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Developing Learning Trajectory For Enhancing Students' Relational Thinking

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Abstract

Algebra is part of Mathematics learning in Indonesian curriculum. It takes one half of the teaching hours in senior high school, and one third in junior high school. However, the learning trajectory of Algebra needs to be improved because teachers teach computational thinking by applying paper-and-pencil strategy combining with the concepts-operations-example-drilling approach. Mathematics textbooks do not give enough guidance for teachers to conduct good activities in the classroom to promote algebraic thinking especially in the primary schools.

To reach Indonesian Mathematics teaching goals, teachers should develop learning trajectories based on pedagogical and theoretical backgrounds. Teachers need to have knowledge of student's developmental progressions and understanding of mathematics concepts and students' thinking. Research shows that teachers' knowledge of student's mathematical development is related to their students' achievement. In fostering a greater emphasis on algebraic thinking, teachers and textbooks need to work more closely together to develop a clearer learning trajectory. Having this algebraic thinking, students developed not only their own character of learning and thinking but also their attitude, attention and discipline.

Key Words: Learning Trajectory, Relational Thinking

I. INTRODUCTION

1. Background

In the Indonesian Mathematics Curriculum (KTSP – Kurikulum Tingkat Satuan Pendidikan, Peraturan Menteri Pendidikan Nasional Indonesia No. 22 Tahun 2006 tentang Standar Isi) learning mathematics aims at having good mathematical thinking ability such as in logical thinking, analytical thinking, systematic thinking, critical thinking and being creative and cooperative in working together in harmony. These competences are needed in order to get, manage, and use information for living in the uncertain circumstances, inconsistent conditions, and competitive situations in their life. By learning mathematics the students can develop their ability to use mathematics in solving problems and communicating idea and thought by using mathematics symbols, tables, diagram, and other media. The mathematics subjects for the Junior High Schools (SMP) students are Algebra, Integers, Geometry and measurement, and Statistics. Algebra has approximately one third of all the teaching time of Mathematics. In Primary Schools (SD), although there is no subject of Algebra, it has been learnt and

included in the subjects of Arithmetic. In Senior High School (SMA), Algebra is one of the six subjects being taught in the classroom (Logics, Geometry, Statistics and Probability, Calculus, and Trigonometry). It takes one half of the teaching time in the classroom every year.

Research on Algebra is urgent to be conducted. According to Mathematical Association of America (2007), Algebra is the gateway to a Technological Future. It is said that "We need a much fuller picture of the essential early algebra ideas, how these ideas are connected to the existing curriculum, how they develop in children's thinking, how to scaffold this development, and what are the critical junctures of this development" (p.2). For this reason it needs a clear learning trajectory supported by good problems in order to foster the development of the students' algebraic thinking in solving problems.

The development of the algebraic thinking depends on the recognition and analysis of patterns that are important components of the young child's mathematical development (Sarama and Clements, 2009, p. 319). Patterns such as subitized patterns, patterns in the number words of counting (Wu, 2007), "one-more" pattern of counting, numerical patterns, arithmetic patterns (Parker & Baldridge, 2004), spatial patterns and array structures are typical early childhood practice of patterning. Patterning is the search for mathematical regularities and structures, to bring order, cohesion, and predictability to seemingly unorganized situations and facilitate generalizations beyond the information directly available (Sarama and Clements, 2009, p. 319). It is a process, a habit of mind in understanding mathematics. Research has showed that understanding of patterns develops gradually during the early childhood years (Clarke et al., 2006; Klein & Starkey, 2004). Rivera (2008) points out that pattern recognition, or as we will refer later to "structural generalisation", is based on an assumption that "patterns are present" and that "things that seem to be the case will continue to be the case". Investigation of patterns helps children to develop their mathematical thinking including their algebraic insight: the conscious understanding that one thing can represent another; sometimes this means seeing "the general in the particular" or at other times seeing a "particular case as an instance of a general mathematical truth" (Mason, Stephens, Watson, 2009). This ability develops considerably over several years, as children interact at first with numbers and later with generalised numbers leading to symbols,

and with other people using these symbols, in a variety of situations, from maps to spatial patterns to numerical patterns.

Algebraic thinking can connect with much of the instruction of all the content of mathematics domains, especially arithmetic in the primary schools. Baroody (1993) mentioned that the students learn to find and extend numerical patterns – extending their knowledge of patterns to thinking algebraically about arithmetic. However, we know that this does not happen automatically for children; and as we shall show later, that it is heavily dependent on how they are taught. In the primary grades, students can learn also to make generalizations and using symbols to represent mathematical ideas and to represent and solve problems (Carpenter & Levi, 1999). The ability of children to invent, learn, apply, justify, and otherwise reason about arithmetic problems, demonstrates that algebraic thinking can be an implicit but significant component of early primary children's learning of mathematics (Sarama and Clements, 2009). For example, students draw upon the fundamental properties of addition, subtraction, multiplication, and division, as well as the relations among the operations. They use the commutatively of addition to create counting-on-from-larger strategies. Similarly, associatively and the inverse relation between addition and subtraction are used extensively in many student's arithmetic strategies.

