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A flexible error estimate for the application of centre manifold theory

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Abstract

In applications of centre manifold theory we need more flexible error estimates than that provided by, for example, the Approximation Theorem 3 by Carr [4, 6]. Here we extend the theory to cover the case where the order of approximation in parameters and that in dynamical variables may be completely different. This allows, for example, the effective evaluation of low-dimensional dynamical models at finite parameter values.

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Contents

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1 Introduction

Interest in the dynamical behaviour of a physical system usually lies in the relatively low-dimensional evolution after heavily damped modes have become insignificant. There are many successful applications of centre manifold techniques to create models of these relatively simple dynamics. We here mention some applications to physical fluid mechanics. Iooss [16, 17], and Laure [18] analysed the dynamics of Taylor vortices in Taylor-Couette flow, whereas Chossat [7] and Hill [13] discuss the non-axisymmetric dynamics involving mode competition by using centre manifold theory. Mode interactions in the dynamics of convection in porous media are analysed with centre manifolds by Neel [26, 27, 28] and Graham & Steen [12]. Arneodo *et al* [1, 2, 3] reduced the dynamics of triple convection down to a set of three coupled ODEs, numerically verified the modelling and then proved the existence of chaos. Roberts *et al* discussed centre manifolds of forced dynamical systems [9], and derived low-dimensional models using centre manifold techniques for contaminant dispersion in channels [24], shear dispersion in pipes [25], thin film fluid dynamics [33], coating flows over a curved substrate in space [35, 20], and Mei, Roberts & Li [22, 23] derived models for turbulent shallow water flow written in terms of vertically averaged quantities derived from the k-ϵ model for turbulent flow. Such applications assure us that centre manifold theory provides a useful route to the low-dimensional modelling of high-dimensional dynamical systems.

Centre manifold theory guarantees the existence of low-dimensional models, matches the solutions of original and the low-dimensional systems, and quantifies errors in the approximation. Algebraic techniques to construct low-dimensional models are based upon the theory. In problems specified in the standard form (8), the centre manifold may be calculated simply by iteration, see Carr [4] for example. For the more directly applicable form $\dot{\boldsymbol{u}} = \boldsymbol{\mathcal{L}}\boldsymbol{u} + \boldsymbol{f}(\boldsymbol{u}, \boldsymbol{\epsilon})$, solutions in the form of an asymptotic power series are found using methods developed by Coullet & Spiegel [8] (and reinvented by Leen [19]). The derivation of initial conditions for such low-dimensional models is given through projecting the initial condition of the system onto the centre manifold [10, 30, 31, 34]. But many of the applications require more flexible errors estimates. For example, physical models recoverd by evaluation at a finite value of a supposedly asymptotically small parameter often need high order approximations in the parameter [24, 32, 22, 33, 23, 21]. Thus asymptotic errors estimates need to be made to high order in some parameters and only low order in other variables. In Section 3 we extend a theorem of Carr & Muncaster [4, 5] to rigorously support such flexible approximations.

2 A simple example

To introduce the issues, consider the well known prototype bifurcation problem

$$\dot{x} = \epsilon x - xy\,, \quad \dot{y} = -y + x^2\,, \tag{1}$$

where ϵ is a parameter. By adjoining the trivial equation, it becomes:

$$\dot{\epsilon} = 0\,, \quad \dot{x} = \epsilon x - xy\,, \quad \dot{y} = -y + x^2\,. \tag{2}$$

According to the distribution of the eigenvalues of the system, we may seek a centre manifold of form $y = h(x, \epsilon)$. Substituting this into the system (2) we deduce that h must satisfy

$$h = x^2 - \frac{\partial h}{\partial x} x(\epsilon - h)\,. \tag{3}$$

Solving this iteratively leads to the approximations

$$h^{(0)} = 0\,, \quad h^{(1)} = x^2\,, \quad h^{(2)} = x^2 - 2\epsilon x^2 + 2x^4\,, \quad \text{etc.} \tag{4}$$

Now elementary calculation shows that the above approximations $h^{(n)}$ satisfy (3) to a residual $\mathcal{O}\left(|(\epsilon, x)|^{n+2}\right)$ as $(\epsilon, x) \to \mathbf{0}$; an error equivalently expressed as $\mathcal{O}\left(\epsilon^{n+2} + x^{n+2}\right)$ since a term $c\epsilon^{p'} x^{q'}$ (for some constant $c \neq 0$) is $\mathcal{O}\left(\epsilon^p + x^q\right)$ only if $p'/p + q'/q \geq 1$. Therefore the centre manifold is $y = h(x, \epsilon) = h^{(n)} + \mathcal{O}\left(|(\epsilon, x)|^{n+2}\right)$ by, for example, Theorem 3 of [5, p264].

