HIGHER ORDER REASONING PRODUCED IN PROOF CONSTRUCTION: HOW WELL DO SECONDARY SCHOOL STUDENTS EXPLAIN AND WRITE MATHEMATICAL PROOFS?

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The culture of mathematical explanations and writings based on conceptual understanding in proof construction is on the focus of the paper. We explore students' attempts to explain construction of mathematical proofs after reading them and write mathematical proofs after working out their own constructions. Two examples of proofs, by induction and by contradiction, are discussed in detail to highlight students' difficulties in proving and possible ways for their resolving.

INTRODUCTION

Despite a consensus on the importance of proof in any mathematically related activities, from the children's first logical reasoning in primary school to mathematicians' research work, its role in the teaching and learning of mathematics, in particular secondary mathematics, has traditionally been neglected in curricula documents for long time. However, recently this situation has changed dramatically. Probably the most demonstrative formal evidence took place in the U.S., where the status of proof has been significantly elevated in the Standards document (NTCM, 2000) with respect to the previous one (NTCM, 1989). Proof has also received a much more prominent role throughout the entire school mathematics curriculum. Evidence of similar actions can be also seen in many other countries throughout the world. The conception of proof seems to be a bridge that connects mathematical research work and teaching of mathematics. Metaphors on the role of proof in mathematics that directly relate to mathematics education (Hanna, 2000; Hanna and Barbeau, 2008; Manin, 1992; Rav, 1999) emphasise the importance of the teaching of proof in school mathematics. Reviews of research on the teaching and learning of proof (Battista & Clements, 1992; Tall, 1991; Yackel & Hanna, 2003) have informed and inspired more recent studies of proof and proving in mathematics education. Nevertheless this area is still not being developed to its maximum potential, and still not enough is known about how students can best be taught proof and proving skills. In one of the latest surveys on the teaching and learning of proof (Harel & Sowder, 2007) the authors stated that

> overall, the performance of students at the secondary and undergraduate levels of proof is weak… Whether the cause lies in the curriculum, the textbooks, the instruction, the teachers' background, or the students themselves, it is clear that the status quo needs, and has needed, improvement. (p.806)

This paper is an attempt to investigate the links between students' abilities in proof construction and their conceptual understanding of mathematical content they deal with. The paper is divided into two parts: the first part elaborates a theoretical model based on Weber's idea (2005) to consider proof construction as a problem-solving task, and the second part presents examples of proofs produced by secondary school students as well as examples of proofs proposed to the same group of students to work on them; and discusses the influence they [examples] may have upon development of students' conceptual understanding and structural knowledge.

ABOUT THE THEORETICAL MODEL OF PROOF CONSTRUCTION

Hanna (1995) emphasised that

> the most important challenge to mathematics educators in the context of proof is to enhance its role in the classroom by finding more effective ways of using it as a vehicle to promote mathematical understanding. (p.42)

We address this challenge in specific conditions, where secondary students possess higher order mathematical thinking and reasoning. We consider these questions with respect to a special group of students, who, for several years, were invited to sit Australian Mathematical Olympiad, which is the highest level of mathematics competitions for school students in Australia. Most high-profile students regularly participate in numerous mathematical competitions and, for them to achieve the best results, their training should be grounded on a comprehensive theoretical base, where the role of proof and proving hardly can be underestimated. In this paper we explore students' attempts to explain construction of mathematical proofs after reading them and write mathematical proofs after working out their own construction. Mathematical reading provides a challenge to understand a text and work up a strategy resolving a given task (Mamona-Downs & Downs, 2005). Mathematical explanations are used to highlight a more general approach that can be applied and elaborated beyond a given task, e.g. to check writing of student's own proof as well as reading of the given proofs. Mathematical explanations allow the reorganisation of the activity of proof construction according to functions of proof (Balacheff, 1988; Bell, 1976; de Villiers, 1990, 1999; Hanna, 1990; Hanna & Jahnke, 1996; Hersh, 1993). Hanna noted (2000) that even for practising mathematicians understanding is more important than rigorous proof, i.e. "they see proofs as primarily conceptual, with the specific technical approach being secondary" (p.7). We consider the mentioned above group of students as potentially prospective candidates, at least some of them, to become professional mathematicians in the future. Therefore, we understand the role of proof in work with gifted students as transitional from the teaching and learning mathematics, at the one hand, to inquiry work in mathematics, at the other hand, i.e. the role which combine both kinds of activities. To analyse this we use a method of simultaneous investigation of both: (1) influence, which proof construction in common, and specific examples in particular, may have upon development of students' abilities to understand proofs in the proper way, and (2) perception of proving process by individuals, which may or may not contribute towards conceptual understanding of mathematical content. We call this method a model of mutual convergence, keeping in mind that mutual impact of both components of the method on each other requires further clarification.