Teachers need to understand the student's development of algebraic thinking. They are willing to promote it in order to enhance a flexible algebraic habit of thinking and minimize student's limited and erroneous ideas. For example, the students believe that the equal sign represents only a one-way operation that produces an answer of the right side. They also can do and recognize that equality is preserved if equivalent transformations are made on both 'sides' of a situation, from balance scales to sets of objects, verbal problems, and written equations (Schliemann, Carraher, & Brizuela, 2007).

In learning mathematics, students follow natural development progressions in learning and development. They follow natural development progressions in learning mathematical ideas and skills in their own way (Clements and Sarama, 2009, p. 2). When teachers understand these developmental progressions, and sequence activities based on them, they build a learning mathematics trajectory for students. Based on students' interpretations, teachers conjecture what the students might be able to learn

from potential educational experiences (Sarama and Clements, 2009, p.17).

This paper will analyze Indonesian students' relational thinking of Algebra and will illustrate a possible learning trajectory for teaching relational thinking. This analysis is based on students' work of Grade 5 and 6 of Indonesian primary schools. By comparing students' textbooks used in Indonesia and Japan, this paper develops a hypothetical learning trajectory for students and teachers in Indonesia.

2. An Indonesian perspective

In learning mathematics, some Indonesian students' difficulties occur because of the mechanistic way of instruction. In doing multiplication for instance, students memorize the operation procedures and try to apply them in answering problem. Using their procedural and computational thinking the pupils mistakenly applied a corrupt operation procedure (Armanto, 2002). The two below figures illustrate the mistakes.
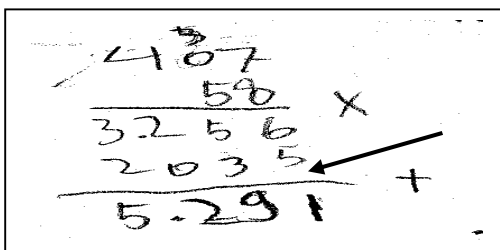
Picture 1. The corrupt multiplication procedure (*arrows by author*)

In our investigation of Indonesian student's capability to think algebraically, the following problems were given to Indonesian students' upper primary and junior secondary years (Stephens, Armanto, & Mailizar, 2009).

How can you find the missing numbers in these mathematical sentences?

$$23 + 15 = 26 + \square$$
$$18 + \blacksquare = 20 + \blacktriangle$$

For students in Indonesia these problems were difficult. Most students used their computational thinking to solve the problems. Moerlands and Anne-Making (2003) mentioned in their study of Indonesian students in Bandung that the format of the sum appears to be very difficult. The formal style surely is more on the problem than on the underlying calculating. They also said that the understanding of the formal

mathematical language seems to depend on the meaningfulness of the problem. In other words, the students have no ideas of the meaning of the unknown numbers included in the second sentence. Other reason was that students can not represent unknown or general numbers by symbols, such as denoted by the use of empty box, circle and triangle. However, Davis (1984), Kaput, Carraher, & Blanton (2008), and Schifter (1999) mentioned that students in primary grades can learn to formulate, represent, and reason about generalizations and conjectures, although their justification do not always adequately validate the conjectures they create.

Stephens (2009) showed that when using computational thinking, students recognize the field the problem belongs to firstly, and then activate the procedure they have already mastered to find the answer. In addition for instance, for the first problem above, a student might answer by having this kind of mathematical sentence.

$$23 + 15 = 26 + \square \quad \rightarrow \quad 38 = 26 + \square \quad \rightarrow \quad \square = 38 - 26 \quad \rightarrow \quad \square = \mathbf{12}$$

Another answer relates to the relation between numbers involved. Since the relation between 23 in the first field and the 26 in the second field is 3 more, then there should be a relation between 15 and the number in the box that is 3 less. So the number in the box must be: $15 - 3 = 12$. This kind of thinking is called *a relational thinking* of mathematics. The following picture illustrates the relational thinking process as mentioned above.

The second question asks for a generalization from students' answer. Initially they were asked to think about a number sentence $18 + \square = 20 + \square$ where the two boxes were appropriately named Box A and Box B. It is simpler here to refer to the number sentence as $18 + \blacksquare = 20 + \blacktriangle$. Students were later given examples of similar number sentences involving subtraction, multiplication and division. In each case, they were asked to use numbers to make three correct examples of the given number sentence.