The limitation in applications is that the established theorem on approximation strongly couples the order of truncation in both parameters and variables—the "weight" of the parameter and variable is the same in the error estimate. Some flexibilty may be introduced by a nonlinear transformation of the parameters; for example, introduce $\delta = \sqrt{\epsilon}$ and instead of (2) study

$$\dot{\delta} = 0\,, \quad \dot{x} = \delta^2 x - xy\,, \quad \dot{y} = -y + x^2\,. \tag{5}$$

The resultant iterative solution of (3) is identical to (4). The only difference is that the approximation theorem asserts the errors in $h^{(n)}$ are $\mathcal{O}\left(|(\delta, x)|^{2n+2}\right)$ as $(\delta, x) \to \mathbf{0}$, that is, $\mathcal{O}\left(\epsilon^{n+1} + x^{2n+2}\right)$. Thus certain trivial nonlinear transformations make no difference to the algebraic analysis but do affect the error estimate. There must be more flexibility in the errors than has so far been proved.

A more flexible error bound to include the effects of both ϵ and x to any desired orders is to express and seek errors as $\mathcal{O}(x^q, \epsilon^p)$. Note that $f = \mathcal{O}(x^q, \epsilon^p)$ means that any terms in f of the form $c\epsilon^{p'} x^{q'}$ (for some constant $c \neq 0$) must satisfy $p' \geq p$ or $q' \geq q$. For example, we may deduce, supported by Theorem 1 herein,

$$h = x^2(1 - 2\epsilon + 4\epsilon^2) + x^4(2 - 16\epsilon + 88\epsilon^2) + \mathcal{O}\left(x^6, \epsilon^3\right) . \tag{6}$$

This kind of error allows us separately to choose the orders of the parameter ϵ and the variable x which we want to include in the centre manifold. For example, here we may compute to higher orders in ϵ, observe the pattern of coefficients in this simple problem, and realise [29] that the above is the low-order Taylor polynomial in ϵ of

$$h = \frac{x^2}{1 + 2\epsilon} + \frac{2x^4}{(1 + 2\epsilon)^2(1 + 4\epsilon)} + \mathcal{O}\left(x^6\right) . \tag{7}$$

Such approimations to high-order in parameters and low-order in dynamical variables were used, before proof, and are essential to the analyses in [24, 32, 22, 33, 23, 21].

3 The flexible extension

Consider dynamical systems expressed in the form

$$\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\epsilon}) , \\ \dot{\boldsymbol{y}} &= B\boldsymbol{y} + \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\epsilon}) , \end{aligned} \tag{8}$$

where $\boldsymbol{x} \in \mathrm{R}^m$, $\boldsymbol{y} \in \mathrm{R}^n$, $\boldsymbol{\epsilon} \in \mathrm{R}^l$ and A and B are constant matrices such that all the eigenvalues of A have zero real parts, and all eigenvalues of B have negative real parts. Functions $\boldsymbol{f}$ and $\boldsymbol{g}$ are nonlinear for $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{\epsilon}$ and $\boldsymbol{f}(\boldsymbol{0},\boldsymbol{0},\boldsymbol{0}) = \boldsymbol{g}(\boldsymbol{0},\boldsymbol{0},\boldsymbol{0}) = \boldsymbol{f}'(\boldsymbol{0},\boldsymbol{0},\boldsymbol{0}) = \boldsymbol{g}'(\boldsymbol{0},\boldsymbol{0},\boldsymbol{0}) = \boldsymbol{0}$ (where $\boldsymbol{f}' = [\boldsymbol{f}_{\boldsymbol{x}}, \boldsymbol{f}_{\boldsymbol{y}}, \boldsymbol{f}_{\boldsymbol{\epsilon}}]$, and similarly for $\boldsymbol{g}'$ and other Jacobians).