According to Weber (ibid.) proof construction is a mathematical task in which a desired conclusion can be deduced from some initial information (assumptions, axioms, definitions) by applying rules of inferences (theorems, previously established facts, etc). Weber (2001) noted that there are dozens of valid inferences in most proving situations, but only a small number of these inferences can be useful in constructing a proof. Our special interest was analysis of the situations in proof construction, where students didn't know how to proceed (in the sense of both kinds of activities, students' mathematical reading with explanations that followed and their own attempts in proving, including writing). The hypothesis was in existence of non-linear complicated dependence between (1) and (2), which under certain conditions may lead to a significant extension of learning opportunities (Weber, 2005) affordable for students as a result of proof construction.

ANALYSIS OF SOME EXAMPLES AND METHODS USED IN PROOF CONSTRUCTION

Below we present two examples of proof construction and discuss them with respect to students' explanations either on the base of their reading or writing. We use Weber and Alcock (2004) terminology of procedural, syntactic and semantic proof productions as components of proof construction.

Proof by mathematical induction

Mathematical induction is an important part of knowledge on proof construction. Many students perceive mathematical induction as a procedural proof production. We observed no difficulties in students' work with direct proofs. Therefore, mostly we focused on the situation, where the procedural or syntactic part of proof was completed, but proof itself wasn't. The following extract (as mathematical reading activity) proposed to students to get their views and explanations, gives a good example of the case. Text in bold italic was unavailable for students.

Example 1 (Euler)

Prove that for each positive integer $n \geq 3$, a number 2^n can be represented as $2^n = 7x^2 + y^2$ where x and y are both odd numbers.

Proof

The beginning of this proof is syntactic.

We prove this statement by induction. For $n = 3$ it is true. Assume that the property is true for a certain *n*, i.e. $2^n = 7x^2 + y^2$, where x and y are both odd numbers.

Semantic part of proof begins here. Direct application of induction doesn't work and informal interpretation of the components of inductive process needs to be done.

Then, for pairs

$\left\{A=\frac{1}{2}(x-y),\ B=\frac{1}{2}(7x+y)\right\}$ and $\left\{C=\frac{1}{2}(x+y),\ D=\frac{1}{2}(7x-y)\right\}$ we have

$2^{n+1}=7A^2+B^2$ and $2^{n+1}=7C^2+D^2$, respectively.

The first gap within semantic part of proof is below. Since it depends on understanding of a certain concept or theorem and may lead to (in)correct application in the construction of proof we call this gap as a conceptual one.

A and B are either odd or even simultaneously. Indeed, if $A=\frac{1}{2}(x-y)=l$ is odd,

then $B=\frac{1}{2}(7x+y)=\frac{1}{2}(7y+14l+y)=4y+7l$ must be odd. If A is even, then B is even, respectively. The same property is valid for C and D.

Another conceptual gap follows.

Moreover, if $A=\frac{1}{2}(x-y)$ is odd, then $C=\frac{1}{2}(x+y)$ is even, and vice versa. This means that both numbers are odd in one of the pairs Q.E.D.

Our observations show that students may fail to provide explanations of proof construction because of limited understanding of the relationships between mathematical objects involved.

Proof by contradiction

Proof by contradiction is a complex activity, where students may experience significant difficulties. The following example was supposed for students' own attempts to construct a proof and provide explanations in writing.

Example 2

Natural numbers from 1 to 99 (not necessarily distinct) are written on 99 cards. It is given that the sum of the numbers on any subset of cards (including the set of all cards) is not divisible by 100. Prove that all the cards contain the same number.