Students were then asked to explain and to generalize their solution by describing the relationship between the numbers in Box A and Box A, here represented by: $18 + \blacksquare = 20 + \blacktriangle$. In our study, this question was given to eighteen Grade 5 students and to twenty four Grade 6 in Medan, Indonesia. Almost all students correctly made up three examples of a correct number sentence using addition. Most Indonesian students write nice solutions with three sentences (not all students used 3, 4, 5 to add on to 18; some used 2; others used different numbers) when asked to describe the relationship between the numbers in Box A and the numbers in Box B, almost always they described (in perfectly correct Bahasa Indonesian) the relationship between the numbers they had used in box A and Box B as follows:

$18 + 3 \ = 20 + 1$
$18 + 4 \ = 20 + 2$ The relation of numbers in box A and B is 2 difference
$18 + 5 \ = 20 + 3$

These students did not see any need to refer to "the number in Box A" and "the number in Box B" as part of their answer. In fact, this kind of answer was still relatively common among students in Grade 8 but by then some students explicitly referred to "the number in Box A" and "the number in Box B" as part of their answer. However, for Grade 5 and 6 students, the prevalence of the answer "two difference" raised a question: Were they describing a generalization between "the numbers in Box A" and "the numbers in Box B", or were they simply commenting on the three pairs of numbers that they had correctly used? If it is the latter, then the description of: "*2 difference*" is very clear for them because it completely describes the relationship between (3, 1); (4, 2); and (5, 3). It might be possible to refer to the first number being two more than the second; but this is not mentioned – nor do the students think it necessary for this to be mentioned because the three pairs of numbers in question are already written down. Almost all grade 5 and 6 students appear happy to use specific numbers to correctly represent solutions, but these specific numbers are not thought of as variables (i.e. numbers that can vary); or to use Rivera's idea that the pattern might need to continue beyond the numbers that they have used. However in this case the question asks for a generalization, where numbers in Box A are not just 3, 4, and 5. Students were asked: *When you make a correct sentence, what is the relationship between the numbers in Box A and Box B?* It can be answered in two ways. The first is simply to comment on the

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three pairs of numbers that have been used. This is the most highly favored response among the grade 5 and 6 students. Alternatively, the question may be interpreted as looking at the pattern exhibited by the three pairs used, knowing that many other pairs of numbers could have been used. Looking at the sentences, this way requires a mathematical relationship between any numbers of possible pairs of numbers that could be used to make a true sentence. From a mathematical point of view, this question calls for a generalization to describe the relationship between all possible numbers.

3. Students' Relational Thinking

According to Molina, M., Castro, E., & Mason, J. (2007), students consider the number sentence as a whole in their mind, then analyze and find the structure and important elements or relationship to generate productive solutions. Other research from Carpenter (2003) and Stephens (2006) refer to relational thinking in the same way, when students see the equals sign as a relational symbol, students can focus on the structure of expression, and students carry out reasonable strategies to solve the number sentence attending to the operations involved.

In mathematics curriculum, thinking relationally includes the following activities:
a. developing the concept of a number that can vary (a variable number),
b. building upon similarities and differences in solving arithmetical and algebraic equations,
c. appreciation for the equals sign as signifying equivalence of expressions, and
d. understanding that the direction of compensation is dependent on the operation.

These key ideas are assumed in most text book treatments of algebra which typically start at Year 7, where it is introduced as generalised arithmetic. An appreciation, understanding, and comprehension for these ideas are necessary for continued development of and success in algebraic thinking throughout secondary school. But the foundations for this kind of thinking need to be laid in the primary school years.

Three categorizations of students' relational understanding based on students' answers to questions like those used above (Stephens, 2008).

a. Emerging relational thinking
b. Consolidating relational thinking
c. Established relational thinking.

Students who have *the emerging relational thinking* have a limited understanding of developing the concept of a variable. They do not have an idea of building upon similarities and differences in solving arithmetical and algebraic equations. They do not even have an understanding of mathematical variable and expressions. The students also do not have an appreciation for the equals sign as signifying equivalence of expressions. As an addition, the students do not understand that the direction of compensation is dependent on the operation.

Students who have *the consolidating relational thinking* can give some reasons about the correct or incorrect mathematical sentence of their answers. They may find another solution or alternative answers of the problems given. The students can also describe the relationships between numbers in the mathematical sentences. In some manners they can also response to generalise the relationship between numbers in the sentences and express the generality in different sentences.

There is a third category of fully-referenced and directed (*established*) *relational thinking*, that gives a correct mathematical generalisation (Stephens, Armanto, Mailizar, 2009). In the sample of Grade 5 and 6 students discussed above, the only response was "Non-directed Relation" in the form of "two difference". That is what leads us to think that many Indonesian students are treating the sentence: *18 +* ☐ *= 20 +* ☐

Box A   Box B

as a calculation, not as a number of sentence that can be instances of a general pattern.