For any function $\boldsymbol{\phi} : \mathrm{R}^m \times \mathrm{R}^l \to \mathrm{R}^n$ which is a continuously differentiable function and $\boldsymbol{\phi}(\boldsymbol{0},\boldsymbol{0}) = \boldsymbol{\phi}'(\boldsymbol{0},\boldsymbol{0}) = \boldsymbol{0}$, define

$$(H\boldsymbol{\phi}) = \boldsymbol{\phi}_{\boldsymbol{x}}(\boldsymbol{x},\boldsymbol{\epsilon})[A\boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\phi}(\boldsymbol{x},\boldsymbol{\epsilon}), \boldsymbol{\epsilon})] - B\boldsymbol{\phi}(\boldsymbol{x},\boldsymbol{\epsilon}) - \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{\phi}(\boldsymbol{x},\boldsymbol{\epsilon}), \boldsymbol{\epsilon}) .$$

Also let $\mathcal{O}(s^q, \epsilon^p)$ denote any terms of the form $c\epsilon_1^{p_1} \ldots \epsilon_l^{p_l} s_1^{q_1} \ldots s_m^{q_m}$ (where the constant $c \neq 0$) which satisfy $p_1 + \cdots + p_l \geq p$ or $q_1 + \cdots + q_m \geq q$ and $p_i, q_j \geq 0$.

Theorem 1 (Approximation) *Suppose that*

$$(H\boldsymbol{\phi})(\boldsymbol{x},\boldsymbol{\epsilon}) = \mathcal{O}(x^q,\epsilon^p) \quad \text{as} \quad (\boldsymbol{x},\boldsymbol{\epsilon}) \to \boldsymbol{0}\,,$$

where $p \geq 1$, $q > 1$, then

$$|\boldsymbol{h}(\boldsymbol{x},\boldsymbol{\epsilon}) - \boldsymbol{\phi}(\boldsymbol{x},\boldsymbol{\epsilon})| = \mathcal{O}\left(x^q,\epsilon^p\right) \quad \text{as} \quad (\boldsymbol{x},\boldsymbol{\epsilon}) \to \boldsymbol{0}\,.$$

That is, the errors in the approximation $\boldsymbol{\phi}$ to a centre manifold $\boldsymbol{h}$ is the same as the order of the residuals of the equations of the dynamical system.

Proof: This proof is adapted from Carr [4, pp25–28].

Let $\boldsymbol{\theta} : \mathrm{R}^m \times \mathrm{R}^l \to \mathrm{R}^n$ be a continuously differentiable function with compact support such that

$$\boldsymbol{\theta}(\boldsymbol{x},\boldsymbol{\epsilon}) = \boldsymbol{\phi}(\boldsymbol{x},\boldsymbol{\epsilon}) \quad \text{for} \quad |(\boldsymbol{x},\boldsymbol{\epsilon})| \quad \text{small}\,.$$

Set

$$\begin{aligned} \boldsymbol{N}(\boldsymbol{x},\boldsymbol{\epsilon}) &= \boldsymbol{\theta}_{\boldsymbol{x}}(\boldsymbol{x},\boldsymbol{\epsilon})\left[A\boldsymbol{x} + \boldsymbol{F}\left(\boldsymbol{x},\boldsymbol{\theta}(\boldsymbol{x},\boldsymbol{\epsilon}),\boldsymbol{\epsilon}\right)\right] \\ &\quad - B\boldsymbol{\theta}(\boldsymbol{x},\boldsymbol{\epsilon}) - \boldsymbol{G}\left(\boldsymbol{x},\boldsymbol{\theta}(\boldsymbol{x},\boldsymbol{\epsilon}),\boldsymbol{\epsilon}\right)\,, \end{aligned} \tag{9}$$

where

$$\begin{aligned} \boldsymbol{F}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\epsilon}) &= \boldsymbol{f}\left(\boldsymbol{x}\psi\left(\frac{\boldsymbol{x}}{\delta}\right),\boldsymbol{y},\boldsymbol{\epsilon}\right)\,, \\ \boldsymbol{G}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\epsilon}) &= \boldsymbol{g}\left(\boldsymbol{x}\psi\left(\frac{\boldsymbol{x}}{\delta}\right),\boldsymbol{y},\boldsymbol{\epsilon}\right)\,, \end{aligned}$$

