Phi}_2 = \Phi_2 - \dot{\pi} - H_N\pi, \quad \tilde{B}_2 = B_2 + \bar{N}\pi - \frac{b^2}{\bar{N}}\dot{\epsilon}, \quad \tilde{\Psi}_2 = \Psi_2 + H_b\pi, \quad \tilde{E}_2 = E_2 - \epsilon, \tag{6.23}$$

where we have defined $H = \dot{a}/a$, $H_b = \dot{b}/b$ and $H_N = \dot{\bar{N}}/\bar{N}$. Note that we have also used the SVT decomposition for the gauge parameter ϵ^μ, and written its two scalar components π and ϵ such that $\epsilon^\mu = (\pi, \delta^{ij}\partial_j \epsilon)$.

In order that the action is diffeomorphism invariant, we need to determine the Noether identities that arise for *both* spatial and temporal linear gauge transformations. These identities are the constraints which, when enforced, make the action gauge invariant. We calculate the Noether identities by first taking an infinitesimal variation of the total quadratic action with regards to each one of the scalar perturbations. This variation can be written as:

$$\hat{\delta}S^{(2)} = \hat{\delta}S_G^{(2)} + \hat{\delta}S_m^{(2)} = \int d^4x \, \left[\mathcal{E}_{\Phi_A}\hat{\delta}\Phi_A + \mathcal{E}_{B_A}\hat{\delta}B_A + \mathcal{E}_{\Psi_A}\hat{\delta}\Psi_A + \mathcal{E}_{E_A}\hat{\delta}E_A + \mathcal{E}_{\delta\varphi}\hat{\delta}\left(\delta\varphi\right)\right] \quad (6.24)$$

where $\mathcal{E}_X$ is the equation of motion of the perturbation field X, and $\hat{\delta}$ stands for functional variation. Here, there is an implicit sum over the subindex A. We then replace the variations of the fields by the corresponding gauge transformations in eq. (6.5) and eq. (6.22)-(6.23), and integrate by parts to end up with:

$$\begin{aligned} \hat{\delta}gS^{(2)} &= \int d^4x \, \left[\mathcal{E}_{B_1} + \bar{N}\mathcal{E}_{B_2} + H\mathcal{E}_{\Psi_1} + H_b\mathcal{E}_{\Psi_2} + \dot{\mathcal{E}}_{\Phi_1} + \dot{\mathcal{E}}_{\Phi_2} - H_N\mathcal{E}_{\Phi_2} - \mathcal{E}_{\delta\varphi}\dot{\bar{\varphi}}\right]\pi \\ &+ \int d^4x \, \left[-\mathcal{E}_{E_1} - \mathcal{E}_{E_2} + \frac{d}{dt}\left(a^2\mathcal{E}_{B_1}\right) + \frac{d}{dt}\left(\frac{b^2}{\bar{N}}\mathcal{E}_{B_2}\right)\right]\epsilon, \end{aligned} \quad (6.25)$$

where the expression $\hat{\delta}_g$ stands for the functional variation of the action due to the gauge transformation. Given that the total quadratic action should be invariant under these gauge transformations, and given that both π and ϵ are arbitrary and independent, each set of brackets should be zero; this gives us two *Noether identities*, one associated to each scalar gauge parameter. Furthermore, each combination of free coefficients, inside each of the brackets, multiplying the perturbation fields and their derivatives such as Φ_A, $\dot{\Phi}_A$, $\partial^2\Phi_A$, Ψ_A, etc, must be individually zero for the Noether identities to be satisfied off-shell. This gives a set of *Noether constraints*. As previously mentioned, these constraints will be, in general, linear ordinary differential equations for the coefficients Ls and Ps. However, for the bimetric case presented in this chapter, these Noether constraints can be solved algebraically. We solve all of these constraints and replace the solutions in the quadratic action, resulting in an action satisfying the Noether identities and, as consequence, gauge invariant. Therefore, the resulting action will be the most general linearly diffeomorphism-invariant local quadratic action, given the field content.

The resulting gauge-invariant bimetric action is lengthy so we do not show it explicitly here but we mention some general characteristics. We find that the final action depends only on 29 free parameters, in addition to the four background functions a, b, $\bar{N}$ and $\bar{\varphi}$, which add three independent free functions due to the constraint in eq. (4.28). Therefore, there are $29+3$ free parameters determining the background and linear cosmological evolution of these bimetric theories. In Appendix E.2 we give expressions for all these parameters in terms of the coefficients Ls and Ps, and we connect them with the well-known parameters present in the scalar-tensor parametrisation EFT of dark energy [161].

In addition, we find that in the final action the eight metric scalar fields Φ_A, B_A, Ψ_A and E_A, have dynamical terms, i.e. time derivatives. However, the fields Φ_A and B_A appear in specific combinations such that if we introduce two new fields Φ_3 and B_3:

$$\Phi_1 = \Phi_3 + \Phi_2, \quad B_1 = B_3 + \frac{a^2 \bar{N}}{b^2} B_2, \tag{6.26}$$

then Φ_3 and B_3 appear as dynamical fields whereas Φ_2 and B_2 become auxiliary fields (without time derivatives). This means that the final gravitational action propagates at most four physical scalar DoFs. The counting goes as follows: there are two auxiliary variables that can be worked out from their own equations of motion, and therefore expressed entirely in terms of the 6 remaining dynamical fields. In addition, we have two scalar gauge parameters that we can use to fix the gauge and eliminate two dynamical fields. Therefore, the final action has at most four physical scalar DoFs, although for specific values of the parameters (and background evolutions) there could be less.

From our results it is not possible to know where the four scalar DoFs are coming from (e.g. massive gravitons or ghosts), but we do know that all well-known healthy bimetric theories propagate at most one scalar DoF, signalling the possible presence of unstable modes in the action found in this chapter. This shows a crucial feature of any approach based on linearized theories solely. The consistency of a full theory requires background, linearized perturbative and higher-order perturbative contributions all to be consistent, i.e. to avoid the propagation of unstable degrees of freedom such as ghosts. And so, crucially, while all well-behaved theories will map onto the free functions in our linearized perturbation theory parametrisation, not all possible functional forms for these seemingly free functions are associated with healthy theories. This happens for the very simple reason that there is more to a full theory than the action it gives rise to for linear perturbations, and that there are additional constraints not captured by any formalism based on linearized perturbations. These extra constraints would reduce the free functions and the number of propagating fields we have found further. A detailed analysis on the construction of possible fundamental consistent theories leading to the parametrised bimetric action found here is beyond the scope of this work, but it is certainly relevant and requires further work.

In the next section, we focus on the specific case when there are no derivative interactions between both metrics. This case is interesting as it encompasses most well-known bimetric theories such as massive bigravity and EiBI. Furthermore, there is a no-go theorem for the existence of ghost-free Lorentz-invariant derivative interactions [132] for massive gravity, rendering the general case of derivative interactions likely to propagate unstable modes. Notwithstanding, we do briefly discuss derivative interactions in Appendix E.4, as they might still be relevant in the context of Lorentz-breaking theories.

6.2 A reduced case: excluding derivative interactions

In this section we study the general structure of the parametrised bimetric action in the absence of derivative interactions between both metrics. The starting point is the general action we found in the previous section, which had 29 free parameters. In this action we impose that all derivative interactions vanish, i.e. $\mathcal{L}^1_{T_1T_2} = \mathcal{L}^2_{T_1T_2} = 0$, which enforces the relations on the free parameters that we present in Appendix E.3. Specifically, we find 26 relations, reducing greatly the number of free parameters to only three. These three parameters are:

$$M_1^2 = 2L_{1KK\times}, \tag{6.27}$$

$$\alpha_L = -\frac{1}{2M_1^2H^2}\left(\bar{L}_1 - 4L_{1hh+} - 8L_{1hh\times}\right), \tag{6.28}$$

$$\alpha_E = -\frac{2}{M_2^2}\left[\bar{N}^2T_{2Nh} - H_b\left(2L_{2hK\times} + 3L_{2hK+}\right)\right]. \tag{6.29}$$

Therefore, if we take into account the background functions, there are, in total, $3+3$ free functions of time determining the evolution of the background and linear perturbations in this subclass of theories: bimetric theories without derivative interactions. In Appendix E.3 we show the quadratic action without derivative interactions.

In this case we find that the four fields B_A and Φ_A appear as auxiliary variables, i.e. do not have any time derivatives. This means that the resulting gravitational action propagates at most two scalar DoFs. This result is consistent with previous analyses of massive gravity, where it has been shown that most potential interactions between two metrics lead to the propagation of a helicity-0 mode for a massive graviton and an extra unstable scalar mode, the Boulware-Deser ghost [65, 77, 78].

In order to get a healthy action we construct actions that propagate only one scalar DoF (although there are some trivial healthy cases that propagate no scalar, as we will see later on). We do this by imposing that one of the dynamical fields is an auxiliary variable, i.e. by setting to zero the coefficients of the kinetic terms of one field, after integrating out the four auxiliary fields B_A and Φ_A. As we will see later on, the resulting action is a generalisation of massive bigravity and thus the only propagating physical field should correspond to the helicity-0 mode of a massive graviton. We find that the only non-trivial situation we can have is when Ψ_2 becomes an auxiliary variable [4], which imposes one extra constraint on the parameters:

$$\alpha_E = \frac{H_b\left(\rho_0 + P_0 + 2\dot{H}M_1^2\right)}{r^3(H - H_b)M_2^2}, \tag{6.30}$$

where we have introduced the scale factor ratio $r = b/a$, and the mass scale $M_2^2 = 2L_{2KK\times}$. Therefore, the most general bimetric quadratic action without derivative interactions *and* propagating only one scalar DoF, depends on $2 + 3$ free functions of time. From now on, we focus on such a subclass of actions. The resulting parametrised action is much simpler in this case, and can be written in the following form:

$$S^{(2)} = S^{(2)}_{T_1} + S^{(2)}_{T_2} + S^{(2)}_{T_1T_2} + S^{(2)}_{\varphi}, \tag{6.31}$$

where $S^{(2)}_{T_A}$ is the action for the self-interaction terms of the metric $g_{A\mu\nu}$, whereas $S^{(2)}_{T_1T_2}$ is the action for the interaction terms between both metrics, and $S^{(2)}_{\varphi}$ includes all the terms involving the matter perturbation $\delta\varphi$ in eq. (4.34). Notice that $S^{(2)}_{T_1}$ does include the quadratic terms of the metric perturbations coming from the matter action $S^{(2)}_m$. These actions are given by:

$$\begin{aligned} S^{(2)}_{T_1} = \int d^4x\, a^3 M_1^2 \Big[& -3\dot{\Psi}_1^2 - 6H\dot{\Psi}_1\Phi_1 + 2a^2\partial^2\dot{E}_1(\dot{\Psi}_1 + H\Phi_1) - 2\dot{\Psi}_1\partial^2 B_1 \\ & - \left(1 + \frac{d\ln M_1^2}{d\ln a}\right)\Psi_1\partial^2\Psi_1 + 2\Phi_1\partial^2\Psi_1 - 2H\Phi_1\partial^2 B_1 - \left(3H^2 - \frac{\dot{\bar{\varphi}}^2}{2M_1^2}\right)\Phi_1^2 \\ & + \frac{r^2 Z}{2(\bar{N} + r)}\left(\partial^i B_1\right)\left(\partial_i B_1\right) + r\left(Z\frac{d\ln M_1^2}{d\ln a} + 2\tilde{Z}\right)\Psi_1\left(\frac{3}{2}\Psi_1 - a^2\partial^2 E_1\right) \\ & + \frac{(\bar{N} - r)}{H - H_b}HZ\Phi_1\left(3\Psi_1 - a^2\partial^2 E_1\right) + a^4\alpha_L H^2\left(\partial^2 E_1\right)^2 \Big], \end{aligned} \tag{6.32}$$

[4] In any other possible case the resulting action will lead to a copy of the linear Einstein-Hilbert action, and thus the gravitational action does not propagate any scalar DoF.

$$
\begin{aligned}
S_{T_2}^{(2)} = \int d^4x\ \bar{N}b^3M_2^2 \Bigg[& -3\frac{\dot{\Psi}_2^2}{\bar{N}^2} - 6\frac{H_b}{\bar{N}^2}\dot{\Psi}_2\Phi_2 + 2\frac{b^2}{\bar{N}^2}\partial^2\dot{E}_2(\dot{\Psi}_2 + H_b\Phi_2) - 2\frac{\dot{\Psi}_2}{\bar{N}}\partial^2 B_2 \\
& - \left(1 + \frac{d\ln M_2^2}{d\ln b}\right)\Psi_2\partial^2\Psi_2 + 2\Phi_2\partial^2\Psi_2 - 2\frac{H_b}{\bar{N}}\Phi_2\partial^2 B_2 - 3\frac{H_b^2}{\bar{N}^2}\Phi_2^2 \\
& + \frac{\nu^2\bar{N}Z}{2r^3(\bar{N}+r)}\left(\partial^i B_2\right)\left(\partial_i B_2\right) + \frac{\nu^2}{\bar{N}r^2}\left(Z\frac{d\ln M_1^2}{d\ln a} + 2\tilde{Z}\right)\Psi_2\left(\frac{3}{2}\Psi_2 - b^2\partial^2 E_2\right) \\
& + H_b\nu^2 Z\frac{(\bar{N}-r)}{\bar{N}r^3(H-H_b)}\Phi_2\left(3\Psi_2 - b^2\partial^2 E_2\right) + \frac{r}{\bar{N}}\alpha_L a^4 H^2\nu^2\left(\partial^2 E_2\right)^2 \Bigg], \qquad (6.33)
\end{aligned}
$$

$$
\begin{aligned}
S_{T_1T_2}^{(2)} = \int d^4x\ M_1^2 a^3 \Bigg[& -r\left(2\tilde{Z} + Z\frac{d\ln M_1^2}{d\ln a}\right)\left(3\Psi_2\Psi_1 - a^2\Psi_2\partial^2 E_1 - b^2\Psi_1\partial^2 E_2\right) \\
& - Z\frac{\bar{N}}{(\bar{N}+r)}\partial_i B_2\partial^i B_1 - Z\frac{(\bar{N}-r)}{(H-H_b)}\Big(H\Phi_1\left(3\Psi_2 - b^2\partial^2 E_2\right) \\
& + H_b\Phi_2\left(3\Psi_1 - a^2\partial^2 E_1\right)\Big) - 2a^2b^2H^2\alpha_L\partial^2 E_1\partial^2 E_2 \Bigg]. \qquad (6.34)
\end{aligned}
$$

Here, we have two mass scales for each metric M_1^2 and M_2^2, and we have introduced the mass ratio $\nu^2 = M_1^2/M_2^2$. In addition, for ease of comparison with massive bigravity, we have introduced two functions Z and $\tilde{Z}$ such that:

$$
M_1^2\left(\bar{N} - r\right)Z = \hat{L}_{1KK\times}, \qquad (6.35)
$$

$$
rM_1^2\left(Z\frac{1}{2}\frac{d\ln M_1^2}{d\ln a} + 2\tilde{Z}\right) = \frac{1}{(H_b - H)}\left(3H + \frac{H_N H_b}{(H-H_b)}\right)\hat{L}_{1KK\times} + \frac{\dot{\hat{L}}_{1KK\times}}{(H_b - H)}, \qquad (6.36)
$$

where

$$
\hat{L}_{1KK\times} = \rho_0 + P_0 + 2\dot{H}M_1^2. \qquad (6.37)
$$

We omit the expression for $S_\varphi^{(2)}$ as it can be straightforwardly obtained from eq. (4.34). We emphasise, even though it may not be obvious, that these actions do depend on $2+3$ free independent functions. There is an explicit dependence on five parameters M_A, Z, $\tilde{Z}$ and α_L, in addition to the four background functions a, b, $\bar{N}$ and $\bar{\varphi}$. However, Z and $\tilde{Z}$ are dependent functions according to eq. (6.35)-(6.36), one background function is dependent through eq. (4.28), and M_2 is also dependent through one of the relations shown in Appendix E.3, which is necessary to avoid derivative interactions. This relation is the following:

$$
M_2^2 = -\frac{\bar{N}}{2r^3}\frac{\left(\rho_0 + P_0 + 2\dot{H}M_1^2\right)}{\left(\dot{H}_b - H_N H_b\right)}. \qquad (6.38)
$$

From the parametrised action shown here we can see that the fields Φ_A and B_A appear as auxiliary variables, whereas the fields Ψ_A and E_A have dynamical terms. A naive counting of DoFs might lead to a total of two physical propagating fields, but as we have mentioned before, this action propagates only one. This is because this action is such that after integrating out the four auxiliary fields Φ_A and B_A, the kinetic terms for Ψ_2 vanish and thus Ψ_2 becomes an auxiliary field. If we also integrated Ψ_2 out, the resulting action would have three dynamical fields E_A and Ψ_1, from which two are gauge freedoms and one is a physical propagating DoF.