4. A Concept of Learning Trajectory

In learning mathematics, students follow natural development progressions in learning and development. They follow development progressions in learning mathematical ideas and skills in their own way (Clements and Sarama, 2009, p. 2). When teachers understand these developmental progressions, and sequence activities based on them, they build mathematical environments that are appropriate and effective. In this manner, teachers create a mathematics learning trajectory for students. Teachers

must be particularly careful, not to assume that students can see situations, problems, or solutions as adults do. Therefore, teachers should interpret learning activities from students' points of view. Based on their interpretations, teachers conjecture what the students might be able to learn from potential educational experiences. Similarly, when they interact with students, teachers also consider their own actions from the students' point of view (Sarama and Clements, 2009, p.17).

Formally, learning trajectories are descriptions of students' thinking as they learn to achieve specific goals in a mathematical domain. It is a related, conjectured path through a set of instructional tasks designed to produce those mental processes or actions hypothesized to move students through a developmental level of thinking progression. It needs to be built upon theoretical construct and pedagogical background. The construction process is very similar to that of the mathematical teaching cycles (Simon, 1995) that serve the development of local instructional theory. It is a process of developing prototypical materials for a specific topic where the researcher constructs a provisional set of instructional activities, that are worked out in an iterative process of (re) designing and testing (Gravemeijer, 1999). The cyclic process aims at designing and testing a conjectured local instruction theory on how to teach specific subject. The activities begin with a preliminary design of the prototypical instructional activities, followed by a teaching experiment and end up with a retrospective analysis.

The core element of building a learning trajectory is classroom teaching experiments in which the local instructional theories and prototypical instructional sequences are developed. In the course of the teaching experiments, the researcher develops sequences of instructional activities that embody conjectures of students' learning routes. The development is based on designing and testing instructional activities in daily basis. During the teaching experiments the researcher also carries out anticipatory thought experiments, in which he/she foresees both how the proposed instructional activities might be realized during the interaction in the classroom and what students might learn as they engage in the activities. These provide useful information to guide the revision of the instructional activities for the next instructional activity. Finally a well-considered and empirically-based instructional sequence is construed by reconstructing the sequence in retrospect. When this process of teaching experiment and revision is repeated a number of times, the rationale of the instructional

sequence can be refined until it acquires the status of a local instructional theory (Gravemeijer, 1994). In fact, there is a reflexive relation between the thought and teaching experiments and the local instruction theory. On one hand the conjectured local instructional theory guides the thought and teaching experiments, and on the other hand, the micro instruction experiments shape the local instruction theory.

Learning trajectories need pedagogical and theoretical constructs (Simon, 1995). Creating good learning trajectories will help teachers to have knowledge of students' developmental progressions (pedagogical backgrounds). It is essential for high-quality teaching that builds upon understanding of mathematics concepts and student's thinking (theoretical constructs). Research shows that teachers' knowledge of students' mathematical development is related to their students' achievement (Carpenter, Fennema, Peterson, and Carey, 1988). Learning trajectories help teachers to answer what objectives should be established, where to start, how to know where to go next, and how to get there. It has three parts: a mathematical goal, a developmental path along which students develop to reach that goal, and a set of instructional activities, or tasks, matched to each of the levels of thinking in that path that help students develop ever higher levels of thinking.
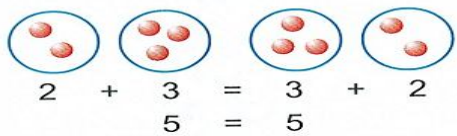
II. DISCUSSION

1. An example of Indonesia Learning Trajectory

The objectives of learning mathematics in Indonesian curriculum are to give experiences to students to promote good mathematical thinking such as in logical thinking, analytical thinking, systematic thinking, critical thinking and being creative and cooperative in working together in harmony. In implementing the curriculum, the teachers are not always well prepared to teach for understanding, how to do the learning process, how to work smoothly, how to turn for assistance, how to implement consistently teaching and learning plans, and how to evaluate the effect on students. These difficulties influence teachers to focus on lower levels of thinking by using paper-and-pencil strategy combine with the concepts-operations-example-drilling approach (Suyono, 1996). This conventional teaching and learning process needs to change to a more active approach where the teachers challenge students with well-selected mathematical problems and a classroom culture that encourages and facilitates learning.

Becker and Shelter (1996) believed that, by learning in this environment, students will improve their learning activities: learning actively, individually, and cooperatively, and learning in strands and contexts.