where $\psi : \mathrm{R}^m \to [0,1]$ is a infinitely differentiable function with $\psi(\boldsymbol{x})$=1 when $|\boldsymbol{x}| \leq 1$ and $\psi(\boldsymbol{x}) = 0$ when $|\boldsymbol{x}| \geq 2$ and δ is a positive real number. The properties of $\boldsymbol{F}$ and $\boldsymbol{G}$ are the same as in [4, p18] for $\boldsymbol{\epsilon}$ small. So $\boldsymbol{N}(\boldsymbol{x},\boldsymbol{\epsilon}) = \mathcal{O}\left(x^q,\epsilon^p\right)$ as $(\boldsymbol{x},\boldsymbol{\epsilon}) \to 0$.

For $a > 0$ and $b > 0$ let Γ be the set of Lipschitz functions $\boldsymbol{h} : \mathrm{R}^m \times \mathrm{R}^l \to \mathrm{R}^n$ with Lipschitz constant b, $|\boldsymbol{h}(\boldsymbol{x},\boldsymbol{\epsilon})| \leq a$ for $(\boldsymbol{x},\boldsymbol{\epsilon}) \in \mathrm{R}^m \times \mathrm{R}^l$ and $\boldsymbol{h}(\boldsymbol{0},\boldsymbol{0}) = \boldsymbol{0}$. With the supremum norm $\|\,.\,\|$, Γ is a complete space.

For $\boldsymbol{h} \in \Gamma$ and $\boldsymbol{x}_0 \in \mathrm{R}^m$, let $\boldsymbol{x}(t,\boldsymbol{x}_0,\boldsymbol{h})$ be the solution of

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{F}(\boldsymbol{x},\boldsymbol{h},\boldsymbol{\epsilon})\,, \quad \boldsymbol{x}(0,\boldsymbol{x}_0,\boldsymbol{h}) = \boldsymbol{x}_0\,.$$

The bounds on $\boldsymbol{F}$ and $\boldsymbol{h}$ ensure that the solutions of the above equation exists for all time t. Define an operator T on Γ by

$$(T\boldsymbol{h})(\boldsymbol{x}_0) = \int_{-\infty}^{0} e^{-Bs}\boldsymbol{G}\left(\boldsymbol{x}(s,\boldsymbol{x}_0,\boldsymbol{h}),\boldsymbol{h}(\boldsymbol{x}(s,\boldsymbol{x}_0,\boldsymbol{h})),\boldsymbol{\epsilon}\right)ds\,.$$

We know that T is a contraction mapping and the centre manifold $\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{\epsilon})$ is a fixed point of T for a, b and δ small enough from the proof of the existence theorem [4, pp.16–19]. Define

$$S\boldsymbol{Z} = T(\boldsymbol{Z} + \boldsymbol{\theta})\,,$$

for $\boldsymbol{Z}$ such that $\boldsymbol{Z} + \boldsymbol{\theta} \in \Gamma$. The domain of S is a closed subset of Γ since $\boldsymbol{\theta} \in \Gamma$. Since $|S\boldsymbol{Z}_1 - S\boldsymbol{Z}_2| = |T(\boldsymbol{Z}_1 + \boldsymbol{\theta}) - T(\boldsymbol{Z}_2 + \boldsymbol{\theta})|$, thus S is also a contraction mapping. For $K > 0$ let[1]

$$Y = \left\{ \boldsymbol{Z} \in \Gamma \mid |\boldsymbol{Z}(\boldsymbol{x}, \boldsymbol{\epsilon})| \le K(x^q, \epsilon^p), \forall (\boldsymbol{x}, \boldsymbol{\epsilon}) \in \boldsymbol{O} \subset \mathrm{R}^m \times \mathrm{R}^l \right\},$$

where $\boldsymbol{O}$ is a neighbourhood of the origin in $\mathrm{R}^m \times \mathrm{R}^l$. Since $\boldsymbol{N}(\boldsymbol{x}, \boldsymbol{\epsilon}) = \mathcal{O}(x^q, \epsilon^p)$ as $(\boldsymbol{x}, \boldsymbol{\epsilon}) \to 0$, then