In addition, from eq. (6.32)-(6.33) we can see that all the kinetic terms have exactly the same structure as linearised Einstein-Hilbert. In fact, the two first lines in both equations correspond to the terms coming from two copies of linearised GR with generalised (time-dependent) Planck masses M_A, in addition to the self-interaction metric terms for $g_{1\mu\nu}$ from the matter action of eq. (4.34). The rest of the two lines in both equations represent then modifications to GR, which depend on the coupling parameters Z, $\tilde{Z}$ and α_L. This last point is clear from the fact that all the interactions terms in eq. (6.34) depend on these three parameters. For this reason, an equivalent yet more intuitive parametrisation would be if we considered the five parameters M_A, Z, $\tilde{Z}$ and α_L to be independent, and all the background functions to be dependent.

This final action is a generalisation of massive bigravity [66, 70, 72], a bimetric theory propagating one massless graviton and one massive graviton. This theory propagates only one scalar field: the helicity-0 mode of the massive graviton. We recover the quadratic action and background of massive bigravity when the three interaction parameters take the following form:

$$\alpha_L = 0, \tag{6.39}$$

$$Z = m^2 \left(\beta_1 + 2\beta_2 r + \beta_3 r^2\right), \tag{6.40}$$

$$\tilde{Z} = m^2 \left(\beta_1 + \beta_2 \left(r + \bar{N}\right) + \beta_3 r \bar{N}\right), \tag{6.41}$$

and when M_1 and M_2 are non-zero constants. Here, $\beta_{1,2,3}$ are dimensionless constants determining the coupling between the two metrics in the dRGT potential [66], and m is a constant mass scale degenerate with the parameters βs. Notice that we have given five constraints to recover massive bigravity, which fix completely the five free functions of time of the general bimetric action. We can also recover massive gravity (with only one dynamical metric) by setting $M_2 = 0$. We note, however, that such model has been ruled out due to the presence of an instability in which the helicity-0 mode of the massive graviton behaves as a ghost (i.e. has a negative kinetic term) [121, 211] when the non-dynamical metric is either FLRW or de-Sitter. Furthermore, if the reference metric is Minkowski, massive gravity does not even allow a spatially flat FLRW solutions [206] (although a generalisation of massive gravity has been found to lead to viable cosmological solutions [212].)

We remark that the dRGT potential of massive gravity depends on two other constants, namely β_0 and β_4, which act as cosmological constants for both metrics. In our results we do not find the presence of such terms explicitly, but instead they appear as integration constants of equations (6.35) and (6.38), that give the time derivatives of the Friedmann equations of both metrics.

We notice that the parameter α_L is not present in massive bigravity, which means that it represents the linear term of a theory that either might propagate a Boulware-Deser ghost at the non-linear level, or that it is not fully diffeomorphism invariant, or that it propagates different DoFs. An example of the last case is the bimetric theory Eddington-inspired Born Infeld (EiBI) [11, 221], whose action is given by:

$$S = \frac{M_{\rm P}^2}{2} \int d^4x \left[\sqrt{-g_2} \left(R(g_2) + \frac{2}{\kappa} \right) - \frac{1}{\kappa} \left(\sqrt{-g_2} g_2^{\mu\nu} g_{1\mu\nu} - 2\sqrt{-g_1} \right) \right], \tag{6.42}$$

where $g_{1\mu\nu}$ is the physical metric to be coupled to matter, $g_{2\mu\nu}$ is an additional metric with a Ricci scalar $R(g_2)$, and κ is an arbitrary coupling constant with dimensions of L^2. EiBI is a bimetric theory for a massless graviton, and thus it does not propagate any scalar DoF, but it does introduce relevant modifications to GR – specifically, in the strong-field regime. We can recover the EiBI quadratic action and background [37] by setting:

$$M_1^2 \alpha_L = \frac{r\bar{N}}{2H^2\kappa}, \tag{6.43}$$

$$\alpha_E = \frac{1}{\kappa r^2 M_2^2}, \tag{6.44}$$

$$M_1^2 Z = -\frac{1}{\kappa} \frac{r}{\bar{N}} \left(r + \bar{N} \right), \tag{6.45}$$

$$M_1^2 \tilde{Z} = -\frac{1}{2r\kappa} \left(r\bar{N} + \frac{H_N^2}{2\bar{N}(H_b - H)^2} \right), \tag{6.46}$$

and when $M_1 = 0$ and M_2 is the constant Planck mass. From these equations we can see that in this theory there are no kinetic terms for the metric $g_{1\mu\nu}$, but there are non-derivative interactions terms. We clarify that the action for EiBI theory does not satisfy the extra constraint in eq. (6.30), but instead α_E takes the value shown in eq. (6.44). In fact, in the EiBI action, the fields Ψ_2 and E_2 are the only dynamical variables, but they can both be fixed by the gauge freedom, leading then to an action with no scalar field propagating. Notice that since EiBI does not satisfy eq. (6.30), it does not fall within the action presented in this section. Instead, EiBI is a specific case of the action in Appendix E.3 with $M_1 = 0$. Such an action depends on five free independent parameters, namely M_2, Z, $\tilde{Z}$, α_L and α_E, or equivalently, α_L, α_E plus three free independent background functions.

So far we have focused only on scalar perturbations, however, the constraints we have found on the parameters are also valid for vector and tensor perturbations (although there could be additional Noether constraints for those two type of perturbations). Thus, from our results it is possible to see that some of the three parameters modifying GR will also affect the vector and tensor perturbations. In particular, from the resulting action we can identify some relevant parameters of eq. (6.12) and see that vector perturbations are coupled through $P_{S_2S_1}$ whereas tensor perturbations have a coupling through $P_{h_1h_2\times}$. Specifically, for the subclass of theories addressed in this section, these two parameters take the following form:

$$P_{S_2S_1} = -a^3M_1^2\frac{Z\bar{N}r^2}{(\bar{N}+r)}, \tag{6.47}$$

$$P_{h_1h_2\times} = \frac{M_1^2a^3}{8r}\left(2\tilde{Z} + Z\frac{d\ln M_1^2}{d\ln a}\right) - \frac{3}{4r^2}M_1^2a^3H^2\alpha_L. \tag{6.48}$$

From this we conclude that Z generates interactions between vector perturbations, while the three parameters Z, $\tilde{Z}$ and α_L generate interactions between tensor perturbations. This result is consistent with previous studies in massive bigravity, and its two branches of solutions. In the so-called branch I, where $Z = 0$, it has been found that scalar and vector perturbations behave in the same way as in GR, while tensor perturbations are coupled and evolve in a different way [115, 203]. This is in fact what we find from our results: vector perturbations are not coupled if $Z = 0$ because of eq. (6.47). While scalar perturbations would have a coupling with $\tilde{Z}$, in this case that coupling happens to be irrelevant after integrating out the auxiliary fields, and thus scalars behave as in GR when $Z = 0$. Finally, tensor perturbations do have a non-trivial coupling with $\tilde{Z}$, which indeed affects their evolution. In the so-called branch II, where $Z \neq 0$, the three types of perturbations are coupled and differ from GR [2, 123].

Finally, we emphasise that even though a detailed analysis in vector and tensor perturbations is necessary, scalar perturbations carry crucial information. From eq. (6.47)-(6.48) we can see that by observing vector perturbations, we can analyse the behaviour of the parameter Z, while from tensor perturbations we cannot discriminate between $\tilde{Z}$ and α_L, as they appear in the same interaction term. This suggests that an observational test to discriminate if α_L is present or not can be done by analysing scalar perturbations alone.

6.3 Discussion

In this chapter we applied the method developed in Chapter 4 to bimetric theories. We calculated the most general diffeomorphism-invariant quadratic action with two metrics, around a homogeneous and isotropic background, and leading to, up to, second-order equations of motion. For simplicity, we assumed that only one of the metrics was coupled to matter, a minimally coupled scalar field, although generalisations to perfect fluids or double couplings should be straightforward. Following the standard SVT decomposition for cosmological perturbations, we focused on scalar perturbations, and found that the final action depends on 29 free parameters (functions of time), in addition to three parameters determining the background evolution, and propagates at most four scalar physical DoFs. Due to the no-go theorem for healthy Lorentz-invariant bimetric theories with derivative interactions, in this chapter we focused on the subclass of bimetric theories without derivative interactions. In this case, we find that the number of free parameters in the quadratic action greatly reduces from 29 to 3, namely M_1, α_L and α_E, in addition to three extra free parameters that determine the evolution of the background. The resulting action propagates at most two scalar DoFs, which suggests the presence of an unstable mode due to the fact that all well-known bigravity theories propagate at most one scalar DoF. For this reason, we focused on subclasses of theories that propagate one or no scalar field.

In order to construct actions with only one propagating DoF, we imposed an extra constraint on the free parameters, which fixed the value of α_E. In this case, the most general action has only two free parameters, and we found that it is a generalisation of the quadratic action of massive bigravity. We recovered massive bigravity when $\alpha_L = 0$ and M_1 is a constant mass scale. We found that the presence of the parameter α_L affects the evolution of scalar and tensor perturbations, and even though it is not present in massive bigravity, it is present in other bimetric models such as EiBI theory. We also looked at cases in which the bimetric action propagates no scalar field. We found that when $M_1 = 0$, all the kinetic terms of one of the metrics vanished and, as result, the gravitational action does not propagate any physical scalar DoF. Such an action depends on two free parameters, and represents a generalisation of the EiBI theory. In Table 6.1 we summarise the results of this chapter.

Combining the results from this chapter with those of the previous one we have been able to extend the widely-used parametrisation of [161] originally proposed for Horndeski theories. It is now possible to construct a complete action for linear perturbations for general gravity with one propagating degree of freedom arising from either a scalar, vector or tensor field. It has been

Fields	Free Functions	Theory
$g_{1\mu\nu}$, $g_{2\mu\nu}$	M_1, $\alpha_L + 3$	Massive bigravity
$g_{1\mu\nu}$, $g_{2\mu\nu}$	α_E, $\alpha_L + 3$	EiBI

Table 6.1: In this table we compile parametrised models for bimetric theories of gravity, that are invariant under linear coordinate transformations, lead to second-order derivative equations. The first column indicates the field content of the gravitational theory. The second column shows the free functions parametrising the quadratic action for cosmological perturbations, while +1, +2 or +3 counts the number of extra free functions determining the background (and in turn affecting the perturbations). The parameter M_1 has dimensions of mass, and the extra parameters α are dimensionless. The third column shows examples of non-linear completions that are encompassed by the corresponding parametrisation.

shown that adding more interacting tensor fields will necessarily lead to more propagating scalar DoFs [140]. We expect the same to be true when adding vector fields, unless further gauge symmetries arise, such as invariance under U(1) transformations. Similarly, the addition of more scalar fields should lead to the propagation of extra DoFs, unless they appear as auxiliary variables. Therefore, even though there may not exist a theorem that would prevent adding more fields to the actions we have studied, while maintaining only one propagating DoF and its diffeomorphism invariance, we have been unable to find non-trivial such examples. Hence there is a possibility that our combined parametrisation for theories with one propagating degree of freedom is complete.

Our action should allow us to identify the subspace of effective parameters in the Parametrised Post-Friedman (PPF) approach [153, 154] which is, at the moment, still the most general parametrisation of gravitational theories currently available. Ultimately it should be possible develop a numerical tool, along the lines of EFTCAMB [222] or HiCLASS [223], which can be used for analysing data from future large-scale structure surveys such as Euclid, SKA, LSST and WFIRST, allowing us to test and compare the performance of these theories.

CHAPTER 7

Conclusions

General relativity is widely accepted as the correct description of gravity. While at Solar System scales GR certainly agrees with observational data to high precision, its viability at cosmological scales has been put in question lately. In this regime GR can fit data only under the assumption of the presence of exotic matter components in the Universe: dark matter, dark energy, and inflationary field. In particular, dark energy is introduced as a constant in the Einstein field equations in order to explain the presently observed accelerated expansion of the Universe. This cosmological constant receives quantum corrections that are many orders of magnitude larger than its observed value, which shows the presence of a fine-tuning problem that poses major tensions between ΛCDM and expectations from robust theories of modern particle physics. This suggests that GR might not be the appropriate theoretical model to describe gravity at large scales, and hence motivates the exploration of alternative gravity theories, which hopefully can offer a dynamical explanation to the accelerated expansion of the Universe, either due to interactions with new fields or modified gravitational self-interactions.

This thesis in concerned with testing gravity at cosmological scales by analysing the consistency and viability of gravity models, and constructing theoretical tools to constrain them in a unified and efficient way with future observational data. In particular, we focus on the analysis of linear cosmological perturbations, which play a crucial role as many gravity theories can predict very similar (or even exactly the same) background evolution as the ΛCDM model, but perturbations help break this degeneracy. Furthermore, with the next generation galaxy surveys such as EUCLID, DES, SKA, WFIRST and LSST, we will be able to reach unprecedented precision on cosmological observables for perturbations, and thus find tight constraints for GR and alternative gravity theories.

In the first part of this thesis we study a specific modified gravity theory called massive bigravity, which can predict a late-time accelerated Universe without a cosmological constant, in a technically natural way (stable under radiative corrections). In particular, we analyse the evolution of linear cosmological perturbations and we show the presence of certain instabilities that could jeopardise the viability of this model. We show that tensor perturbations grow as a power-law in time, and hence depending on their initial conditions they could generate a large amplitude of primordial gravitational waves, and in principle be incompatible with present bounds from CMB data. However, later papers showed that inflation could naturally lead to the appropriate initial conditions for tensor modes to fit data [125, 126]. Nevertheless, we also show that while scalar perturbations behave well (have a viable growth rate), they in fact propagate a Higuchi ghost, as the helicity-0 mode of the massive graviton has negative energy and then the model is likely to show instabilities beyond the classical linear regime of perturbations.

In the second part of this thesis we take a broader approach, in which we analyse entire classes of gravity theories. Specifically, we develop a method for constructing the most general parametrised action for linear cosmological perturbations for a given class of gravity theories, invariant under particular gauge symmetries. Our method allows us to describe a broad range of theories on cosmological scales in a unified manner, and enables us to ultimately test and compare gravity models by constraining the parametrised actions with relevant observational data such as the CMB or measures of large-scale structures such as weak lensing and galaxy redshift surveys. Our proposal contributes to develop the work done by a number of groups who have focused on linear perturbation theory in scalar-tensor theories and other variants of modified gravity [154–161, 169, 224, 225].

The method we develop is general and systematic and thus can be applied to a wide range of cases. The main ingredient of the method are the Noether identities, which allow us to systematically impose any gauge symmetry. Since most gravity theories are diffeomorphism invariant, we decide to focus on constructing linearly diffeomorphism invariant actions for perturbations around a homogeneous and isotropic background. In Chapter 5, we apply it to scalar-tensor and vector-tensor gravity theories. We show that the form of the quadratic action, crucially, depends on the gauge transformation properties of any extra fields that may arise in a modified gravity theory, and for this reason the parametrised action can be very different depending on the field content of gravity. For instance, for scalar-tensor theories there are only four free parameters determining the general form of the quadratic action for cosmological perturbations, whereas for vector-tensor theories there are ten. Then, in Chapter 6 we extend our results and apply the method to bimetric theories of gravity. We show that, in this case, the most general quadratic action is determined by 29 free parameters, and propagates at most four scalar degrees of freedom. However, if we do not allow derivative interactions between both metrics, the number of free parameters reduces to three. Furthermore, if we focus on actions that propagate only one DoF, the action has only two free parameters. We summarise our results in Table 7.1, omitting the bimetric case with derivative interactions.