Other difficulty is coming from the facts that teachers have minimum resources to make up a particular topic, are not able to spend the long period of time needed to prepare the instructional materials, and are not financially adequate to provide instructional aids. The text books used make the instruction complicated. There is learning trajectory given inside the textbook, but with limited guidance for enhancing students' meaningful comprehension. This learning trajectory should be analyzed by teacher in order to find a good activity to transfer the learning process. The following examples are taken from a learning trajectory of Grade 5 students' mathematics textbook (Khafid and Suyati, 2004).


Picture 2. above shows that the commutative property is taught in Grade 5, in the first semester. *Firstly*, students learn about set of balls, two sets of two and three balls inside in the left hand and two other sets with three and two balls in the right hand. Then by counting each set they can see and put the numbers below the set: *2 + 3 = 3 + 2*. Then students can see the sentence: *5 = 5*. Since it has the same result then it can be said as commutative property in mathematics. Here, the main focus of the learning is based on the calculation process. The following example has greater potential for developing ideas of equivalence as well as calculation.

is the same as
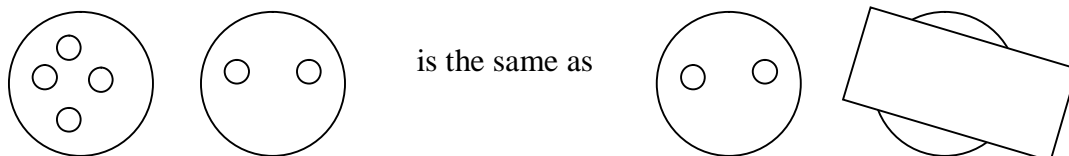
More complex examples could be constructed following this pattern.

Students are asked to calculate the right and left sentences. By comparing them, the conclusion can be taken, *the commutative property is met*. Then the book gives an

example of the property.

a. 3 + (–7) = –4
(–7) + 3 = –4
Jadi, 3 + (–7) = (–7) + 3

b. 8 + (–5) = 3
(–5) + 8 = 3
Jadi, 8 + (–5) = (–5) + 8

These examples differ than the book illustrated in the first place. The examples use negative numbers. Students are asked to see and analyze the sentence and see the calculation and later on having an understanding that the commutative property is met: *8 + (-5) = (-5) + 8*. After then, they are asked to solve the following problems on the blackboards.

B. Isilah titik-titik berikut ini. Kerjakan di papan tulis!

1. 3 + 5   = 5 + ...
2. 4 + (–2) = (–2) + ...
3. (–5) + 1 = . . . + (–5)

4. (–8) + (–2) = (–2) + ...
5. (–9) + 2   = . . . + ...

For teachers, there is no guidance for conducting the learning activity. Should teachers deliver the examples or should they ask the students to read the examples? Should they ask students to create further examples? There is no guidance for students as well. The only guidance is that students are asked to see and read the examples and try to understand them as the commutative property. And later on, they are asked to solve five problems in the blackboards. The next activity (see the picture 5. below) illustrates that students will solve problems for the rest of their time during the mathematics lesson.

1. (–8) + 5 = 5 + ...
2. 6 + (–4) = . . . + 6
3. (–7) + 5 = 5 + ...
4. 11 + (–2) = (–2) + ...
5. (–12) + 4 = 4 + ...

k book

This "learning trajectories" of commutative property become a main menu for Grade 5 Indonesian students. The activities are very much the same activities as described above here in learning distributive property (see Picture 6. below). It begins by giving explanation of the distributive property. Teachers practice the property based on computational skill.

$$(86 \times 140) - (86 \times 40) = 86 \times (140 - 40) \quad \text{(angka pengali yang}$$
$$\text{sama dipisahkan)}$$
$$= 86 \times 100 \quad \text{(The same multiplication}$$
$$\text{number is isolated)}$$

1. $(9 \times 6) + (9 \times 4)$ $\quad = \ldots \times (\ldots + \ldots) = \ldots \times \ldots = \ldots$
2. $(12 \times 7) + (12 \times 3)$ $\quad = \ldots \times (\ldots + \ldots) = \ldots \times \ldots = \ldots$
3. $(15 \times 20) + (15 \times 80)$ $\quad = \ldots \times (\ldots + \ldots) = \ldots \times \ldots = \ldots$
4. $(71 \times 18) + (71 \times 82)$ $\quad = \ldots \times (\ldots + \ldots) = \ldots \times \ldots = \ldots$
5. $(23 \times 64) + (23 \times 36)$ $\quad = \ldots \times (\ldots + \ldots) = \ldots \times \ldots = \ldots$