$$|\boldsymbol{N}(\boldsymbol{x}, \boldsymbol{\epsilon})| \le C_1(x^q, \epsilon^p)\,, \quad (\boldsymbol{x}, \boldsymbol{\epsilon}) \in \boldsymbol{O} \tag{10}$$

where C_1 is a constant. Thus Y is not empty because $\boldsymbol{N} \in Y \subset \Gamma$ by defining $\boldsymbol{\theta}(\boldsymbol{x}, \boldsymbol{\epsilon})$ such that $C_1 \le K$. If we can find a constant K such that S maps Y into Y, then $\exists\ \boldsymbol{Z}_0 \in Y$ is a fixed point of S, and

$$\begin{aligned} \boldsymbol{Z}_0 = S(\boldsymbol{Z}_0) = T(\boldsymbol{Z}_0 + \boldsymbol{\theta}) - \boldsymbol{\theta}\,, \\ \text{that is,} \quad T(\boldsymbol{Z}_0 + \boldsymbol{\theta}) = \boldsymbol{Z}_0 + \boldsymbol{\theta}\,, \end{aligned}$$

i.e., $\boldsymbol{Z}_0 + \boldsymbol{\theta}$ is a centre manifold of (8), let $\boldsymbol{h} = \boldsymbol{Z}_0 + \boldsymbol{\theta}$,

$$|\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{\epsilon}) - \boldsymbol{\theta}(\boldsymbol{x}, \boldsymbol{\epsilon})| = \boldsymbol{Z}_0 \le K(x^q, \epsilon^p)\,.$$

To finish the proof define

$$\begin{aligned} \boldsymbol{Q}(\boldsymbol{x}, \boldsymbol{Z}, \boldsymbol{\epsilon}) \;&=\; \boldsymbol{\theta}_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{\epsilon}) \left[\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{\theta} + \boldsymbol{Z}, \boldsymbol{\epsilon}) - \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{\theta}, \boldsymbol{\epsilon}) \right] - \boldsymbol{N}(\boldsymbol{x}, \boldsymbol{\epsilon}) \\ &\quad + \boldsymbol{G}(\boldsymbol{x}, \boldsymbol{\theta} + \boldsymbol{Z}, \boldsymbol{\epsilon}) - \boldsymbol{G}(\boldsymbol{x}, \boldsymbol{\theta}, \boldsymbol{\epsilon})\,. \end{aligned}$$

Then

$$\begin{aligned} |\boldsymbol{Q}(\boldsymbol{x}, \boldsymbol{Z}, \boldsymbol{\epsilon})| \;&\le\; |\boldsymbol{Q}(\boldsymbol{x}, \boldsymbol{0}, \boldsymbol{\epsilon})| + |\boldsymbol{Q}(\boldsymbol{x}, \boldsymbol{Z}, \boldsymbol{\epsilon}) - \boldsymbol{Q}(\boldsymbol{x}, \boldsymbol{0}, \boldsymbol{\epsilon})| \\ &= |\boldsymbol{N}(\boldsymbol{x}, \boldsymbol{\epsilon})| + |\boldsymbol{Q}(\boldsymbol{x}, \boldsymbol{Z}, \boldsymbol{\epsilon}) - \boldsymbol{Q}(\boldsymbol{x}, \boldsymbol{0}, \boldsymbol{\epsilon})|\,. \end{aligned} \tag{11}$$

From the properties of $\boldsymbol{F}$ and $\boldsymbol{G}$ on p18 in [4] and $\boldsymbol{\theta}'(\boldsymbol{0}, \boldsymbol{0}) = \boldsymbol{0}$, we have

$$|\boldsymbol{Q}(\boldsymbol{x}, \boldsymbol{Z}, \boldsymbol{\epsilon}) - \boldsymbol{Q}(\boldsymbol{x}, \boldsymbol{0}, \boldsymbol{\epsilon})| \le k(\delta)|\boldsymbol{Z}| \quad \text{for} \quad |\boldsymbol{Z}| \le \delta\,. \tag{12}$$

[1] $K(x^q, \epsilon^p)$ denotes the product of K and sum of finite terms $c\epsilon_1^{p_1} \cdots \epsilon_l^{p_l} x_1^{q_1} \cdots x_m^{q_m}$, where $p_1 + \cdots + p_l \ge p$ or $q_1 + \cdots + q_m \ge q$ and p_i,$q_j \ge 0$, c is a non-zero constant.