Fields	Der.	Free Functions	Theories
$g_{\mu\nu}$	2	M	GR
$g_{\mu\nu}$, χ	2	M, $\alpha_{\{K,T,B\}}+2$	Horndeski
$g_{\mu\nu}$, χ	3	M, $\alpha_{\{K,T,B,H\}}+2$	Beyond Horndeski
$g_{\mu\nu}$, χ	4	M, $\alpha_{\{K,T,B,H,P\}}$, $\alpha_{Q_{\{1,2,3,4,5\}}}+2$	4^{th} Scalar-Tensor
$g_{\mu\nu}$, A^μ	2	M, $\alpha_{\{T,H\}}$, $\alpha_{D_{\{1,2,3\}}}$, $\alpha_{V_{\{0,1,2,3\}}}+2$	Generalised Proca
$g_{\mu\nu}$, A^μ, λ	2	M, $\alpha_{V_{\{3,4,5\}}}+1$	Einstein-Aether
$g_{1\mu\nu}$, $g_{2\mu\nu}$	2	M_1, α_L+3	Massive bigravity
$g_{1\mu\nu}$, $g_{2\mu\nu}$	2	α_E, α_L+3	EiBI

Table 7.1: In this table we summarise the results shown in this thesis on parametrised cosmological models, which are all linearly diffeomorphism invariant. In all cases the gravitational quadratic action propagates one scalar DoF, except in the generalised vector-tensor theories, in which we can have two, and EiBI in which we have none. The first column indicates the field content of the gravitational action. In all cases we also added a matter scalar field φ whose presence is omitted in this table. The second column indicates the maximum number of derivatives of the perturbation fields allowed in the equations of motion. Note that in the cases where this number is higher than 2, we assumed a maximum of two time derivatives, but allowed higher spatial derivatives. The third column shows the free coefficients parametrising the quadratic action, while the +1 or +2 counts the number of extra free background functions. In most cases there is one free mass parameter M or M_1, and extra dimensionless parameters that we term α. Note that, as explained in Chapter 4, even in theories with one single metric we find a free mass scale, and hence a model more general than GR. The fourth column shows examples of non-linear completions that are encompassed by the corresponding parametrisation.

The ultimate goal is to construct an action that spans as large a swathe of the landscape of gravitational theories as possible. To do so, in the future we hope to bring all the results in Table 7.1 together, and propose a completely general parametrisation for theories of gravity with one propagating scalar degree of freedom. This extends the widely-used parametrisation of [178] that arises in Horndeski theories, and is a substantial step towards achieving a general parametrisation which transcends scalar-tensor theories. It will also allow us to identify the subspace of effective parameters in the PPF approach [154] which is, at the moment, still the most general parametrisation of gravitational theories currently available. In addition, we will analyse the quasi-static limit of our new formalism, map out the region of stability of these theories, and ultimately develop a numerical tool which can be used for analysing data from forthcoming large-scale structure surveys such as Euclid, SKA, LSST and WFIRST.

Special care must be taken when constraining modified gravity theories with cosmological data in the context of linear perturbation theory as non-linear effects will become relevant on small to intermediate scales. In fact, in modified gravity theories non linearities can become relevant at much larger scales than those in the standard ΛCDM model; the linear approximation of a modified gravity theory can give an inaccurate prediction of the Universe even at scales as large as $k \sim 0.05h/\mathrm{Mpc}$ at present (see [226, 227]). This happens because many modified gravity theories propagate more DoFs than GR and these extra DoFs undergo a non-linear process known as screening (such as the Chameleon mechanism in scalar-tensor theories or the Vainshtein mechanism in bimetric theories), which, depending on the specific theory, can have a substantial effect in regimes which seem, a priori, linear. While we emphasise that the tools presented in this thesis can be used to predict the evolution of perturbations and thus constrain modified gravity theories at sufficiently large scales, a more detailed and accurate understanding of the effects of screening (such as in [228–232]) must also be used in order to improve and extend these results (see [55] for an attempt at including these effects yet using linear perturbations for scalar-tensor theories).

The work presented in this thesis contributes to the ongoing effort of developing appropriate theoretical models to describe gravity at cosmological scales. In general, we would like the new candidate theory to explain the accelerated expansion of the Universe in a dynamical and technically natural way, agree with observations, and be simple, to some degree. Most gravity models proposed fail to achieve these characteristics in one way or another. For instance, as we have seen, massive bigravity can certainly fit the expansion history of the Universe, is equipped with a screening mechanism to fit Solar System constraints, but it is plagued by instabilities at the linear level of cosmological perturbations. Different extensions to this model have been considered to circumvent this problem (such as double matter couplings) but they have been shown to be also unstable [144, 145]. For this reason, we have seen a number of new proposals taking a different approach in which generic modifications of gravity are parametrised, staying agnostic regarding the specific underlying model, which are then used to test and falsify GR. Such approaches are being considered not only for linear cosmology but also to constrain possible modifications of gravity in other regimes [233]. In the case of cosmology, preliminary estimates have been done to determine how future experimental data can constrain parametrised modified gravity models [54–58]. They have shown that while current data precision and systematic errors cannot place strong constraints on gravity models, future data could improve the precision in a factor of order 10. If future observations were to find deviations from GR, the work presented here could help substantially towards finding viable gravity models, and understanding the possible physics that governs the dynamics of the

Universe and its constituents.

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APPENDIX A

Introduction

A.1 Gauge transformations

In this section we present the linear gauge transformation rules for a metric, vector and scalar field under linear coordinate transformations.

In general, the linear transformations of any field can easily be derived from the general transformation laws. For a 2-rank tensor field $g^{\mu\nu}$, the general transformation law from a set of coordinates x^μ to another coordinates $\tilde{x}^\mu$ is given by:

$$\tilde{g}^{\mu\nu}(\tilde{x}) = \frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\frac{\partial \tilde{x}^\nu}{\partial x^\beta} g^{\alpha\beta}(x), \tag{A.1}$$

where $\tilde{g}^{\mu\nu}$ represents the tensor in the $\tilde{x}^\mu$ coordinates. If we now consider linear transformations where $\tilde{x}^\mu = x^\mu + \epsilon^\mu$, with ϵ^μ being an arbitrary small 4-vector, the transformation law becomes:

$$\tilde{g}^{\mu\nu}(x) \approx g^{\alpha\beta}(x)\left(\delta^\mu{}_\alpha + \partial_\alpha \epsilon^\mu\right)\left(\delta^\nu{}_\beta + \partial_\beta \epsilon^\nu\right) - \epsilon^\alpha \partial_\alpha \tilde{g}^{\mu\nu}(x), \tag{A.2}$$

where we have Taylor expanded the coordinates of $\tilde{g}^{\mu\nu}(\tilde{x})$ up to first order in ϵ^μ. Here all partial derivatives are with respect to the x^μ coordinates. Then, up to linear order in ϵ^μ, we get

$$\tilde{g}^{\mu\nu}(x) = g^{\mu\nu}(x) + g^{\mu\beta}\partial_\beta \epsilon^\nu + g^{\beta\nu}\partial_\beta \epsilon^\mu - \epsilon^\alpha \partial_\alpha g^{\mu\nu}(x), \tag{A.3}$$

which can be seen as a gauge transformation where the fields change but the coordinates are kept fixed. Notice that in our last step we have used eq. (A.2) recursively to write $\epsilon^\alpha \partial_\alpha \tilde{g}^{\mu\nu}(x) = \epsilon^\alpha \partial_\alpha g^{\mu\nu}(x) + \mathcal{O}(\epsilon^2)$.

Now we consider linear perturbations of the metric $\delta g_{\mu\nu}$ around some background metric $\bar{g}_{\mu\nu}$, and find the gauge transformation rule for $\delta g_{\mu\nu}$ under linear coordinate transformation.

From eq. (A.3), at zeroth order the metric does not change, while at linear order we find:

$$\delta\tilde{g}^{\mu\nu} = \delta g^{\mu\nu} + \bar{g}^{\mu\beta}\partial_\beta\epsilon^\nu + \bar{g}^{\beta\nu}\partial_\beta\epsilon^\mu - \epsilon^\alpha\partial_\alpha\bar{g}^{\mu\nu}, \tag{A.4}$$

where $\delta g^{\mu\nu}$ are the linear perturbations of the inverse metric $g^{\mu\nu}$. Here it is understood that all the fields depend on the coordinates x^μ. Finally, from eq. (A.4) we can find the transformation rule for $\delta g_{\mu\nu}$ by using that $g^{\mu\alpha}g_{\alpha\nu} = \delta^\mu{}_\nu$. We find that:

$$\delta\tilde{g}_{\mu\nu} = \delta g_{\mu\nu} - \bar{g}_{\mu\beta}\partial_\nu\epsilon^\beta - \bar{g}_{\beta\nu}\partial_\mu\epsilon^\beta + \epsilon^\alpha\bar{g}_{\mu\beta}\bar{g}_{\nu\gamma}\partial_\alpha\bar{g}^{\beta\gamma}. \tag{A.5}$$

If we focus on the scalar-type perturbations, defined in eq. (E.1), around a spatially-flat homogeneous and isotropic background metric, from eq. (A.5) we find that:

$$\begin{aligned}
\tilde{\Phi} &= \Phi - \dot{\pi}, \\
\tilde{B} &= B + \pi - a^2\dot{\epsilon}, \\
\tilde{\Psi} &= \Psi + \frac{\dot{a}}{a}\pi, \\
\tilde{E} &= E - \epsilon,
\end{aligned} \tag{A.6}$$

where $a(t)$ is the scale factor. Here the dots denote derivatives with regards to the physical time t. Notice we have also rewritten the gauge parameter ϵ^μ in terms of its scalar-type components as $\epsilon^\mu = (\pi, \delta^{ij}\partial_j\epsilon)$.

Finally, we emphasise that the same kind of analysis can be done for any type of field. For a linear perturbation of a scalar field χ, expanded as $\chi = \chi_0 + \delta\chi$, the transformation under linear coordinate transformations is given by:

$$\delta\tilde{\chi} = \delta\chi - \epsilon^\mu\left(\partial_\mu\chi_0\right), \tag{A.7}$$

where χ_0 is the background solution of the scalar field and $\delta\chi$ its first-order perturbation. In the case of a homogeneous and isotropic background, the transformation becomes:

$$\delta\tilde{\chi} = \delta\chi - \dot{\chi}_0\pi, \tag{A.8}$$

where we have assumed that $\chi_0 = \chi_0(t)$. For linear perturbations of a vector field, expanded as $A^\mu = A_0^\mu + \alpha^\mu$, the transformation under linear coordinate transformations is given by:

$$\tilde{\alpha}^\mu = \alpha^\mu + A_0^\nu \left(\partial_\nu \epsilon^\mu\right) - \epsilon^\nu \left(\partial_\nu A_0^\mu\right), \tag{A.9}$$

where A_0^μ is the background solution of the vector field, and α^μ its first-order perturbation. If we focus on scalar-type perturbations around a homogeneous and isotropic background, the transformation becomes:

$$\begin{aligned} \tilde{\alpha}^0 &= \alpha^0 + \dot{\pi} A - \dot{A}\pi, \\ \tilde{\alpha} &= \alpha + a^2 A \dot{\epsilon}, \end{aligned} \tag{A.10}$$

where $(A(t), \vec{0})$ is the homogeneous and isotropic background solution, and the scalar-type perturbations are such that $\alpha^\mu = (\alpha^0, \alpha^i) = (\alpha^0, \bar{h}^{ij}\partial_j \alpha)$, where $\bar{h}^{ij}$ is the 3-spatial metric from a spatially-flat FLRW background metric.

APPENDIX B

Cosmological perturbations in massive bigravity

B.1 Scalar perturbation equations

In this section we present the relevant analysis and equations related to scalar perturbations.

B.1.1 Auxiliary variables

As explained in Section 3.1, the fields B_1, B_2, Φ_1 and Φ_2 appear as auxiliary variables in the equations of motion, and therefore they can be worked out in terms of the remaining fields E_1, E_2 and Ψ_2, by using their own equations of motion-namely eq. (3.21), (3.22), (3.25) and (3.26). The explicit expressions for the four auxiliary variables are:

$$\begin{aligned}
B_2 = \frac{1}{D_a}\Big[&k^2\left(\frac{3}{2}Za^2m^2X + k^2N(1+X)\right)\mathcal{H}E_2' + \frac{3}{2}\mathcal{H}k^2N^2a^2E_1'm^2Z \\
&+ \frac{3}{4}\rho_* m^2XZ(1+w)(3\Psi_2 + k^2E_2)a^4 + \frac{1}{2}\big(m^2Z(1+X)(k^2E_2 - k^2E_1 + 3\Psi_2)N^2 \\
&+\rho_*(1+X)(1+w)\left(3\Psi_2 + k^2E_2\right)N + 3m^2XZ\Psi_2\big)k^2a^2 + \Psi_2k^4N(1+X)\Big],
\end{aligned} \tag{B.1}$$

$$\begin{aligned}
B_1 = \frac{1}{4XD_a}\Bigg[&4Nk^2\mathcal{H}E_1'\left(\frac{3}{2}a^2\left(\rho_*X(1+w) + Nm^2Z + \rho_*(1+w)\right) + k^2(1+X)\right) \\
&- 2Za^2Xm^2\Bigg(-3k^2\mathcal{H}E_2' + \frac{3}{2}\left((k^2E_2 - k^2E_1 + 3\Psi_2)X - k^2E_1\right)(1+w)\rho_*a^2 \\
&+k^2\left((k^2E_2 - k^2E_1 + 3\Psi_2)X + k^2(E_2 - E_1)\right)\Bigg)\Bigg],
\end{aligned} \tag{B.2}$$

$$\Phi_2 = \frac{-1}{8\mathcal{H}D_a}\Big[8\mathcal{H}\Psi_2'\Big(\frac{9}{4}\rho_* m^2 XZ(1+w)a^4 + \frac{3}{2}k^2a^2\left(\rho_*(1+X)(1+w)N + m^2Z(X+N^2)\right)$$
$$+k^4N(1+X)\Big) + 2a^2\Big(2k^2\mathcal{H}\Big(\frac{3}{2}\rho_* m^2XZ(1+w)a^2 + N(\rho_*(1+X)(1+w))k^2$$
$$+Nm^2Z\Big)E_2' - 2m^2N^2Z\mathcal{H}k^4E_1' + \frac{3}{2}ZX(1+w)\rho_* m^2a^4\left(\rho_*(1+w)(3\Psi_2+k^2E_2)\right.$$
$$\left.+m^2Z(k^2E_2 - k^2E_1 + 3\Psi_2)N\right) + k^2a^2\left(N(1+w)^2(1+X)(3\Psi_2+k^2E_2)\rho_*^2+\right.$$
$$Z(1+w)m^2\rho_*\left(3X\Psi_2 + N^2(X(k^2E_2 - k^2E_1 + 3\Psi_2) + 2k^2E_2 - k^2E_1 + 6\Psi_2)\right)$$
$$\left.+Nm^4Z^2(X+N^2)(k^2E_2 - k^2E_1 + 3\Psi_2)\right) + 2N\left(Nm^2Z + \rho_*(1+X)(1+w)\right)\Psi_2k^4\Big)\Big], \tag{B.3}$$

$$\Phi_1 = \frac{Za^2m^2}{4\mathcal{H}ND_a}\Big[-Nk^2\mathcal{H}E_1'(3\rho_*(1+w)a^2 + 2k^2) + \frac{3}{2}\rho_* m^2XZ(1+w)(k^2E_2 - k^2E_1 + 3\Psi_2)a^4$$
$$+ k^2a^2\left(m^2Z(k^2E_2 - k^2E_1 + 3\Psi_2)N^2 + \rho_*(1+w)(3\Psi_2 + k^2E_2)N\right.$$
$$\left.+m^2XZ(k^2E_2 - k^2E_1 + 3\Psi_2)\right) + 2\mathcal{H}k^4E_2'N + 2\Psi_2k^4N\Big], \tag{B.4}$$

where D_a is given by:

$$D_a = \mathcal{H}\left[\frac{3}{2}k^2a^2\left(m^2N^2Z + \rho_*(1+X)(1+w)N + m^2XZ\right) + k^4N(1+X) + \frac{9}{4}\rho_* m^2XZ(1+w)a^4\right] \tag{B.5}$$