This is a "paper-and-pencil" teaching strategy combine with "concepts-operations-example-drilling" approach. Students learn the subject by seeing, hearing, trying out, and trying to understand, and later drilling and practicing the problem. The next activities are solving problems as an application for the use of properties being learnt. These activities appear to focus on learning through computation, not through relational thinking. (Although it is clear that the numbers have been carefully chosen to relate to multiples of 10 or 100.) Some students may indeed learn to see the structure implicit in these calculations, but to make sure that they do requires a more explicit treatment. The answer to each question above appears to be the result of calculation. Of course, various relationships are used in obtaining the answer but these may not be seen by students as the focus of their attention and learning. For example, in the case of $(71 \times 18) + (71 \times 82) = 71 \times (18 + 82) = 71 \times 100$, instead of going to the answer (7100), it might be important to ensure that students attend to the mathematical structure (i.e. the relational thinking) embodied in this example, by asking them to work backwards from 71×100 to create three or four other instances that might lead to this result. Then, it would be important to ask students what they noticed about the numbers that could be used to give 71×100. That is, what different numbers could be used in $(71 \times \square) + (71 \times \blacktriangle)$ if the answer is 71×100? Please try to describe these numbers in as many ways as you can. What numbers would need to be used if the answer was: 65×100; or if it was: 71×95? These opportunities to investigate and describe mathematical structure are hidden from the above examples. Some students may pick up the key ideas that the writers clearly had in mind in choosing the numbers that they did, but many will not do so because it has not been brought to their attention. The examples shown in Picture 6

above have high potential for inviting students to explore mathematical structure. Attending to the mathematical structure, however, requires a different form of questioning.

2. Building a stronger learning trajectory in Indonesia

In order to fulfil the objectives of learning mathematics in Indonesian curriculum, relational thinking should be part of the learning process towards fostering students' thinking ability such as logical, analytic, systematic, and critical thinking. Teachers should be encouraged to apply active approach where the teachers challenge students with well-selected mathematical problems and a classroom culture that facilitates learning. Becker and Shelter (1996) believed that learning in this environment, students will improve their learning activities: learning actively, learning individually, learning cooperatively, and learning in strands and contexts.

The activities would be worthy when; *firstly*, learning activity involves some problems of finding more examples of the properties by using □, ▲, and ● representing different numbers based on number sentences that students are familiar with. In the sentence, $5 \times 6 = 5 \times 7 - 5 \times \square$, the "empty box" represents 1 because the multiplication sentence is quite specific. However, as in Picture 14, these "symbols" can also be used to represent miscellaneous numbers – in this case the numbers represent different possibilities for the length, width and height of a cuboid. In other multiplication sentences, such as $5 \times 6 = 5 \times ● + 5 \times ▲$, students could be invited to think what numbers might be used to make a correct sentence. They will want to know if they have to use different numbers, like 5 and 1. How many different numbers can be used? Some will realise that 3 and 3 can be used. Others may need to be convinced that the same number can be used when the "symbols" look different. Other students may even wonder if it is possible to use fractions or decimals. Whatever they decide, the structural principle is the same: the two missing numbers have to add up to 6; or some may express this as $● + ▲ = 6$.

Examples like these have high potential to lead students towards a generalization pattern of the numbers in the algebraic sentences. These also lead students to the

generalization of operation properties. These activities should focus on generalizing the properties involving various numbers, such as integers, negative numbers, fractions, decimals, and so on. This helps students to think beyond the particular. Students get an understanding of relational thinking based on many possible numbers. Relational thinking enables students to meet the Indonesian mathematics learning goal that is having logical thinking and a better strategy for solving mathematics problems.

Secondly, in each grade, students need to undergo a process of relational thinking in each algebra subject. They learn how to develop their understanding towards the process by comparing and distinguishing relational thinking and computational thinking. As the Japanese examples show, computational thinking has the potential to lead students to relational thinking; and relational thinking has the potential to help students understand computation better and to perform it more fluently. In the textbook, there should be some problems involving relational thinking along the line of practicing problem solving. There should be some activities for students to practice and enhance their understanding of relational thinking. Three activities described in the Japanese textbook might be the main menu, for example, of finding the same answer for algebraic sentences, discussing numbers for a new sentence, and working cooperatively to generate different patterns of numbers.

Thirdly, the learning environment is designed developmentally for the use of varieties of relational words, such as "equivalent to" or "means the same as" or "has the same value as". Other relational words need to be used such as, difference of ...; more than …..; bigger than …..;   larger than …..; be ….. part; a ….. difference; a distance of …..; higher than the other; and is ….. more than the other. The examples we have discussed show that there has to be a learning trajectory where students, as they move through the primary school and into the junior high school, use these relational words in more complex and more varied sentences, such as "The answer will be the same (in a subtraction) if each number is increased by the same amount", or "The difference remains the same", or "We can make this addition (or subtraction, multiplication or division) easier to calculate by changing the numbers in the following way", "The answers are equivalent" We expect all students to be able to think this way – some will be more fluent than others, of course, but learning to think relationally should be something that all students appreciate to some degree. The teacher's role is facilitating

this learning environment. The learning process should be based on understanding toward the curriculum, the mathematics contents, and the teaching strategy. The last but not least, teachers should develop learning trajectories in their lesson plan based on pedagogical and theoretical backgrounds. Teachers should have knowledge of student's developmental progressions and understanding of mathematics concepts and student's thinking. Research shows that teachers' knowledge of student's mathematical development is related to their students' achievement.