Using (10), (11) and (12),

$$|\boldsymbol{Q}(\boldsymbol{x},\boldsymbol{Z},\boldsymbol{\epsilon})| \le (C_1+Kk(\delta))(x^q,\epsilon^p)\,, \quad \text{for} \quad \boldsymbol{Z}\in Y\,. \tag{13}$$

Using the same calculations as (2.5.9) on p27 [4], for each $r>0$, there is a constant $M(r)$ such that

$$|\boldsymbol{x}(t,\boldsymbol{x}_0,\boldsymbol{\epsilon})| \le M(r)|\boldsymbol{x}_0|e^{-\gamma t}\,, \quad t\le 0 \tag{14}$$

where $\gamma = r+2M(r)k(\delta)$ and $\boldsymbol{x}(t,\boldsymbol{x}_0,\boldsymbol{\epsilon})$ is the solution of

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}+\boldsymbol{F}(\boldsymbol{x},\boldsymbol{Z}(\boldsymbol{x},\boldsymbol{\epsilon})+\boldsymbol{\theta}(\boldsymbol{x},\boldsymbol{\epsilon}),\boldsymbol{\epsilon})\,, \quad \boldsymbol{x}(0,\boldsymbol{x}_0,\boldsymbol{\epsilon})=\boldsymbol{x}_0\,.$$

Using (2.3.6) on p18 and (2.5.3) on p26 in [4], and (13), (14), if $\boldsymbol{Z}\in Y$

$$\begin{aligned} |(S\boldsymbol{Z})(\boldsymbol{x}_0,\boldsymbol{\epsilon}) &\le \left|\int_{-\infty}^0 e^{-Bs}(C_1+Kk(\delta))(x^q,\epsilon^p)\,ds\right| \\ &\le \left|\int_{-\infty}^0 e^{-Bs}(C_1+Kk(\delta))M(r)^q e^{-q\gamma s}(x_0^q,\epsilon^p)\,ds\right| \\ &\le C(C_1+Kk(\delta))M(r)^q(\beta-\gamma q)^{-1}(x_0^q,\epsilon^p)\,. \end{aligned}$$

provided δ and r small enough so that $\beta-\gamma q>0$. Choose $\boldsymbol{O}$ and δ small enough and K large enough such that $K\ge C(C_1+Kk(\delta))M(r)^q(\beta-\gamma q)^{-1}$. Therefore $S:Y\to Y$. Hence Theorem 1 holds. ♠

More general theorems allowing varying orders of truncations within parameters and dynamical variables may be also useful. However, most such cases can be easily established by simple nonlinear transformations of the parameters along the same lines as the example in (5).

In applications, such as many fluid dynamics problems, we need theory not only dealing with infinite dimensional problems, but also infinite dimensional centre manifolds. Carr [6] presented the corresponding results for infinite dimensional problems, and analysed two problems arising from partial differential equations for finite dimensional centre manifolds. The restrictions upon the nonlinear terms $\boldsymbol{f}$ is that $\boldsymbol{f}$ has order 2 continuous derivative and $\boldsymbol{f}(\boldsymbol{0})=\boldsymbol{f}'(\boldsymbol{0})=\boldsymbol{0}$. More recently, Gallay [11] gave an extension of the existence theorem to infinite dimensional centre manifolds, but a bounded restriction on the nonlinear terms is required. This condition limits its rigorous application. Scarpellini [36] apparently places significantly less restrictions upon the nonlinearities in the dynamical equations, but while he addresses infinite dimensional centre manifolds, the results are severely constrained by requiring finite dimensional stable dynamics. Hărăguş [14, 15] has developed theory supporting infinite dimensional models, such as the Korteweg-de Vries equation, but only by placing extreme restrictions upon the linear operators. We identify the extension of the theorems to infinite dimensional centre manifolds as a significant problem for future research.

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