At a first glance, one might expect that the original system of equations (3.21)-(3.27), with seven scalar fields, has three dynamical degrees of freedom, as the equations of motion for E_1, E_2 and Ψ_2 are independent and contain second derivatives. However, when eliminating the four auxiliary variables, and replacing them in the three remaining equations, we get that, in eq. (3.23) all first and second derivatives of Ψ_2 cancel out, so that Ψ_2 becomes an explicit auxiliary variable. Therefore, it can be written in terms of E_1 and E_2. Next, we show the expression for Ψ_2 when worked out from eq. (3.23):

$$\Psi_2 = \frac{k^2}{2D_p}\Big[-2\mathcal{H}k^2N^2E_1'\left(\frac{3}{2}m^2Za^2(X-1)N^2 + (-3\mathcal{H}^2X + 3\mathcal{H}^2 + k^2)N + \frac{3}{2}a^2m^2Z(X-1)\right)$$
$$+ 2k^4\mathcal{H}N^3E_2' + \frac{3}{2}k^2X^2Z^2a^4m^4(E_1 - E_2X)N^5 + (E_1 - E_2X)m^2a^2\Big(\frac{9}{4}m^4Xa^4(X-1)Z^2$$
$$+k^2(k^2X - 6X^2\mathcal{H}^2 + 3\mathcal{H}^2)\Big)ZN^4 + \Big(\frac{3}{2}m^4a^4\left(-2X^3E_2(k^2 - 3\mathcal{H}^2)\right.$$
$$\left.+\left(k^2E_1 - 6\mathcal{H}^2(E_2+E_1)\right)X^2 + 2E_1(3\mathcal{H}^2 + k^2)X - k^2E_1\right)Z^2 - 2\mathcal{H}^2k^2\left(3X^3E_2\mathcal{H}^2\right.$$

$$
\begin{aligned}
&-(k^2E_2+3\mathcal{H}^2E_1)X^2+(k^2E_1-3E_2\mathcal{H}^2)X+3\mathcal{H}^2E_1+k^2(-E_2+E_1))\Big)N^3\\
&+\Bigg(\frac{9}{2}m^4Xa^4(X-1)(E_1-E_2X)Z^2+(-9E_2\mathcal{H}^4+6k^2\mathcal{H}^2E_2)X^3\\
&-\left(k^2-3\mathcal{H}^2\right)\left(3\mathcal{H}^2(E_1+E_2)+k^2E_2\right)X^2+\left(-9\mathcal{H}^4E_1-6k^2\mathcal{H}^2\left(E_1+\frac{1}{2}E_2\right)+k^4E_2\right)X\\
&+\left(6\mathcal{H}^2E_1+k^2(-E_2+E_1)\right)k^2\Bigg)m^2a^2ZN^2+3m^4a^4\Bigg(-\frac{1}{2}E_2(k^2-6\mathcal{H}^2)X^3-\frac{1}{2}k^2E_1\\
&-3\mathcal{H}^2(E_1+E_2)X^2+E_1\left(3\mathcal{H}^2+k^2\right)X\Bigg)Z^2N+\frac{9}{4}m^6XZ^3a^6(X-1)(E_1-E_2X)\Bigg], \qquad \text{(B.6)}
\end{aligned}
$$

where D_p is given by

$$
\begin{aligned}
D_p=X\Bigg[&3m^2a^2\left(\frac{9}{8}m^4Xa^4(X-1)Z^2+\left(k^2X+\frac{3}{2}\mathcal{H}^2-3X^2\mathcal{H}^2\right)k^2\right)ZN^4\\
&+\frac{9}{4}X^2m^4Z^2a^4k^2N^5+\Bigg(\frac{9}{2}m^4a^4\left((k^2-3\mathcal{H}^2)X^2+\left(\frac{1}{2}k^2+3\mathcal{H}^2\right)X-\frac{1}{2}k^2\right)Z^2\\
&-6k^4X\mathcal{H}^2-9k^2\mathcal{H}^4+9X^2k^2\mathcal{H}^4+k^6\Bigg)N^3+3m^2a^2ZN^2\Bigg(\frac{9}{4}m^4Xa^4(X-1)Z^2\\
&+\left(\frac{9}{2}\mathcal{H}^4-3\mathcal{H}^2k^2\right)X^2+\left(k^4-\frac{9}{2}\mathcal{H}^4-\frac{3}{2}\mathcal{H}^2k^2\right)X+3\mathcal{H}^2k^2\Bigg)\\
&+\frac{9}{4}m^4a^4\left(\left(k^2-6\mathcal{H}^2\right)X^2+\left(k^2+6\mathcal{H}^2\right)X-k^2\right)Z^2N+\frac{27}{8}m^6XZ^3a^6(X-1)\Bigg]. \qquad \text{(B.7)}
\end{aligned}
$$

Therefore, as expected, only two degrees of freedoms are remain: E_1 and E_2.

B.1.2 Complete equations of motion

In this subsection we present the full equations of motion for the two propagating, physical scalar degrees of freedom: E_1 and E_2. The equation for E_2 takes the following form:

$$
\begin{aligned}
&E_2''-\frac{27}{4D_2}\left(w+\frac{1}{3}\right)k^2N^2a^4\rho_*m^2\mathcal{H}Z(1+w)E_1'-\frac{3\mathcal{H}}{D_2}\Bigg[-\frac{3}{2}(1+w)^2a^4\Bigg(\frac{3}{2}a^2Xm^2Z\\
&+k^2N(1+X)\Bigg)\rho_*^2+\frac{1}{2}(1+w)\Bigg(\frac{3}{2}m^2a^2Z\left(\left(3w-1\right)X-2N^2\right)+k^2N\left(3(w-1)X+3w\right.\\
&\left.+1\right)\Bigg)a^2k^2\rho_*+(X-1)k^6N\left(w-\frac{1}{3}\right)\Bigg]E_2'+\frac{k^2}{D_2}\Big[k^6Nw(X-1)\\
&-\frac{3}{4}(1+w)^2a^4\left(\frac{3}{2}a^2Xm^2Z+k^2N(1+X)\right)\rho_*^2+\frac{1}{2}a^2(1+w)k^2\rho_*\Bigg\{\frac{3}{2}Zm^2a^2\left(3Xw\right.\\
&\left.+N^2\left(\left(3w+1\right)X-1\right)\right)+Nk^2\left(\left(3w-1\right)X+3w+1\right)\Bigg\}\Big]E_2\\
&+\frac{m^2}{D_2}a^2N\Bigg[\frac{9}{4}k^2\tilde{Z}\rho_*m^2a^4Z(1+w)N^2+k^2\Bigg\{3(1+w)\left(\tilde{Z}-(1+3w)\frac{Z}{4}\right)k^2X\rho_*a^2
\end{aligned}
$$

$$+\frac{9}{4}\tilde{Z}\rho_*^2(1+w)^2(1+X)a^4+\left(\tilde{Z}-\frac{3}{2}\left(w+\frac{1}{3}\right)Z\right)k^4(X-1)\Big\}N$$
$$+\frac{9}{4}(1+w)\tilde{Z}ZX\rho_*m^2a^4\left(\frac{3}{2}\rho_*(1+w)a^2+k^2\right)\Big](E_2-E_1)=0, \tag{B.8}$$

and the equation for E_1 takes the following form:

$$E_1''+\frac{2\mathcal{H}}{D_1}\Big[\frac{9}{4}a^4\Big\{\frac{3}{4}a^2m^2\left(4X^2+(3w-1)X-(1+3w)\right)Z^2+\frac{1}{2}\left(3X^2+(3w+1)X\right.$$
$$\left.-3w\right)k^2NZ+N\tilde{Z}k^2(1+X)\Big\}(1+w)^3N\rho_*^3+3a^2\Big\{-\frac{9}{16}a^4m^4(1+X)(X-1)^2(N^2+1)Z^3$$
$$-\frac{9}{8}a^2m^2Z^2\left(\frac{1}{3}k^2N\left(N^2X(X^2-X-1)+X^3+2+3w(1-X+N^2)-5X^2\right)\right.$$
$$\left.-m^2\tilde{Z}(X-1)^2(N^2+1)a^2\right)+\frac{3}{4}k^2\left(m^2\tilde{Z}(X-1)^2(N^2+1)a^2\right.$$
$$\left.+\left(2X^2+\left(w-\frac{1}{3}\right)X-\left(w+\frac{1}{3}\right)\right)k^2N\right)NZ+\tilde{Z}k^4N^2X\Big\}(1+w)^2\rho_*^2$$
$$+k^2(X-1)\Big\{-\frac{9}{8}a^4m^4(X-1)(1+X)(N^2+1)Z^3-\frac{3}{2}a^2(N^2+1)(X-1)m^2\left(Nk^2(1+X)\right.$$
$$\left.-\frac{3}{2}m^2a^2\tilde{Z}\right)Z^2+\frac{1}{2}\left(6m^2\tilde{Z}(N^2+1)(X-1)a^2+(1+3X)k^2N\right)k^2NZ+k^4\tilde{Z}N^2\Big\}(1+w)\rho_*$$
$$+(N^2+1)k^6(X-1)^2Zm^2N\left(\tilde{Z}-\frac{1}{2}(X+1)Z\right)\Big]E_1'$$
$$-2\frac{\mathcal{H}k^2}{D_1}\Big[-\frac{9}{8}a^4\rho_*m^4(1+X)(X-1)^2(N^2+1)(1+w)Z^3-\frac{1}{2}m^2Z^2\Big\{k^4N(X-1)^2(1$$
$$+X)(N^2+1)-\frac{27}{2}\left(\frac{2}{3}X^2+\left(w-\frac{1}{3}\right)X-\frac{1}{2}w-\frac{1}{6}\right)a^4(1+w)^2N\rho_*^2+\frac{3}{2}\left(Nk^2(1+X)\right.$$
$$\left.-3m^2a^2\tilde{Z}\right)a^2(1+w)(N^2+1)(X-1)^2\rho_*\Big\}+k^2NZ\left(\frac{9}{2}a^2(1+w)^2\left(Xw+\frac{1}{2}X^2-\frac{1}{6}\right)N\rho_*^2\right.$$
$$+\frac{3}{2}(1+w)(X-1)\left(m^2a^2\tilde{Z}(X-1)N^2+k^2(X+w)N+m^2a^2\tilde{Z}(X-1)\right)\rho_*$$
$$\left.+k^2\tilde{Z}m^2(X-1)^2(N^2+1)\right)+\left(\frac{3}{2}a^2(1+X)(1+w)\rho_*+k^2(X-1)\right)\rho_*\tilde{Z}(1+w)k^2N^2\Big]E_2'$$
$$+\frac{\rho_*k^2(1+w)}{D_1}\Big[-\frac{3}{4}m^4Z^2\left((1+X)Z-2\tilde{Z}\right)a^4k^2(X-1)^2N^4-\frac{1}{2}\Big\{-3\rho_*\left(\left(\left(-\frac{1}{2}X^3+2X^2\right.\right.\right.$$
$$\left.\left.\left.+\left(3w-\frac{1}{2}\right)X-\frac{3}{2}w-1\right)Z+\tilde{Z}(X-1)^2\right)(1+w)a^2+k^2(X-1)^2((1+X)Z$$
$$-2\tilde{Z})\Big\}m^2Za^2k^2N^3+\Big\{\left(\left(-\frac{3}{2}m^4(1+X)(X-1)^2Z^3+3m^4\tilde{Z}(X-1)^2Z^2+\frac{3}{4}\left(X^2-3w\right.\right.\right.$$
$$\left.\left.+(3w-2)X-1\right)\rho_*^2(1+w)^2Z-\frac{3}{2}\rho_*^2X\tilde{Z}(1+w)^2(1+X)\right)a^4+\frac{1}{2}\rho_*a^2k^2(1+w)\left((-1\right.$$
$$\left.+(2+3w)X^2+(3w-1)X\right)Z-4\tilde{Z}\left(\frac{1}{2}+X\right)(X-1)\Big)+\frac{1}{3}((-1+(1+3w)X)Z$$

$$
\begin{aligned}
&-2\tilde{Z}(X-1)\Big)\, k^4(X-1)\Big\}\, k^2N^2 - \frac{1}{2}m^2Za^2\left\{\frac{9}{4}ZX^3\rho_*^2(1+w)^2a^4 - 3k^2(1+w)\left(\tilde{Z}(X-1)^2\right.\right.\\
&\left.\left.+Z\left(-X^3+\frac{5}{2}X^2+\left(-\frac{1}{2}+3w\right)X-\frac{3}{2}w-1\right)\right)\rho_*a^2+((1+X)Z-2\tilde{Z})k^4(X-1)^2\right\}N\\
&\left.-\frac{3}{4}m^4Z^2\left((1+X)Z-2\tilde{Z}\right)a^4k^2(X-1)^2\right]E_1+\frac{1}{D_1}\left[-a^2(X-1)^2m^4Z^2k^6N^4\left(\tilde{Z}\right.\right.\\
&\left.-(X+1)\frac{Z}{2}\right)-\frac{2}{3}m^2Zk^4\left\{-\frac{1}{2}(X-1)k^2Z\left(-\frac{3}{2}\rho_*\left(1+3X-X^2+3w\right)(1+w)a^2+(X^2\right.\right.\\
&\left.\left.-1)k^2\right)+\left(\frac{9}{4}\rho_*^2(1+w)^2(1+X)a^4+\frac{3}{2}\rho_*k^2X(X-1)(1+w)a^2+k^4(X-1)^2\right)\tilde{Z}\right\}N^3\\
&-\frac{2}{3}k^2\left\{-\frac{3}{2}k^4m^4a^2(1+X)(X-1)^2Z^3+3a^2m^4\left(\frac{9}{8}\rho_*^2X(1+w)^2a^4+k^4(X-1)^2\right)\tilde{Z}Z^2\right.\\
&-\frac{3}{2}\rho_*\left(\frac{3}{2}\rho_*\left(X^2w+\frac{2}{3}X+w\right)(1+w)a^2+(X-1)k^2\left(Xw-\frac{1}{3}\right)\right)k^4(1+w)Z\\
&\left.+\rho_*\left(\frac{3}{2}a^2(1+X)(1+w)\rho_*+k^2(X-1)\right)\tilde{Z}k^2\left(\frac{3}{2}a^2(1+X)(1+w)\rho_*+k^2X\right)(1+w)\right\}N^2\\
&-\frac{2}{3}m^2Zk^2\left\{-\frac{1}{2}\left(\frac{9}{4}\rho_*^2X\left(X^2-1+(1+3w)X\right)(1+w)^2a^4+\frac{9}{2}\rho_*(X-1)\left(\frac{2}{3}X^2-\frac{2}{3}-w\right.\right.\right.\\
&\left.\left.+\left(w-\frac{2}{3}\right)X\right)k^2(1+w)a^2+k^4(1+X)(X-1)^2\right)k^2Z+\tilde{Z}\left(\frac{3}{2}\rho_*(1+w)a^2+k^2\right)\\
&\left.\cdot\left(\frac{9}{2}\rho_*^2X(1+w)^2(1+X)a^4+3\rho_*k^2X(X-1)(1+w)a^2+k^4(X-1)^2\right)\right\}N\\
&-a^2\left(\tilde{Z}\left(\frac{27}{8}X^2\rho_*^3(1+w)^3a^6+\frac{9}{4}k^2\rho_*^2X^2(1+w)^2a^4+k^6(X-1)^2\right)\right.\\
&\left.\left.-\frac{k^6}{2}(1+X)(X-1)^2Z\right)m^4Z^2\right](E_2-E_1)=0,
\end{aligned}
\tag{B.9}
$$

where D_1 and D_2 are given by:

$$
\begin{aligned}
D_1 =& Z\rho_*(1+w)N\left[\frac{9}{4}\rho_*m^2\left(N^2k^2+\left(\frac{3}{2}\rho_*(1+w)a^2+k^2\right)X\right)a^4(1+w)Z\right.\\
&\left.+\left(\frac{3}{2}\rho_*(1+X)(1+w)a^2+k^2(X-1)\right)\left(\frac{3}{2}\rho_*(1+w)a^2+k^2\right)Nk^2\right]
\end{aligned}
\tag{B.10}
$$

$$
\begin{aligned}
D_2 =& \frac{27}{8}m^2ZX\rho_*^2(1+w)^2a^6+\frac{9}{4}k^2\rho_*\left(m^2(X+N^2)Z+\rho_*(1+X)(1+w)N\right)(1+w)a^4\\
&+3k^4\rho_*NX(1+w)a^2+k^6N(X-1).
\end{aligned}
\tag{B.11}
$$

B.1.3 Exponential instabilities

As mentioned previously, in the expanding branch scalar perturbations have an exponential instability at early times for sub-horizon scales; the instability is independent of the particular values of the parameters βs. However, during the bouncing branch, different solutions can be found for different parameters. For this reason, we distinguish the following cases: (a) $\beta_3 \neq 0$;

(b) $\beta_3 = 0$ and $(\beta_4 - 3\beta_2) \neq 0$; (c)$\beta_3 = 0$ and $(\beta_4 - 3\beta_2) = 0$; (d) $\beta_3 = \beta_2 = 0$.