The following path is an example of building learning trajectory; i.e. teaching multiplication properties. The goal is developing relational thinking toward operation properties (addition, subtraction, multiplication, and division). It begins by giving students, around Grade 4 or 5, the following activities of solving three problems.

Fill in the box any numbers that make the sentence true.

$5 \times 6 = 5 \times 7 - \square$

$5 \times 6 = 5 \times 5 + \square$

$5 \times 6 = 5 \times 7 + \square$

Many students will use calculation thinking, the answer might be like this.

$30 = 35 - 5$

$30 = 25 + 5$

$30 = 20 + 10$

This kind of answer depends much upon the skill of calculation. However, this calculation is hard to generalize because every time the numbers change and the answers adjust too. This is the type of response focuses on calculation. We expect that many students will at first see this number sentences as requiring them to calculate the unknown number. In a recommended learning trajectory however, we want students to leave the sentences in uncalculated form in order that they can begin generalize fundamental distributive properties of multiplication. This requires them to write 5 x 1 instead of 5, and 5 x 2 instead of 10.

The important of leaving in uncalculated form is shown below in the examples taken from the main Japanese textbook (in this case for Grade 4).

For Japanese students, the learning trajectory should focus on generalization. Students need to use *uncalculated form of answer*. Students need to move from;

$5 \times 6 = 5 \times 7 - 5$ $\qquad\qquad\qquad\qquad\qquad$ $5 \times 6 = 5 \times 7 - 5 \times 1$

$5 \times 6 = 5 \times 5 + \quad 5$ ***to*** $5 \times 6 = 5 \times 5 + 5 \times 1$

$5 \times 6 = 5 \times 4 + 10$ $5 \times 6 = 5 \times 4 + 5 \times 2$

This type of answer can lead toward several generalizations. For example, teachers might ask students to think of how they might represent 5×13 in order to make calculation simpler. Students who understand the structural principle embedded in the above examples might say: $5 \times 13 = 5 \times 10 + 5 \times 3$, or they could say $5 \times 13 = 5 \times 15 - 5 \times 2$. Students who can think this way should have no difficulty in considering how to simplify 5×99 as $5 \times 100 - 5 \times 1$. Having more complex generalisations becomes possible when students become confident and competent in this kind of thinking, such as $5 \times (\bigcirc + \square) = 5 \times \bigcirc + 5 \times \square$. Here, $\bigcirc$ and $\square$ can be used to represent any number. Clearly at this stage of learning trajectory, students have move beyond calculation. However, this kind of thinking is very powerful and necessary in helping student in simplifying calculation. At the same time this learning trajectory is already move students into algebraic representation that will later be express more formally but not too differently as $5 \times (a + b)$.

If students can find answers like these than the next learning activities are easier for them to do. Students see the problems as a relationship between numbers because $6 = 7 - 1$, so $5 \times 6 = 5 \times 7 - 5 \times 1$. Or, since $6 = 4 + 2$, so $5 \times 6 = 5 \times 4 + 5 \times 2$ or $99 = 100 - 1$ or $102 = 100 + 2$. These number relationships are important for calculation and make calculation a lot easier. Since then, the building of relational thinking of numbers in the operational properties

The Japanese textbook highlights the important of moving from (or beyond) calculation thinking to relational (structural) thinking in the deliberate use of representative symbolic "terms", such as ▲, ■, and ●, as a way of summarising multiple numerical expressions that students have already met (Fujii and Stephens, 2008, p.138). Instances of representative symbolic "terms" are introduced quite early in the elementary schools from Grade 3 upward to help students to focus on patterns and relationships as distinct from calculation.

These problems use ■, ●, and ▲ to represent generalized numbers. We notice that in the Japanese books when representative symbolic "terms" are introduced, students are asked to check the calculation by giving specific values to this terms that

can be used to represent numbers. We notice that these representative symbolic "terms" are never referred to in Japanese textbook as "triangle", and "box". They are always referred to in terms of numbers, such as "this number" and "that number", or "the first number" and "the second number", or "one number" and "another number" or "these two numbers", according to context. They represent numbers that children have used or could use. They are not yet formal symbolic terms like x and y which have a special meaning in algebra. These representative symbolic "terms" only have meaning in relation to actual number sentences that the students already met. If students are not sure what these terms mean then the teachers have to take them back to the number sentence or sentences from which they receive their meaning. Later in Algebra, symbols like x and y come to have a meaning of their own. This is not yet the case.