Analysis of Example 2 and students' writings

The first part of proof construction (syntactic one) is easy to follow – to assume the opposite, which means that at least two cards contain distinct numbers, e.g. $n_{98}\neq n_{99}$ using standard notation, where n_i is a number written on the card i. The next step is to identify and apply a method (technique) that leads to a contradiction. The main idea of the semantic part of proof is to investigate different remainders x_i of $n_1+n_2+...+n_i$ upon division by 100, which guarantees the result that all x_i must be distinct for $i=1,2,\dots,99$. After that, making comparison of the sum $n_1+n_2+\cdots+n_{97}+n_{99}$ (just one of the two distinct numbers needs to be omitted) with another sum having the same remainder (conceptual gap) gives three possible results, each of which leads to a contradiction.

Our observations show that students may have difficulty with their own approach and explanations of proof construction due to lack of understanding of which mathematical objects can be used. Consequently, some invalid conceptual gaps (we call them pseudo-conceptual gaps) within semantic part of proof may appear in writing. It leads to the vague construction of a proof, where actual information about mathematical objects may be replaced with desirable property.

CONCLUDING REMARKS

We observed that in writing their own explanations on proof construction students are more aware about the gaps between different parts of proof, i.e. syntactic and semantic ones, than in the case of explanations based on reading. It can be connected with students' perception of mathematical reading as more instructional and prescriptive part of learning activities than writing. At the same time representation of formal mathematical concepts as components of proof makes reading more beneficial than writing, if students can identify some conceptual gaps properly (those gaps that often constitute the style and culture of formal mathematical texts used in textbooks and monographs). We suggest that focusing teacher's actions on such transitional and conceptual gaps within proof construction will influence the ways in which students attempt to construct proofs. In other words, transitions between different parts of proof in Weber's terms together with local components of semantic part of proof are the places, where significant learning potential can be accumulated. It may lead to further positive impact on development of conceptual understanding and optimization of learning process in the context of proof construction.

REFERENCES

Balacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics. In D. Pimm (Ed.), *Mathematics, teachers and children*, (pp. 216–235). London: Holdder & Stoughton.

Battista, M., & Clements, D. (1992). Geometry and spatial reasoning. In D. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning*, (pp. 420–464). New York: Macmillan.

Bell, A. (1976). A study of pupils' proof-explanations in mathematical situations. *Educational Studies in Mathematics*, 7, 23–40.

de Villiers, M. D. (1990). The role and function of proof in mathematics. *Pythagoras*, 24, 17–24.

de Villiers, M. D. (1999). *Rethinking proof with the Geometer's Sketchpad*. Emeryville, CA: Key Curriculum Press.

Hanna, G. (1990). Some pedagogical aspects of proof. *Interchange*, 21, 6–13.

Hanna, G. (1995). Challenges to the importance of proof. *For the Learning of Mathematics*, 15, 42–49.

Hanna, G. (2000). Proof, explanation and exploration: An overview. *Educational Studies in Mathematics*, 44(1), 5–23.

Hanna, G., & Barbeau, E. (2008). Proofs as bearers of mathematical knowledge. *ZDM The International Journal on Mathematics Education*.

Hanna, G., & Jahnke, H. N. (1995). Proof and proving. In A. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International Handbook of Mathematics Education*, (Part 2, pp. 877–908). Dordrecht, The Netherlands: Kluwer.

Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In F. K. Lester, Jr. (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning*, (pp. 805–842). NCTM, Information Age Publishing Inc.

Hersh, R. (1993). Proving is convincing and explaining. *Educational Studies in Mathematics*, 24(4), 389–399.

Mamona-Downs, J., & Downs, M. (2005). The identity of problem solving. *Journal of Mathematical Behaviour*, In Special Issue: Mathematical problem solving: What we know and where we are going, v.24, 385–401.

Manin, Y. (1992). The theory and practice of proof. *Proceedings of the 7th International Congress on Mathematical Education*. Montreal, Canada.

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: NCTM.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Rav, Y. (1999). Why do we prove theorems? *Philosophia Mathematica*, 7, 5–41.

Tall, D. (Ed.) (1991). *Advanced mathematical thinking*. Dordrecht, The Netherlands: Kluwer.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics*, 48(1), 101–119.

Weber, K. (2005). Problem-solving, proving, and learning: The relationship between problem-solving processes and learning opportunities in the activity of proof construction. *Journal of Mathematical Behaviour*, In Special Issue: Mathematical problem solving: What we know and where we are going, v.24, 351–360.

Weber, K. & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics*, 56, 209–234.

Yackel, E., & Hanna, G. (2003). Reasoning and proof. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to Principles and standards for school mathematics*, (pp. 227–236). Reston, VA: NCTM.