In what follows we will see that in cases (a), (b) and (c), E_1 develop an exponential instability at early times. For simplicity, let us study the equations of motion during the radiation-dominated era for sub-horizon scales. Generically, the equations of motion can be written as

$$E_a'' + f_{ab}(x, N)E_b' + g_{ab}(x, N)E_b = 0;\ x = k\mathcal{H}^{-1}, \tag{B.12}$$

but when approximated at early times in the bouncing branch ($N \gg 1$) and for sub-horizon scales ($x \gg 1$), these coefficients become:

Case (a):

$$f_{11} = \frac{16}{3}\frac{\beta_4}{\beta_3}\frac{\mathcal{H}}{N}x^2,\ f_{12} = \frac{8}{9}\frac{\beta_4^2}{\beta_3^2}\frac{\mathcal{H}}{N^2}x^2,\ f_{22} = 2\mathcal{H},\ f_{21} = -18\mathcal{H}, \tag{B.13}$$

$$g_{11} = -\frac{1}{9}x^2\mathcal{H}^2,\ g_{12} = \frac{4}{27}x^2\mathcal{H}^2,\ g_{22} = -\frac{\beta_3}{\beta_4}N\mathcal{H}^2,\ g_{21} = \frac{\beta_3}{\beta_4}N\mathcal{H}^2, \tag{B.14}$$

Case (b):

$$f_{11} = -2\mathcal{H},\ f_{12} = 2\mathcal{H},\ f_{22} = \frac{12}{x^2}\frac{(\beta_4 - 3\beta_2)}{\beta_4}\mathcal{H},\ f_{21} = \frac{54}{x^4}\frac{(\beta_4 - 3\beta_2)\beta_2}{\beta_4^2}\mathcal{H}, \tag{B.15}$$

$$g_{11} = -\frac{1}{3}x^2\mathcal{H}^2,\ g_{12} = -\frac{2}{3}x^2\mathcal{H}^2,\ g_{22} = \frac{1}{3}x^2\mathcal{H}^2,\ g_{21} = 6\frac{\beta_2}{\beta_4}\mathcal{H}^2, \tag{B.16}$$

Case (c):

$$f_{11} = -6\mathcal{H},\ f_{12} = 6\mathcal{H},\ f_{22} = -\frac{36}{Nx^2}\frac{\beta_1}{\beta_4}\mathcal{H},\ f_{21} = -\frac{27}{Nx^4}\frac{\beta_1}{\beta_4}\mathcal{H}, \tag{B.17}$$

$$g_{11} = -\frac{1}{3}x^2\mathcal{H}^2,\ g_{12} = -\frac{14}{3}x^2\mathcal{H}^2,\ g_{22} = \frac{1}{3}x^2\mathcal{H}^2,\ g_{21} = 4\mathcal{H}^2. \tag{B.18}$$

As we can see in all cases, the coefficient g_{11} has a negative sign, which will induce an exponential instability in the solutions for E_1.

B.1.4 Density contrast

The explicit form of the density contrast δ_{GIk} as a function of E_i is:

$$
\begin{aligned}
\delta_{GIk} = \frac{2(1+w)}{D_d} &\left[-27k^2a^4\rho_* m^2\mathcal{H}N^2Z(1+w)E_1' + 9\rho_* k^4 m^2 ZX(1+w)(N^2E_1+E_2)a^4\right. \\
&+\frac{9}{2}\rho_*\mathcal{H}(1+w)a^2E_2'\left(9\rho_* m^2XZ(1+w)a^4 + 6k^2Na^2\left((1+X)(1+w)\rho_* + Zm^2N\right)\right. \\
&\left.+4k^4N(X-1)\right) + 6k^6N\left(E_2(1+X)(1+w)\rho_* - m^2NZ(X-1)(E_2-E_1)\right)a^2 \\
&\left.+4k^8E_2N(X-1)\right], \qquad \text{(B.19)}
\end{aligned}
$$

where D_d is given by:

$$
\begin{aligned}
D_d = &\ 27m^2ZX\rho_*^2(1+w)^2a^6 + 18k^2\rho_*(1+w)a^4\left(\rho_*(1+X)(1+w)N + m^2Z(X+N^2)\right) \\
&+ 24k^4\rho_*NX(1+w)a^2 + 8k^6N(X-1). \qquad \text{(B.20)}
\end{aligned}
$$

B.2 Ghost-like instabilities

As it was shown in [68], bimetric massive gravity given by eq. (2.20) is said to be ghost-free in the sense that it propagates the right number of degrees of freedom: five for a massive graviton and two for a massless graviton, and avoids an extra ghost-like scalar field (with negative sign in its kinetic term). However, as realised for the first time by Higuchi in [121], the helicity-0 mode of the massive graviton might behave as a ghost for some values of the parameters of the theory in de-Sitter spacetime, leading to instabilities on the solutions beyond the classical linear regime. The condition to have positive kinetic terms only in the action is known as the Higuchi bound. In addition, the helicity-1 vector field could also propagate as a ghost for some parameters, while the tensor fields are always safe from becoming ghosts (see [234]).

In the case of FLRW backgrounds, described by eq. (3.1)-(3.2), a Higuchi bound for scalar and vector fields was found in [194] for the bimetric massive gravity model addressed in this chapter, by analysing the quadratic action for linear perturbations. In this section, we analyse the satisfiability of these Higuchi bounds for the relevant cases considered in this chapter.

B.2.1 Scalar fields

According to [194], the Higuchi bound for the helicity-0 mode in the second branch of background solutions, satisfying $X\mathcal{H} = h$, is:

$$\tilde{m}^2 \left(1 + \frac{1}{N^2}\right) - 2H^2 \geq 0, \tag{B.21}$$

where H is the Hubble parameter and $\tilde{m}$ is given by:

$$\tilde{m}^2 = m^2 N Z, \tag{B.22}$$

where Z was defined previously as $Z = \beta_1 + 2\beta_2 N + \beta_3 N^2$.

In what follows, we consider the expanding and bouncing branches, and analyse the Higuchi bound in two relevant limit cases: early and late times.

Expanding branch: In this branch we have $\beta_1 > 0$. Using the Friedmann equation given by eq. (3.4), the bound (B.21) becomes:

$$m^2 N(\beta_1 + 2\beta_2 N + \beta_3 N^2)\left(1 + \frac{1}{N^2}\right) - \frac{2}{3}\left[\rho_0 + m^2\left(\beta_0 + 3N\beta_1 + 3\beta_2 N^2 + \beta_3 N^3\right)\right] \geq 0, \tag{B.23}$$

or equivalently, using the constraint (3.12),

$$N(\beta_1 + 2\beta_2 N + \beta_3 N^2)\left(1 + \frac{1}{N^2}\right) - \frac{2}{3}\left(\frac{\beta_1}{N} + 3\beta_2 + 3\beta_3 N + \beta_4 N^2\right) \geq 0. \tag{B.24}$$

1. **Early times:** At early times, $N \ll 1$. Considering only the leading terms in $1/N$, the bound (B.24) becomes:

$$\frac{\beta_1}{3N} \geq 0, \tag{B.25}$$

which is satisfied for the cases considered in this chapter, as it was assumed that $\beta_1 > 0$ and $N > 0$.

2. **Late times:** At late times we approach a de-Sitter spacetime where $\rho_0 \to 0$ and $N \to \bar{N}$, where $\bar{N}$ satisfies eq. (3.12) with $\rho_0 = 0$. In this regime the bound (B.23) becomes:

$$\bar{N}(\beta_1 + 2\beta_2 \bar{N} + \beta_3 \bar{N}^2)\left(1 + \frac{1}{\bar{N}^2}\right) - \frac{2}{3}\left(\beta_0 + 3\bar{N}\beta_1 + 3\beta_2 \bar{N}^2 + \beta_3 \bar{N}^3\right) \geq 0. \tag{B.26}$$

This bound can be satisfied for different values of the parameters. One interesting case is when β_1 is the only non-zero parameter. In this case the bound becomes:

$$\left(\frac{1}{\bar{N}^2} - 1\right) \geq 0 \quad \Rightarrow \quad \bar{N} < 1, \tag{B.27}$$

which is actually satisfied, as in this β_1-only model, $\bar{N} = 1/\sqrt{3}$.

Finally, we have found that the Higuchi bound can be satisfied in the expanding branch for appropriate values of the parameters at early times and late times[1]. However, this does not guarantee instability-free solutions, as we could have tachyonic instabilities, which is what happens in this branch as described in Section 3.2, where growing exponential solutions were found.

Bouncing branch: In this branch we have $\beta_3 = \beta_2 = 0$ and $\beta_4 \neq 0$ with $\beta_1 \neq 0$. Here, $Z = \beta_1$. Using the Friedmann equation given by eq. (3.4), the bound (B.21) becomes:

$$m^2 N \beta_1 \left(1 + \frac{1}{N^2}\right) - \frac{2}{3}\left[\rho_0 + m^2\left(\beta_0 + 3N\beta_1\right)\right] \geq 0, \tag{B.28}$$

or equivalently, using the constraint (3.12),

$$N\beta_1 \left(1 + \frac{1}{N^2}\right) - \frac{2}{3}\left(\frac{\beta_1}{N} + \beta_4 N^2\right) \geq 0. \tag{B.29}$$

1. **Early times:** At early times, $N \gg 1$. Using eq. (B.29) and considering leading terms in N, the bound becomes:

$$m^2 N \beta_1 - \frac{2}{3} m^2 \beta_4 N^2 \geq 0 \tag{B.30}$$

$$\Rightarrow \quad \approx -\frac{2}{3} m^2 \beta_4 N^2 \geq 0, \tag{B.31}$$

which can only be satisfied if $\beta_4 < 0$, which is not viable as we would have negative energy density (see eq. (3.64)).

1. **Late times:** At late times we approach a de-Sitter spacetime where $\rho_0 \to 0$ and $N \to \bar{N}$, where $\bar{N}$ satisfies eq. (3.12) with $\rho_0 = 0$. Using eq. (B.28), the bound

[1] A more careful analysis is needed to check that the Higuchi bound is satisfied at all times, which will be left as future work.

becomes:

$$m^2\bar{N}\beta_1\left(1+\frac{1}{\bar{N}^2}\right)-\frac{2}{3}m^2(\beta_0+3\beta_1\bar{N})\geq 0 \tag{B.32}$$

$$\Rightarrow \quad \frac{\beta_1}{\bar{N}}\left(1-\bar{N}^2\right)-\frac{2}{3}\beta_0\geq 0. \tag{B.33}$$

For the interesting case of self-acceleration, where $\beta_0 = 0$, this bound is generically not satisfied as $\bar{N} \geq 1$ (see [111]). It can only be satisfied if $\beta_4 = 2\beta_1$, where $\bar{N} = 1$.

Finally, we have found that the Higuchi bound is not satisfied in the bouncing branch. This means that in the quadratic action for perturbations, the helicity-0 mode has a negative kinetic term, becoming a ghost-like degree of freedom. In this case, this does not translate into instabilities in the solutions as we found well-behaved solutions in Section 3.2. However, instabilities might appear when studying higher order classical perturbations or in semi-classical analyses.

B.2.2 Vector fields

According to [194], the Higuchi bound for the vector modes in the second branch of background solutions, satisfying $X\mathcal{H} = h$, is:

$$\tilde{m}^2 > 0. \tag{B.34}$$

For the relevant cases considered in this chapter $m^2 > 0$ and $N > 0$, so this condition becomes:

$$Z = \beta_1 + 2\beta_2 N + \beta_3 N^2 > 0. \tag{B.35}$$

Analogously to the scalar modes, we now consider the expanding and bouncing branches, and analyse the Higuchi bound in two relevant limit cases: early and late times.

Expanding branch: In this branch $\beta_1 > 0$.

Early times: Early times are characterised by $N \ll 1$. Then, in this regime eq. (B.35) becomes simply $Z \approx \beta_1$, which is satisfied.

Late times: At late times we approach a de-Sitter spacetime where $\rho_0 \to 0$ and $N \to \bar{N}$, where $\bar{N}$ satisfies eq. (3.12) with $\rho_0 = 0$. Condition (B.35) becomes:

$$\beta_1 + 2\beta_2\bar{N} + \beta_3\bar{N}^2 > 0, \tag{B.36}$$

which can be satisfied for appropriate values for βs. In particular, for the β_1-only model, this condition will be satisfied.

Bouncing branch: This branch is characterised for $\beta_2 = \beta_3 = 0$, and therefore $Z = \beta_1$. This means that at all times, the condition (B.34) is satisfied if $\beta_1 > 0$, which corresponds to the case considered in Subsection 3.3.2, as there it was shown that $\beta_1 < 0$ introduced exponential instabilities in scalar, vector and tensor modes, and therefore that case was ruled out.

Finally, we have found that the Higuchi bound for vector modes can be satisfied at early and late times for appropriate values of parameters in the expanding branch, while it is always satisfied in the bouncing branch.

APPENDIX C

Tools for testing gravity: Noether identities

C.1 3+1 decomposition

In this section we present the 3+1 decomposition of the metric used throughout Chapter 4. The spacetime metric $g_{\mu\nu}$ can be decomposed as follows:

$$g_{\mu\nu} = -n_\mu n_\nu + h_{\mu\nu}, \tag{C.1}$$

where n^μ is a time-like unit vector satisfying $n^\mu n^\nu g_{\mu\nu} = -1$. Note that this means that $h_{\mu\nu} n^\nu = 0$, and then $h_{\mu\nu}$ describes 3-dimensional space-like hypersurfaces normal to n^μ.

If we define the *lapse* and *shift* functions through:

$$n^0 = \frac{1}{N}, \tag{C.2}$$

$$n^i = -\frac{N^i}{N}, \tag{C.3}$$

then the metric components become:

$$g_{00} = -N^2 + h_{ij} N^i N^j, \tag{C.4}$$

$$g_{0i} = h_{ij} N^j, \tag{C.5}$$

$$g_{ij} = h_{ij}. \tag{C.6}$$

In this setting, we can construct the Ricci curvature for the 3-dimensional space, $R_{\mu\nu}$ in terms of h_{ij} and the corresponding three dimensional covariant derivatives, as well as the extrinsic curvature $K^{\mu}{}_{\nu}$:

$$K^{\mu}{}_{\nu} \equiv h_{\nu}{}^{\rho}\nabla_{\rho}n^{\mu}, \tag{C.7}$$

which satisfies

$$K^{\mu}{}_{\nu}n^{\nu} = K^{\mu}{}_{\nu}n_{\mu} = 0. \tag{C.8}$$

Specifically, in terms of the lapse and shift functions, the extrinsic curvature can be rewritten as:

$$K_{ij} = \frac{1}{2N}\left(\dot{h}_{ij} - D_i N_j - D_j N_i\right), \tag{C.9}$$

where $\dot{h} \equiv dh/dt$ and D_i denotes the covariant derivatives in the 3-dimensional space described by h_{ij}.

For completeness, we also show the Gauss-Codazzi relation, which relates the standard 4-dimensional curvature ${}^{(4)}R_{\mu\nu}$ to the 3-dimensional curvature $R_{\mu\nu}$:

$${}^{(4)}R = K_{\mu\nu}K^{\mu\nu} - K^2 + R + 2\nabla_{\mu}\left(Kn^{\mu} - n^{\rho}\nabla_{\rho}n^{\mu}\right), \tag{C.10}$$

or, alternatively, through

$$R_{\mu\nu} = h^{\rho}_{\mu}h^{\sigma}_{\nu}\left[{}^{(4)}R_{\sigma\rho} + n^{\alpha}n^{\beta(4)}R_{\rho\alpha\sigma\beta}\right] - KK_{\mu\nu} + K_{\mu\rho}K^{\rho}_{\nu}. \tag{C.11}$$

C.2 Scalar perturbations

In this section we show relevant quantities in terms of the four linear scalar perturbations of the metric.

Following the standard SVT decomposition, we consider linear perturbations around a FRW background, and write the metric components in terms of four scalar perturbation fields B, Φ, Ψ and E in the following way:

$$\begin{aligned} g_{00} &= -\left(1 + 2\Phi\right), \\ g_{0i} &= \partial_i B, \\ g_{ij} &= a^2\left[\left(1 - 2\Psi\right)\delta_{ij} + 2\partial_i\partial_j E\right], \end{aligned} \tag{C.12}$$

where a is the scale factor and depends only on the time t, while the four perturbations depend on time and space in general.