Students are asked to use their relational thinking to see the pattern and find the numbers that relates to each other. They should find those numbers and fill it in the given (■, ●, ▲). They can use calculation just only for checking the answer whether it is correct or not. Here the same symbol is used to denote the same number.

$$5 \text{ x } (\blacktriangle) = 5 \text{ x } (\blacktriangle - 1) + 5 \text{ x } 1$$
$$5 \text{ x } (\blacksquare) = 5 \text{ x } (\blacksquare + 1) - 5 \text{ x } 1$$

$$5 \text{ x } (\blacksquare + \bullet) = 5 \text{ x } \blacksquare + 5 \text{ x } \bullet$$
$$5 \text{ x } (\blacksquare - \bullet) = 5 \text{ x } \blacksquare - 5 \text{ x } \bullet$$

Students might think of the following answer related to *properties of operations*:

$$(\blacksquare + \bullet) \text{ x } \blacktriangle = \blacksquare \text{ x } \blacktriangle + \bullet \text{ x } \blacktriangle \quad \text{and} \quad (\blacksquare - \bullet) \text{ x } \blacktriangle = \blacksquare \text{ x } \blacktriangle - \bullet \text{ x } \blacktriangle$$

In the Japanese textbook these may appear to be symbolic expressions but their meaning is based in very specific number sentences that they represent. For example, students have been attending to $(5 + 2) \times 4 = 5 \times 4 + 2 \times 4$. And then students are asked to about the expression $(\blacksquare + \bullet) \times \blacktriangle = \blacksquare \times \blacktriangle + \bullet \times \blacktriangle$. Students have to think that $\blacksquare$ is standing in the place of the 5, $\bullet$ is standing in the place of 2 and $\blacktriangle$ is standing in the place of 4. But these symbol-like terms can represent other whole numbers and students are asked to check if the number relationship represented by this expression could be true for fraction numbers. The symbol-like expression has no truth on its own. Students in Grade 4 and 5 need to check to see that $(5.5 + 2.5) \times 0.5$ can be re-expressed as $5.5 \times$

0.5 + 2.5 × 0.5. The decimal calculation should be familiar to upper primary students who can calculate the value of this expression as 2.75 + 1.25 which is equal to 4. They also need to see that the original expression (5.5 + 2.5) × 0.5 can be simplified to be (8.0) × 0.5 which is easily calculates to 4.0. In this learning trajectory we can say how students need to give their meaning to expression, such as (■ + ●) x ▲ = ■ x ▲ + ● x ▲ by using whole numbers (small and large), decimal numbers, and fractions.

In learning division, the learning trajectory begins by discussing and making a generalization of relation between varieties division sentences and the answer of a number. For instance teachers can use a division sentence, such as 4 : 1 = 4. What are other sentences that have the same result? What is the relation among number? Students can see that when they double the dividend (the number is being divided), what number should the divisor be in order to keep the same answer. In this example if the dividend (first number) is doubled then the divisor (second number) must be doubled for the expression to remain equivalent. When they triple the dividend, what is number in the divisor? As a whole activity, students can discuss and see the link between calculation and relational thinking for understanding the mathematical structures. Then the next activity is about finding a hidden number. When we get a sentence: 40 are divided by 10 equals to 4, what the numbers are in the next division sentence if all numbers are divided by 5. Students are asked to find a sentence related to the sentence. Again, it is about the relationship of the numbers rather than only a calculation sentences. In the end, students will come up with the idea of: *whenever the dividend is divided with a number, so is the divisor.*

III. CONCLUSION AND SUGGESTION

The Indonesian national curriculum definitely places a high importance on developing mathematical thinking and logical argument in all students. We have argued that these objectives are not well achieved in practice. In particular, we have identified many missed opportunities for textbook writers to focus the attention of teachers and students on the mathematical structure that is present and embodied in number sentences. We have shown that in many of the examples used in Indonesian textbooks it is easy for

students and teachers to believe that the goal is to carry out a correct computation. Yet, with just a few additional questions, the structural principles embedded in these examples could be explored quite easily. No doubt, some bright students pick up and grasp these structural ideas, but many clearly do not.

For developing a good relational thinking, the mathematics textbooks should give just as much attention to computation and correct calculation, but they must have a much clearer learning trajectory which progressively develops relational ways of thinking about arithmetic and calculation. This learning trajectory is continuous and systematic throughout the primary years. We can assume that not all students are ready to understand relational thinking the first time it is introduced, but they will have many other occasions to think relationally about number sentences. The textbooks clearly aim to develop strong patterns of relational thinking by the end of primary school. They do this not by having longer and heavier textbooks but by carefully chosen examples and a well developed learning and teaching trajectory. This is an area that Indonesia could well consider for the teachers and students.

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