Using the expressions given in Appendix C.1 in the 3+1 decomposition, we can express all the relevant quantities used throughout Chapter 4 in terms of the four scalar metric fluctuations:

$$\begin{aligned}
\delta N &= \Phi, && \text{(C.13)}\\
\delta N^i &= \bar{h}^{ij}\partial_j B, && \text{(C.14)}\\
\delta h_{ij} &= a^2\left[-2\Psi\delta_{ij} + 2\partial_i\partial_j E\right], && \text{(C.15)}\\
\delta_2 N &= -\frac{1}{2}\Phi^2 + \frac{1}{2}\bar{h}^{ij}\partial_i B\partial_j B, && \text{(C.16)}\\
\delta\sqrt{|h|} &= a^3\left[-3\Psi - a^2\partial^2 E\right], && \text{(C.17)}\\
\delta_2\sqrt{|h|} &= a^3\left[\frac{3}{2}\Psi^2 - \frac{1}{2}a^4(\partial^2 E)(\partial^2 E) - a^2\Psi\partial^2 E\right], && \text{(C.18)}\\
\delta K^i{}_j &= -(\dot{\Psi} + H\Phi)\delta^i{}_j + a^2\bar{h}^{il}\partial_l\partial_j\dot{E} - \bar{h}^{il}\partial_l\partial_j B, && \text{(C.19)}\\
\delta K &= -3(\dot{\Psi} + H\Phi) + a^2\partial^2\dot{E} - \partial^2 B, && \text{(C.20)}\\
\delta R^i{}_j &= \delta^i{}_j\partial^2\Psi + \bar{h}^{il}\partial_l\partial_j\Psi, && \text{(C.21)}\\
\delta R &= 4\partial^2\Psi, && \text{(C.22)}\\
\delta_2 R &= 2\left[4\Psi\partial^2\Psi - \bar{h}^{ij}(\partial_i\Psi)(\partial_j\Psi)\right] - 4a^2\partial^2\Psi\partial^2 E, && \text{(C.23)}
\end{aligned}$$

where $\bar{h}^{ij} = \frac{\delta^{ij}}{a^2}$ represents the background spatial metric, and also $\partial^2 = \bar{h}^{ij}\partial_i\partial_j$. Here, a single δ stands for linear perturbations, while δ_2 stands for quadratic perturbations, which will be needed to calculate the second-order action.

C.3 New coefficients

The coefficients T_* in the quadratic action for the metric are related to the original coefficients (i.e. the functional derivatives of the gravitational Lagrangian, L_*) via:

$$T_{hh+} = L_{hh+} + \frac{1}{2}L_h \quad \text{(C.24)}$$

$$T_{hh\times} = L_{hh\times} + \frac{1}{2}L_h \quad \text{(C.25)}$$

$$\bar{T} = \bar{L} - 3HL_K - \dot{L}_K + 2L_h \quad \text{(C.26)}$$

$$T_{SS} = L_{SS} - 5HL_{S\dot{S}} - \dot{L}_{S\dot{S}} \quad \text{(C.27)}$$

$$T_{NN} = 2L_N + L_{NN} - \dot{L}_{\dot{N}} - 9HL_{\dot{N}} - 12HL_{\dot{N}h} - 3HL_{N\dot{N}} - \dot{L}_{N\dot{N}} \tag{C.28}$$

$$T_{Nh} = \frac{1}{2}L_h + L_{Nh} - \dot{L}_{\dot{N}h} - 3HL_{\dot{N}h} \tag{C.29}$$

$$T_N = \bar{L} + L_N - 3HL_{\dot{N}} - \dot{L}_{\dot{N}} - 3HL_K \tag{C.30}$$

$$T_{\partial Sh+} = L_{\partial Sh+} + \frac{1}{2}L_{\partial S} \tag{C.31}$$

$$T_{hR+} = L_{hR+} + \frac{1}{2}L_R \tag{C.32}$$

$$T_{NK} = L_{NK} - L_{\dot{N}} - 2L_{\dot{N}h} \tag{C.33}$$

$$T_{N\partial S} = L_{\partial S} - L_{\dot{N}} - 2L_{\dot{N}h} + L_{N\partial S} - L_{\partial NS} - 3HL_{\dot{N}\partial S} - \dot{L}_{\dot{N}\partial S} + 3HL_{\partial\dot{N}S} + \dot{L}_{\partial\dot{N}S} \tag{C.34}$$

$$T_{\partial\dot{S}h+} = \frac{1}{2}L_{\partial\dot{S}} + L_{\partial\dot{S}h+} \tag{C.35}$$

$$\begin{aligned} T_{\partial S\partial S+} &= L_{\partial S\partial S+} + L_{\partial S\partial S\times 1} - 6H\left(L_{\partial S\partial\dot{S}+} + L_{\partial S\partial\dot{S}\times 1}\right) - 2\left(\dot{L}_{\partial S\partial\dot{S}+} + \dot{L}_{\partial S\partial\dot{S}\times 1}\right) \\ &\quad - 4L_{S\partial^2 S\times 1} + 12H\left(L_{S\partial^2\dot{S}\times 1} + L_{\dot{S}\partial^2 S\times 1}\right) + 4\left(\dot{L}_{S\partial^2\dot{S}\times 1} + \dot{L}_{\dot{S}\partial^2 S\times 1}\right) \end{aligned} \tag{C.36}$$

$$\begin{aligned} T_{\partial S\partial S\times} &= L_{\partial S\partial S\times 2} + 6H(L_{S\partial^2\dot{S}\times 2} + L_{\dot{S}\partial^2 S\times 2} - L_{\partial S\partial\dot{S}\times 2}) - 2\dot{L}_{\partial S\partial\dot{S}\times 2} - 2L_{S\partial^2 S\times 2} \\ &\quad + 2\dot{L}_{S\partial^2\dot{S}\times 2} + 2\dot{L}_{\dot{S}\partial^2 S\times 2} \end{aligned} \tag{C.37}$$

$$\begin{aligned} T_{\partial N\partial N} &= -2L_{N\partial^2 N} + H(L_{N\partial^2\dot{N}} + L_{\dot{N}\partial^2 N} - L_{\partial N\partial\dot{N}}) + \dot{L}_{N\partial^2\dot{N}} + \dot{L}_{\dot{N}\partial^2 N} + L_{\partial N\partial N} \\ &\quad - \dot{L}_{\partial N\partial\dot{N}} - 2L_{\partial^2 N} + 7HL_{\partial^2\dot{N}} + \dot{L}_{\partial^2\dot{N}} \end{aligned} \tag{C.38}$$

$$2T_{h\partial^2 N+} = L_{\partial^2 N} - HL_{\partial^2\dot{N}} - \dot{L}_{\partial^2\dot{N}} + 2L_{h\partial^2 N+} \tag{C.39}$$

$$T_{NR} = L_R + L_{NR} \tag{C.40}$$

$$T_{N\partial\dot{S}} = L_{\partial\dot{S}} + L_{N\partial\dot{S}} - L_{\dot{N}\partial S} - L_{\partial N\dot{S}} + L_{\partial\dot{N}S} \tag{C.41}$$

$$T_{h\partial^3 S+} = L_{h\partial^3 S+} + \frac{1}{2}L_{\partial^3 S} \tag{C.42}$$

$$T_{K\partial^2 N+} = L_{K\partial^2 N+} - L_{\partial^2\dot{N}} \tag{C.43}$$

$$T_{\dot{N}\partial\dot{S}} = L_{\dot{N}\partial\dot{S}} - L_{\partial\dot{N}\dot{S}} \tag{C.44}$$

$$T_{\partial^2 N\partial S} = L_{\partial^3 SN} + L_{\partial^3 S} - L_{\partial^2\dot{N}} - L_{\partial N\partial^2 S} + L_{\partial^2 N\partial S} - L_{\partial^3 NS} \tag{C.45}$$

C.4 Background equations

In this section we show the derivation of the metric background equations of motion, for a spatially-flat FRW metric. We do this by calculating the Taylor expansion of the fundamental total action (gravity and matter) up to first order on the metric perturbation fields. Let us start by finding the linear terms in the expansion of the fundamental non-perturbed gravitational

Lagrangian L_G. From eq. (4.12) we find:

$$L_G^{(1)} = L_h\delta h + L_N\delta N + L_K\delta K + L_R\delta R + L_{\dot{N}}\delta\dot{N} + L_{\partial^2 N}\partial^2\delta N + L_{\partial S}\partial_i\delta N^i + L_{\partial\dot{S}}\partial_i\delta\dot{N}^i, \quad (C.46)$$

where δ stands for first-order perturbations only. Thus, the linear terms of the gravitational action S_{G} will be:

$$S_{\mathrm{G}}^{(1)} = \int d^4x\ \bar{L}\left(a^3\delta N + \delta\sqrt{|h|}\right) + a^3\left(L_h\delta h + L_N\delta N + L_{\dot{N}}\delta\dot{N} + L_K\delta K\right), \quad (C.47)$$

where $\bar{L} = L_G^{(0)}$, and we have eliminated many terms that formed a total derivative. Now we make use of the following relations:

$$\delta\sqrt{|h|} = \frac{1}{2}a^3\delta h; \quad \delta K = -3H\delta N + \frac{1}{2}\delta\dot{h} - \partial_i\delta N^i, \quad (C.48)$$

to rewrite the linear action as:

$$\begin{aligned} S_{\mathrm{G}}^{(1)} = \int d^4x\ a^3 \Big[&\delta N\left(\bar{L} + L_N - 3HL_{\dot{N}} - \dot{L}_{\dot{N}} - 3HL_K\right) \\ &+ \frac{1}{2}\delta h\left(\bar{L} + 2L_h - 3HL_K - \dot{L}_K\right)\Big]. \end{aligned} \quad (C.49)$$

Now we proceed to find the linear terms from some matter action S_{M}. If we consider as matter a general perfect fluid with a stress-energy tensor $T^{\mu\nu}$, the linear expansion leads to:

$$S_{\mathrm{M}}^{(1)} = \frac{1}{2}\int d^4x\ a^3\bar{T}^{\mu\nu}\delta g_{\mu\nu} = \int d^4x\ a^3\left(-\rho_0\delta N + \frac{P_0}{2}\delta h\right), \quad (C.50)$$

where $\bar{T}^{\mu\nu}$ is the diagonal background stress-energy tensor for the fluid with rest-energy density ρ_0 and pressure P_0. Notice that here we have also ignored terms that formed total derivatives.

Finally, the total first-order action will be:

$$\begin{aligned} S_{\mathrm{G}}^{(1)} + S_{\mathrm{M}}^{(1)} = \int d^4x\ a^3 \Big[&\delta N\left(\bar{L} + L_N - 3HL_{\dot{N}} - \dot{L}_{\dot{N}} - 3HL_K - \rho_0\right) \\ &+ \frac{1}{2}\delta h\left(\bar{L} + 2L_h - 3HL_K - \dot{L}_K + P_0\right)\Big]. \end{aligned} \quad (C.51)$$

Now we notice that the equations of motion of the perturbation fields will have zeroth-order terms coming from the total linear action, and first-order terms coming from the total quadratic action. Since the resulting equations of motion must be satisfied order-by-order, we will have, in particular, that the total contribution from zeroth-order terms will vanish. Therefore, both brackets in eq. (C.51) must be zero, leading to the following two metric background equations:

$$\begin{aligned}\bar{L} + L_N - 3HL_{\dot{N}} - \dot{L}_{\dot{N}} - 3HL_K &= \rho_0 \\ \bar{L} + 2L_h - 3HL_K - \dot{L}_K &= -P_0.\end{aligned} \tag{C.52}$$

Notice that the total linear action will always be zero then, given the background equations.

APPENDIX D

A general theory of linear cosmological perturbations: scalar-tensor and vector-tensor theories

D.1 New coefficients for scalar-tensor action

The coefficients in the quadratic action of the scalar field χ are related to the original coefficients (i.e. the derivatives of the Lagrangian, L_*) via

$$T_{\chi\chi} = L_{\chi\chi} - 3HL_{\chi\dot{\chi}} - \dot{L}_{\chi\dot{\chi}} \tag{D.1}$$

$$T_{\chi h} = (2L_{\chi h} + L_{\chi})/2 \tag{D.2}$$

$$T_{\chi N} = L_{\chi} + L_{\chi N} - 3HL_{\chi\dot{N}} - \dot{L}_{\chi\dot{N}} \tag{D.3}$$

$$T_{\dot{\chi}h} = \frac{1}{2}L_{\dot{\chi}} + L_{\dot{\chi}h} \tag{D.4}$$

$$T_{\chi\partial S} = L_{\chi\partial S} - 3HL_{\chi\partial\dot{S}} - \dot{L}_{\chi\partial\dot{S}} - L_{S\partial\chi} + 3HL_{\partial\chi\dot{S}} + \dot{L}_{\partial\chi\dot{S}} \tag{D.5}$$

$$T_{\dot{\chi}N} = L_{\dot{\chi}} + L_{\dot{\chi}N} - L_{\chi\dot{N}} \tag{D.6}$$

$$T_{\partial^2\chi h+} = \frac{1}{2}L_{\partial^2\chi} + L_{\partial^2\chi h+} \tag{D.7}$$

$$T_{\partial\chi\partial\chi} = L_{\partial\chi\partial\chi} - 2L_{\chi\partial^2\chi} + HL_{\chi\partial^2\dot{\chi}} + \dot{L}_{\chi\partial^2\dot{\chi}} + HL_{\partial^2\chi\dot{\chi}} + \dot{L}_{\partial^2\chi\dot{\chi}} - HL_{\partial\chi\partial\dot{\chi}} - \dot{L}_{\partial\chi\partial\dot{\chi}} \tag{D.8}$$

$$T_{\dot{\chi}\partial S} = L_{\dot{\chi}\partial S} - L_{\chi\partial\dot{S}} + L_{\partial\chi\dot{S}} - L_{\partial\dot{\chi}S} \tag{D.9}$$

$$\begin{aligned} T_{\partial\chi\partial N} = {} & -L_{\partial^2\chi} - L_{\chi\partial^2 N} + HL_{\chi\partial^2\dot{N}} + \dot{L}_{\chi\partial^2\dot{N}} + L_{\partial\chi\partial N} - HL_{\partial\chi\partial\dot{N}} - \dot{L}_{\partial\chi\partial\dot{N}} \\ & - L_{\partial^2\chi N} + HL_{\partial^2\chi\dot{N}} + \dot{L}_{\partial^2\chi\dot{N}} \end{aligned} \tag{D.10}$$

$$T_{\partial^2\dot{\chi}h+} = L_{\partial^2\dot{\chi}h+} + \frac{1}{2}L_{\partial^2\dot{\chi}} \tag{D.11}$$

$$T_{\partial^2\dot{\chi}N} = L_{\partial^2\dot{\chi}} - L_{\chi\partial^2\dot{N}} + L_{\dot{\chi}\partial^2 N} + L_{\partial\chi\partial\dot{N}} - L_{\partial\dot{\chi}\partial N} - L_{\partial^2\chi\dot{N}} + L_{\partial^2\dot{\chi}N} \tag{D.12}$$

$$T_{\dot{\chi}\partial\dot{S}} = L_{\dot{\chi}\partial\dot{S}} - L_{\partial\dot{\chi}\dot{S}} \tag{D.13}$$

$$T_{\partial^2\chi\partial S} = L_{\chi\partial^3 S} - L_{\partial\chi\partial^2 S} + L_{\partial^2\chi\partial S} - L_{\partial^3\chi S} \tag{D.14}$$

D.2 Fourth order action for scalar-tensor theories

In this section we show the most general quadratic Lagrangians of a scalar-tensor theory involving four derivatives of the perturbation fields (at most two time derivatives, though).

$$\begin{aligned}
\mathcal{L}_T^4 = a^3 \Bigg[& \frac{1}{2} L_{RR+} (\delta R)^2 + L_{RR\times} \delta R^i_j \delta R^j_i + L_{R\partial \dot{S}+} \delta R \partial_i \delta \dot{N}^i + L_{R\partial \dot{S}\times} \delta R^i_j \partial_i \delta \dot{N}^j \\
& + T_{\partial \dot{S} \partial \dot{S}+} \partial_i \delta \dot{N}^i \partial_j \delta \dot{N}^j + T_{\partial \dot{S} \partial \dot{S}\times} \bar{h}_{ik} \partial^j \delta \dot{N}^k \partial_j \delta \dot{N}^i + T_{\partial^2 N \partial \dot{S}} \partial^2 \delta N \partial_j \delta \dot{N}^j \\
& + T_{h\partial^3 \dot{S}+} \delta h \partial^2 \partial_i \delta \dot{N}^i + L_{h\partial^3 \dot{S}\times} \delta h_{ij} \partial^2 \partial^j \dot{N}^i + L_{h\partial^3 \dot{S}\odot} \delta h_{ij} \partial^i \partial^j \partial_k \delta \dot{N}^k \\
& + \frac{1}{2} T_{\partial^2 N \partial^2 N} \partial^2 \delta N \partial^2 \delta N + T_{\partial^2 S \partial^2 S \times} \bar{h}_{ij} \partial^2 \delta N^i \partial^2 \delta N^j + T_{\partial^2 S \partial^2 S+} \partial_i \partial_k \delta N^i \partial^k \partial_j \delta N^j \\
& + T_{h\partial^4 N+} \delta h \partial^4 \delta N + L_{h\partial^4 N\times} \delta h_{ij} \partial^i \partial^j \partial^2 \delta N + L_{K\partial^3 S+} \delta K \partial^2 \partial_j \delta N^j \\
& + L_{K\partial^3 S\odot} \delta K^i_j \partial^j \partial_i \partial_k \delta N^k + L_{K\partial^3 S\times} \delta K^i_j \partial^2 \partial_i \delta N^j + L_{K\partial^2 \dot{N}+} \delta K \partial^2 \delta \dot{N} \\
& + L_{K\partial^2 \dot{N}\times} \delta K^i_j \partial^j \partial_i \delta \dot{N} + \frac{1}{2} L_{\partial \dot{N} \partial \dot{N}} \partial_i \delta \dot{N} \partial^i \delta \dot{N} \Bigg] ,
\end{aligned} \tag{D.15}$$

$$\begin{aligned}
\mathcal{L}_\chi^4 = a^3 \Bigg[& L_{\partial^2 \chi R+} \delta R \partial^2 \delta \chi + 2 L_{\partial^2 \chi R\times} \delta R^i_j \partial^j \partial_i \delta \chi + T_{\partial^4 \chi h+} \delta h \partial^4 \delta \chi + \frac{1}{2} T_{\partial^2 \chi \partial^2 \chi} (\partial^2 \delta \chi)^2 \\
& + 4 L_{\partial^4 \chi h\times} \delta h_{ij} \partial^i \partial^j \partial^2 \delta \chi + \frac{1}{2} T_{\partial \dot{\chi} \partial \dot{\chi}} \partial_i \delta \dot{\chi} \partial^i \delta \dot{\chi} + T_{\partial^2 N \partial^2 \chi} \partial^2 \delta N \partial^2 \delta \chi \\
& + T_{\partial^2 \chi \partial \dot{S}} \partial^2 \delta \chi \partial_i \delta \dot{N}^i + T_{\partial \dot{\chi} \partial \dot{N}} \partial_i \delta \dot{\chi} \partial^i \delta \dot{N} + L_{K\partial^2 \dot{\chi}+} \delta K \partial^2 \delta \dot{\chi} + L_{K\partial^2 \dot{\chi}\times} \delta K^i_j \partial^j \partial_i \delta \dot{\chi} \Bigg] ,
\end{aligned} \tag{D.16}$$

where we have made integrations by parts and grouped some coefficients L_* together into new coefficients T_*, for simplicity.

D.3 New coefficients for vector-tensor action

The coefficients in the quadratic action for the vector field are related to the original coefficients (i.e. the derivatives of the Lagrangian, L_*) via:

$$T_{\alpha S} = L_{\alpha S} - 3H L_{\alpha \dot{S}} - \dot{L}_{\alpha \dot{S}}, \tag{D.17}$$

$$T_{\partial \alpha h+} = L_{\partial \alpha h+} + \frac{1}{2} L_{\partial \alpha}, \tag{D.18}$$

$$T_{\alpha \alpha} = L_{\alpha \alpha} - H L_{\alpha \dot{\alpha}} - \dot{L}_{\alpha \dot{\alpha}}, \tag{D.19}$$

$$T_{\dot{\alpha} S} = L_{\dot{\alpha} S} - L_{\alpha \dot{S}}, \tag{D.20}$$

$$T_{\alpha\partial N} = L_{\alpha\partial N} - L_{\partial\alpha} - HL_{\alpha\partial\dot{N}} - \dot{L}_{\alpha\partial\dot{N}} - L_{\partial\alpha N} + HL_{\partial\alpha\dot{N}} + \dot{L}_{\partial\alpha\dot{N}}, \tag{D.21}$$

$$T_{\partial\dot{\alpha}h+} = L_{\partial\dot{\alpha}h+} + \frac{1}{2}L_{\partial\dot{\alpha}}, \tag{D.22}$$

$$T_{\partial\dot{\alpha}N} = L_{\partial\dot{\alpha}} + L_{\alpha\partial\dot{N}} - L_{\dot{\alpha}\partial N} - L_{\partial\alpha\dot{N}} + L_{\partial\dot{\alpha}N}, \tag{D.23}$$

$$T_{\partial\alpha\partial S} = L_{\partial\alpha\partial S} - L_{\alpha\partial^2 S} - L_{\partial^2\alpha S}, \tag{D.24}$$

$$T_{\alpha\partial\alpha^0} = L_{\alpha\partial\alpha^0} - L_{\partial\alpha\alpha^0}, \tag{D.25}$$

$$T_{\alpha\partial\dot{\alpha}^0} = L_{\alpha\partial\dot{\alpha}^0} - L_{\partial\alpha\dot{\alpha}^0} - H\left(L_{\dot{\alpha}\partial\alpha^0} - L_{\partial\dot{\alpha}\alpha^0}\right) - \dot{L}_{\dot{\alpha}\partial\alpha^0} + \dot{L}_{\partial\dot{\alpha}\alpha^0}, \tag{D.26}$$

$$T_{\partial\alpha\partial\alpha+} = L_{\partial\alpha\partial\alpha+} + L_{\partial\alpha\partial\alpha\times} \tag{D.27}$$

APPENDIX E

A general theory of linear cosmological perturbations: bimetric theories

E.1 Scalar perturbations

In this section we show relevant quantities in terms of the four linear scalar perturbations of each metric. Following the standard SVT decomposition, we consider linear perturbations around a homogeneous and isotropic background, and write the metrics in the following way:

$$ds_1^2 = -\left(1+\Phi_1\right)dt^2 + 2\partial_i B_1 dt dx^i + a^2\left[\left(1-2\Psi_1\right)\delta_{ij} + 2\partial_i\partial_j E_1\right]dx^i dx^j, \tag{E.1}$$

$$ds_2^2 = -\bar{N}^2\left(1+\Phi_2\right)dt^2 + 2\bar{N}\partial_i B_2 dt dx^i + b^2\left[\left(1-2\Psi_2\right)\delta_{ij} + 2\partial_i\partial_j E_2\right]dx^i dx^j, \tag{E.2}$$

where a, b and $\bar{N}$ are background quantities and depend only on the time t, whereas the 8 perturbations Φ_A, B_A, Ψ_A and E_A (for $A=\{1,2\}$) depend on time and space. From the ADM decomposition of eq. (6.6), we can find the results of eq. (6.20)-(6.21), and also express all the relevant quantities used throughout Chapter 6 in terms of the scalar metric perturbations. Here we give a list of quantities, that appear in the quadratic gravitational action $S_G^{(2)}$, in terms of the scalar perturbations:

$$\begin{aligned}
\delta_2 N_1 &= -\frac{1}{2}\left(\Phi_1^2 + \bar{h}_1^{ij}\partial_i B_1\partial_j B_1\right),\\
\delta\sqrt{h_1} &= a^3\left[-3\Psi_1 - a^2\partial^2 E_1\right],\\
\delta_2\sqrt{h_1} &= a^3\left[\frac{3}{2}\Psi_1^2 - \frac{1}{2}a^4(\partial^2 E_1)(\partial^2 E_1) - a^2\Psi_1\partial^2 E_1\right],\\
\delta K^i{}_{1j} &= -(\dot{\Psi}_1 + H\Phi_1)\delta^i{}_j + a^2\bar{h}_1^{il}\partial_l\partial_j\dot{E}_1 - \bar{h}_1^{il}\partial_l\partial_j B_1,\\
\delta K_1 &= -3(\dot{\Psi}_1 + H\Phi_1) + a^2\partial^2\dot{E}_1 - \partial^2 B_1,
\end{aligned} \tag{E.3}$$

$$
\begin{aligned}
\delta_2 N_2 &= -\frac{\bar{N}}{2}\left(\Phi_2^2 + \bar{h}_2^{ij}\partial_i B_2 \partial_j B_2\right), \\
\delta\sqrt{h_2} &= b^3\left[-3\Psi_2 - b^2\partial^2 E_2\right], \\
\delta_2\sqrt{h_2} &= b^3\left[\frac{3}{2}\Psi_2^2 - \frac{1}{2}b^4(\partial^2 E_2)(\partial^2 E_2) - b^2\Psi_2\partial^2 E_2\right], \\
\delta K^i_{2\ j} &= -\frac{1}{\bar{N}}\left(\dot{\Psi}_2 + H_b\Phi_2\right)\delta^i{}_j + \frac{b^2}{\bar{N}}\bar{h}_2^{il}\partial_l\partial_j \dot{E}_2 - \bar{h}_2^{il}\partial_l\partial_j B_2, \\
\delta K_2 &= -\frac{3}{\bar{N}}(\dot{\Psi}_2 + H_b\Phi_2) + \frac{b^2}{\bar{N}}\partial^2\dot{E}_2 - \partial^2 B_2,
\end{aligned}
\tag{E.4}
$$

where $\bar{h}_1^{ij} = \frac{\delta^{ij}}{a^2}$ and $\bar{h}_2^{ij} = \frac{\delta^{ij}}{b^2}$ represent the background spatial metrics, and $\partial^2 = \partial^i\partial_i$, where the indices are lowered and raised using the background metric of the corresponding field the derivative is acting on. Also, a single δ stands for linear perturbations, while δ_2 stands for quadratic perturbations.

E.2 Dictionary of parameters

In this section we give expressions for the 29 parameters the final bimetric action depends on. The names we have given to those parameters are the following:

$$
M_A^2,\ \alpha_{H_A},\ \alpha_{T_A},\ \alpha_{B_A},\ \alpha_{K_A},\ \alpha_L,\ \alpha_E,\ \alpha_{i,A},\ \alpha_j, \tag{E.5}
$$

where $A = \{1,2\}$, $i = \{1..4\}$ and $j = \{5..13\}$. In terms of the original coefficients Ls and Ps these parameters can be expressed as:

$$
M_A^2 = 2L_{AKK\times}, \tag{E.6}
$$

$$
\alpha_{H_A} = -\frac{1}{M_A^2}\left(2L_{Ah\partial^2 N\times} + 3L_{Ah\partial^2 N+}\right) - 1, \tag{E.7}
$$

$$
\alpha_{T_A} = -\frac{4}{M_A^2}\left(L_{Ah\partial^2 h\odot} + 3L_{Ah\partial^2 h\times} + 9L_{Ah\partial^2 h+}\right) - 1, \tag{E.8}
$$

$$
\alpha_{B_A} = \frac{1}{2H_A M_A^2}\left[H_{AN}L_{AK\dot{N}} + T_{ANK} - \frac{H_A}{\bar{N}_A^2}(2L_{AKK\times} + 3L_{AKK+})\right] - 1, \tag{E.9}
$$

$$
\begin{aligned}
\alpha_{K_A} = \frac{1}{M_A^2 H_A^2 \bar{N}_A^2}&\left[\bar{N}_A^4 L_{ANN} - 9H_A^2\left(L_{AKK+} + 2L_{AKK\times}\right)\right. \\
&\left.-12H_A^2\left(\bar{N}_A^2 - 1\right)L_{AKK\times} + \bar{N}_A^3 H_{AN} L_{A\dot{N}\dot{N}}\right],
\end{aligned}
\tag{E.10}
$$

$$
\alpha_L = -\frac{1}{2M_1^2 H^2}\left(\bar{L}_1 - 4L_{1hh+} - 8L_{1hh\times}\right), \tag{E.11}
$$

$$
\alpha_E = -\frac{2}{M_2^2}\left[\bar{N}^2 T_{2Nh} - H_b\left(2L_{2hK\times} + 3L_{2hK+}\right)\right], \tag{E.12}
$$

$$\alpha_{1,A} = -\frac{1}{M_A^2} L_{AK\dot{N}}, \tag{E.13}$$

$$\alpha_{2,A} = -\frac{1}{M_A^2} \left(L_{Ah\partial^2 h\odot} + 2L_{Ah\partial^2 h\times} + 3L_{Ah\partial^2 h+} \right), \tag{E.14}$$

$$\alpha_{3,A} = \frac{\bar{N}_A}{M_A^2} \left(2L_{Ah\partial^2 N\times} + L_{Ah\partial^2 N+} \right), \tag{E.15}$$

$$\alpha_{4,A} = \frac{1}{M_A^2} L_{A\partial N\partial N}, \tag{E.16}$$

$$\alpha_5 = \frac{1}{M_1^2} L_{1\dot{S}\dot{S}} = \frac{\bar{N} r^5}{M_1^2} L_{2\dot{S}\dot{S}} = -\frac{1}{M_1^2 a^3} P_{\dot{S}_1 \dot{S}_2}, \tag{E.17}$$

$$\alpha_6 = \frac{1}{M_1^2} L_{1\dot{N}\dot{N}} = \frac{\bar{N}^3 r^3}{M_1^2} L_{2\dot{N}\dot{N}} = -\frac{\bar{N}}{M_1^2 a^3} P_{\dot{N}_1 \dot{N}_2}, \tag{E.18}$$

$$\alpha_7 = -\frac{2}{M_1^2} \left(L_{1\partial\dot{S}h+} + 2L_{1\partial\dot{S}h\times} \right) + \frac{1}{M_1^2 H} \left(L_{1\partial Sh+} + 2L_{1\partial Sh\times} - L_{1hK+} - 2L_{1hK\times} \right), \tag{E.19}$$

$$\alpha_8 = \frac{1}{M_1^2 a^3} (r^2 P_{h_2\partial^2 N_1\times} + 3P_{h_2\partial^2 N_1+}), \tag{E.20}$$

$$\alpha_9 = \frac{1}{M_1^2 a^3} P_{h1\partial^2 N2+}, \tag{E.21}$$

$$\alpha_{10} = \frac{1}{M_1^2} L_{1h\partial^2 h\times}, \tag{E.22}$$

$$\alpha_{11} = \frac{1}{M_1^2 a^3} P_{K_1 K_2+}, \tag{E.23}$$

$$\alpha_{12} = \frac{1}{M_1^2 a^3} \left(3P_{K_1 K_2+} + P_{K_1 K_2\times} \right), \tag{E.24}$$

$$\alpha_{13} = \frac{1}{M_1^2 a^3} \left[r^6 \left(P_{h_2\partial^2 h_1\odot} + 3P_{h_2\partial^2 h_1\times 2} \right) + 9r^4 P_{h_2\partial^2 h_1+} + 3P_{h_2\partial^2 h_1\times 1} \right], \tag{E.25}$$

where we have defined $\bar{N}_A$ such that $\bar{N}_1 = 1$ and $\bar{N}_2 = \bar{N}$, thus $H_{1N} = 0$, $H_{1N} = H_N$, and also $H_1 = H$ and $H_2 = H_b$. Here, we have also introduced the ratio of the scale factors $r = b/a$. The parameters M_A have mass dimensions and appear multiplying the whole quadratic action, whereas all the parameters αs are dimensionless and are the couplings coefficients of the different interactions terms for the fields. For instance, the parameters α_{H_A}, α_{T_A}, α_{B_A}, and α_{K_A} determine the interactions terms given by $\Psi_A \partial^2 \Phi_A$, $\Psi_A \partial^2 \Psi_A$, $\Phi_A \dot{\Psi}_A$ and Φ_A^2, respectively. These parameters are generalisations of those present in the parametrisation of dark energy models of [161].

Finally, we comment on the fact that these 29 parameters can have different expressions in terms of the functions Ls and Ps, if we use relations between them given by the Noether constraints. This is why parameters α_5 and α_6 have three equivalent expressions in equations (E.17) and (E.18). For these specific parameters, these three expressions show that the dynamical terms of the fields B_1 with B_2 and Φ_1 with Φ_2 are related to each other, signalling the fact that the dynamical terms of these fields appear in a specific combination in the action, and thus there is a field redefinition such as eq. (6.26), that can make two fields appear as auxiliary fields (without time derivatives) instead of dynamical fields.

E.3 Complete action without derivative interactions

If we avoid derivative interactions between both metrics, then 26 of the 29 previous parameters are fixed. Specifically, we find that the following 26 constraints:

$$M_2^2 = -\frac{\bar{N}}{2r^3}\frac{\left(2\dot{H}M_1^2 + \rho_0 + P_0\right)}{\left(\dot{H}_b - H_N H_b\right)}, \tag{E.26}$$

$$\alpha_{H_1} = 0,\ \alpha_{H_2} = -\frac{(\bar{N}-1)}{\bar{N}}, \tag{E.27}$$

$$\alpha_{T_A} = \frac{d\ln M_A^2}{d\ln a_A}, \tag{E.28}$$

$$\alpha_{B_1} = 0,\ \alpha_{B_2} = -\frac{\left(\bar{N}-1\right)\left(\bar{N}+1\right)}{\bar{N}^2}, \tag{E.29}$$

$$\alpha_{K_1} = \frac{6H_b\dot{H}}{H_N H^2} + \frac{1}{2H^2}(2\rho_0 + P_0) + \frac{3}{2}(\rho_0 + P_0)\frac{H_b}{H_N H^2} - 3\alpha_E\frac{(H - H_b)}{\nu^2\bar{N}H_N H^2}, \tag{E.30}$$

$$\alpha_{K_2} = -\frac{6}{\bar{N}^2 H_N H_b}\left[H_b H_N\left(\bar{N}^2 - 2\right) + \dot{H}_b\right] - 3\alpha_E\frac{(H - H_b)}{\bar{N}H_N H_b^2}, \tag{E.31}$$

$$\alpha_{1,A} = \alpha_{2,A} = \alpha_{3,A} = \alpha_{4,A} = 0, \tag{E.32}$$

$$\alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = \alpha_9 = 0, \tag{E.33}$$

$$\alpha_{10} = \frac{1}{8}\left(\frac{d\ln M_1^2}{d\ln a} + 1\right), \tag{E.34}$$

$$\alpha_{11} = \alpha_{12} = \alpha_{13} = 0, \tag{E.35}$$

where we have defined the scale factor a_A such that $a_1 = a$ and $a_2 = b$, and we have introduced the ratio of the mass scales $\nu^2 = M_1^2/M_2^2$. We can see that the quadratic bimetric action without derivative interactions depends only on three independent free parameters: α_E, α_L and M_1, in addition to the four background functions a, b, $\bar{N}$ and $\bar{\varphi}$, which give three additional independent free functions, due to the background equation (4.28).

The resulting quadratic action can be written as:

$$S^{(2)} = S^{(2)}_{T_1} + S^{(2)}_{T_2} + S^{(2)}_{T_1T_2} + S^{(2)}_{\varphi}, \tag{E.36}$$

where $S^{(2)}_{T_A}$ is the action for the self-interaction terms of the metric $g_{A\mu\nu}$, whereas $S^{(2)}_{T_1T_2}$ is the action for the interaction terms between both metrics, and $S^{(2)}_{\varphi}$ includes all the terms involving the matter perturbation $\delta\varphi$ in eq. (4.34). Notice that $S^{(2)}_{T_1}$ does include the quadratic terms of the metric perturbations coming from the matter action $S^{(2)}_m$. These actions are given by:

$$
\begin{aligned}
S^{(2)}_{T_1} = \int d^4x\, a^3 M_1^2 \Big[& -3\dot{\Psi}_1^2 - 6H\dot{\Psi}_1\Phi_1 + 2a^2\partial^2\dot{E}_1(\dot{\Psi}_1 + H\Phi_1) - 2\dot{\Psi}_1\partial^2 B_1 \\
& - \left(1 + \frac{d\ln M_1^2}{d\ln a}\right)\Psi_1\partial^2\Psi_1 + 2\Phi_1\partial^2\Psi_1 - 2H\Phi_1\partial^2 B_1 - \left(3H^2 - \frac{\dot{\varphi}^2}{2M_1^2}\right)\Phi_1^2 \\
& + \frac{r^2 Z}{2(\bar{N}+r)}\left(\partial^i B_1\right)\left(\partial_i B_1\right) + r\left(Z\frac{d\ln M_1^2}{d\ln a} + 2\tilde{Z}\right)\Psi_1\left(\frac{3}{2}\Psi_1 - a^2\partial^2 E_1\right) \\
& + \frac{(\bar{N}-r)}{H-H_b} HZ\Phi_1\left(3\Psi_1 - a^2\partial^2 E_1\right) + a^4\alpha_L H^2\left(\partial^2 E_1\right)^2 \\
& + \left(H_b\left(\bar{N}-r\right)Z - \frac{(H-H_b)}{\nu^2} r^3\alpha_E\right)\left(\frac{3}{2}\frac{1}{H_N}\Phi_1^2 + \frac{H_N}{(H-H_b)^2}\Psi_1\left(\frac{3}{2}\Psi_1 - a^2\partial^2 E_1\right)\right. \\
& \left.\left. - \frac{1}{(H-H_b)}\Phi_1\left(3\Psi_1 - a^2\partial^2 E_1\right)\right)\right], \qquad \text{(E.37)}
\end{aligned}
$$

$$
\begin{aligned}
S^{(2)}_{T_2} = \int d^4x\, \bar{N} b^3 M_2^2 \Bigg[& -3\frac{\dot{\Psi}_2^2}{\bar{N}^2} - 6\frac{H_b}{\bar{N}^2}\dot{\Psi}_2\Phi_2 + 2\frac{b^2}{\bar{N}^2}\partial^2\dot{E}_2(\dot{\Psi}_2 + H_b\Phi_2) - 2\frac{\dot{\Psi}_2}{\bar{N}}\partial^2 B_2 \\
& - \left(1 + \frac{d\ln M_2^2}{d\ln b}\right)\Psi_2\partial^2\Psi_2 + 2\Phi_2\partial^2\Psi_2 - 2\frac{H_b}{\bar{N}}\Phi_2\partial^2 B_2 - 3\frac{H_b^2}{\bar{N}^2}\Phi_2^2 \\
& + \frac{\nu^2\bar{N}Z}{2r^3(\bar{N}+r)}\left(\partial^i B_2\right)\left(\partial_i B_2\right) + \frac{\nu^2}{\bar{N}r^2}\left(Z\frac{d\ln M_1^2}{d\ln a} + 2\tilde{Z}\right)\Psi_2\left(\frac{3}{2}\Psi_2 - b^2\partial^2 E_2\right) \\
& + H_b\nu^2 Z\frac{(\bar{N}-r)}{\bar{N}r^3(H-H_b)}\Phi_2\left(3\Psi_2 - b^2\partial^2 E_2\right) + \frac{r}{\bar{N}}\alpha_L a^4 H^2\nu^2\left(\partial^2 E_2\right)^2 \\
& + \left(H_b\left(\bar{N}-r\right)Z - \frac{(H-H_b)}{\nu^2} r^3\alpha_E\right)\left(\frac{3}{2}\frac{1}{H_N}\Phi_2^2 + \frac{H_N}{(H-H_b)^2}\Psi_2\left(\frac{3}{2}\Psi_2 - b^2\partial^2 E_2\right)\right. \\
& \left. - \frac{1}{(H-H_b)}\Phi_2\left(3\Psi_2 - b^2\partial^2 E_2\right)\right)\frac{1}{r^3\nu^2\bar{N}}\Bigg], \qquad \text{(E.38)}
\end{aligned}
$$

$$
\begin{aligned}
S^{(2)}_{T_1T_2} = \int d^4x\, M_1^2 a^3 \Bigg[& -r\left(2\tilde{Z} + Z\frac{d\ln M_1^2}{d\ln a}\right)\left(3\Psi_2\Psi_1 - a^2\Psi_2\partial^2 E_1 - b^2\Psi_1\partial^2 E_2\right) \\
& - Z\frac{\bar{N}}{(\bar{N}+r)}\partial_i B_2\partial^i B_1 - Z\frac{(\bar{N}-r)}{(H-H_b)}\left(H\Phi_1\left(3\Psi_2 - b^2\partial^2 E_2\right)\right. \\
& \left. + H_b\Phi_2\left(3\Psi_1 - a^2\partial^2 E_1\right)\right) - 2a^2b^2H^2\alpha_L\partial^2 E_1\partial^2 E_2 \\
& + \left(H_b\left(\bar{N}-r\right)Z - \frac{(H-H_b)}{\nu^2} r^3\alpha_E\right)\left(\frac{3}{H_N}\Phi_1\Phi_2 + (H_b - H)\left(\Phi_1\left(3\Psi_2 - b^2\partial^2 E_2\right)\right.\right. \\
& \left.\left.\left. + \Phi_2\left(3\Psi_1 - a^2\partial^2 E_1\right)\right) + H_N\left(3\Psi_2\Psi_1 - a^2\Psi_2\partial^2 E_1 - b^2\Psi_1\partial^2 E_2\right)\right)\frac{1}{(H_b - H)^2}\right]. \qquad \text{(E.39)}
\end{aligned}
$$

Here, we have two mass scales for each metric M_1^2 and M_2^2, and we have introduced the mass ratio $\nu^2 = M_1^2/M_2^2$, and the scale factor ratio $r = b/a$. In addition, for ease of comparison with massive bigravity, we have introduced two functions Z and $\tilde{Z}$ such that:

$$M_1^2\left(\bar{N}-r\right)Z = \hat{L}_{1KK\times}, \tag{E.40}$$

$$rM_1^2\left(Z\frac{1}{2}\frac{d\ln M_1^2}{d\ln a}+2\tilde{Z}\right) = \frac{1}{(H_b-H)}\left(3H+\frac{H_N H_b}{(H-H_b)}\right)\hat{L}_{1KK\times}+\frac{\dot{\hat{L}}_{1KK\times}}{(H_b-H)}, \tag{E.41}$$

where

$$\hat{L}_{1KK\times} = \rho_0 + P_0 + 2\dot{H}M_1^2. \tag{E.42}$$

We omit the expression for $S_\varphi^{(2)}$ as it can be straightforwardly obtained from eq. (4.34). We emphasise that the total parametrised action depends on $3+3$ free independent functions of time. There is a dependence on six parameters M_A, Z, $\tilde{Z}$, α_L and α_E, in addition to the four background functions a, b, $\bar{N}$ and $\bar{\varphi}$. However, Z and $\tilde{Z}$ are dependent functions according to eq. (E.40)-(E.41), one background function is dependent through eq. (4.28), and M_2 is also dependent through eq. (E.26). Equivalently, we can consider the six independent parameters to be M_A, Z, $\tilde{Z}$, α_L and α_E, while the background functions would be dependent.

From the parametrised action shown here we can see that the fields Φ_A and B_A appear as auxiliary variables, whereas the fields Ψ_A and E_A have dynamical terms. This means that, in general, this action propagates two physical scalar fields. Nevertheless, there are some trivial cases in which no scalar is propagated. This happens if either M_1 or M_2 vanishes, and thus all the dynamical terms of one of the metrics vanish (although these metrics can still have non-derivative terms as long as the quantities ZM_A^2, $\tilde{Z}M_A^2$, $\alpha_L M_A^2$ or $\alpha_E M_A^2$ are finite). As it is shown in Section 6.2, Eddington-inspired Born Infeld theory is an example of a gravity model with $M_1 = 0$ that does not propagate any scalar DoF. On the other hand, there are also cases that propagate only one scalar DoF, such as massive bigravity, which is presented in Section 6.2.

E.4 The effect of derivative interactions

As it is shown in Section 6.2, when there are no derivative interactions between the metrics, the kinetic terms of each metric correspond to linearized Einstein-Hilbert with a generalised Planck mass. In the context of massive gravity, work towards theories that go beyond the ordinary Einstein-Hilbert terms include [129–131], although later on in [132] it was shown that it is not possible to non-linearly complete the specific terms studied in previous analyses without reintroducing the Boulware-Deser ghost below the cutoff-scale of the effective field theory, concluding that there can be no new healthy Lorentz-invariant derivative interactions in the metric formulation.

For completeness, in this section we briefly discuss the case in which we do allow derivative interactions, which might be relevant in the context of Lorentz-breaking theories (although keeping linearised diffeomorphism invariance) due to the no-go theorem of [132]. Lorentz-breaking massive gravity in flat space has been studied before as it avoids the VDVz discontinuity and improves the strong coupling scale of the effective field theory [235–237], giving an interesting alternative to the standard Fierz-Pauli theory of massive gravity. Possible generalisations to curved space have also been studied [238], but they break linear diffeomorphism invariance, and thus they would not be included in the class of theories studied in Chapter 6.

In what follows, we will study theories with derivative interactions that propagate only one scalar DoF. Such actions can be constructed in different ways, but as an example we mention a case that has a similar structure to massive bigravity, that is, where the fields Φ_A and B_A are auxiliary variables. For this to happen the 29 parameters presented on Appendix E.2 must satisfy the following constraints:

$$\alpha_5 = \alpha_6 = 0, \ \alpha_{1,A} = 0, \ \alpha_{4,1}\nu^2 = r\bar{N}^3\alpha_{4,2}. \tag{E.43}$$

Thus, the most general action satisfying these constraints will depend on 24 free parameters, in addition the background free functions, and will propagate at most two scalar DoFs. In such actions the fields E_A and Ψ_A will appear as dynamical fields. In order to construct ghost-free actions propagating only one scalar DoF we impose that one of the dynamical fields is an auxiliary variable after integrating out the four auxiliary fields B_A and Φ_A. Again, following the structure of massive bigravity, we impose that Ψ_2 is an auxiliary variable. We find that this can happen when different sets of constraints for the parameters are satisfied. For instance, this happens if:

$$\begin{aligned}
\alpha_{4,1} &= \alpha_{12} = 0, \\
\alpha_{K_2} H_N H_b^2 &= 3\left[2H_b^2 H_N(2\alpha_{B_2}+1) - 2H_b\dot{H}_b(\alpha_{B_2}+1) - \alpha_E(H-H_b)\right], \\
\alpha_{3,2} r^3 H &= H_b\bar{N}\frac{(6\alpha_{B_2}-\alpha_{K_2})(\alpha_{H_2}+1)}{6(\alpha_{B_2}+1)} + H_b r^2\nu^2\alpha_8\frac{(\alpha_{K_2}-12\alpha_{B_2}-6)}{6\bar{N}(\alpha_{B_2}+1)} + 2H\nu^2\alpha_9, \\
\alpha_{K_1} H^2 M_1^2 &= (2\rho_0 + P_0) + \bar{N}r^3 M_2^2\frac{(\alpha_{K_2}-6\alpha_{B_2})^2}{(\alpha_{K_2}-12\alpha_{B_2}-6)} + 6H^2 M_1^2\alpha_{B_1}.
\end{aligned} \tag{E.44}$$

Actions satisfying these five constraints propagate only one scalar DoF and, in general, have derivative interactions between both metrics. In addition, we notice that the values for the parameters in Appendix E.3 set to avoid derivative interactions, along with the extra constraint in eq. (6.30), are a particular case of the constraints presented here. Therefore, actions satisfying eq. (E.44) are a direct generalisation of the non-derivative action shown in Section 6.2.

In order to illustrate the form that the action can take now, equation (E.45) shows the extra interaction terms that appear in the quadratic action, compared to the terms in eq. (6.31), due to the new set of constraints given eq. (E.44), when α_9 is a non-zero constant (and thus $\alpha_{3,2} \neq 0$ due to the constraints) and when the rest of the parameters take the same value as those of Appendix E.3:

$$\begin{aligned}\Delta S^{(2)}_{T_1T_2} = \int d^4x\, 16a^5\alpha_9 M_1^2 \Bigg[& \frac{\bar{N}}{\left(\bar{N}^2 - r^2\right)} \left(\bar{N}\partial^2 B_2 \partial^2 \dot{E}_1 + \partial^2 B_1 \partial^2 \dot{E}_2\right) + H_b \partial^2 B_2 \partial^2 E_1 \\ & + 3\frac{\bar{N}^2 \left(H - H_b\right)}{\left(\bar{N}^2 - r^2\right) a^2} \partial^2 B_2 \left(\Psi_1 + \frac{H r^2}{\left(\bar{N}^2 - r^2\right)} B_1\right) - \bar{N}\partial^2 \phi_2 \partial^2 E_1 \\ & - \frac{a^2 r^2}{\bar{N}} H_b \partial^2 E_2 \partial^2 \dot{E}_1 + \frac{a^2 r^2}{\bar{N}} \left(\dot{H}_b - H_N H_b + 3H_b^2\right) \partial^2 E_1 \partial^2 E_2 \Bigg] ,\end{aligned} \tag{E.45}$$

where, for simplicity, we have assumed M_1 and M_2 to be constants. We can see that different derivative interactions appear, including time and space derivatives. All these terms arise because of the non-zero value of α_9 and $\alpha_{3,2}$ solely.

It is important to mention that we have just shown one of the simplest cases that we can have with derivative interactions. The most general model satisfying the set of constraints given in eq. (E.44) has a large number of free parameters, namely 19. This shows that there is a broad class of models with derivative interactions propagating only one scalar DoF at the linear level around homogeneous and isotropic backgrounds. Further restrictions on the parameters could be found by analysing the stability of the evolution of perturbations, as well as by looking for healthy non-linear completions.